Lecture 1
Introduction and overview

- linear programming
- example
- course topics
- software
- integer linear programming

Linear program (LP)

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\sum_{j=1}^{n} c_j x_j \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
& \displaystyle\sum_{j=1}^{n} c_{ij} x_j = d_i, \quad i = 1, \ldots, p
\end{array}
$$

variables: x_j
problem data: the coefficients c_j, a_{ij}, b_i, c_{ij}, d_i

- can be solved very efficiently (several 10,000 variables, constraints)
- widely available general-purpose software
- extensive, useful theory (optimality conditions, sensitivity analysis, . . .)

Example. Open-loop control problem

single-input/single-output system (with input u, output y)

$$y(t) = h_0 u(t) + h_1 u(t-1) + h_2 u(t-2) + h_3 u(t-3) + \cdots$$

output tracking problem: minimize deviation from desired output $y_{\mathrm{des}}(t)$

$$\max_{t=0,\ldots,N} |y(t) - y_{\mathrm{des}}(t)|$$

subject to input amplitude and slew rate constraints:

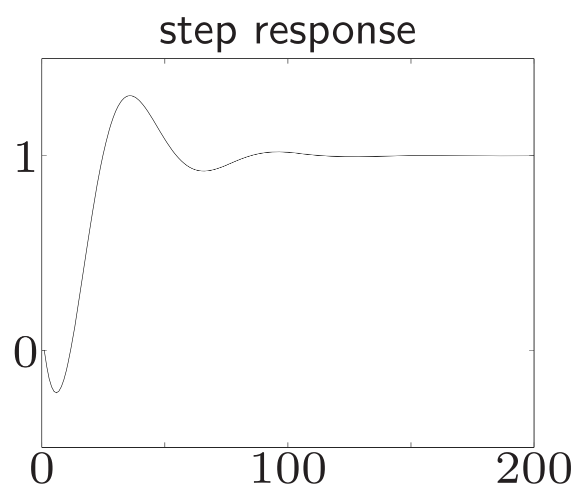
$$|u(t)| \leq U, \qquad |u(t+1) - u(t)| \leq S$$

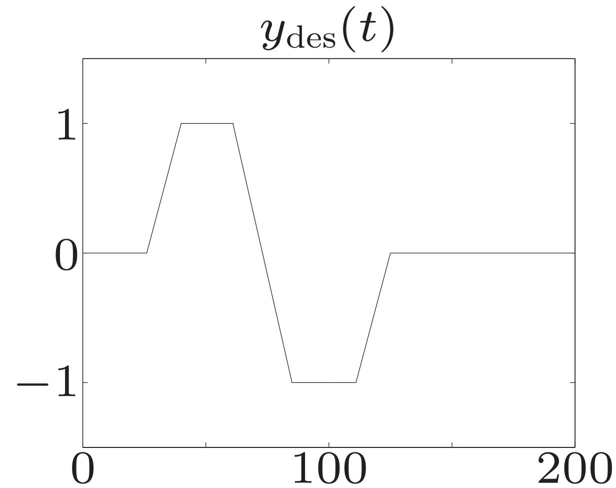
variables: $u(0)$, . . . , $u(M)$ (with $u(t) = 0$ for $t < 0$, $t > M$)

solution: can be formulated as an LP, hence easily solved (more later)

example

step response ($s(t) = h_t + \cdots + h_0$) and desired output:

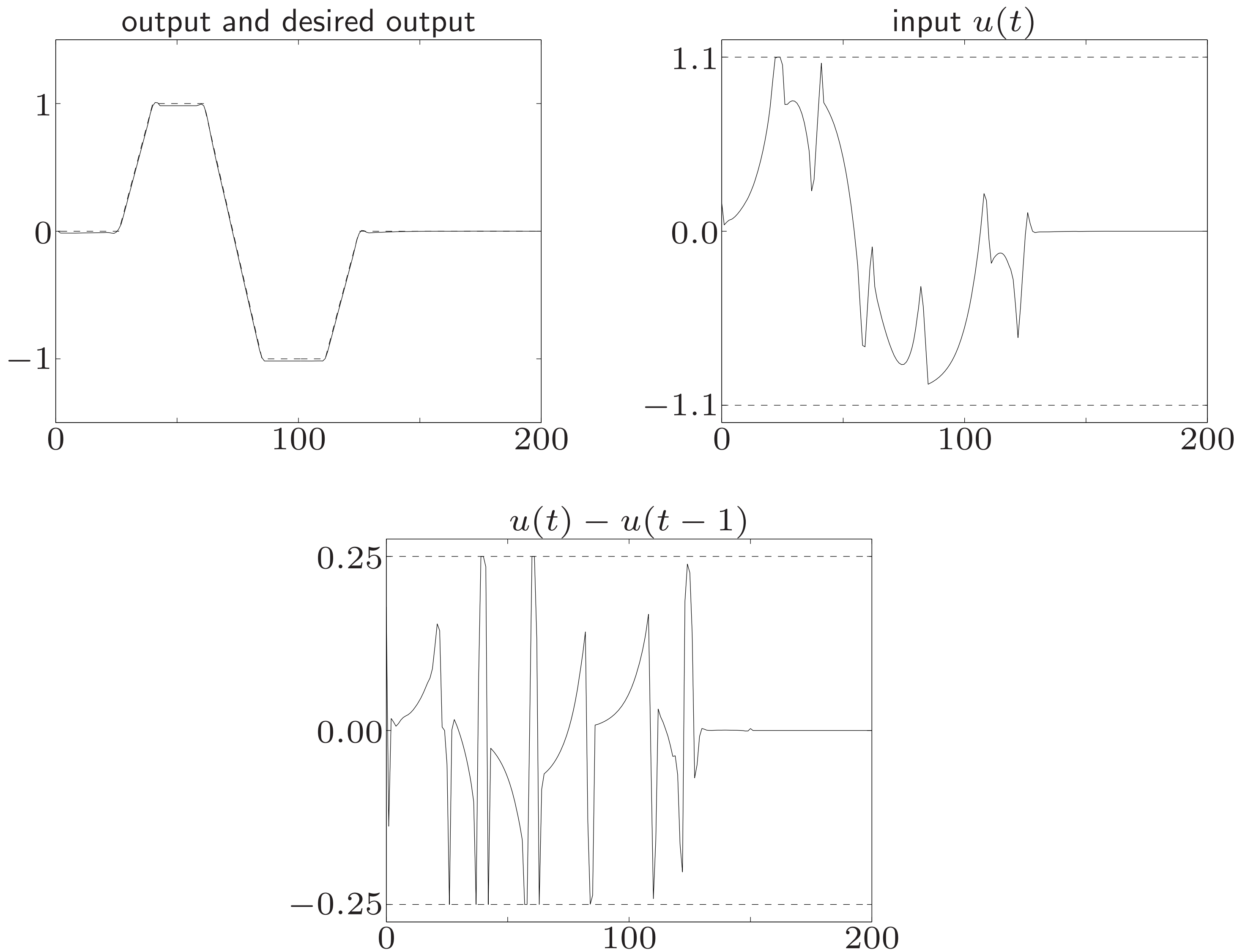




amplitude and slew rate constraint on u:

$$|u(t)| \leq 1.1, \qquad |u(t) - u(t-1)| \leq 0.25$$

optimal solution



Brief history

- **1930s** (Kantorovich): economic applications
- **1940s** (Dantzig): military logistics problems during WW2;
  1947: simplex algorithm
- **1950s–60s** discovery of applications in many other fields (structural optimization, control theory, filter design, . . .)
- **1979** (Khachiyan) ellipsoid algorithm: more efficient (polynomial-time) than simplex in worst case, but slower in practice
- **1984** (Karmarkar): projective (interior-point) algorithm: polynomial-time worst-case complexity, and efficient in practice
- **1984–today**. many variations of interior-point methods (improved complexity or efficiency in practice), software for large-scale problems

Course outline

the linear programming problem
linear inequalities, geometry of linear programming

engineering applications
signal processing, control, structural optimization . . .

duality

algorithms
the simplex algorithm, interior-point algorithms

large-scale linear programming and network optimization
techniques for LPs with special structure, network flow problems

integer linear programming
introduction, some basic techniques

Software

solvers: solve LPs described in some standard form

modeling tools: accept a problem in a simpler, more intuitive, notation and convert it to the standard form required by solvers

software for this course (see class website)

- platforms: Matlab, Octave, Python
- solvers: `linprog` (Matlab Optimization Toolbox),
- modeling tools: CVX (Matlab), YALMIP (Matlab),
- Thanks to Lieven Vandenberghe at UCLA for his slides

Integer linear program

integer linear program

$$
\begin{array}{ll}
\text{minimize} & \sum_{j=1}^n c_j x_j \\
\text{subject to} & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
& \sum_{j=1}^n c_{ij} x_j = d_i, \quad i = 1, \ldots, p \\
& x_j \in \mathbf{Z}
\end{array}
$$

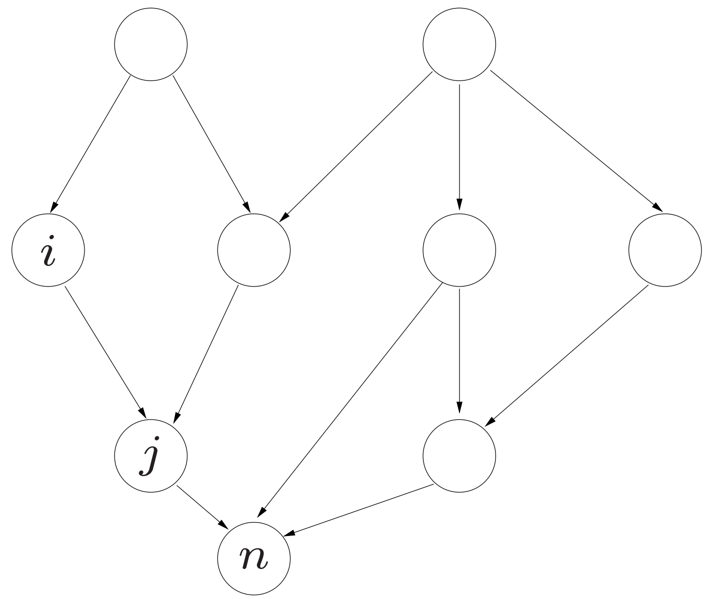
Boolean linear program

$$
\begin{array}{ll}
\text{minimize} & \sum_{j=1}^n c_j x_j \\
\text{subject to} & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
& \sum_{j=1}^n c_{ij} x_j = d_i, \quad i = 1, \ldots, p \\
& x_j \in \{0, 1\}
\end{array}
$$

- very general problems; can be extremely hard to solve
- can be solved as a sequence of linear programs

Example. Scheduling problem

scheduling graph $\mathcal{V}$:



- nodes represent operations ($e.g.$, jobs in a manufacturing process, arithmetic operations in an algorithm)
- $(i, j) \in \mathcal{V}$ means operation j must wait for operation i to be finished
- M identical machines/processors; each operation takes unit time

problem: determine fastest schedule

Boolean linear program formulation

variables: x_{is}, $i = 1, \ldots, n$, $s = 0, \ldots, T$:

$$x_{is} = 1 \text{ if job } i \text{ starts at time } s\ , \qquad x_{is} = 0 \text{ otherwise}$$

constraints:

1. $x_{is} \in \{0, 1\}$

2. job i starts exactly once:
$$\sum_{s=0}^{T} x_{is} = 1$$

3. if there is an arc (i, j) in $\mathcal{V}$, then
$$\sum_{s=0}^{T} s x_{js} - \sum_{s=0}^{T} s x_{is} \geq 1$$

4. limit on capacity (M machines) at time s:

$$\sum_{i=1}^{n} x_{is} \leq M$$

cost function (start time of job n):

$$\sum_{s=0}^{T} s x_{ns}$$

Boolean linear program

$$\begin{array}{ll}
\text{minimize} & \sum_{s=0}^{T} s x_{ns} \\
\text{subject to} & \sum_{s=0}^{T} x_{is} = 1, \quad i = 1, \ldots, n \\
& \sum_{s=0}^{T} s x_{js} - \sum_{s=0}^{T} s x_{is} \geq 1, \quad (i, j) \in \mathcal{V} \\
& \sum_{i=1}^{n} x_{is} \leq M, \quad s = 0, \ldots, T \\
& x_{is} \in \{0, 1\}, \quad i = 1, \ldots, n, \quad s = 0, \ldots, T
\end{array}$$

Lecture 2
Linear inequalities

- vectors
- inner products and norms
- linear equalities and hyperplanes
- linear inequalities and halfspaces
- polyhedra

Vectors

(column) vector $x \in \mathbf{R}^n$:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- $x_i \in \mathbf{R}$: ith *component* or *element* of x
- also written as $x = (x_1, x_2, \ldots, x_n)$

some special vectors:

- $x = 0$ (*zero vector*): $x_i = 0$, $i = 1, \ldots, n$
- $x = \mathbf{1}$: $x_i = 1$, $i = 1, \ldots, n$
- $x = e_i$ (ith *basis vector* or ith *unit vector*): $x_i = 1$, $x_k = 0$ for $k \neq i$

(n follows from context)

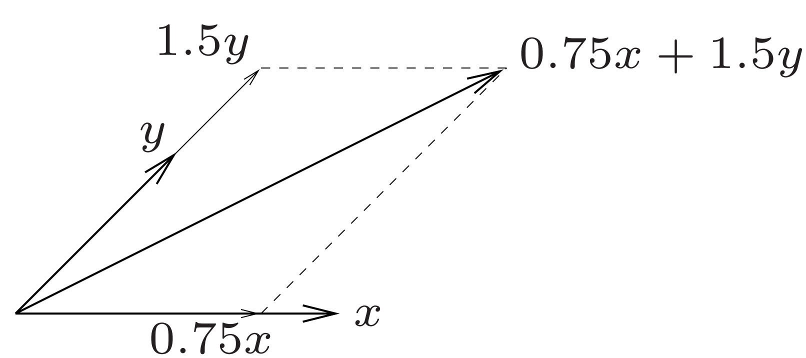
Vector operations

multiplying a vector $x \in \mathbf{R}^n$ with a scalar $\alpha \in \mathbf{R}$:

$$\alpha x = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

adding and subtracting two vectors x, $y \in \mathbf{R}^n$:

$$x + y = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}, \qquad x - y = \begin{bmatrix} x_1 - y_1 \\ \vdots \\ x_n - y_n \end{bmatrix}$$



Inner product

x, $y \in \mathbf{R}^n$

$$\langle x, y \rangle := x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = x^T y$$

important properties

- $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle x, x \rangle \geq 0$
- $\langle x, x \rangle = 0 \iff x = 0$

linear function: $f : \mathbf{R}^n \to \mathbf{R}$ is linear, *i.e.*

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y),$$

if and only if $f(x) = \langle a, x \rangle$ for some a

Euclidean norm

for $x \in \mathbf{R}^n$ we define the (Euclidean) norm as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^T x}$$

$\|x\|$ measures *length* of vector (from origin)

important properties:

- $\|\alpha x\| = |\alpha| \|x\|$ (homogeneity)
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)
- $\|x\| \geq 0$ (nonnegativity)
- $\|x\| = 0 \iff x = 0$ (definiteness)

distance between vectors: $\mathbf{dist}(x, y) = \|x - y\|$

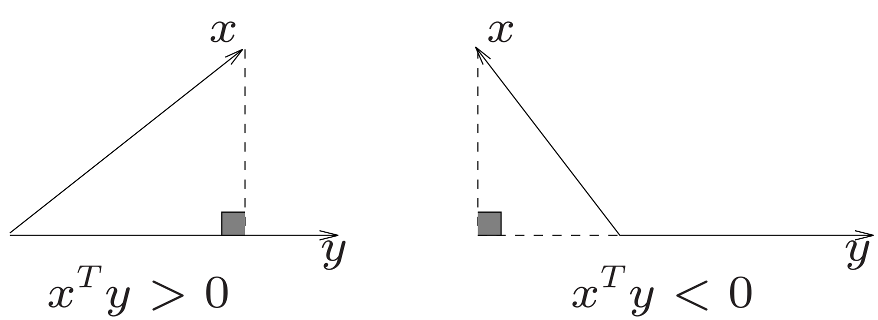
Inner products and angles

angle between vectors in $\mathbf{R}^n$:

$$\theta = \angle(x, y) = \cos^{-1} \frac{x^T y}{\|x\| \|y\|}$$

i.e., $x^T y = \|x\| \|y\| \cos\theta$

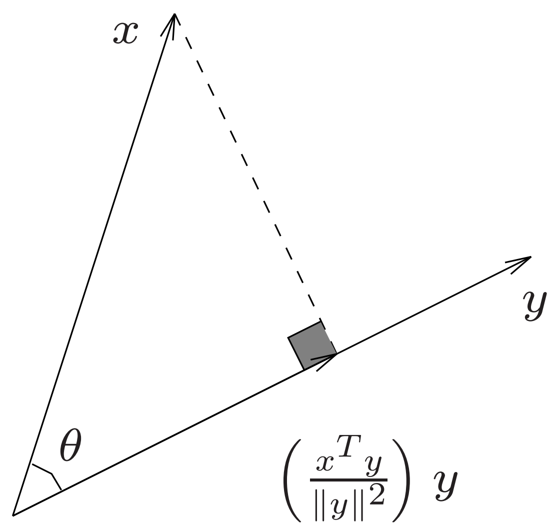
- x and y *aligned*: $\theta = 0$; $x^T y = \|x\| \|y\|$
- x and y *opposed*: $\theta = \pi$; $x^T y = -\|x\| \|y\|$
- x and y *orthogonal*: $\theta = \pi/2$ or $-\pi/2$; $x^T y = 0$ (denoted $x \perp y$)
- $x^T y > 0$ means $\angle(x, y)$ is acute; $x^T y < 0$ means $\angle(x, y)$ is obtuse



Cauchy-Schwarz inequality:

$$|x^T y| \leq \|x\| \|y\|$$

projection of x on y



projection is given by

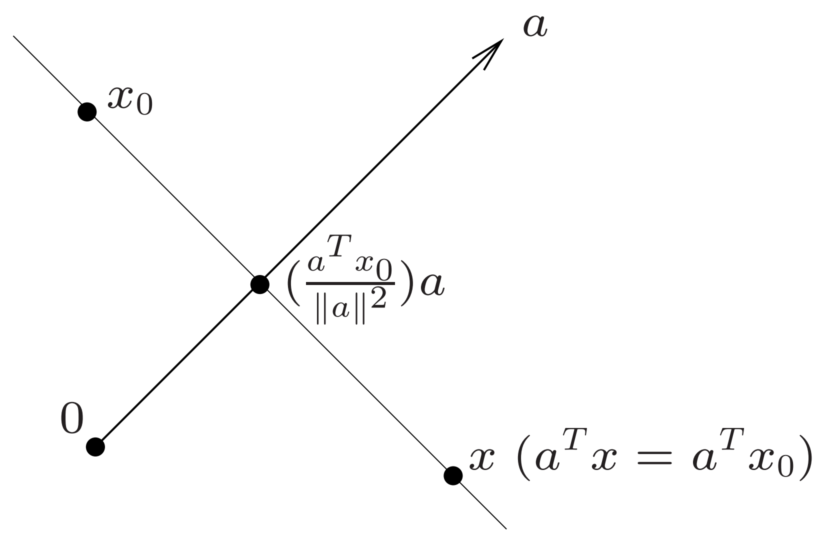
$$\left(\frac{x^T y}{\|y\|^2}\right) y$$

Hyperplanes

hyperplane in $\mathbf{R}^n$:

$$\{x \mid a^T x = b\} \quad (a \neq 0)$$

- solution set of one linear equation $a_1x_1 + \cdots + a_nx_n = b$ with at least one $a_i \neq 0$
- set of vectors that make a constant inner product with vector $a = (a_1, \ldots, a_n)$ (the *normal* vector)



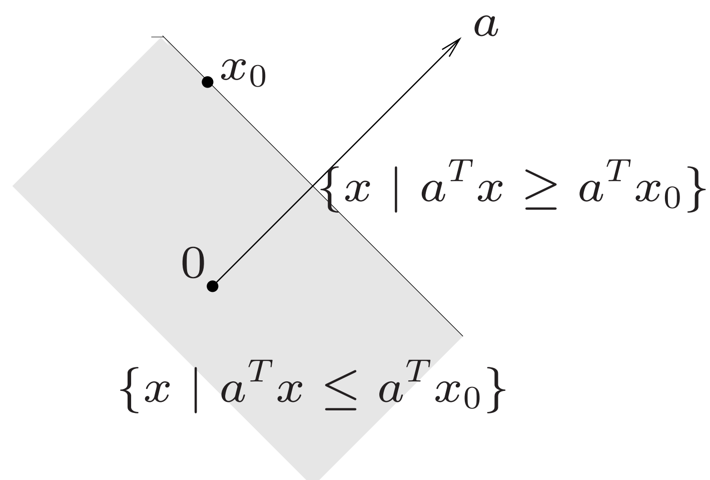
in $\mathbf{R}^2$: a line, in $\mathbf{R}^3$: a plane, . . .

Halfspaces

(closed) halfspace in $\mathbf{R}^n$:

$$\{x \mid a^T x \leq b\} \quad (a \neq 0)$$

- solution set of one linear inequality $a_1 x_1 + \cdots + a_n x_n \leq b$ with at least one $a_i \neq 0$
- $a = (a_1, \ldots, a_n)$ is the (outward) *normal*



- $\{x \mid a^T x < b\}$ is called an *open* halfspace

Affine sets

solution set of a set of linear equations

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &=& b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &=& b_1 \\ &\vdots& \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &=& b_m \end{array}$$

intersection of m hyperplanes with normal vectors $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ (w.l.o.g., all $a_i \neq 0$)

in matrix notation:

$$Ax = b$$

with

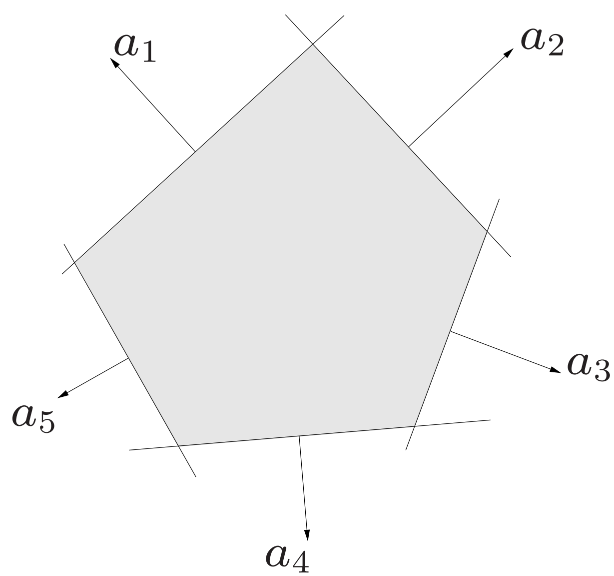
$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array} \right], \qquad b = \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_m \end{array} \right]$$

Polyhedra

solution set of system of linear inequalities

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \leq & b_1 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \leq & b_m \end{array}$$

intersection of m halfspaces, with normal vectors $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ (w.l.o.g., all $a_i \neq 0$)



matrix notation

$$Ax \leq b$$

with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$Ax \leq b$ stands for *componentwise* inequality, *i.e.*, for y, $z \in \mathbf{R}^n$,

$$y \leq z \quad \Longleftrightarrow \quad y_1 \leq z_1, \ldots, y_n \leq z_n$$

Examples of polyhedra

- a hyperplane $\{x \mid a^T x = b\}$:

$$a^T x \leq b, \qquad a^T x \geq b$$

- solution set of system of linear equations/inequalities

$$a_i^T x \leq b_i, \quad i = 1, \ldots, m, \qquad c_i^T x = d_i, \quad i = 1, \ldots, p$$

- a slab $\{x \mid b_1 \leq a^T x \leq b_2\}$
- the probability simplex $\{x \in \mathbf{R}^n \mid \mathbf{1}^T x = 1,\ \ x_i \geq 0,\ i = 1, \ldots, n\}$
- (hyper)rectangle $\{x \in \mathbf{R}^n \mid l \leq x \leq u\}$ where $l < u$

Lecture 3
Geometry of linear programming

- subspaces and affine sets, independent vectors
- matrices, range and nullspace, rank, inverse
- polyhedron in inequality form
- extreme points
- degeneracy
- the optimal set of a linear program

Subspaces

$\mathcal{S} \subseteq \mathbf{R}^n$ ($\mathcal{S} \neq \emptyset$) is called a *subspace* if

$$x, y \in \mathcal{S}, \quad \alpha, \beta \in \mathbf{R} \quad \Longrightarrow \quad \alpha x + \beta y \in \mathcal{S}$$

$\alpha x + \beta y$ is called a *linear combination* of x and y

examples (in $\mathbf{R}^n$)

- $\mathcal{S} = \mathbf{R}^n$, $\mathcal{S} = \{0\}$
- $\mathcal{S} = \{\alpha v \mid \alpha \in \mathbf{R}\}$ where $v \in \mathbf{R}^n$ ($i.e.$, a line through the origin)
- $\mathcal{S} = \mathrm{span}(v_1, v_2, \ldots, v_k) = \{\alpha_1 v_1 + \cdots + \alpha_k v_k \mid \alpha_i \in \mathbf{R}\}$, where $v_i \in \mathbf{R}^n$
- set of vectors orthogonal to given vectors $v_1, \ldots, v_k$:

$$\mathcal{S} = \{x \in \mathbf{R}^n \mid v_1^T x = 0, \ldots, v_k^T x = 0\}$$

Independent vectors

vectors $v_1, v_2, \ldots, v_k$ are *independent* if and only if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0 \quad \Longrightarrow \quad \alpha_1 = \alpha_2 = \cdots = 0$$

some equivalent conditions:

- coefficients of $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$ are uniquely determined, $i.e.$,

  $$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k$$

  implies $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_k = \beta_k$

- no vector v_i can be expressed as a linear combination of the other vectors $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k$

Basis and dimension

$\{v_1, v_2, \ldots, v_k\}$ is a *basis* for a subspace $\mathcal{S}$ if

- $v_1, v_2, \ldots, v_k$ span $\mathcal{S}$, *i.e.*, $\mathcal{S} = \mathrm{span}(v_1, v_2, \ldots, v_k)$
- $v_1, v_2, \ldots, v_k$ are independent

equivalently: every $v \in \mathcal{S}$ can be uniquely expressed as

$$v = \alpha_1 v_1 + \cdots + \alpha_k v_k$$

fact: for a given subspace $\mathcal{S}$, the number of vectors in any basis is the same, and is called the *dimension* of $\mathcal{S}$, denoted $\dim \mathcal{S}$

Affine sets

$\mathcal{V} \subseteq \mathbf{R}^n$ ($\mathcal{V} \neq \emptyset$) is called an *affine set* if

$$x, y \in \mathcal{V},\ \alpha + \beta = 1 \quad \Longrightarrow \quad \alpha x + \beta y \in \mathcal{V}$$

$\alpha x + \beta y$ is called an *affine combination* of x and y

examples (in $\mathbf{R}^n$)

- subspaces
- $\mathcal{V} = b + \mathcal{S} = \{x + b \mid x \in \mathcal{S}\}$ where $\mathcal{S}$ is a subspace
- $\mathcal{V} = \{\alpha_1 v_1 + \cdots + \alpha_k v_k \mid \alpha_i \in \mathbf{R}, \sum_i \alpha_i = 1\}$
- $\mathcal{V} = \{x \mid v_1^T x = b_1, \ldots, v_k^T x = b_k\}$ (if $\mathcal{V} \neq \emptyset$)

every affine set $\mathcal{V}$ can be written as $\mathcal{V} = x_0 + \mathcal{S}$ where $x_0 \in \mathbf{R}^n$, $\mathcal{S}$ a subspace ($e.g.$, can take any $x_0 \in \mathcal{V}$, $\mathcal{S} = \mathcal{V} - x_0$)

$\dim(\mathcal{V} - x_0)$ is called the dimension of $\mathcal{V}$

Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbf{R}^{m \times n}$$

some special matrices:

- $A = 0$ (zero matrix): $a_{ij} = 0$
- $A = I$ (identity matrix): $m = n$ and $A_{ii} = 1$ for $i = 1, \ldots, n$, $A_{ij} = 0$ for $i \neq j$
- $A = \mathbf{diag}(x)$ where $x \in \mathbf{R}^n$ (diagonal matrix): $m = n$ and

$$A = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix}$$

Matrix operations

- addition, subtraction, scalar multiplication

- transpose:

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \in \mathbf{R}^{n\times m}$$

- multiplication: $A \in \mathbf{R}^{m\times n}$, $B \in \mathbf{R}^{n\times q}$, $AB \in \mathbf{R}^{m\times q}$:

$$AB = \begin{bmatrix} \sum_{i=1}^n a_{1i}b_{i1} & \sum_{i=1}^n a_{1i}b_{i2} & \cdots & \sum_{i=1}^n a_{1i}b_{iq} \\ \sum_{i=1}^n a_{2i}b_{i1} & \sum_{i=1}^n a_{2i}b_{i2} & \cdots & \sum_{i=1}^n a_{2i}b_{iq} \\ \vdots & \vdots & & \vdots \\ \sum_{i=1}^n a_{mi}b_{i1} & \sum_{i=1}^n a_{mi}b_{i2} & \cdots & \sum_{i=1}^n a_{mi}b_{iq} \end{bmatrix}$$

Rows and columns

rows of $A \in \mathbf{R}^{m\times n}$:

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}$$

with $a_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \in \mathbf{R}^n$

columns of $B \in \mathbf{R}^{n\times q}$:

$$B = \begin{bmatrix} b_1 & b_2 & \cdots & b_q \end{bmatrix}$$

with $b_i = (b_{1i}, b_{2i}, \ldots, b_{ni}) \in \mathbf{R}^n$

for example, can write AB as

$$AB = \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_q \\ a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_q \\ \vdots & \vdots & & \vdots \\ a_m^T b_1 & a_m^T b_2 & \cdots & a_m^T b_q \end{bmatrix}$$

Range of a matrix

the *range* of $A \in \mathbf{R}^{m \times n}$ is defined as

$$\mathcal{R}(A) = \{Ax \mid x \in \mathbf{R}^n\} \subseteq \mathbf{R}^m$$

- a subspace
- set of vectors that can be 'hit' by mapping $y = Ax$
- the span of the columns of $A = [a_1 \ \cdots \ a_n]$

$$\mathcal{R}(A) = \{a_1 x_1 + \cdots + a_n x_n \mid x \in \mathbf{R}^n\}$$

- the set of vectors y s.t. $Ax = y$ has a solution

$\mathcal{R}(A) = \mathbf{R}^m \iff$

- $Ax = y$ can be solved in x for any y
- the columns of A span $\mathbf{R}^m$
- $\dim \mathcal{R}(A) = m$

Interpretations

$v \in \mathcal{R}(A)$, $w \notin \mathcal{R}(A)$

- $y = Ax$ represents output resulting from input x
  - v is a possible result or output
  - w cannot be a result or output

  $\mathcal{R}(A)$ characterizes the *achievable outputs*

- $y = Ax$ represents measurement of x
  - $y = v$ is a *possible* or *consistent* sensor signal
  - $y = w$ is *impossible* or *inconsistent*; sensors have failed or model is wrong

  $\mathcal{R}(A)$ characterizes the *possible results*

Nullspace of a matrix

the *nullspace* of $A \in \mathbf{R}^{m\times n}$ is defined as

$$\mathcal{N}(A) = \{ \, x \in \mathbf{R}^n \mid Ax = 0 \, \}$$

- a subspace
- the set of vectors mapped to zero by $y = Ax$
- the set of vectors orthogonal to all rows of A:

$$\mathcal{N}(A) = \left\{ \, x \in \mathbf{R}^n \mid a_1^T x = \cdots = a_m^T x = 0 \right\}$$

  where $A = [a_1 \ \cdots \ a_m]^T$

zero nullspace: $\mathcal{N}(A) = \{0\} \iff$

- x can always be uniquely determined from $y = Ax$
  ($i.e.$, the linear transformation $y = Ax$ doesn't 'lose' information)
- columns of A are independent

Interpretations

suppose $z \in \mathcal{N}(A)$

- $y = Ax$ represents output resulting from input x
  - z is input with no result
  - x and $x + z$ have same result

  $\mathcal{N}(A)$ characterizes *freedom of input choice* for given result

- $y = Ax$ represents measurement of x
  - z is undetectable — get zero sensor readings
  - x and $x + z$ are indistinguishable: $Ax = A(x + z)$

  $\mathcal{N}(A)$ characterizes *ambiguity* in x from $y = Ax$

Inverse

$A \in \mathbf{R}^{n \times n}$ is *invertible* or *nonsingular* if $\det A \neq 0$

equivalent conditions:

- columns of A are a basis for $\mathbf{R}^n$
- rows of A are a basis for $\mathbf{R}^n$
- $\mathcal{N}(A) = \{0\}$
- $\mathcal{R}(A) = \mathbf{R}^n$
- $y = Ax$ has a unique solution x for every $y \in \mathbf{R}^n$
- A has an inverse $A^{-1} \in \mathbf{R}^{n \times n}$, with $AA^{-1} = A^{-1}A = I$

Rank of a matrix

we define the *rank* of $A \in \mathbf{R}^{m \times n}$ as

$$\mathbf{rank}(A) = \dim \mathcal{R}(A)$$

(nontrivial) facts:

- $\mathbf{rank}(A) = \mathbf{rank}(A^T)$
- $\mathbf{rank}(A)$ is maximum number of independent columns (or rows) of A, hence
$$\mathbf{rank}(A) \leq \min\{m, n\}$$
- $\mathbf{rank}(A) + \dim \mathcal{N}(A) = n$

Full rank matrices

for $A \in \mathbf{R}^{m \times n}$ we have $\mathbf{rank}(A) \leq \min\{m, n\}$

we say A is *full rank* if $\mathbf{rank}(A) = \min\{m, n\}$

- for *square* matrices, full rank means nonsingular
- for *skinny* matrices ($m > n$), full rank means columns are independent
- for *fat* matrices ($m < n$), full rank means rows are independent

Sets of linear equations

$$Ax = y$$

given $A \in \mathbf{R}^{m \times n}$, $y \in \mathbf{R}^m$

- solvable if and only if $y \in \mathcal{R}(A)$
- unique solution if $y \in \mathcal{R}(A)$ and $\mathbf{rank}(A) = n$
- general solution set:

$$\{x_0 + v \mid v \in \mathcal{N}(A)\}$$

  where $Ax_0 = y$

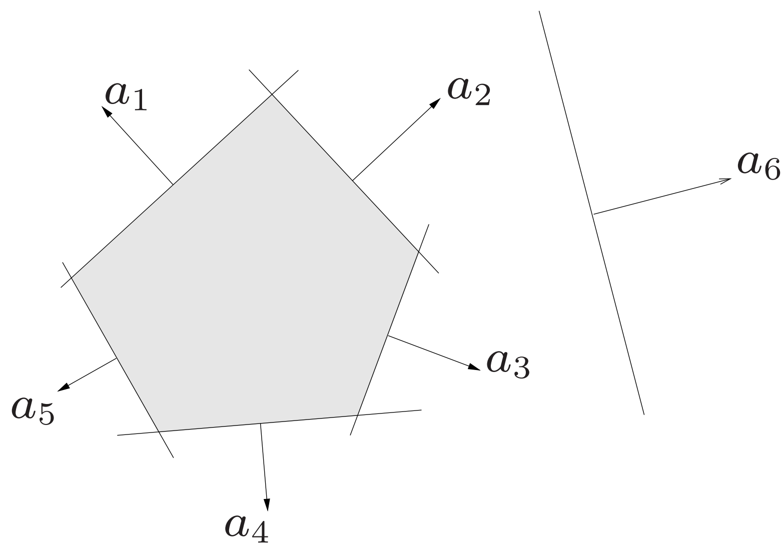
A square and invertible: unique solution for every y:

$$x = A^{-1}y$$

Polyhedron (inequality form)

$A = [a_1 \ \cdots \ a_m]^T \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$

$$\mathcal{P} = \{x \mid Ax \le b\} = \{x \mid a_i^T x \le b_i, \ \ i = 1, \ldots, m\}$$



$\mathcal{P}$ is convex:

$$x, y \in \mathcal{P}, \ \ 0 \le \lambda \le 1 \quad \Longrightarrow \quad \lambda x + (1-\lambda) y \in \mathcal{P}$$

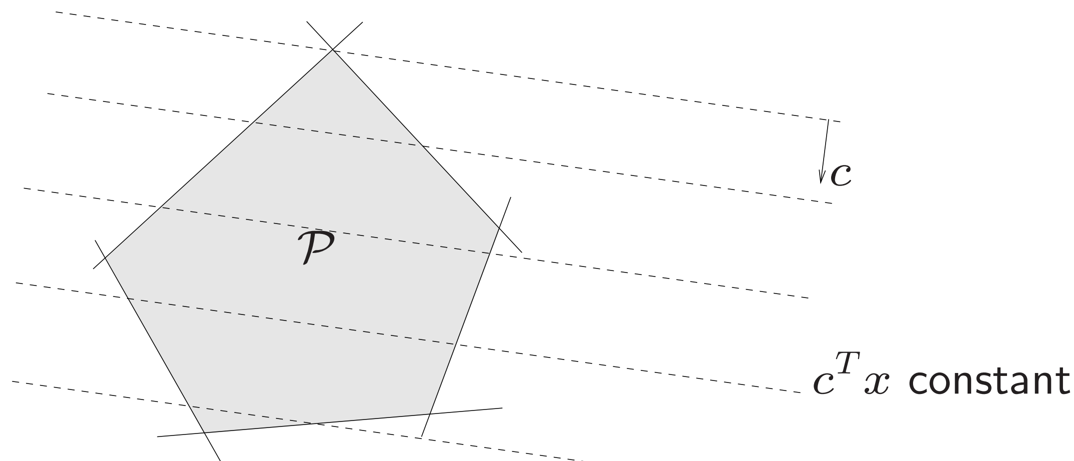
i.e., the *line segment* between any two points in $\mathcal{P}$ lies in $\mathcal{P}$

Extreme points and vertices

$x \in \mathcal{P}$ is an **extreme point** if it cannot be written as

$$x = \lambda y + (1 - \lambda) z$$

with $0 \leq \lambda \leq 1$, $y, z \in \mathcal{P}$, $y \neq x$, $z \neq x$



$x \in \mathcal{P}$ is a **vertex** if there is a c such that $c^T x < c^T y$ for all $y \in \mathcal{P}$, $y \neq x$

fact: x is an extreme point $\Longleftrightarrow$ x is a vertex (proof later)

Basic feasible solution

define I as the set of indices of the *active* or *binding* constraints (at $x^\star$):

$$a_i^T x^\star = b_i, \quad i \in I, \qquad a_i^T x^\star < b_i, \quad i \notin I$$

define $\bar{A}$ as

$$\bar{A} = \begin{bmatrix} a_{i_1}^T \\ a_{i_2}^T \\ \vdots \\ a_{i_k}^T \end{bmatrix}, \qquad I = \{i_1, \ldots, i_k\}$$

$x^\star$ is called a *basic feasible solution* if

$$\mathbf{rank}\, \overline{A} = n$$

fact: $x^\star$ is a vertex (extreme point) $\Longleftrightarrow$ $x^\star$ is a basic feasible solution (proof later)

Example

$$\begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix}$$

- (1,1) is an extreme point
- (1,1) is a vertex: unique minimum of $c^T x$ with $c = (-1, -1)$
- (1,1) is a basic feasible solution: $I = \{2, 4\}$ and $\mathbf{rank}\, \overline{A} = 2$, where

$$\overline{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Equivalence of the three definitions

vertex $\Longrightarrow$ extreme point

let $x^\star$ be a vertex of $\mathcal{P}$, $i.e.$, there is a $c \neq 0$ such that

$$c^T x^\star < c^T x \quad \text{for all } x \in \mathcal{P},\ x \neq x^\star$$

let $y, z \in \mathcal{P}$, $y \neq x^\star$, $z \neq x^\star$:

$$c^T x^\star < c^T y, \qquad c^T x^\star < c^T z$$

so, if $0 \leq \lambda \leq 1$, then

$$c^T x^\star < c^T(\lambda y + (1-\lambda)z)$$

hence $x^\star \neq \lambda y + (1-\lambda)z$

extreme point $\Longrightarrow$ basic feasible solution

suppose $x^\star \in \mathcal{P}$ is an extreme point with

$$a_i^T x^\star = b_i, \quad i \in I, \qquad a_i^T x^\star < b_i, \quad i \notin I$$

suppose $x^\star$ is not a basic feasible solution; then there exists a $d \neq 0$ with

$$a_i^T d = 0, \quad i \in I$$

and for small enough $\epsilon > 0$,

$$y = x^\star + \epsilon d \in \mathcal{P}, \quad z = x^\star - \epsilon d \in \mathcal{P}$$

we have

$$x^\star = 0.5y + 0.5z,$$

which contradicts the assumption that $x^\star$ is an extreme point

basic feasible solution $\Longrightarrow$ vertex

suppose $x^\star \in \mathcal{P}$ is a basic feasible solution and

$$a_i^T x^\star = b_i \quad i \in I, \qquad a_i^T x^\star < b_i \quad i \notin I$$

define $c = -\sum_{i \in I} a_i$; then

$$c^T x^\star = -\sum_{i \in I} b_i$$

and for all $x \in \mathcal{P}$,

$$c^T x \geq -\sum_{i \in I} b_i$$

with equality only if $a_i^T x = b_i$, $i \in I$

however the only solution to $a_i^T x = b_i$, $i \in I$, is $x^\star$; hence $c^T x^\star < c^T x$ for all $x \in \mathcal{P}$

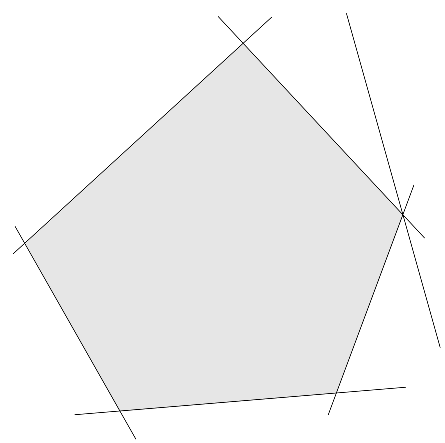
Degeneracy

set of linear inequalities $a_i^T x \leq b_i$, $i = 1, \ldots, m$

a basic feasible solution $x^\star$ with

$$a_i^T x^\star = b_i, \quad i \in I, \qquad a_i^T x^\star < b_i, \quad i \notin I$$

is *degenerate* if $\#$indices in I is greater than n



- a property of the *description* of the polyhedron, not its geometry
- affects the performance of some algorithms
- disappears with small perturbations of b

Unbounded directions

$\mathcal{P}$ contains a **half-line** if there exists $d \neq 0$, x_0 such that

$$x_0 + td \in \mathcal{P} \text{ for all } t \geq 0$$

equivalent condition for $\mathcal{P} = \{x \mid Ax \leq b\}$:

$$Ax_0 \leq b, \quad Ad \leq 0$$

fact: $\mathcal{P}$ unbounded $\Longleftrightarrow$ $\mathcal{P}$ contains a half-line

$\mathcal{P}$ contains a **line** if there exists $d \neq 0$, x_0 such that

$$x_0 + td \in \mathcal{P} \text{ for all } t$$

equivalent condition for $\mathcal{P} = \{x \mid Ax \leq b\}$:

$$Ax_0 \leq b, \quad Ad = 0$$

fact: $\mathcal{P}$ has no extreme points $\Longleftrightarrow$ $\mathcal{P}$ contains a line

Optimal set of an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

- optimal value: $p^\star = \min\{c^T x \mid Ax \leq b\}$ ($p^\star = \pm\infty$ is possible)
- optimal point: $x^\star$ with $Ax^\star \leq b$ and $c^T x^\star = p^\star$
- optimal set: $X_{\text{opt}} = \{x \mid Ax \leq b,\ c^T x = p^\star\}$

example

$$\begin{array}{ll} \text{minimize} & c_1 x_1 + c_2 x_2 \\ \text{subject to} & -2x_1 + x_2 \leq 1 \\ & x_1 \geq 0, \quad x_2 \geq 0 \end{array}$$

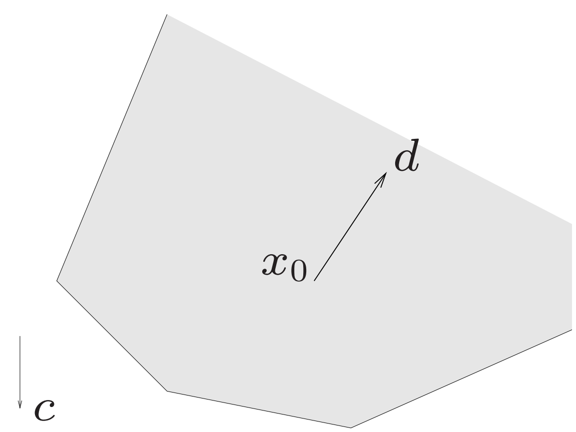
- $c = (1,1)$: $X_{\text{opt}} = \{(0,0)\}$, $p^\star = 0$
- $c = (1,0)$: $X_{\text{opt}} = \{(0, x_2) \mid 0 \leq x_2 \leq 1\}$, $p^\star = 0$
- $c = (-1,-1)$: $X_{\text{opt}} = \emptyset$, $p^\star = -\infty$

Existence of optimal points

- $p^\star = -\infty$ if and only if there exists a feasible half-line

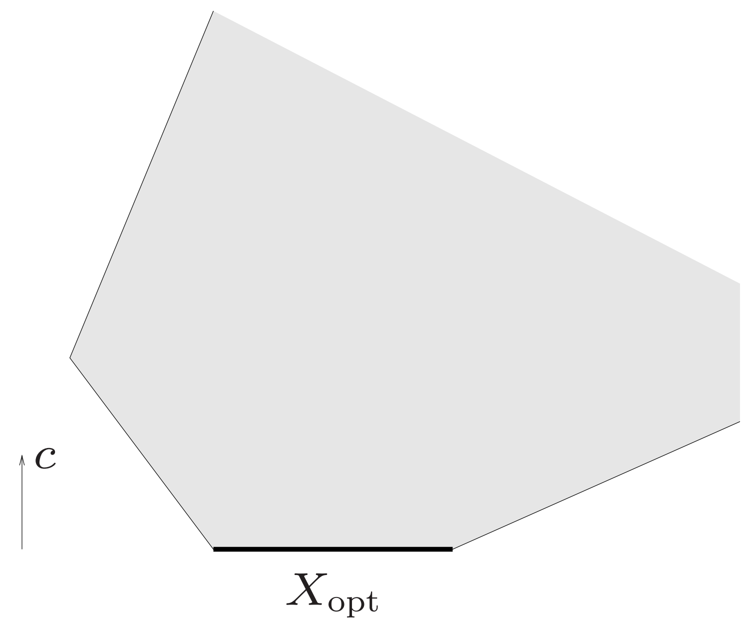
$$\{x_0 + td \mid t \geq 0\}$$

with $c^T d < 0$



- $p^\star = +\infty$ if and only if $\mathcal{P} = \emptyset$
- $p^\star$ is finite if and only if $X_{\rm opt} \neq \emptyset$

property: if $\mathcal{P}$ has at least one extreme point and $p^\star$ is finite, then there exists an extreme point that is optimal



Lecture 4
The linear programming problem: variants and examples

- variants of the linear programming problem
- LP feasibility problem
- examples and some general applications
- linear-fractional programming

Variants of the linear programming problem

general form

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \\
& g_i^T x = h_i, \quad i = 1, \ldots, p
\end{array}
$$

in matrix notation:

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax \leq b \\
& Gx = h
\end{array}
$$

where

$$
A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} \in \mathbf{R}^{m \times n}, \qquad G = \begin{bmatrix} g_1^T \\ g_2^T \\ \vdots \\ g_p^T \end{bmatrix} \in \mathbf{R}^{p \times n}
$$

inequality form LP

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m\end{array}$$

in matrix notation:

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \leq b\end{array}$$

standard form LP

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & g_i^T x = h_i, \quad i = 1, \ldots, m \\ & x \geq 0\end{array}$$

in matrix notation:

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Gx = h \\ & x \geq 0\end{array}$$

Reduction of general LP to inequality/standard form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \\ & g_i^T x = h_i, \quad i = 1, \ldots, p \end{array}$$

reduction to inequality form:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \\ & g_i^T x \geq h_i, \quad i = 1, \ldots, p \\ & g_i^T x \leq h_i, \quad i = 1, \ldots, p \end{array}$$

in matrix notation (where A has rows a_i^T, G has rows g_i^T)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \begin{bmatrix} A \\ -G \\ G \end{bmatrix} x \leq \begin{bmatrix} b \\ -h \\ h \end{bmatrix} \end{array}$$

reduction to standard form:

$$\begin{array}{ll} \text{minimize} & c^T x^+ - c^T x^- \\ \text{subject to} & a_i^T x^+ - a_i^T x^- + s_i = b_i, \quad i = 1, \ldots, m \\ & g_i^T x^+ - g_i^T x^- = h_i, \quad i = 1, \ldots, p \\ & x^+, x^-, s \geq 0 \end{array}$$

- variables x^+, x^-, s
- recover x as $x = x^+ - x^-$
- $s \in \mathbf{R}^m$ is called a *slack* variable

in matrix notation:

$$\begin{array}{ll} \text{minimize} & \widetilde{c}^T \widetilde{x} \\ \text{subject to} & \widetilde{G}\widetilde{x} = \widetilde{h} \\ & \widetilde{x} \geq 0 \end{array}$$

where

$$\widetilde{x} = \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix}, \qquad \widetilde{c} = \begin{bmatrix} c \\ -c \\ 0 \end{bmatrix}, \qquad \widetilde{G} = \begin{bmatrix} A & -A & I \\ G & -G & 0 \end{bmatrix}, \qquad \widetilde{h} = \begin{bmatrix} b \\ h \end{bmatrix}$$

LP feasibility problem

feasibility problem: find x that satisfies $a_i^T x \leq b_i$, $i = 1, \ldots, m$

solution via LP (with variables t, x)

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x \leq b_i + t, \quad i = 1, \ldots, m \end{array}$$

- variables t, x
- if minimizer $x^\star$, $t^\star$ satisfies $t^\star \leq 0$, then $x^\star$ satisfies the inequalities

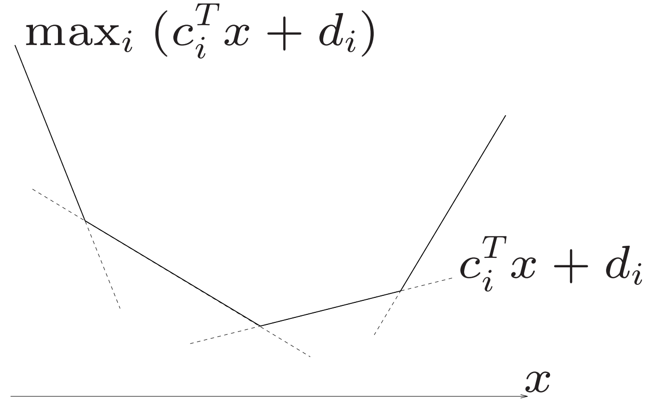
LP in matrix notation:

$$\begin{array}{ll} \text{minimize} & \widetilde{c}^T \widetilde{x} \\ \text{subject to} & \widetilde{A}\widetilde{x} \leq \widetilde{b} \end{array}$$

$$\widetilde{x} = \begin{bmatrix} x \\ t \end{bmatrix}, \qquad \widetilde{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \widetilde{A} = \begin{bmatrix} A & -\mathbf{1} \end{bmatrix}, \qquad \widetilde{b} = b$$

Piecewise-linear minimization

piecewise-linear minimization: minimize $\max_{i=1,\ldots,m}(c_i^T x + d_i)$



equivalent LP (with variables $x \in \mathbf{R}^n$, $t \in \mathbf{R}$):

$$
\begin{array}{ll}
\text{minimize} & t \\
\text{subject to} & c_i^T x + d_i \le t, \quad i = 1, \ldots, m
\end{array}
$$

in matrix notation:

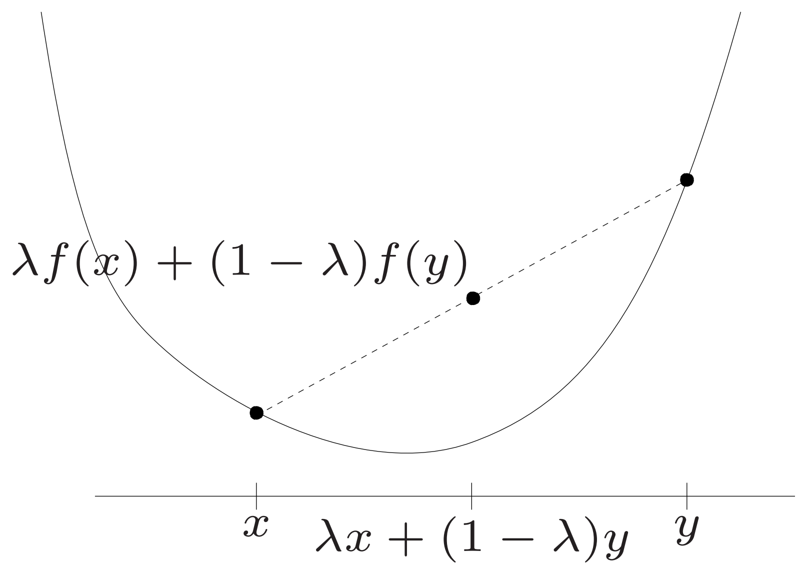
$$
\begin{array}{ll}
\text{minimize} & \tilde{c}^T \tilde{x} \\
\text{subject to} & \tilde{A}\tilde{x} \le \tilde{b}
\end{array}
$$

$$
\tilde{x} = \begin{bmatrix} x \\ t \end{bmatrix}, \qquad \tilde{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \tilde{A} = \begin{bmatrix} C & -\mathbf{1} \end{bmatrix}, \qquad \tilde{b} = \begin{bmatrix} -d \end{bmatrix}
$$

Convex functions

$f : \mathbf{R}^n \to \mathbf{R}$ is convex if for $0 \leq \lambda \leq 1$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

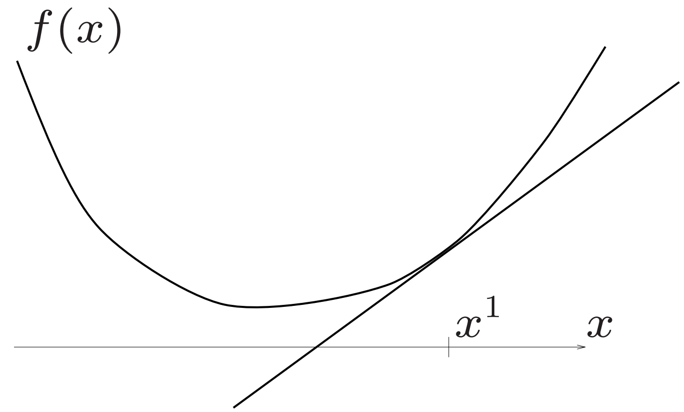


Piecewise-linear approximation

assume $f : \mathbf{R}^n \to \mathbf{R}$ differentiable and convex

- 1st-order approximation at x^1 is a *global lower bound* on f:

$$f(x) \geq f(x^1) + \nabla f(x^1)^T (x - x^1)$$



- evaluating f, ∇f at several x^i yields a *piecewise-linear* lower bound:

$$f(x) \geq \max_{i=1,\ldots,K} \left(f(x^i) + \nabla f(x^i)^T (x - x^i) \right)$$

Convex optimization problem

$$\text{minimize} \quad f_0(x)$$

(f_i convex and differentiable)

LP approximation (choose points x^j, $j = 1, \ldots, K$):

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & f_0(x^j) + \nabla f_0(x^j)^T(x - x^j) \leq t, \quad j = 1, \ldots, K \end{array}$$

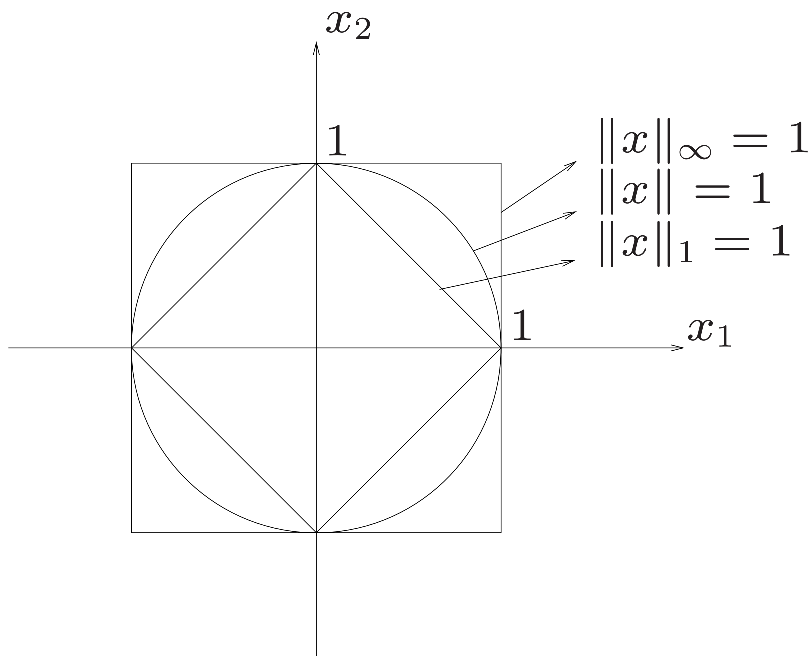
(variables x, t)

- yields lower bound on optimal value
- can be extended to nondifferentiable convex functions
- more sophisticated variation: cutting-plane algorithm (solves convex optimization problem via sequence of LP approximations)

Norms

norms on $\mathbf{R}^n$:

- Euclidean norm $\|x\|$ (or $\|x\|_2$) $= \sqrt{x_1^2 + \cdots + x_n^2}$
- ℓ_1-norm: $\|x\|_1 = |x_1| + \cdots + |x_n|$
- ℓ_∞- (or Chebyshev-) norm: $\|x\|_\infty = \max_i |x_i|$



Norm approximation problems

$$\text{minimize} \quad \|Ax - b\|_p$$

- $x \in \mathbf{R}^n$ is variable; $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$ are problem data
- $p = 1, 2, \infty$
- $r = Ax - b$ is called *residual*
- $r_i = a_i^T x - b_i$ is ith residual (a_i^T is ith row of A)
- usually overdetermined, *i.e.*, $b \notin \mathcal{R}(A)$ (*e.g.*, $m > n$, A full rank)

interpretations:

- approximate or fit b with linear combination of columns of A
- b is corrupted measurement of Ax; find 'least inconsistent' value of x for given measurements

examples:

- $\|r\| = \sqrt{r^T r}$: least-squares or ℓ_2-approximation (a.k.a. regression)
- $\|r\| = \max_i |r_i|$: Chebyshev, ℓ_∞, or minimax approximation
- $\|r\| = \sum_i |r_i|$: absolute-sum or ℓ_1-approximation

solution:

- ℓ_2: closed form expression

$$x_{\text{opt}} = (A^T A)^{-1} A^T b$$

  (assume $\mathbf{rank}(A) = n$)

- ℓ_1, ℓ_∞: no closed form expression, but readily solved via LP

ℓ_1-approximation via LP

ℓ_1-approximation problem

$$\text{minimize} \quad \|Ax - b\|_1$$

write as

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^m y_i \\ \text{subject to} & -y \leq Ax - b \leq y \end{array}$$

an LP with variables y, x:

$$\begin{array}{ll} \text{minimize} & \tilde{c}^T \tilde{x} \\ \text{subject to} & \tilde{A}\tilde{x} \leq \tilde{b} \end{array}$$

with

$$\tilde{x} = \left[\begin{array}{c} x \\ y \end{array}\right], \qquad \tilde{c} = \left[\begin{array}{c} 0 \\ \mathbf{1} \end{array}\right], \qquad \tilde{A} = \left[\begin{array}{rr} A & -I \\ -A & -I \end{array}\right], \qquad \tilde{b} = \left[\begin{array}{r} b \\ -b \end{array}\right]$$

ℓ_∞-approximation via LP

ℓ_∞-approximation problem

$$\text{minimize} \quad \|Ax - b\|_\infty$$

write as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & -t\mathbf{1} \leq Ax - b \leq t\mathbf{1} \end{array}$$

an LP with variables t, x:

$$\begin{array}{ll} \text{minimize} & \widetilde{c}^T \widetilde{x} \\ \text{subject to} & \widetilde{A}\widetilde{x} \leq \widetilde{b} \end{array}$$

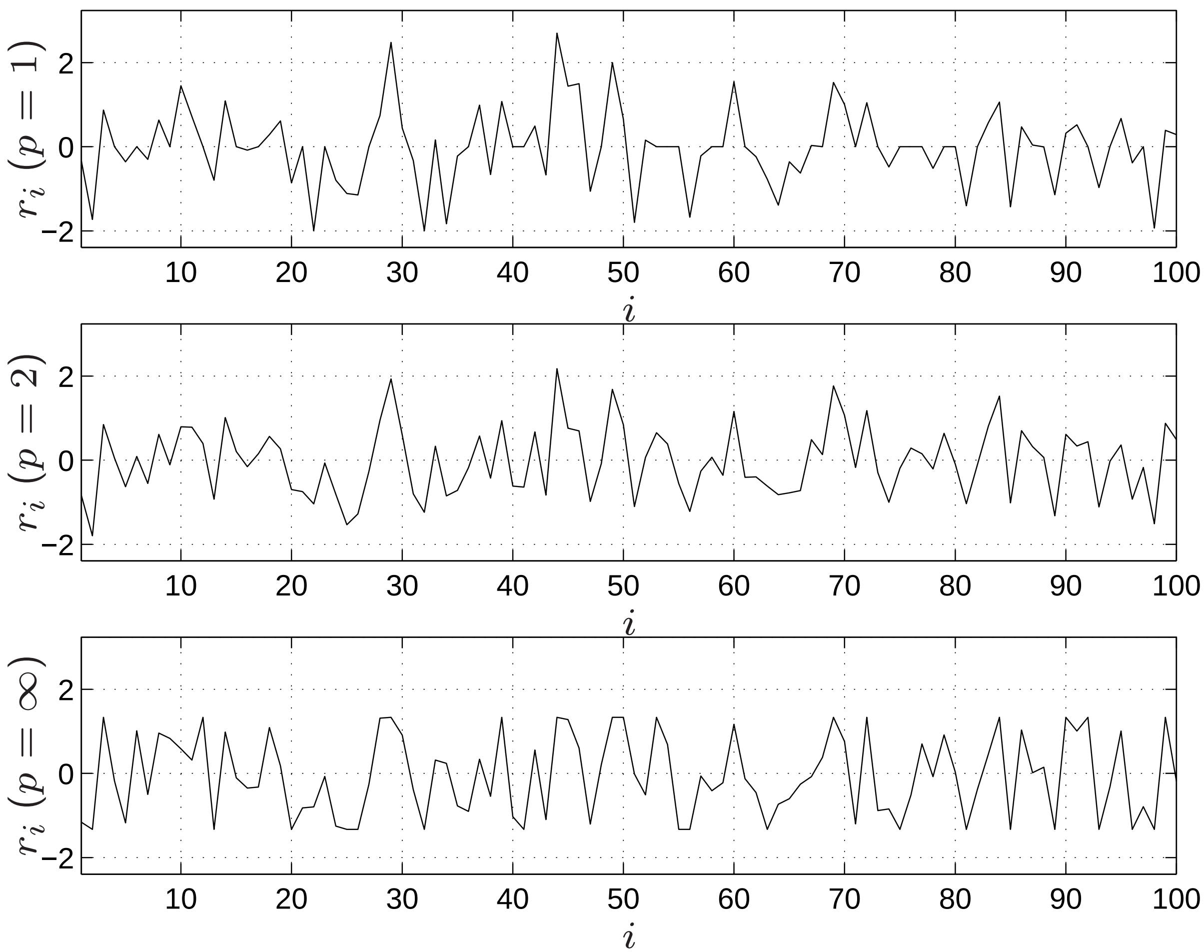
with

$$\widetilde{x} = \left[\begin{array}{c} x \\ t \end{array}\right], \qquad \widetilde{c} = \left[\begin{array}{c} 0 \\ 1 \end{array}\right], \qquad \widetilde{A} = \left[\begin{array}{rr} A & -\mathbf{1} \\ -A & -\mathbf{1} \end{array}\right], \qquad \widetilde{b} = \left[\begin{array}{r} b \\ -b \end{array}\right]$$

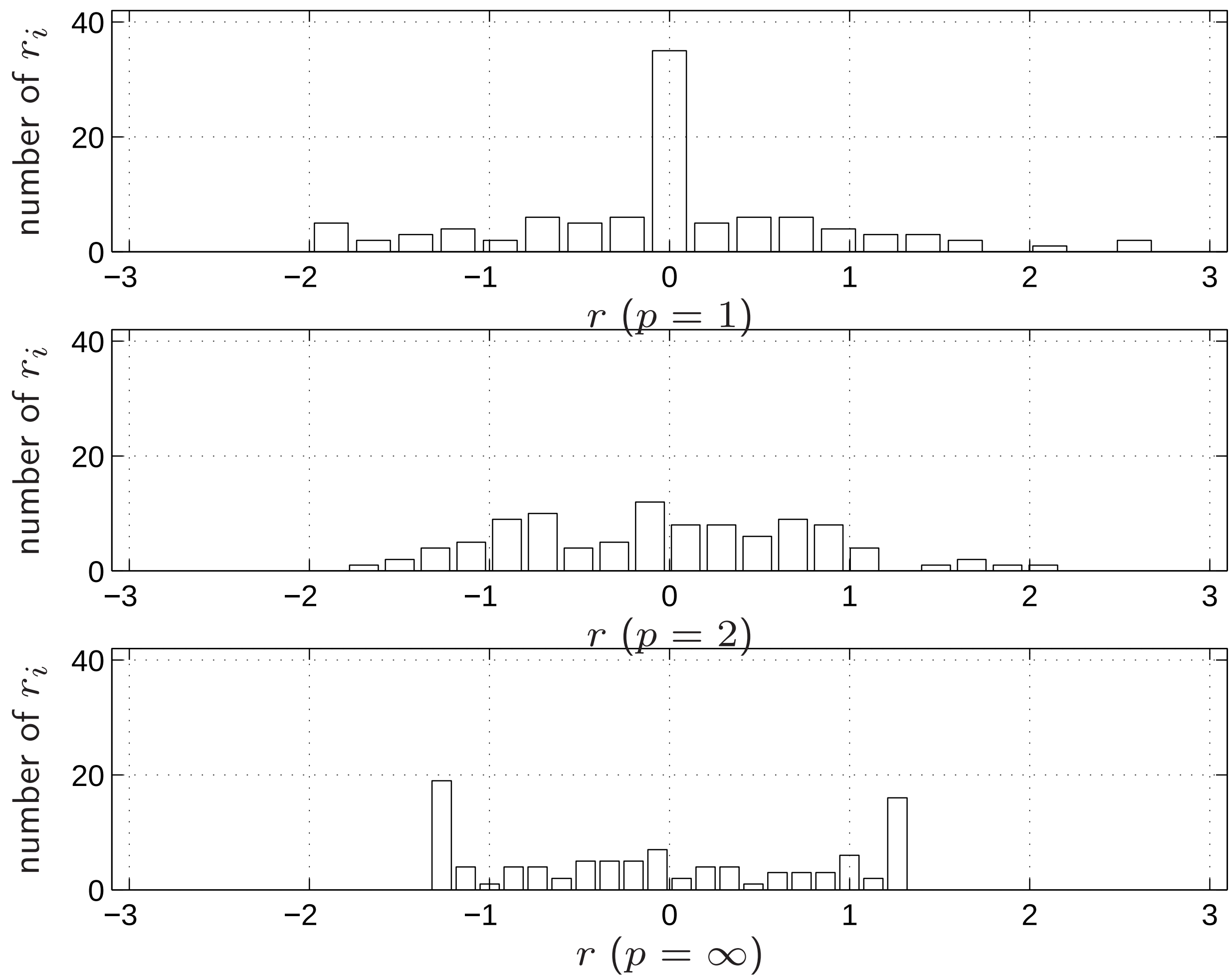
Example

minimize $\|Ax - b\|_p$ for $p = 1,\ 2,\ \infty$ ($A \in \mathbf{R}^{100 \times 30}$)

resulting residuals:



histogram of residuals:



- $p = \infty$ gives 'thinnest' distribution; $p = 1$ gives widest distribution
- $p = 1$ most very small (or even zero) r_i

Interpretation: maximum likelihood estimation

m linear measurements $y_1, \ldots, y_m$ of $x \in \mathbf{R}^n$:

$$y_i = a_i^T x + v_i, \quad i = 1, \ldots, m$$

- v_i: measurement noise, IID with density p
- y is a random variable with density $p_x(y) = \prod_{i=1}^m p(y_i - a_i^T x)$

log-likelihood function is defined as

$$\log p_x(y) = \sum_{i=1}^m \log p(y_i - a_i^T x)$$

maximum likelihood (ML) estimate of x is

$$\hat{x} = \operatorname*{argmax}_x \sum_{i=1}^m \log p(y_i - a_i^T x)$$

examples

- v_i Gaussian: $p(z) = 1/(\sqrt{2\pi}\sigma)e^{-z^2/2\sigma^2}$

  ML estimate is ℓ_2-estimate $\hat{x} = \mathop{\rm argmin}_x \|Ax - y\|_2$

- v_i double-sided exponential: $p(z) = (1/2a)e^{-|z|/a}$

  ML estimate is ℓ_1-estimate $\hat{x} = \mathop{\rm argmin}_x \|Ax - y\|_1$

- v_i is one-sided exponential: $p(z) = \begin{cases} (1/a)e^{-z/a} & z \geq 0 \\ 0 & z < 0 \end{cases}$

  ML estimate is found by solving LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T(y - Ax) \\ \text{subject to} & y - Ax \geq 0 \end{array}$$

- v_i are uniform on $[-a, a]$: $p(z) = \begin{cases} 1/(2a) & -a \leq z \leq a \\ 0 & \text{otherwise} \end{cases}$

  ML estimate is any x satisfying $\|Ax - y\|_\infty \leq a$

Linear-fractional programming

$$\begin{array}{ll} \text{minimize} & \dfrac{c^Tx+d}{f^Tx+g} \\ \text{subject to} & Ax \leq b \\ & f^Tx+g \geq 0 \end{array}$$

(asume $a/0 = +\infty$ if $a > 0$, $a/0 = -\infty$ if $a \leq 0$)

- nonlinear objective function
- like LP, can be solved very efficiently

equivalent form with linear objective (vars. x, γ):

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & c^Tx+d \leq \gamma(f^Tx+g) \\ & f^Tx+g \geq 0 \\ & Ax \leq b \end{array}$$

Bisection algorithm for linear-fractional programming

given: interval $[l, u]$ that contains optimal γ
repeat: solve feasibility problem for $\gamma = (u + l)/2$

$$\begin{array}{l} c^T x + d \leq \gamma(f^T x + g) \\ f^T x + g \geq 0 \\ Ax \leq b \end{array}$$

if feasible $u := \gamma$; if infeasible $l := \gamma$
until $u - l \leq \epsilon$

- each iteration is an LP feasibility problem
- accuracy doubles at each iteration
- number of iterations to reach accuracy ϵ starting with initial interval of width $u - l = \epsilon_0$:

$$k = \lceil \log_2(\epsilon_0/\epsilon) \rceil$$

Generalized linear-fractional programming

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\max_{i=1,\ldots,K} \frac{c_i^T x + d_i}{f_i^T x + g_i} \\
\text{subject to} & Ax \leq b \\
& f_i^T x + g_i \geq 0, \quad i = 1, \ldots, K
\end{array}
$$

equivalent formulation:

$$
\begin{array}{ll}
\text{minimize} & \gamma \\
\text{subject to} & Ax \leq b \\
& c_i^T x + d_i \leq \gamma(f_i^T x + g_i), \quad i = 1, \ldots, K \\
& f_i^T x + g_i \geq 0, \quad i = 1, \ldots, K
\end{array}
$$

- efficiently solved via bisection on γ
- each iteration is an LP feasibility problem

Von Neumann economic growth problem

simple model of an economy: m goods, n economic sectors

- $x_i(t)$: 'activity' of sector i in current period t
- $a_i^T x(t)$: amount of good i consumed in period t
- $b_i^T x(t)$: amount of good i produced in period t

choose $x(t)$ to maximize *growth rate* $\min_i x_i(t+1)/x_i(t)$:

$$\begin{array}{ll} \text{maximize} & \gamma \\ \text{subject to} & Ax(t+1) \leq Bx(t), \quad x(t+1) \geq \gamma x(t), \quad x(t) \geq \mathbf{1} \end{array}$$

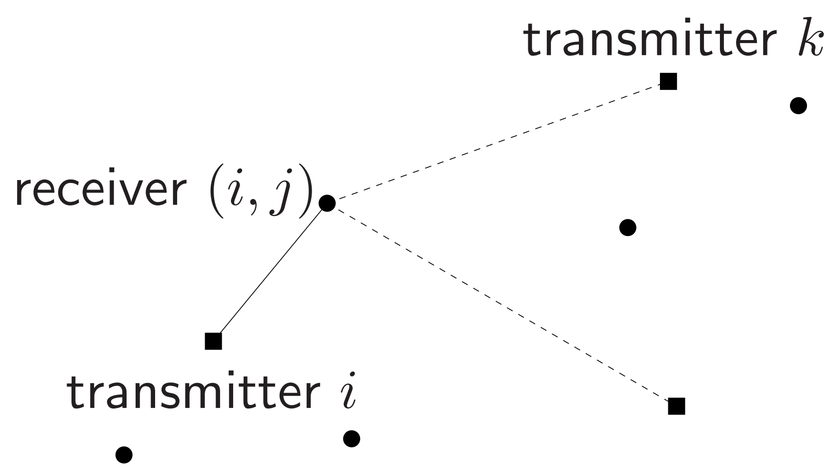
or equivalently (since $a_{ij} \geq 0$):

$$\begin{array}{ll} \text{maximize} & \gamma \\ \text{subject to} & \gamma Ax(t) \leq Bx(t), \quad x(t) \geq \mathbf{1} \end{array}$$

(linear-fractional problem with variables $x(0)$, γ)

Optimal transmitter power allocation

- m transmitters, mn receivers all at same frequency
- transmitter i wants to transmit to n receivers labeled (i, j), $j = 1, \ldots, n$



- A_{ijk} is path gain from transmitter k to receiver (i, j)
- N_{ij} is (self) noise power of receiver (i, j)
- variables: transmitter powers p_k, $k = 1, \ldots, m$

at receiver (i, j):

- signal power: $S_{ij} = A_{iji} p_i$
- noise plus interference power: $I_{ij} = \sum_{k \neq i} A_{ijk} p_k + N_{ij}$
- signal to interference/noise ratio (SINR): S_{ij}/I_{ij}

problem: choose p_i to maximize smallest SINR:

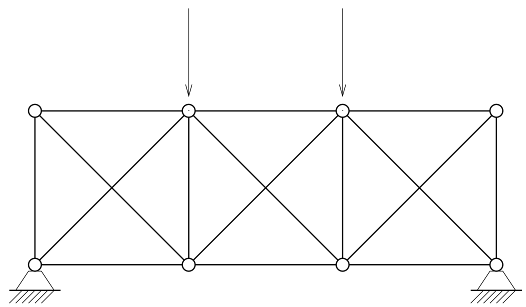
$$\begin{array}{ll} \text{maximize} & \min\limits_{i,j} \dfrac{A_{iji} p_i}{\sum_{k \neq i} A_{ijk} p_k + N_{ij}} \\ \text{subject to} & 0 \leq p_i \leq p_{\max} \end{array}$$

- a (generalized) linear-fractional program
- special case with analytical solution: $m = 1$, no upper bound on p_i (see exercises)

Lecture 5
Structural optimization

- minimum weight truss design
- truss topology design
- limit analysis
- design with minimum number of bars

Truss



- m bars with lengths l_i and cross-sectional areas x_i
- N nodes; nodes $1, \ldots, n$ are free, nodes $n+1, \ldots, N$ are anchored
- external load: forces $f_i \in \mathbf{R}^2$ at nodes $i = 1, \ldots, n$

design problems:

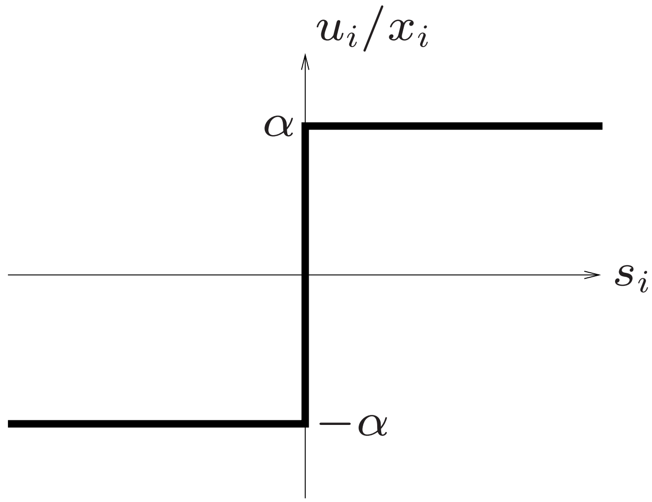
- given the topology (*i.e.*, location of bars and nodes), find the lightest truss that can carry a given load (vars: bar sizes x_k, cost: total weight)
- same problem, where cost $\propto$ #bars used
- find best topology
- find lightest truss that can carry several given loads

analysis problem: for a given truss, what is the largest load it can carry?

Material characteristics

- $u_i \in \mathbf{R}$ is force in bar i ($u_i > 0$: tension, $u_i < 0$: compression)
- $s_i \in \mathbf{R}$ is deformation of bar i ($s_i > 0$: lengthening, $s_i < 0$: shortening)

we assume the material is *rigid/perfectly plastic*:

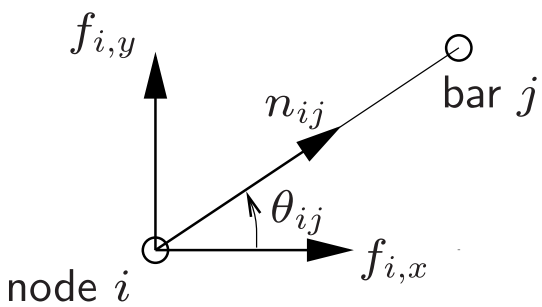


$$
\begin{array}{ll}
s_i = 0 & \text{if } -\alpha < u_i/x_i < \alpha \\
u_i/x_i = \alpha & \text{if } s_i > 0 \\
u_i/x_i = -\alpha & \text{if } s_i < 0
\end{array}
$$

(α is a material constant)

Minimum weight truss for given load

force equilibrium for (free) node i: $\sum_{j=1}^m u_j \begin{bmatrix} n_{ij,x} \\ n_{ij,y} \end{bmatrix} + \begin{bmatrix} f_{i,x} \\ f_{i,y} \end{bmatrix} = 0$



n_{ij} depends on topology:

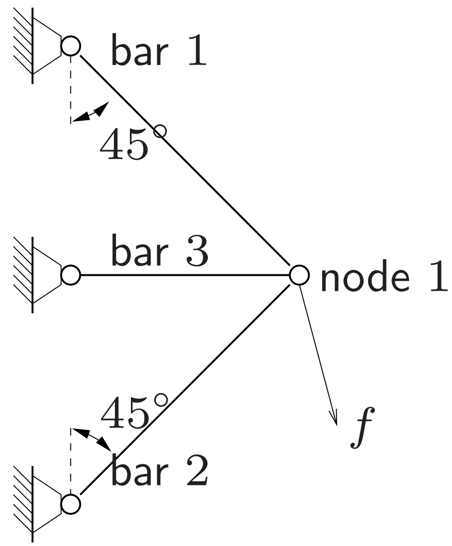
- $n_{ij} = 0$ if bar j is not connected to node i
- $n_{ij} = (\cos\theta_{ij}, \sin\theta_{ij})$ otherwise

minimum weight truss design via LP:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^m l_i x_i \\ \text{subject to} & \sum_{j=1}^m u_j n_{ij} + f_i = 0, \quad i = 1, \ldots, n \\ & -\alpha x_j \leq u_j \leq \alpha x_j, \quad j = 1, \ldots, m \end{array}$$

(variables x_j, u_j)

example



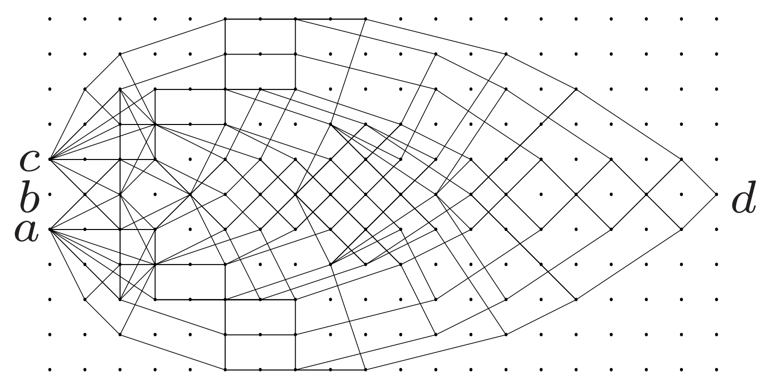
$$
\begin{array}{ll}
\text{mimimize} & l_1x_1 + l_2x_2 + l_3x_3 \\
\text{subject to} & -u_1/\sqrt{2} - u_2/\sqrt{2} - u_3 + f_x = 0 \\
& u_1/\sqrt{2} - u_2/\sqrt{2} + f_y = 0 \\
& -\alpha x_1 \leq u_1 \leq \alpha x_1 \\
& -\alpha x_2 \leq u_2 \leq \alpha x_2 \\
& -\alpha x_3 \leq u_3 \leq \alpha x_3
\end{array}
$$

Truss topology design

- grid of nodes; bars between any pair of nodes
- design minimum weight truss: $u_i = 0$ for most bars
- optimal topology: only use bars with $u_i \neq 0$

example:

- 20×11 grid, *i.e.*, 220 (potential) nodes, 24,090 (potential) bars
- nodes a, b, c are fixed; unit vertical force at node d
- optimal topology has 289 bars



Multiple loading scenarios

minimum weight truss that can carry M possible loads $f_i^1, \ldots, f_i^M$:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^m l_i x_i \\ \text{subject to} & \sum_{j=1}^m u_j^k n_{ij} + f_i^k = 0, \quad i = 1, \ldots, n, \quad k = 1, \ldots, M \\ & -\alpha x_j \le u_j^k \le \alpha x_j, \quad j = 1, \ldots, m, \quad k = 1, \ldots, M \end{array}$$

(variables x_j, $u_j^1, \ldots, u_j^M$)

adds *robustness*: truss can carry any load

$$f_i = \lambda_1 f_i^1 + \cdots + \lambda_M f_i^M$$

with $\lambda_k \ge 0$, $\sum_k \lambda_k \le 1$

Limit analysis

- truss with given geometry (including given cross-sectional areas x_i)
- load f_i is given up to a constant multiple: $f_i = \gamma g_i$, with given $g_i \in \mathbf{R}^2$ and $\gamma > 0$

find largest load that the truss can carry:

$$\begin{array}{ll} \text{maximize} & \gamma \\ \text{subject to} & \sum_{j=1}^m u_j n_{ij} + \gamma g_i = 0, \quad i = 1, \ldots, n \\ & -\alpha x_j \leq u_j \leq \alpha x_j, \quad j = 1, \ldots, m \end{array}$$

an LP in γ, u_j

maximum allowable γ is called the *safety factor*

Design with smallest number of bars

integer LP formulation (assume wlog $x_i \leq 1$):

$$\begin{array}{ll} \text{minimize} & \sum_{j=1}^m z_j \\ \text{subject to} & \sum_{j=1}^m u_j n_{ij} + f_i = 0, \quad i = 1, \dots, n \\ & -\alpha x_j \leq u_j \leq \alpha x_j, \quad j = 1, \dots, m \\ & x_j \leq z_j, \quad j = 1, \dots, m \\ & z_j \in \{0, 1\}, \quad j = 1, \dots, m \end{array}$$

- variables z_j, x_j, u_j
- extremely hard to solve; we may have to enumerate all 2^m possible values of z

heuristic: replace $z_j \in \{0, 1\}$ by $0 \leq z_j \leq 1$

- yields an LP; at the optimum many (but not all) z_j's will be 0 or 1
- called *LP relaxation* of the integer LP

Lecture 6
FIR filter design

- FIR filters
- linear phase filter design
- magnitude filter design
- equalizer design

FIR filters

finite impulse response (FIR) filter:

$$y(t) = \sum_{\tau=0}^{n-1} h_\tau u(t-\tau), \quad t \in \mathbf{Z}$$

- $u : \mathbf{Z} \to \mathbf{R}$ is *input signal*; $y : \mathbf{Z} \to \mathbf{R}$ is *output signal*
- $h_i \in \mathbf{R}$ are called *filter coefficients*; n is filter *order* or *length*

filter frequency response: $H : \mathbf{R} \to \mathbf{C}$

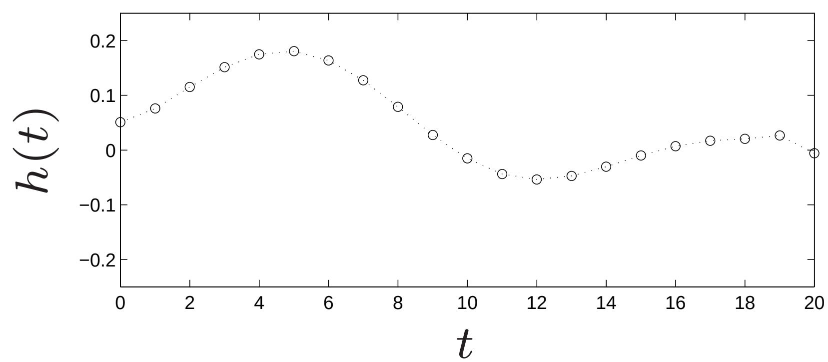
$$\begin{aligned} H(\omega) &= h_0 + h_1 e^{-j\omega} + \cdots + h_{n-1} e^{-j(n-1)\omega} \\ &= \sum_{t=0}^{n-1} h_t \cos t\omega - j \sum_{t=0}^{n-1} h_t \sin t\omega \qquad (j = \sqrt{-1}) \end{aligned}$$

periodic, conjugate symmetric, so only need to know/specify for $0 \le \omega \le \pi$

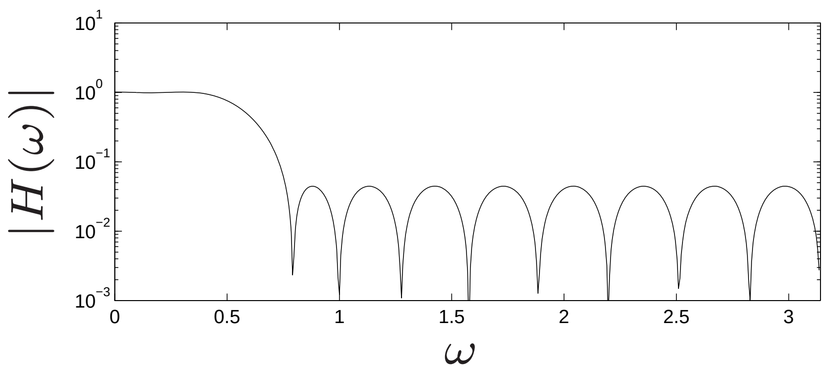
FIR filter design problem: choose h so H and h satisfy/optimize specs

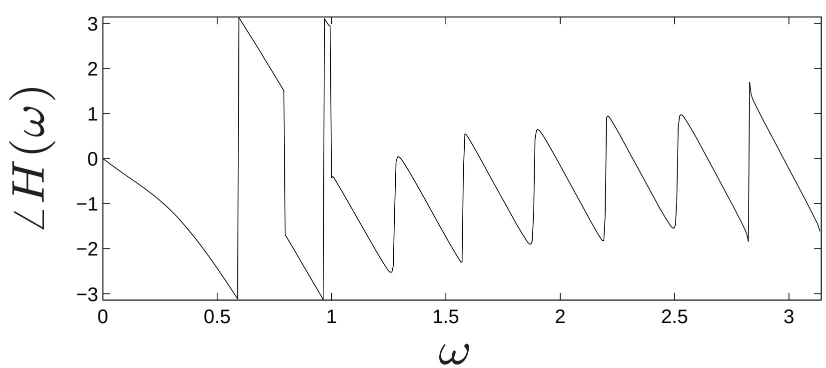
example: (lowpass) FIR filter, order $n = 21$

impulse response h:



frequency response magnitude $|H(\omega)|$ and phase $\angle H(\omega)$:





Linear phase filters

suppose $n = 2N + 1$ is odd and impulse response is symmetric about midpoint:

$$h_t = h_{n-1-t}, \quad t = 0, \ldots, n-1$$

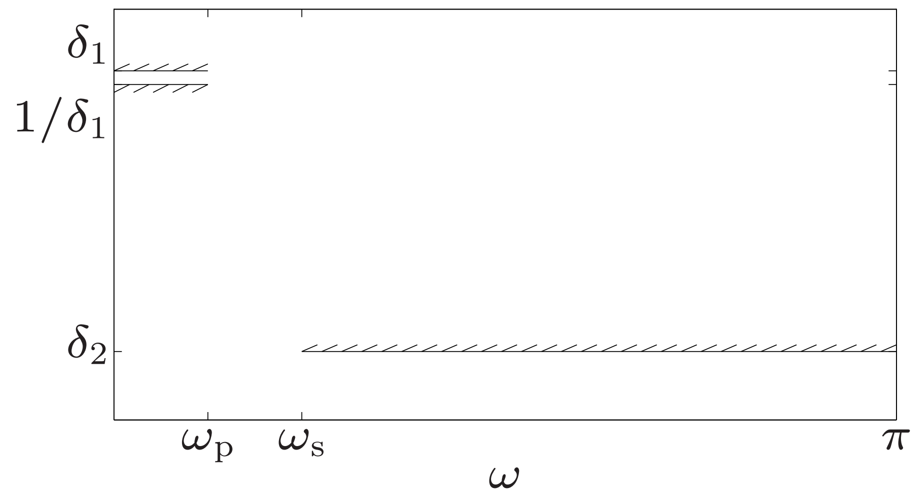
then

$$\begin{aligned} H(\omega) &= h_0 + h_1 e^{-j\omega} + \cdots + h_{n-1} e^{-j(n-1)\omega} \\ &= e^{-jN\omega} \left(2h_0 \cos N\omega + 2h_1 \cos(N-1)\omega + \cdots + h_N \right) \\ &= e^{-jN\omega} \widetilde{H}(\omega) \end{aligned}$$

- term $e^{-jN\omega}$ represents N-sample delay
- $\widetilde{H}(\omega)$ is **real**
- $|H(\omega)| = |\widetilde{H}(\omega)|$

called **linear phase** filter ($\angle H(\omega)$ is linear except for jumps of $\pm\pi$)

Lowpass filter specifications



specifications:

- maximum *passband ripple* ($\pm 20 \log_{10} \delta_1$ in dB):

$$1/\delta_1 \leq |H(\omega)| \leq \delta_1, \quad 0 \leq \omega \leq \omega_p$$

- minimum *stopband attenuation* ($-20 \log_{10} \delta_2$ in dB):

$$|H(\omega)| \leq \delta_2, \quad \omega_s \leq \omega \leq \pi$$

Linear phase lowpass filter design

- sample frequency ($\omega_k = k\pi/K$, $k = 1, \ldots, K$)
- can assume wlog $\widetilde{H}(0) > 0$, so ripple spec is

$$1/\delta_1 \leq \widetilde{H}(\omega_k) \leq \delta_1$$

design for **maximum stopband attenuation:**

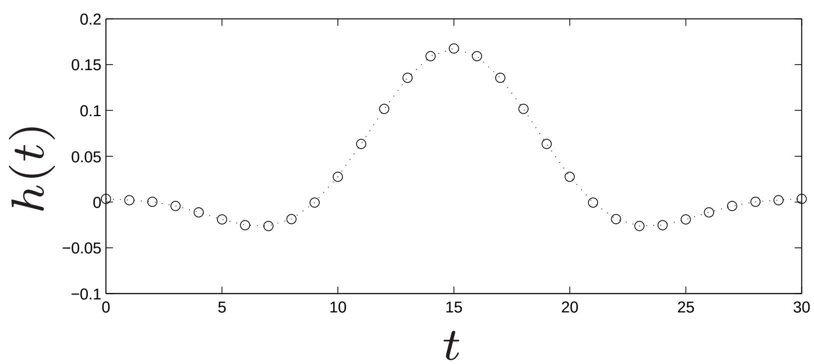
$$\begin{array}{ll} \text{minimize} & \delta_2 \\ \text{subject to} & 1/\delta_1 \leq \widetilde{H}(\omega_k) \leq \delta_1, \quad 0 \leq \omega_k \leq \omega_{\mathrm{p}} \\ & -\delta_2 \leq \widetilde{H}(\omega_k) \leq \delta_2, \quad \omega_{\mathrm{s}} \leq \omega_k \leq \pi \end{array}$$

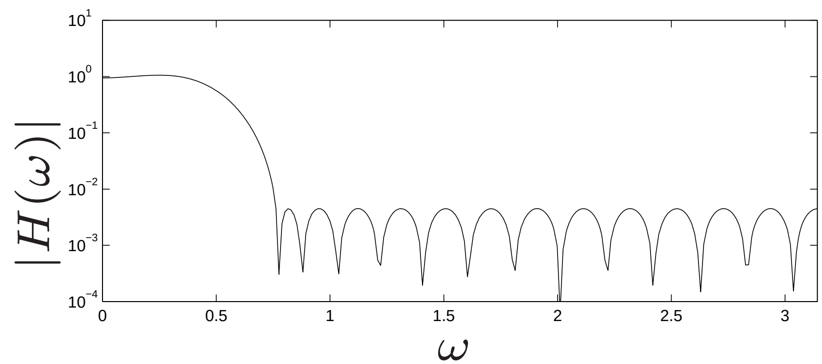
- passband ripple δ_1 is given
- an LP in variables h, δ_2
- known (and used) since 1960's
- can add other constraints, *e.g.*, $|h_i| \leq \alpha$

example

- linear phase filter, $n = 31$
- passband $[0, 0.12\pi]$; stopband $[0.24\pi, \pi]$
- max ripple $\delta_1 = 1.059$ (± 0.5dB)
- design for maximum stopband attenuation

impulse response h and frequency response magnitude $|H(\omega)|$





Some variations

$$\widetilde{H}(\omega) = 2h_0 \cos N\omega + 2h_1 \cos(N-1)\omega + \cdots + h_N$$

minimize passband ripple (given δ_2, ω_s, ω_p, N)

$$\begin{array}{ll} \text{minimize} & \delta_1 \\ \text{subject to} & 1/\delta_1 \leq \widetilde{H}(\omega_k) \leq \delta_1, \quad 0 \leq \omega_k \leq \omega_p \\ & -\delta_2 \leq \widetilde{H}(\omega_k) \leq \delta_2, \quad \omega_s \leq \omega_k \leq \pi \end{array}$$

minimize transition bandwidth (given $\delta_1, \delta_2, \omega_p, N$)

$$\begin{array}{ll} \text{minimize} & \omega_s \\ \text{subject to} & 1/\delta_1 \leq \widetilde{H}(\omega_k) \leq \delta_1, \quad 0 \leq \omega_k \leq \omega_p \\ & -\delta_2 \leq \widetilde{H}(\omega_k) \leq \delta_2, \quad \omega_s \leq \omega_k \leq \pi \end{array}$$

minimize filter order (given δ_1, δ_2, ω_s, ω_p)

$$\begin{array}{ll} \text{minimize} & N \\ \text{subject to} & 1/\delta_1 \leq \widetilde{H}(\omega_k) \leq \delta_1, \quad 0 \leq \omega_k \leq \omega_p \\ & -\delta_2 \leq \widetilde{H}(\omega_k) \leq \delta_2, \quad \omega_s \leq \omega_k \leq \pi \end{array}$$

- can be solved using bisection
- each iteration is an LP feasibility problem

Filter magnitude specifications

transfer function *magnitude spec* has form

$$L(\omega) \leq |H(\omega)| \leq U(\omega), \quad \omega \in [0, \pi]$$

where $L, U : \mathbf{R} \to \mathbf{R}_+$ are given and

$$H(\omega) = \sum_{t=0}^{n-1} h_t \cos t\omega - j \sum_{t=0}^{n-1} h_t \sin t\omega$$

- arises in many applications, $e.g.$, audio, spectrum shaping
- not equivalent to a set of linear inequalities in h (lower bound is not even convex)
- can change variables and convert to set of linear inequalities

Autocorrelation coefficients

autocorrelation coefficients associated with impulse response $h = (h_0, \ldots, h_{n-1}) \in \mathbf{R}^n$ are

$$r_t = \sum_{\tau=0}^{n-1-t} h_\tau h_{\tau+t} \quad \text{(with } h_k = 0 \text{ for } k < 0 \text{ or } k \geq n\text{)}$$

$r_t = r_{-t}$ and $r_t = 0$ for $|t| \geq n$; hence suffices to specify $r = (r_0, \ldots, r_{n-1})$

Fourier transform of autocorrelation coefficients is

$$R(\omega) = \sum_\tau e^{-j\omega\tau} r_\tau = r_0 + \sum_{t=1}^{n-1} 2 r_t \cos \omega t = |H(\omega)|^2$$

can express magnitude specification as

$$L(\omega)^2 \leq R(\omega) \leq U(\omega)^2, \quad \omega \in [0, \pi]$$

. . . linear inequalities in r

Spectral factorization

question: when is $r \in \mathbf{R}^n$ the autocorrelation coefficients of some $h \in \mathbf{R}^n$?

answer (*spectral factorization theorem*): if and only if $R(\omega) \geq 0$ for all ω

- spectral factorization condition is convex in r (a linear inequality for each ω)
- many algorithms for spectral factorization, $i.e.$, finding an h such that $R(\omega) = |H(\omega)|^2$

magnitude design via autocorrelation coefficients:

- use r as variable (instead of h)
- add spectral factorization condition $R(\omega) \geq 0$ for all ω
- optimize over r
- use spectral factorization to recover h

Magnitude lowpass filter design

maximum stopband attenuation design with variables r becomes

$$\begin{array}{ll} \text{minimize} & \tilde{\delta}_2 \\ \text{subject to} & 1/\tilde{\delta}_1 \leq R(\omega) \leq \tilde{\delta}_1, \quad \omega \in [0, \omega_{\rm p}] \\ & R(\omega) \leq \tilde{\delta}_2, \quad \omega \in [\omega_{\rm s}, \pi] \\ & R(\omega) \geq 0, \quad \omega \in [0, \pi] \end{array}$$

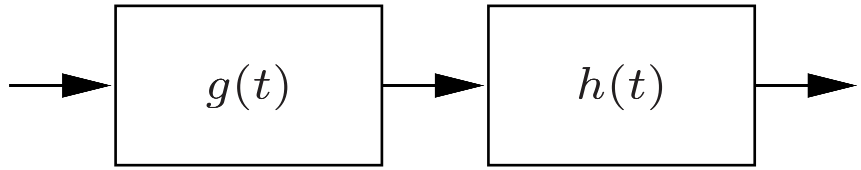
($\tilde{\delta}_i$ corresponds to δ_i^2 in original problem)

now discretize frequency:

$$\begin{array}{ll} \text{minimize} & \tilde{\delta}_2 \\ \text{subject to} & 1/\tilde{\delta}_1 \leq R(\omega_k) \leq \tilde{\delta}_1, \quad 0 \leq \omega_k \leq \omega_{\rm p} \\ & R(\omega_k) \leq \tilde{\delta}_2, \quad \omega_{\rm s} \leq \omega_k \leq \pi \\ & R(\omega_k) \geq 0, \quad 0 \leq \omega_k \leq \pi \end{array}$$

. . . an LP in r, $\tilde{\delta}_2$

Equalizer design



(time-domain) equalization: given

- g (unequalized impulse response)
- g_{des} (desired impulse response)

design (FIR equalizer) h so that $\widetilde{g} = h * g \approx g_{\mathrm{des}}$

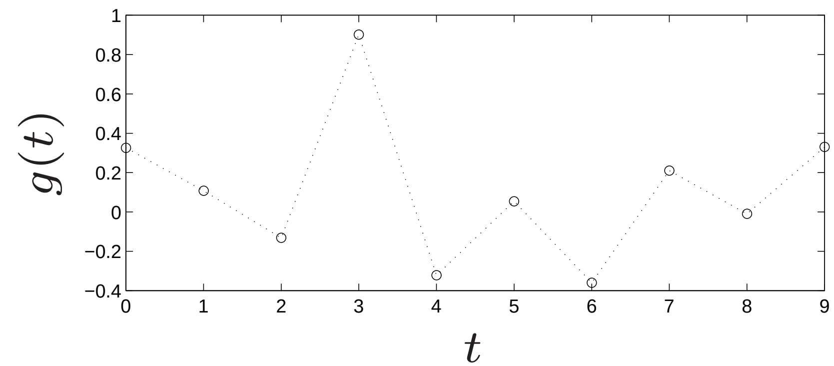
common choice: pure delay D: $g_{\mathrm{des}}(t) = \begin{cases} 1 & t = D \\ 0 & t \neq D \end{cases}$

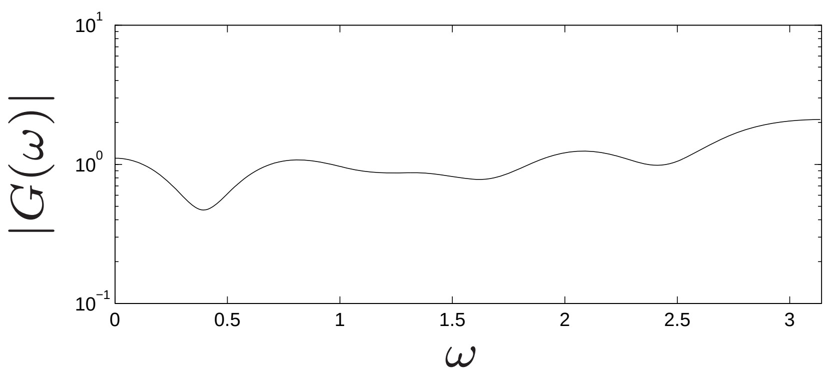
as an LP:

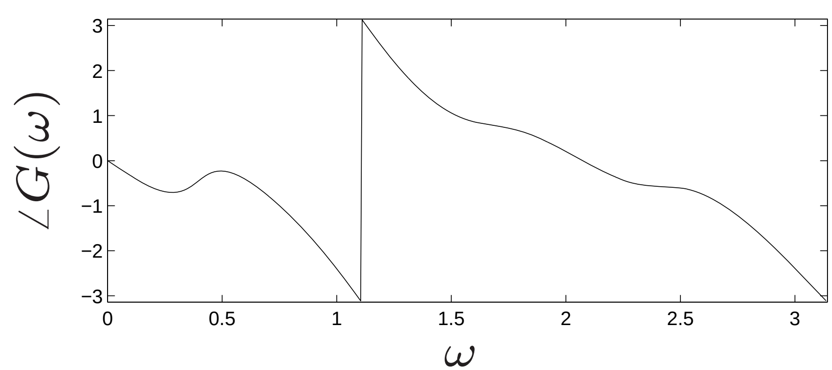
$$\begin{array}{ll} \text{minimize} & \max_{t \neq D} |\tilde{g}(t)| \\ \text{subject to} & \tilde{g}(D) = 1 \end{array}$$

example

unequalized system G is 10th order FIR:



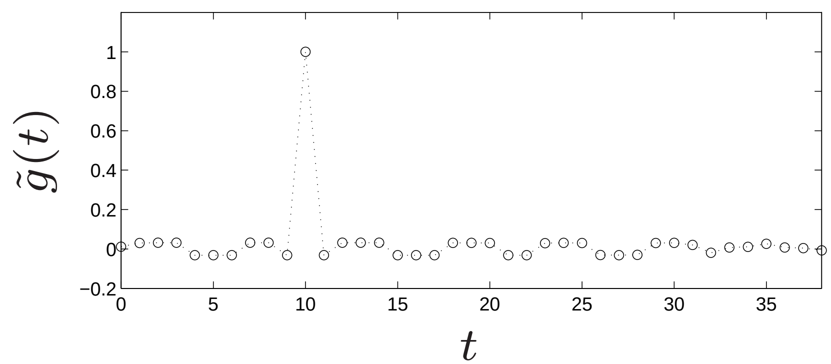




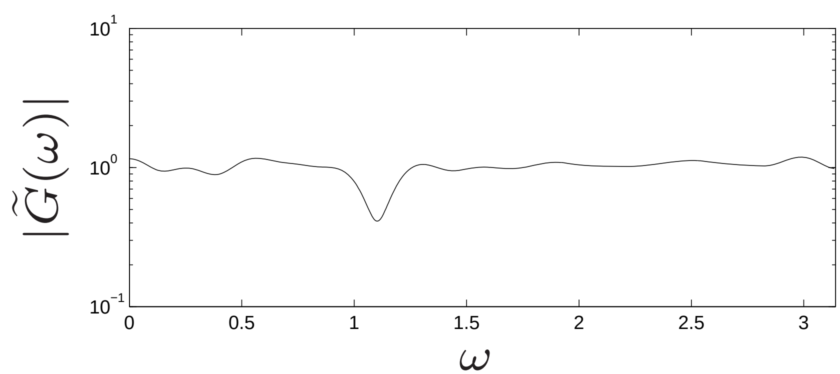
design 30th order FIR equalizer with $\widetilde{G}(\omega) \approx e^{-j10\omega}$

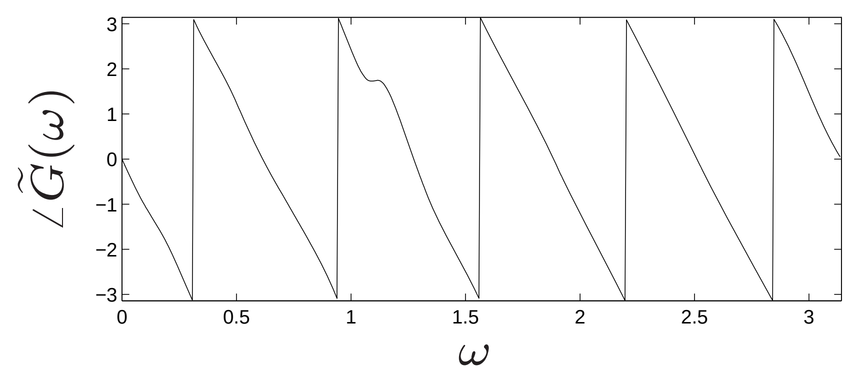
$$\text{minimize} \quad \max_{t \neq 10} |\tilde{g}(t)|$$

equalized system impulse response $\tilde{g}$

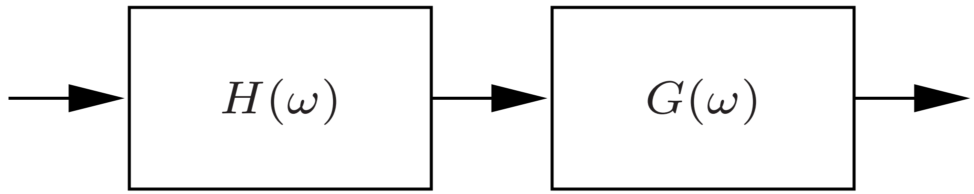


equalized frequency response magnitude $|\widetilde{G}|$ and phase $\angle\widetilde{G}$





Magnitude equalizer design



- given system frequency response $G : [0, \pi] \to \mathbf{C}$
- design FIR equalizer H so that $|G(\omega)H(\omega)| \approx 1$:

$$\text{minimize} \quad \max_{\omega \in [0,\pi]} \left| |G(\omega)H(\omega)|^2 - 1 \right|$$

use autocorrelation coefficients as variables:

$$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & \left| |G(\omega)|^2 R(\omega) - 1 \right| \leq \alpha, \quad \omega \in [0, \pi] \\ & R(\omega) \geq 0, \quad \omega \in [0, \pi] \end{array}$$

when discretized, an LP in r, α, . . .

Multi-system magnitude equalization

- given M frequency responses $G_k : [0, \pi] \to \mathbf{C}$
- design FIR equalizer H so that $|G_k(\omega)H(\omega)| \approx$ constant:

$$\begin{array}{ll} \text{minimize} & \max_{k=1,\ldots,M} \max_{\omega\in[0,\pi]} \left| \, |G_k(\omega)H(\omega)|^2 - \gamma_k \, \right| \\ \text{subject to} & \gamma_k \geq 1, \quad k = 1, \ldots, M \end{array}$$

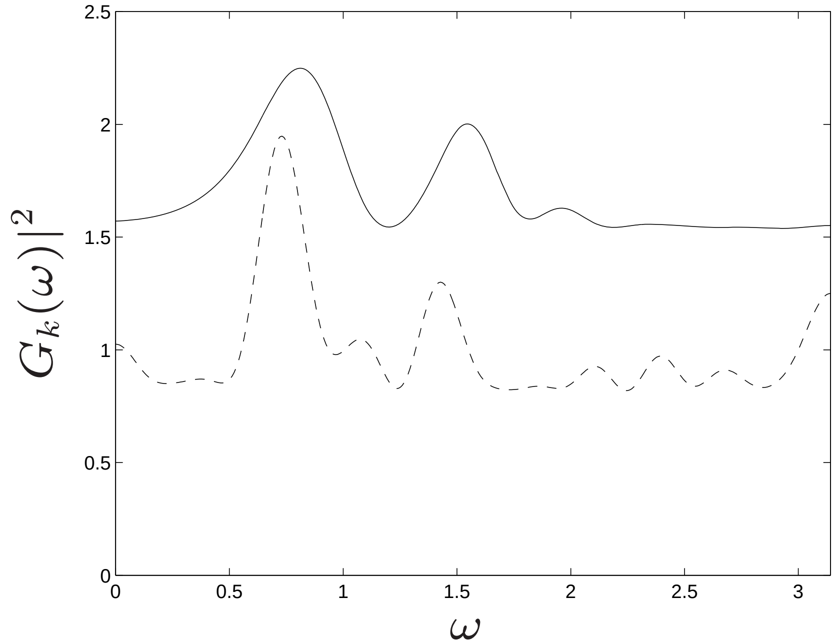
use autocorrelation coefficients as variables:

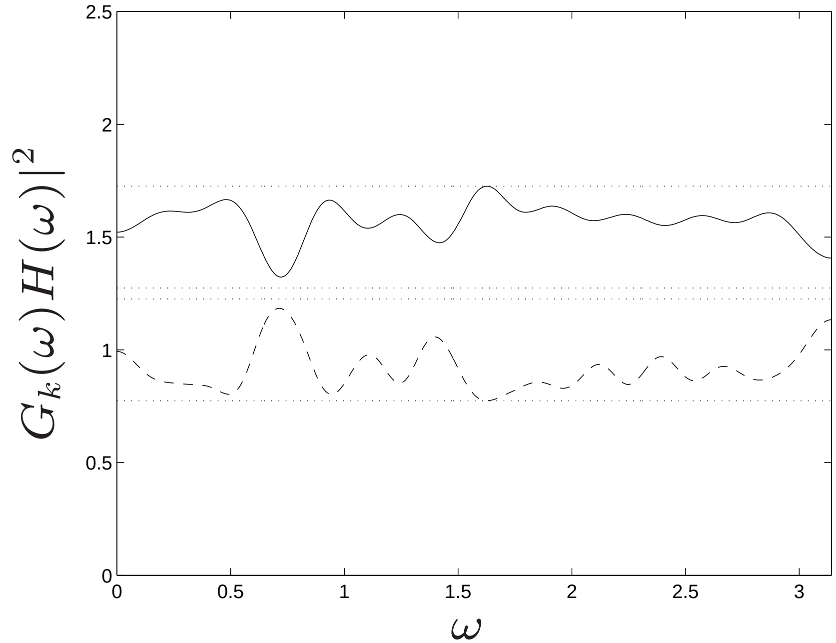
$$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & \left| \, |G_k(\omega)|^2 R(\omega) - \gamma_k \, \right| \leq \alpha, \quad \omega \in [0, \pi], \quad k = 1, \ldots, M \\ & R(\omega) \geq 0, \quad \omega \in [0, \pi] \\ & \gamma_k \geq 1, \quad k = 1, \ldots, M \end{array}$$

. . . when discretized, an LP in γ_k, r, α

example. $M = 2$, $n = 25$, $\gamma_k \geq 1$

unequalized and equalized frequency responses





Lecture 7
Applications in control

- optimal input design
- robust optimal input design
- pole placement (with low-authority control)

Linear dynamical system

$$y(t) = h_0 u(t) + h_1 u(t-1) + h_2 u(t-2) + \cdots$$

- single input/single output: input $u(t) \in \mathbf{R}$, output $y(t) \in \mathbf{R}$
- h_i are called *impulse response* coefficients
- *finite impulse response* (FIR) system of order k: $h_i = 0$ for $i > k$

if $u(t) = 0$ for $t < 0$,

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} h_0 & 0 & 0 & \cdots & 0 \\ h_1 & h_0 & 0 & \cdots & 0 \\ h_2 & h_1 & h_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_N & h_{N-1} & h_{N-2} & \cdots & h_0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(N) \end{bmatrix}$$

a linear mapping from input to output sequence

Output tracking problem

choose inputs $u(t)$, $t = 0, \ldots, M$ ($M < N$) that

- minimize *peak deviation* between $y(t)$ and a desired output $y_{\mathrm{des}}(t)$, $t = 0, \ldots, N$,

$$\max_{t=0,\ldots,N} |y(t) - y_{\mathrm{des}}(t)|$$

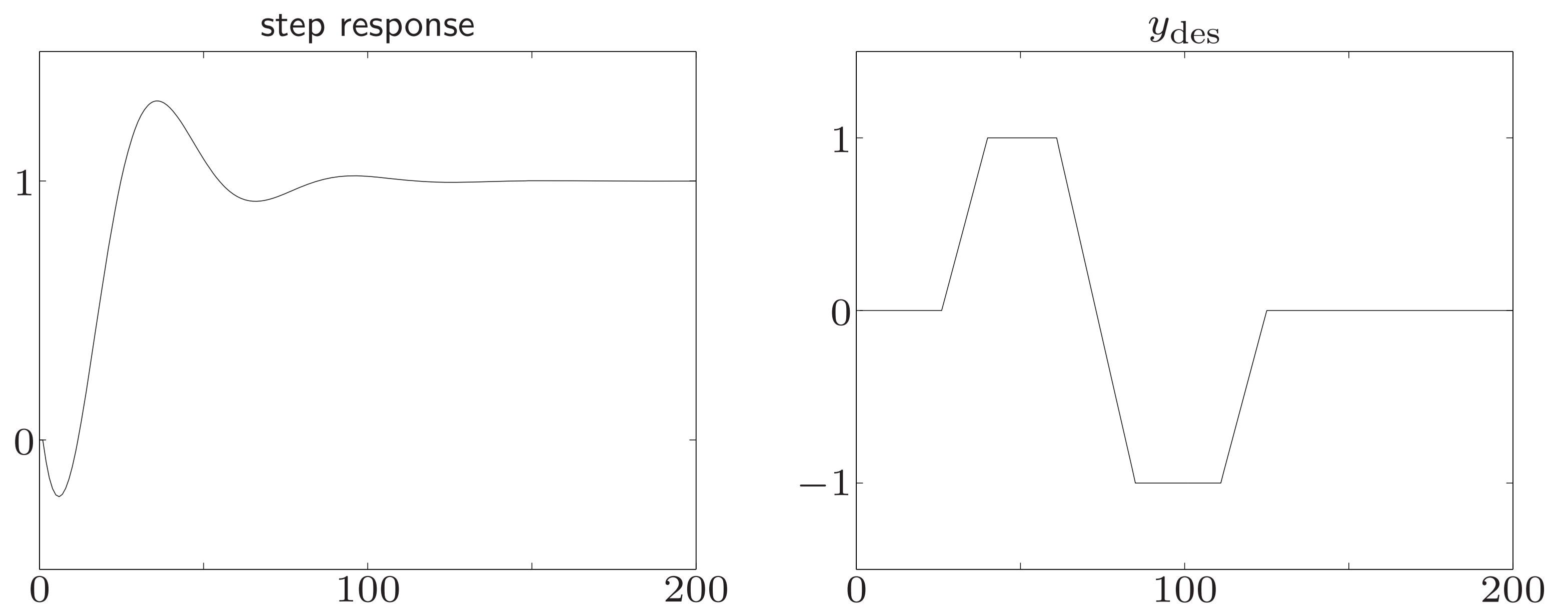
- satisfy amplitude and slew rate constraints:

$$|u(t)| \leq U, \quad |u(t+1) - u(t)| \leq S$$

as a linear program (variables: w, $u(0)$, . . . , $u(N)$):

$$\begin{array}{ll} \text{minimize.} & w \\ \text{subject to} & -w \leq \sum_{i=0}^{t} h_i u(t-i) - y_{\mathrm{des}}(t) \leq w, \quad t = 0, \ldots, N \\ & u(t) = 0, \quad t = M+1, \ldots, N \\ & -U \leq u(t) \leq U, \quad t = 0, \ldots, M \\ & -S \leq u(t+1) - u(t) \leq S, \quad t = 0, \ldots, M+1 \end{array}$$

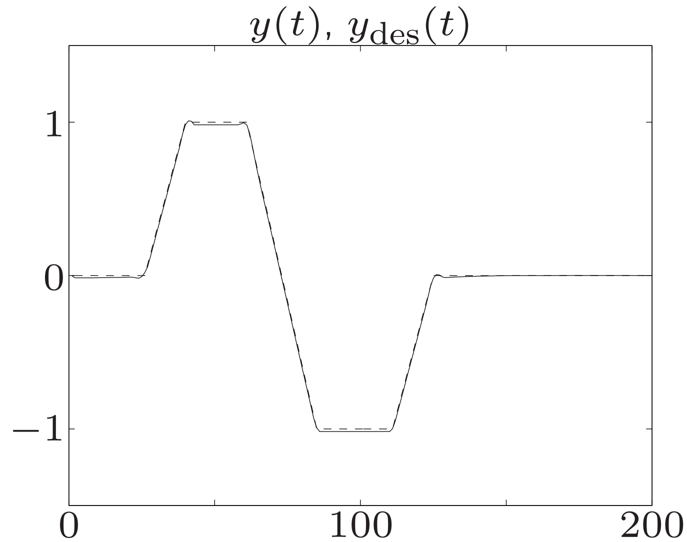
example. single input/output, $N = 200$



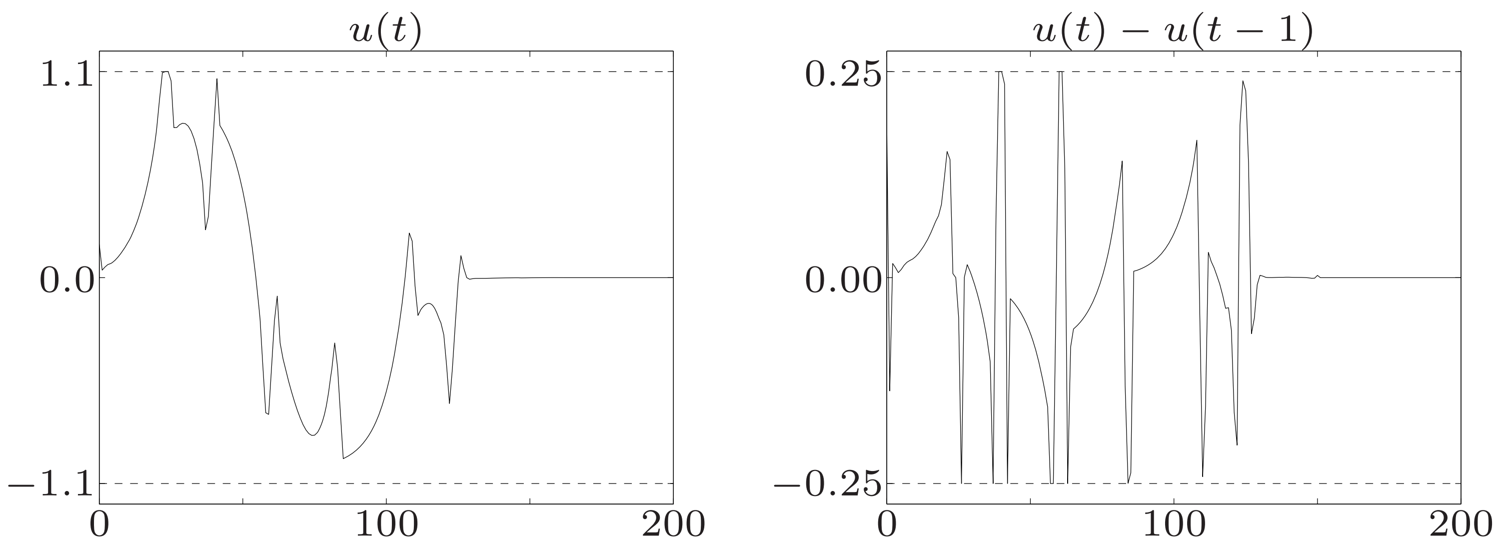
constraints on u:

- input horizon $M = 150$
- amplitude constraint $|u(t)| \leq 1.1$
- slew rate constraint $|u(t) - u(t-1)| \leq 0.25$

output and desired output:



optimal input sequence u:



Robust output tracking (1)

- impulse response is not exactly known; it can take two values:

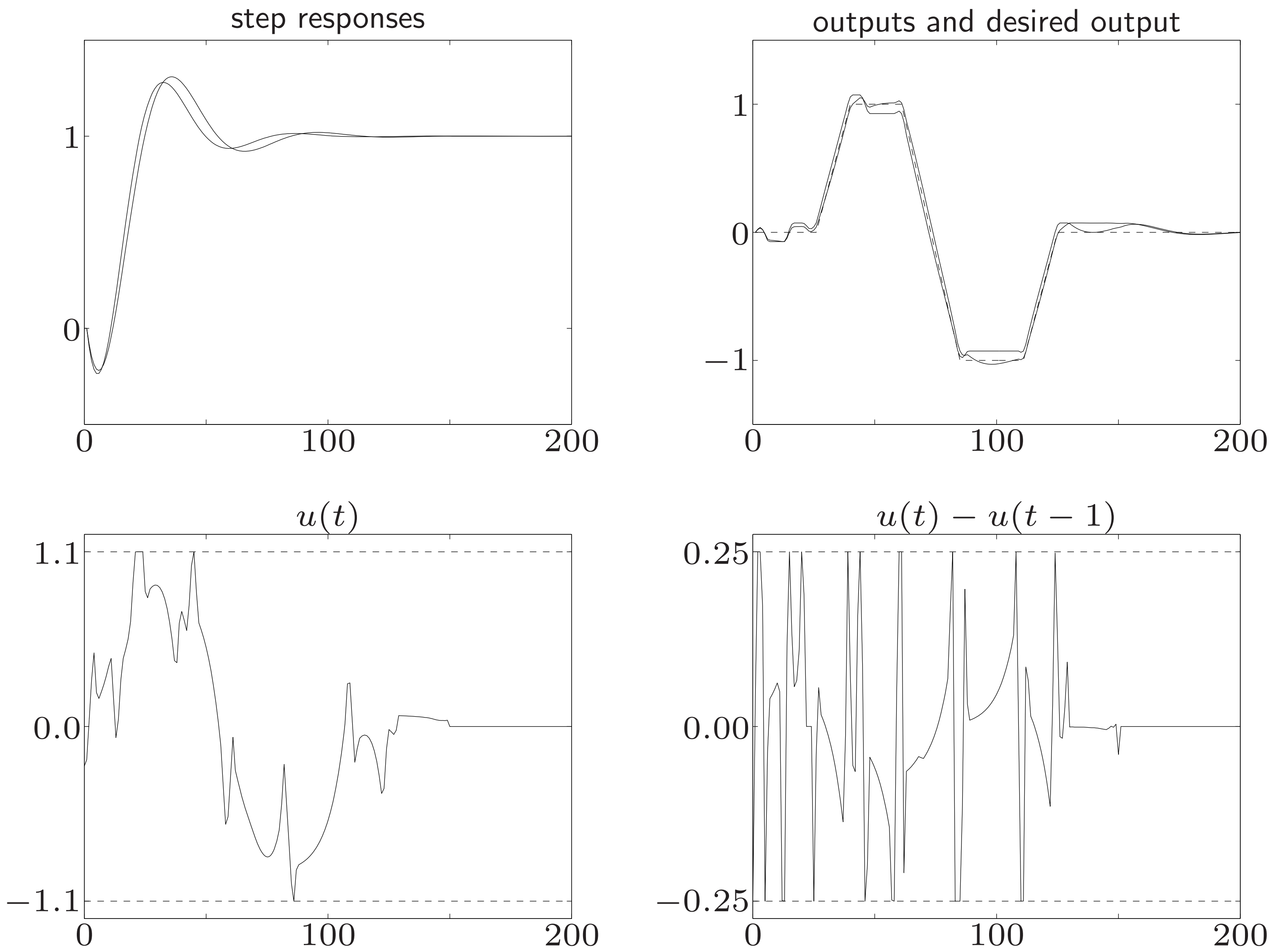
$$(h_0^{(1)}, h_1^{(1)}, \ldots, h_k^{(1)}), \qquad (h_0^{(2)}, h_1^{(2)}, \ldots, h_k^{(2)})$$

- design an input sequence that minimizes the worst-case peak tracking error

$$\begin{array}{ll} \text{minimize} & w \\ \text{subject to} & -w \leq \sum_{i=0}^{t} h_i^{(1)} u(t-i) - y_{\mathrm{des}}(t) \leq w, \quad t = 0, \ldots, N \\ & -w \leq \sum_{i=0}^{t} h_i^{(2)} u(t-i) - y_{\mathrm{des}}(t) \leq w, \quad t = 0, \ldots, N \\ & u(t) = 0, \quad t = M+1, \ldots, N \\ & -U \leq u(t) \leq U, \quad t = 0, \ldots, M \\ & -S \leq u(t+1) - u(t) \leq S, \quad t = 0, \ldots, M+1 \end{array}$$

an LP in the variables w, $u(0)$, . . . , $u(N)$

example



step responses
1
0
0
100
200
outputs and desired output
1
0
−1
0
100
200
$u(t)$
1.1
0.0
−1.1
0
100
200
$u(t) - u(t-1)$
0.25
0.00
−0.25
0
100
200

Robust output tracking (2)

$$\begin{bmatrix} h_0(s) \\ h_1(s) \\ \vdots \\ h_k(s) \end{bmatrix} = \begin{bmatrix} \bar{h}_0 \\ \bar{h}_1 \\ \vdots \\ \bar{h}_k \end{bmatrix} + s_1 \begin{bmatrix} v_0^{(1)} \\ v_1^{(1)} \\ \vdots \\ v_k^{(1)} \end{bmatrix} + \cdots + s_K \begin{bmatrix} v_0^{(K)} \\ v_1^{(K)} \\ \vdots \\ v_k^{(K)} \end{bmatrix}$$

$\bar{h}_i$ and $v_i^{(j)}$ are given; $s_i \in [-1, +1]$ is unknown

robust output tracking problem (variables w, $u(t)$):

$$\begin{array}{ll} \text{min.} & w \\ \text{s.t.} & -w \leq \sum_{i=0}^{t} h_i(s)u(t-i) - y_{\text{des}}(t) \leq w, \quad t = 0, \ldots, N, \quad \forall s \in [-1, 1]^K \\ & u(t) = 0, \quad t = M+1, \ldots, N \\ & -U \leq u(t) \leq U, \quad t = 0, \ldots, M \\ & -S \leq u(t+1) - u(t) \leq S, \quad t = 0, \ldots, M+1 \end{array}$$

straightforward (and very inefficient) solution: enumerate all 2^K extreme values of s

simplification: we can express the 2^{K+1} linear inequalities

$$-w \leq \sum_{i=0}^{t} h_i(s) u(t-i) - y_{\mathrm{des}}(t) \leq w \text{ for all } s \in \{-1,1\}^K$$

as two nonlinear inequalities

$$\sum_{i=0}^{t} \bar{h}_i u(t-i) + \sum_{j=1}^{K} \left| \sum_{i=0}^{t} v_i^{(j)} u(t-i) \right| \leq y_{\mathrm{des}}(t) + w$$

$$\sum_{i=0}^{t} \bar{h}_i u(t-i) - \sum_{j=1}^{K} \left| \sum_{i=0}^{t} v_i^{(j)} u(t-i) \right| \geq y_{\mathrm{des}}(t) - w$$

proof:

$$
\begin{aligned}
&\max_{s\in\{-1,1\}^K} \sum_{i=0}^{t} h_i(s)u(t-i) \\
&\quad = \sum_{i=0}^{t} \bar{h}_i u(t-i) + \sum_{j=1}^{K} \max_{s_j\in\{-1,+1\}} s_j \sum_{i=0}^{t} v_i^{(j)} u(t-i) \\
&\quad = \sum_{i=0}^{t} \bar{h}_i u(t-i) + \sum_{j=1}^{K} \left| \sum_{i=0}^{t} v_i^{(j)} u(t-i) \right|
\end{aligned}
$$

and similarly for the lower bound

robust output tracking problem reduces to:

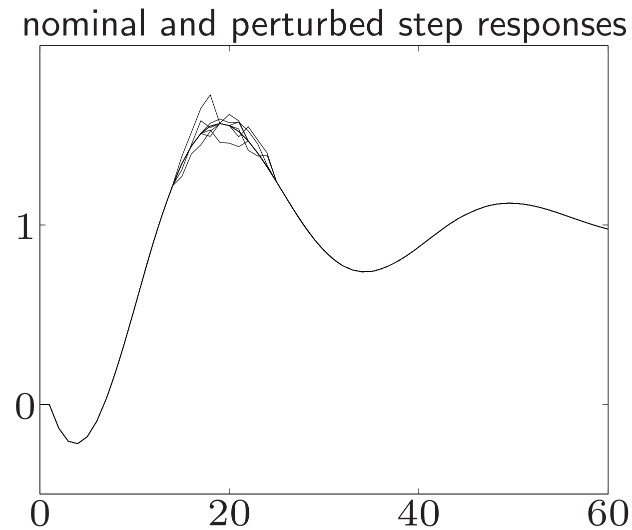
$$
\begin{array}{ll}
\text{min.} & w \\
\text{s.t.} & \sum_{i=0}^{t} \bar{h}_i u(t-i) + \sum_{j=1}^{K} \left| \sum_{i=0}^{t} v_i^{(j)} u(t-i) \right| \leq y_{\mathrm{des}}(t) + w, \quad t = 0, \ldots, N \\
& \sum_{i=0}^{t} \bar{h}_i u(t-i) - \sum_{j=1}^{K} \left| \sum_{i=0}^{t} v_i^{(j)} u(t-i) \right| \geq y_{\mathrm{des}}(t) - w, \quad t = 0, \ldots, N \\
& u(t) = 0, \quad t = M+1, \ldots, N \\
& -U \leq u(t) \leq U, \quad t = 0, \ldots, M \\
& -S \leq u(t+1) - u(t) \leq S, \quad t = 0, \ldots, M+1
\end{array}
$$

(variables $u(t)$, w)

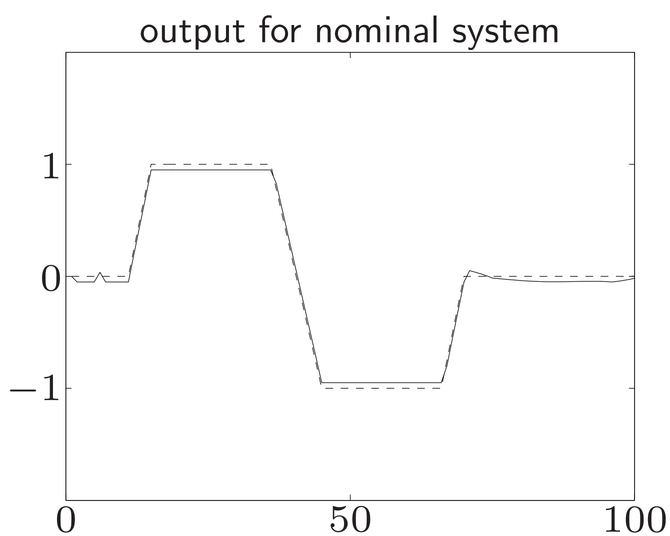
to express as an LP:

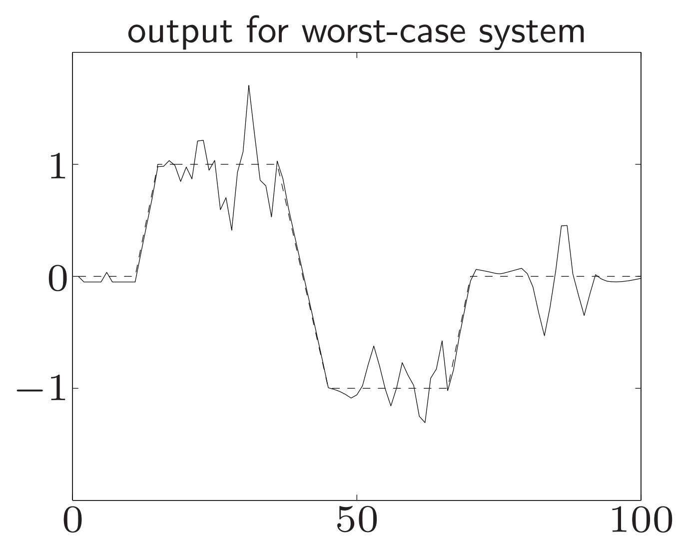
- for $t = 0, \ldots, N$, $j = 1, \ldots, K$, introduce new variables $p^{(j)}(t)$ and constraints
$$-p^{(j)}(t) \leq \sum_{i=0}^{t} v_i^{(j)} u(t-i) \leq p^{(j)}(t)$$
- replace $|\sum_i v_i^{(j)} u(t-i)|$ by $p^{(j)}(t)$

example $(K = 6)$

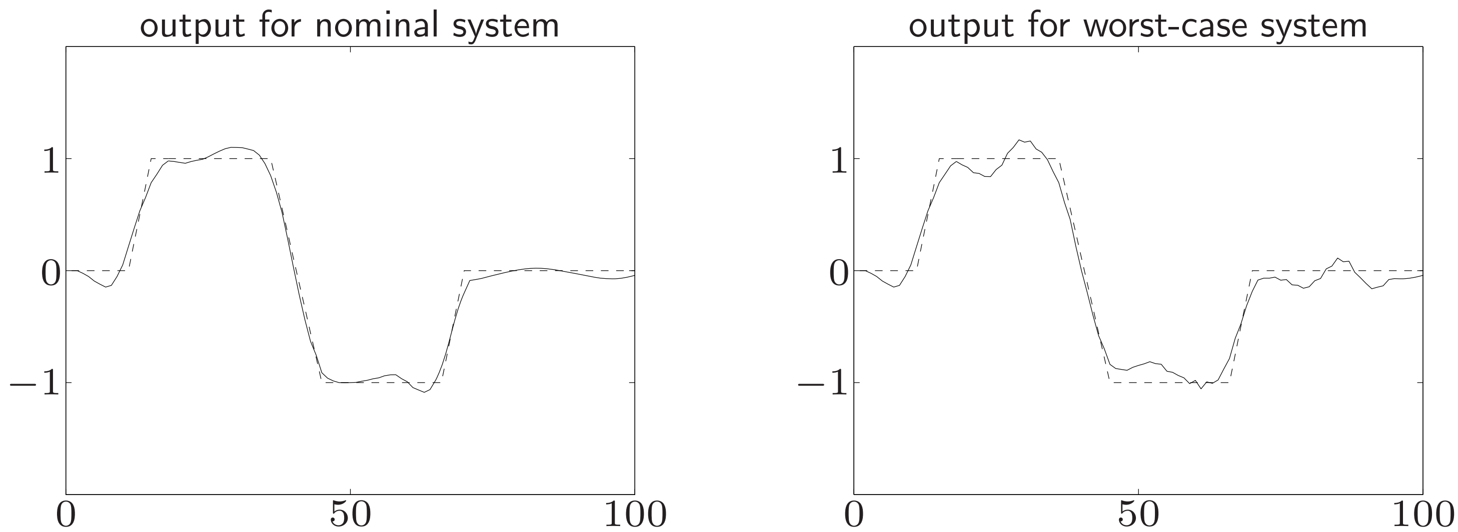


design for nominal system





robust design



State space description

input-output description:

$$y(t) = H_0 u(t) + H_1 u(t-1) + H_2 u(t-2) + \cdots$$

if $u(t) = 0$, $t < 0$:

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(N) \end{bmatrix} = \begin{bmatrix} H_0 & 0 & 0 & \cdots & 0 \\ H_1 & H_0 & 0 & \cdots & 0 \\ H_2 & H_1 & H_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_N & H_{N-1} & H_{N-2} & \cdots & H_0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(N) \end{bmatrix}$$

block Toeplitz structure (constant along diagonals)

state space model:

$$x(t+1) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t)$$

with $H_0 = D$, $H_i = CA^{i-1}B$ $(i > 0)$

$x(t) \in \mathbf{R}^n$ is *state sequence* alternative description:

$$
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ y(0) \\ y(1) \\ \vdots \\ y(N) \end{bmatrix}
=
\begin{bmatrix}
A & -I & 0 & \cdots & 0 & B & 0 & \cdots & 0 \\
0 & A & -I & \cdots & 0 & 0 & B & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -I & 0 & 0 & \cdots & B \\
C & 0 & 0 & \cdots & 0 & D & 0 & \cdots & 0 \\
0 & C & 0 & \cdots & 0 & 0 & D & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & C & 0 & 0 & \cdots & D
\end{bmatrix}
\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N) \\ u(0) \\ u(1) \\ \vdots \\ u(N) \end{bmatrix}
$$

- we don't eliminate the intermediate variables $x(t)$
- matrix is larger, but very sparse (interesting when using general-purpose LP solvers)

Pole placement

linear system

$$\dot{z}(t) = A(x)z(t), \qquad z(0) = z_0$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_p A_p \in \mathbf{R}^{n \times n}$

- solutions have the form

$$z_i(t) = \sum_k \beta_{ik} e^{\sigma_k t} \cos(\omega_k t - \phi_{ik})$$

  where $\lambda_k = \sigma_k \pm j\omega_k$ are the eigenvalues of $A(x)$

- $x \in \mathbf{R}^p$ is the design parameter
- goal: place eigenvalues of $A(x)$ in a desired region by choosing x

Low-authority control

eigenvalues of $A(x)$ are very complicated (nonlinear, nondifferentiable) functions of x

first-order perturbation: if $\lambda_i(A_0)$ is *simple*, then

$$\lambda_i(A(x)) = \lambda_i(A_0) + \sum_{k=1}^{p} \frac{w_i^* A_k v_i}{w_i^* v_i} x_k + o(\|x\|)$$

where w_i, v_i are the left and right eigenvectors:

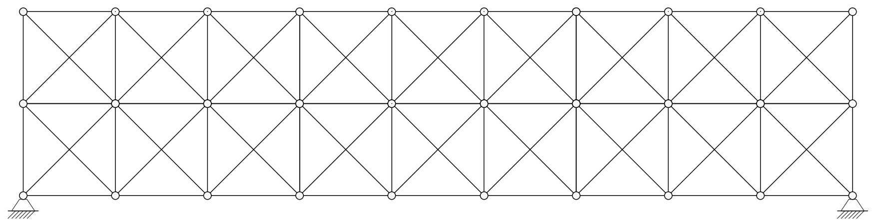
$$w_i^* A_0 = \lambda_i(A_0) w_i^*, \quad A_0 v_i = \lambda_i(A_0) v_i$$

'low-authority' control:

- use linear first-order approximations for λ_i
- can place λ_i in a polyhedral region by imposing linear inequalities on x
- we expect this to work only for small shifts in eigenvalues

Example

truss with 30 nodes, 83 bars



$$M\ddot{d}(t) + D\dot{d}(t) + Kd(t) = 0$$

- $d(t)$: vector of horizontal and vertical node displacements
- $M = M^T > 0$ (mass matrix): masses at the nodes
- $D = D^T > 0$ (damping matrix); $K = K^T > 0$ (stiffness matrix)

to increase damping, we attach dampers to the bars:

$$D(x) = D_0 + x_1 D_1 + \cdots + x_p D_p$$

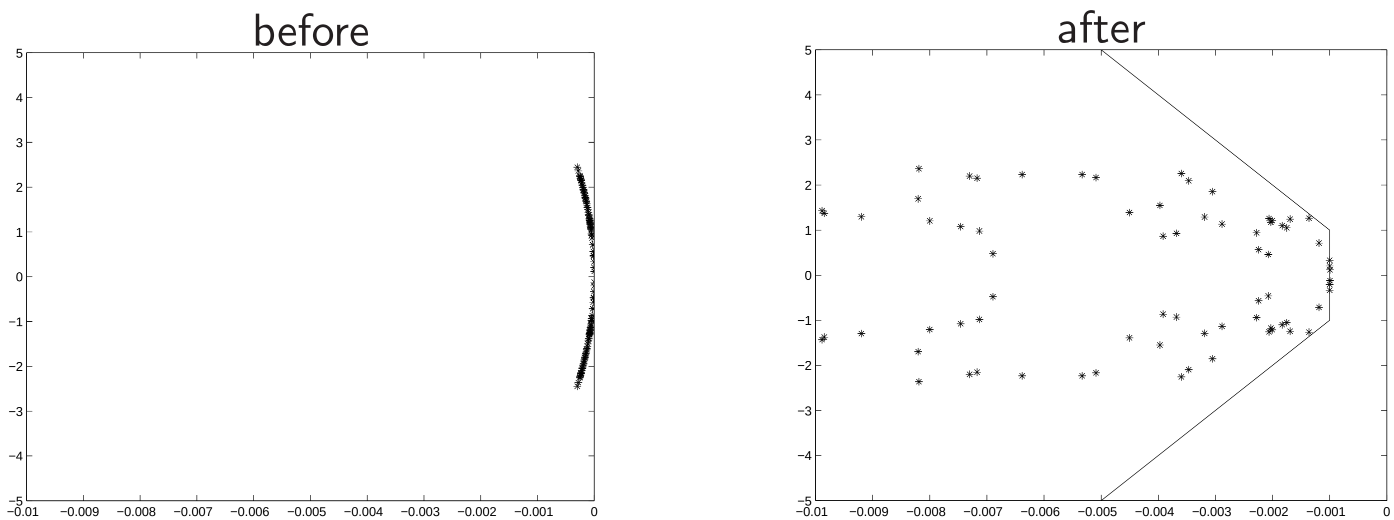
$x_i > 0$: amount of external damping at bar i

eigenvalue placement problem

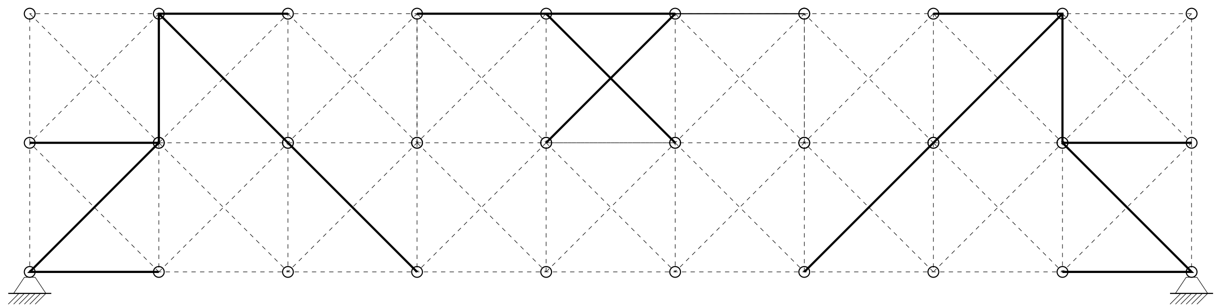
$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{p} x_i \\ \text{subject to} & \lambda_i(M, D(x), K) \in \mathcal{C}, \quad i = 1, \ldots, n \\ & x \geq 0 \end{array}$$

an LP if $\mathcal{C}$ is polyhedral and we use the 1st order approximation for λ_i

eigenvalues



location of dampers



Lecture 8
Duality (part 1)

- the dual of an LP in inequality form
- weak duality
- examples
- optimality conditions and complementary slackness
- Farkas' lemma and theorems of alternatives
- proof of strong duality

The dual of an LP in inequality form

LP in inequality form:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i, \quad i = 1, \ldots, m \end{array}$$

- n variables, m inequality constraints, optimal value $p^\star$
- called *primal problem* (in context of duality)

the dual LP (with $A = [a_1 \; a_2 \; \ldots \; a_m]^T$):

$$\begin{array}{ll} \text{maximize} & -b^T z \\ \text{subject to} & A^T z + c = 0 \\ & z \ge 0 \end{array}$$

- an LP in standard form with m variables, n equality constraints
- optimal value denoted $d^\star$

main property: $p^\star = d^\star$ (if primal or dual is feasible)

Weak duality

lower bound property:

if x is primal feasible and z is dual feasible, then

$$c^T x \geq -b^T z$$

proof: $c^T x \geq c^T x + \sum_{i=1}^m z_i(a_i^T x - b_i) = -b^T z$

$c^T x + b^T z$ is called the *duality gap* associated with x and z

weak duality: minimize over x, maximize over z:

$$p^\star \geq d^\star$$

always true (even when $p^\star = +\infty$ and/or $d^\star = -\infty$)

example

primal problem

$$\begin{array}{ll} \text{minimize} & -4x_1 - 5x_2 \\ \text{subject to} & \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} \end{array}$$

optimal point: $x^\star = (1, 1)$, optimal value $p^\star = -9$

dual problem

$$\begin{array}{ll} \text{maximize} & -3z_2 - 3z_4 \\ \text{subject to} & \begin{bmatrix} -1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} -4 \\ -5 \end{bmatrix} = 0 \\ & z_1 \geq 0, \; z_2 \geq 0, \; z_3 \geq 0, \; z_4 \geq 0 \end{array}$$

$z = (0, 1, 0, 2)$ is dual feasible with objective value -9

conclusion (by weak duality):

- z is a certificate that $x^\star$ is (primal) optimal
- $x^\star$ is a certificate that z is (dual) optimal

Piecewise-linear minimization

$$\text{minimize} \quad \max_{i=1,\ldots,m}(a_i^T x - b_i)$$

lower bounds for optimal value $p^\star$?

LP formulation (variables x, t)

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \left[\begin{array}{cc} A & -\mathbf{1} \end{array}\right] \left[\begin{array}{c} x \\ t \end{array}\right] \preceq b \end{array}$$

dual LP

$$\begin{array}{ll} \text{maximize} & -b^T z \\ \text{subject to} & \left[\begin{array}{c} A^T \\ -\mathbf{1}^T \end{array}\right] z + \left[\begin{array}{c} 0 \\ 1 \end{array}\right] = 0 \\ & z \succeq 0 \end{array}$$

(same optimal value)

Interpretation

lemma: if $z \geq 0$, $\sum_i z_i = 1$, then for all y,

$$\max_i y_i \geq \sum_i z_i y_i$$

hence, $\max_i(a_i^T x - b_i) \geq z^T(Ax - b)$

this yields a lower bound on $p^\star$:

$$p^\star = \min_x \max_i (a_i^T x - b_i) \geq \min_x z^T(Ax - b) = \begin{cases} -b^T z & \text{if } A^T z = 0 \\ -\infty & \text{otherwise} \end{cases}$$

to get best lower bound:

$$\begin{array}{ll} \text{maximize} & -b^T z \\ \text{subject to} & A^T z = 0 \\ & \mathbf{1}^T z = 1 \\ & z \geq 0 \end{array}$$

ℓ_∞-approximation

$$\text{minimize} \quad \|Ax - b\|_\infty$$

LP formulation

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} A & -\mathbf{1} \\ -A & -\mathbf{1} \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \leq \begin{bmatrix} b \\ -b \end{bmatrix} \end{array}$$

LP dual

$$\begin{array}{ll} \text{maximize} & -b^T w + b^T v \\ \text{subject to} & A^T w - A^T v = 0 \\ & \mathbf{1}^T w + \mathbf{1}^T v = 1 \\ & w, v \geq 0 \end{array} \tag{1}$$

can be expressed as

$$\begin{array}{ll} \text{maximize} & b^T z \\ \text{subject to} & A^T z = 0 \\ & \|z\|_1 \leq 1 \end{array} \tag{2}$$

proof of equivalence of (1) and (2)

- assume w, v feasible in (1), $i.e.$, $w \geq 0$, $v \geq 0$, $\mathbf{1}^T(w+v) = 1$
  - $z = v - w$ is feasible in (2):

$$\|z\|_1 = \sum_i |v_i - w_i| \leq \mathbf{1}^T v + \mathbf{1}^T w = 1$$

  - same objective value: $b^T z = b^T v - b^T w$

- assume z is feasible in (2), $i.e.$, $A^T z = 0$, $\|z\|_1 \leq 1$
  - $w_i = \max\{z_i, 0\} + \alpha$, $v_i = \max\{-z_i, 0\} + \alpha$, with $\alpha = (1 - \|z\|_1)/(2m)$, are feasible in (1):

$$v, w \geq 0, \qquad \mathbf{1}^T w + \mathbf{1}^T v = 1$$

  - same objective value: $b^T v - b^T w = b^T z$

Interpretation

lemma: $u^T v \leq \|u\|_1 \|v\|_\infty$

hence, for every z with $\|z\|_1 \leq 1$, we have a lower bound on $\|Ax - b\|_\infty$:

$$\|Ax - b\|_\infty \geq z^T(Ax - b)$$

$$p^\star = \min_x \|Ax - b\|_\infty \geq \min_x z^T(Ax - b) = \begin{cases} -b^T z & \text{if } A^T z = 0 \\ -\infty & \text{otherwise} \end{cases}$$

to get best lower bound

$$\begin{array}{ll} \text{maximize} & -b^T z \\ \text{subject to} & A^T z = 0 \\ & \|z\|_1 \leq 1 \end{array}$$

Optimality conditions

primal feasible x is optimal if and only if there is a dual feasible z with

$$c^T x = -b^T z$$

i.e., associated duality gap is zero

complementary slackness: for x, z optimal,

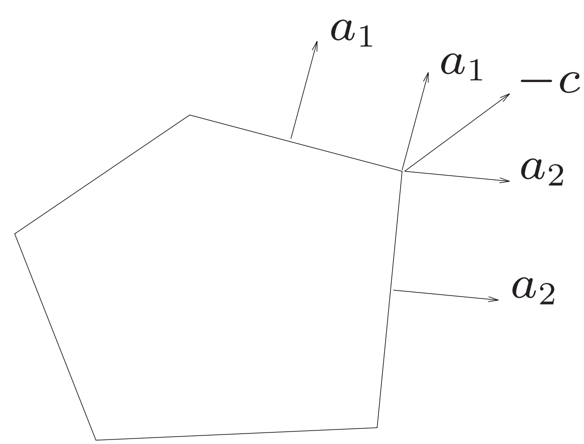
$$c^T x + b^T z = \sum_{i=1}^{m} z_i (b_i - a_i^T x) = 0$$

hence for each i, $a_i^T x = b_i$ or $z_i = 0$:

- $z_i > 0 \Longrightarrow a_i^T x = b_i$ (ith inequality is active at x)
- $a_i^T x < b_i \Longrightarrow z_i = 0$

Geometric interpretation

example in $\mathbf{R}^2$:



- two active constraints at optimum ($a_1^T x^\star = b_1$, $a_2^T x^\star = b_2$)
- optimal dual solution satisfies

$$-c = A^T z, \qquad z \geq 0, \qquad z_i = 0 \text{ for } i \neq 1, 2,$$

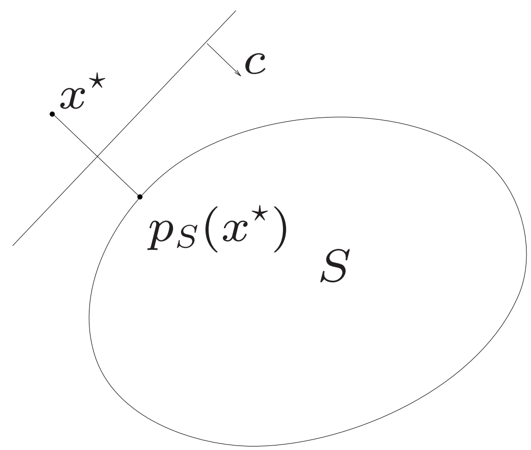
  i.e., $-c = a_1 z_1 + a_2 z_2$
- geometrically, $-c$ lies in the cone generated by a_1 and a_2

Separating hyperplane theorem

if $S \subseteq \mathbf{R}^n$ is a nonempty, closed, convex set, and $x^\star \notin S$, then there exists $c \neq 0$ such that

$$c^T x^\star < c^T x \text{ for all } x \in S,$$

$i.e.$, for some value of d, the hyperplane $c^T x = d$ *separates* $x^\star$ from S



idea of proof: use $c = p_S(x^\star) - x^\star$, where $p_S(x^\star)$ is the projection of $x^\star$ on S, $i.e.$,

$$p_S(x^\star) = \operatorname*{argmin}_{x \in S} \|x^\star - x\|$$

Farkas' lemma

given A, b, exactly one of the following two statements is true:

1. there is an $x \geq 0$ such that $Ax = b$
2. there is a y such that $A^T y \geq 0$, $b^T y < 0$

very useful in practice: any y in 2 is a *certificate* or *proof* that $Ax = b$, $x \geq 0$ is infeasible, and vice-versa

proof (easy part): we have a contradiction if 1 and 2 are both true:

$$0 = y^T(Ax - b) \geq -b^T y > 0$$

proof (difficult part): $\neg 1 \implies 2$

- $\neg 1$ means $b \notin S = \{Ax \mid x \geq 0\}$
- S is nonempty, closed, and convex (the image of the nonnegative orthant under a linear mapping)
- hence there exists a y s.t.

$$y^T b < y^T Ax \text{ for all } x \geq 0$$

  implies:

  - $y^T b < 0$ (choose $x = 0$)
  - $A^T y \geq 0$ (if $(A^T y)_k < 0$ for some k, we can choose $x_i = 0$ for $i \neq k$, and $x_k \to +\infty$; then $y^T Ax \to -\infty$)

i.e., 2 is true

Theorems of alternatives

many variations on Farkas' lemma: *e.g.*, for given $A \in \mathbf{R}^{m \times n}$ $b \in \mathbf{R}^m$, exactly one of the following statements is true:

1. there is an x with $Ax \preceq b$
2. there is a $y \succeq 0$ with $A^T y = 0$, $b^T y < 0$

proof
(easy half): 1 and 2 together imply $0 \leq (b - Ax)^T y = b^T y < 0$

(difficult half): if 1 does not hold, then

$$b \notin S = \{Ax + s \mid x \in \mathbf{R}^n, s \in \mathbf{R}^m, s \succeq 0\}$$

hence, there is a separating hyperplane, *i.e.*, $y \neq 0$ subject to

$$y^T b < y^T (Ax + s) \text{ for all } x \text{ and all } s \succeq 0$$

equivalent to $b^T y < 0$, $A^T y = 0$, $y \succeq 0$ (*i.e.*, 2 is true)

Proof of strong duality

strong duality: $p^\star = d^\star$ (except possibly when $p^\star = +\infty$, $d^\star = -\infty$)

suppose $p^\star$ is finite, and $x^\star$ is optimal with

$$a_i^T x^\star = b_i, \quad i \in I, \qquad a_i^T x^\star < b_i, \quad i \notin I$$

we'll show there is a dual feasible z with $-b^T z = c^T x^\star$

- $x^\star$ optimal implies that the set of inequalities

$$a_i^T d \leq 0, \quad i \in I, \qquad c^T d < 0 \tag{1}$$

is infeasible; otherwise we would have for small $t > 0$

$$a_i^T (x^\star + td) \leq b_i, \quad i = 1, \ldots, m, \qquad c^T (x^\star + td) < c^T x^\star$$

- from Farkas' lemma: (1) is infeasible if and only if there exists λ_i, $i \in I$,

$$\lambda_i \geq 0, \qquad \sum_{i \in I} \lambda_i a_i = -c$$

this yields a dual feasible z:

$$z_i = \lambda_i, \quad i \in I, \qquad z_i = 0, \;\; i \notin I$$

- z is dual optimal:

$$-b^T z = -\sum_{i \in I} b_i z_i = -\sum_{i \in I} (a_i^T x^\star) z_i = -z^T A x^\star = c^T x^\star$$

this proves: $p^\star$ finite $\Longrightarrow d^\star = p^\star$

exercise: $p^\star = +\infty \Longrightarrow d^\star = +\infty$ or $d^\star = -\infty$

Summary

possible cases:

- $p^\star = d^\star$ and finite: primal and dual optima are attained
- $p^\star = d^\star = +\infty$: primal is infeasible; dual is feasible and unbounded
- $p^\star = d^\star = -\infty$: primal is feasible and unbounded; dual is infeasible
- $p^\star = +\infty$, $d^\star = -\infty$: primal and dual are infeasible

uses of duality:

- dual optimal z provides a proof of optimality for primal feasible x
- dual feasible z provides a lower bound on $p^\star$ (useful for stopping criteria)
- sometimes it is easier to solve the dual
- modern interior-point methods solve primal and dual simultaneously

Lecture 9
Duality (part 2)

- duality in algorithms
- sensitivity analysis via duality
- duality for general LPs
- examples
- mechanics interpretation
- circuits interpretation
- two-person zero-sum games

Duality in algorithms

many algorithms produce at iteration k

- a primal feasible $x^{(k)}$
- and a dual feasible $z^{(k)}$

with $c^T x^{(k)} + b^T z^{(k)} \to 0$ as $k \to \infty$

hence at iteration k we **know** $p^\star \in \left[-b^T z^{(k)}, c^T x^{(k)}\right]$

- useful for stopping criteria
- algorithms that use dual solution are often more efficient

Nonheuristic stopping criteria

- (absolute error) $c^T x^{(k)} - p^\star$ is less than ϵ if

$$c^T x^{(k)}) + b^T z^{(k)} < \epsilon$$

- (relative error) $(c^T x^{(k)} - p^\star)/|p^\star|$ is less than ϵ if

$$-b^T z^{(k)} > 0 \quad \& \quad \frac{c^T x^{(k)} + b^T z^{(k)}}{-b^T z^{(k)}} \leq \epsilon$$

or

$$c^T x^{(k)} < 0 \quad \& \quad \frac{c^T x^{(k)}) + b^T z^{(k)}}{-c^T x^{(k)}} \leq \epsilon$$

- target value ℓ is achievable ($p^\star \leq \ell$) if

$$c^T x^{(k)}) \leq \ell$$

- target value ℓ is unachievable ($p^\star > \ell$) if

$$-b^T z^{(k)} > \ell$$

Sensitivity analysis via duality

perturbed problem:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b + \epsilon d \end{array}$$

$A \in \mathbf{R}^{m \times n}$; $d \in \mathbf{R}^m$ given; optimal value $p^\star(\epsilon)$

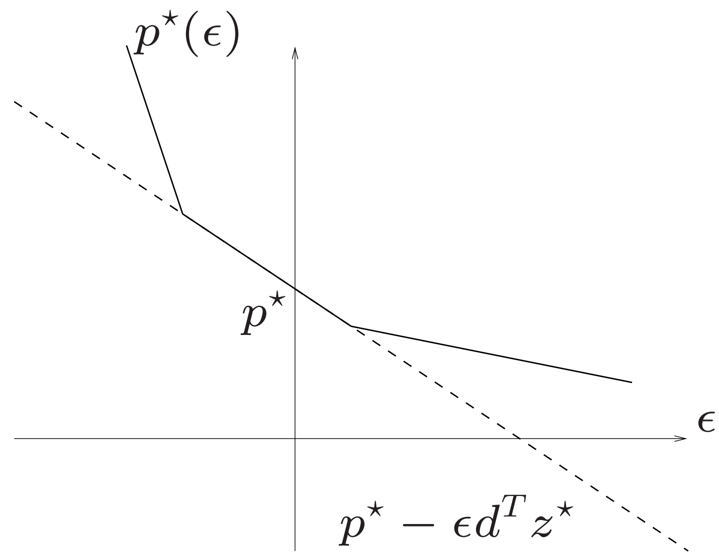
global sensitivity result: if $z^\star$ is (any) dual optimal solution for the unperturbed problem, then for all ϵ

$$p^\star(\epsilon) \geq p^\star - \epsilon d^T z^\star$$

proof. $z^\star$ is dual feasible for all ϵ; by weak duality,

$$p^\star(\epsilon) \geq -(b + \epsilon d)^T z^\star = p^\star - \epsilon d^T z^\star$$

interpretation



- $d^T z^\star > 0$: $\epsilon < 0$ increases $p^\star$
- $d^T z^\star > 0$ and large: $\epsilon < 0$ greatly increases $p^\star$
- $d^T z^\star > 0$ and small: $\epsilon > 0$ does not decrease $p^\star$ too much
- $d^T z^\star < 0$: $\epsilon > 0$ increases $p^\star$
- $d^T z^\star < 0$ and large: $\epsilon > 0$ greatly increases $p^\star$
- $d^T z^\star < 0$ and small: $\epsilon > 0$ does not decrease $p^\star$ too much

Local sensitivity analysis

assumption: there is a nondegenerate optimal vertex $x^\star$, *i.e.*,

- $x^\star$ is an optimal *vertex*: $\mathbf{rank}\,\bar{A} = n$, where

$$\bar{A} = \begin{bmatrix} a_{i_1} & a_{i_2} & \cdots & a_{i_K} \end{bmatrix}^T$$

  and $I = \{i_1, \ldots, i_K\}$ is the set of active constraints at $x^\star$

- $x^\star$ is nondegenerate: $\bar{A} \in \mathbf{R}^{n \times n}$

w.l.o.g. we assume $I = \{1, 2, \ldots, n\}$

consequence: dual optimal $z^\star$ is unique

proof: by complementary slackness, $z_i^\star = 0$ for $i > n$

by dual feasibility,

$$\sum_{i=1,\ldots,n} a_i z_i^\star = \bar{A}^T \begin{bmatrix} z_1^\star \\ \vdots \\ z_n^\star \end{bmatrix} = -c \quad \Longrightarrow \quad z^\star = \begin{bmatrix} -\bar{A}^{-T} c \\ 0 \end{bmatrix}$$

optimal solution of the perturbed problem (for small ϵ):

$$x^\star(\epsilon) = x^\star + \epsilon \bar{A}^{-1}\bar{d} \quad \text{(with } \bar{d} = (d_1, \ldots, d_n)\text{)}$$

- $x^\star(\epsilon)$ is feasible for small ϵ:

$$a_i^T x^\star(\epsilon) = b_i + \epsilon d_i, \quad i = 1, \ldots, n, \qquad a_i^T x^\star(\epsilon) < b_i + \epsilon d_i, \quad i = n+1, \ldots, m$$

- $z^\star$ is dual feasible and satisfies complementary slackness:

$$(b + \epsilon d - A x^\star(\epsilon))^T z^\star = 0$$

optimal value of perturbed problem (for small ϵ):

$$p^\star(\epsilon) = c^T x^\star(\epsilon) = p^\star + \epsilon c^T \bar{A}^{-1}\bar{d} = p^\star - \epsilon d^T z^\star$$

- $z_i^\star$ is sensitivity of cost w.r.t. righthand side of ith constraint
- $z_i^\star$ is called *marginal cost* or *shadow price* associated with ith constraint

Dual of a general LP

method 1: express as LP in inequality form and take its dual

example: standard form LP

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & \begin{bmatrix} -I \\ A \\ -A \end{bmatrix} x \leq \begin{bmatrix} 0 \\ b \\ -b \end{bmatrix}
\end{array}
$$

dual:

$$
\begin{array}{ll}
\text{maximize} & -b^T(v-w) \\
\text{subject to} & -u + A^T(v-w) + c = 0 \\
& u \geq 0, \quad v \geq 0, \quad w \geq 0
\end{array}
$$

method 2: apply Lagrange duality (this lecture)

Lagrangian

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i, \quad i = 1, \ldots, m \end{array}$$

Lagrangian $L : \mathbf{R}^{n+m} \to \mathbf{R}$

$$L(x, \lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

- λ_i are called *Lagrange multipliers*
- objective is *augmented* with weighted sum of constraint functions

lower bound property: if $Ax \le b$ and $\lambda \ge 0$, then

$$c^T x \ge L(x, \lambda) \ge \min_{\tilde{x}} L(\tilde{x}, \lambda)$$

hence, $p^\star \ge \min_{\tilde{x}} L(\tilde{x}, \lambda)$ for $\lambda \ge 0$

Lagrange dual problem

Lagrange dual function $g : \mathbf{R}^m \to \mathbf{R} \cup \{-\infty\}$

$$\begin{aligned} g(\lambda) = \min_x L(x,\lambda) &= \min_x \left(-b^T\lambda + (A^T\lambda + c)^T x\right) \\ &= \begin{cases} -b^T\lambda & \text{if } A^T\lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

(Lagrange) dual problem

$$\begin{array}{ll} \text{maximize} & g(\lambda) \\ \text{subject to} & \lambda \geq 0 \end{array}$$

yields the dual LP:

$$\begin{array}{ll} \text{maximize} & -b^T\lambda \\ \text{subject to} & A^T\lambda + c = 0, \quad \lambda \geq 0 \end{array}$$

finds best lower bound $g(\lambda)$

Lagrangian of a general LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \\ & g_i^T x = h_i, \quad i = 1, \ldots, p \end{array}$$

define **Lagrangian** $L : \mathbf{R}^{n+m+p} \to \mathbf{R}$ as

$$L(x, \lambda, \nu) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i) + \sum_{i=1}^{p} \nu_i (g_i^T x - h_i)$$

lower bound property: if x is feasible and $\lambda \geq 0$,

$$c^T x \geq L(x, \lambda, \nu) \geq \min_{\tilde{x}} L(\tilde{x}, \lambda, \nu)$$

hence, $p^\star \geq \min_x L(x, \lambda, \nu)$ if $\lambda \geq 0$

multipliers associated with equality constraints can have either sign

Lagrange dual function:

$$
\begin{aligned}
g(\lambda,\nu) = \min_x L(x,\lambda,\nu) &= \min_x \left(c^T x + \lambda^T(Ax-b) + \nu^T(Gx-h)\right) \\
&= \min_x \left(-b^T\lambda - h^T\nu + x^T(c + A^T\lambda + G^T\nu)\right) \\
&= \begin{cases} -b^T\lambda - h^T\nu & \text{if } A^T\lambda + G^T\nu + c = 0 \\ -\infty & \text{otherwise} \end{cases}
\end{aligned}
$$

Lagrange dual problem:

$$
\begin{array}{ll}
\text{maximize} & -b^T\lambda - h^T\nu \\
\text{subject to} & A^T\lambda + G^T\nu + c = 0 \\
& \lambda \geq 0
\end{array}
$$

variables λ, ν; optimal value $d^\star$

- an LP (in general form)
- weak duality $p^\star \geq d^\star$
- strong duality holds: $p^\star = d^\star$ (except when both problems are infeasible)

Example: standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \geq 0 \end{array}$$

Lagrangian: $L(x, \nu, \lambda) = c^T x + \nu^T (Ax - b) - \lambda^T x$

dual function

$$g(\lambda, \nu) = \min_x L(x, \nu, \lambda) = \begin{cases} -b^T \nu & \text{if } A^T \nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

dual problem

$$\begin{array}{ll} \text{maximize} & -b^T \nu \\ \text{subject to} & A^T \nu - \lambda + c = 0, \quad \lambda \geq 0 \end{array}$$

equivalent to dual on page 9–9

$$\begin{array}{ll} \text{maximize} & b^T z \\ \text{subject to} & A^T z \leq c \end{array}$$

Price or tax interpretation

- x: describes how an enterprise operates; $c^T x$: cost of operating at x
- $a_i^T x \leq b_i$: limits on resources, regulatory limits

optimal operating point is solution of

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \dots, m \end{array}$$

optimal cost: $p^\star$

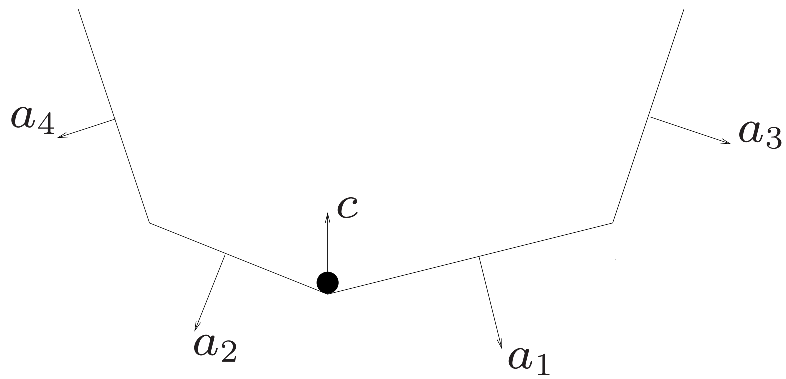
scenario 2: constraint violations can be bought or sold at unit cost $\lambda_i \geq 0$

$$\text{minimize} \quad c^T x + \sum_{i=1}^m \lambda_i (a_i^T x - b_i)$$

optimal cost: $g(\lambda)$

interpretation of strong duality: there exist prices $\lambda_i^\star$ s.t. $g(\lambda^\star) = p^\star$, $i.e.$, there is no advantage in selling/buying constraint violations

Mechanics interpretation



- mass subject to gravity, can move freely between walls described by $a_i^T x = b_i$
- equilibrium position minimizes potential energy, *i.e.*, solves

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

optimality conditions:

1. $a_i^T x^\star \leq b_i$, $i = 1. \ldots, m$

2. $\sum_{i=1}^m z_i^\star a_i + c = 0$, $z^\star \geq 0$

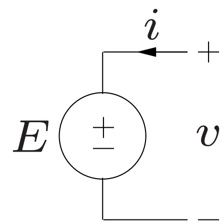
3. $z_i^\star (b_i - a_i^T x^\star) = 0$, $i = 1, \ldots, m$

interpretation: $-z_i a_i$ is contact force with wall i; nonzero only if the ball touches the ith wall

Circuits interpretation

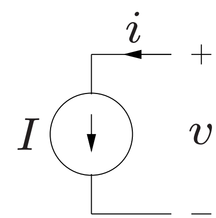
circuit components:

ideal voltage source



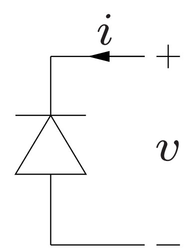
$$v = E$$

ideal current source



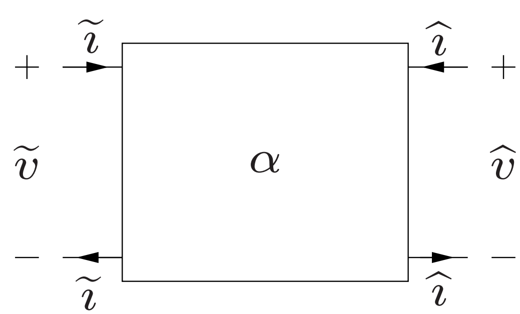
$$i = I$$

ideal diode



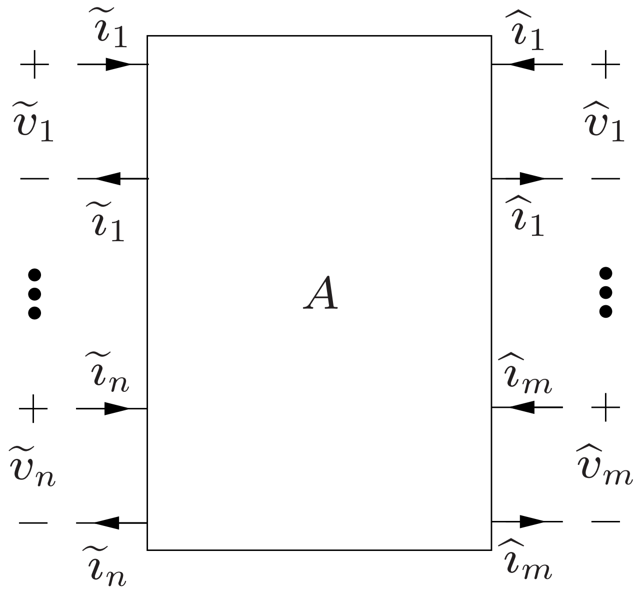
$$v \geq 0, \quad i \leq 0$$
$$vi = 0$$

ideal transformer



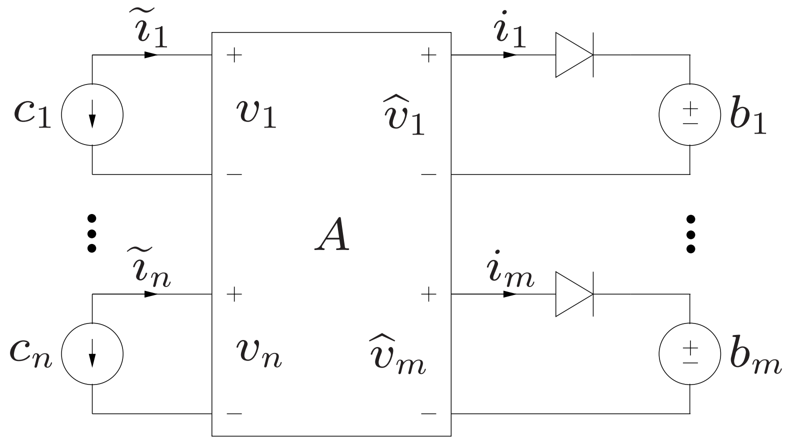
$$\begin{aligned} \hat{v} &= \alpha \tilde{v} \\ \tilde{\imath} &= -\alpha \hat{\imath} \end{aligned}$$

ideal multiterminal transformer ($A \in \mathbf{R}^{m \times n}$)



$$\widehat{v} = A\widetilde{v}$$
$$\widetilde{\imath} = -A^T\widehat{\imath}$$

example



circuit equations:

$$\widehat{v} = Av \le b, \qquad i \ge 0, \qquad \widetilde{i} + c = A^T i + c = 0$$

$$i_k(b_k - a_k^T v) = 0, \quad k = 1, \ldots, m$$

i.e., optimality conditions for LP

$$\begin{array}{ll} \text{minimize} & c^T v \\ \text{subject to} & Av \le b \end{array} \qquad \begin{array}{ll} \text{maximize} & -b^T i \\ \text{subject to} & A^T i + c = 0 \\ & i \ge 0 \end{array}$$

interpretation: two 'potential functions'

- *content* (a function of the voltages)
- *co-content* (a function of the currents)

contribution of each component (notation of page 9–18)

- content of current source is Iv
  co-content is 0 if $i = I$, $-\infty$ otherwise
- content of voltage source is 0 if $v = E$, ∞ otherwise
  co-content is $-Ei$
- content of diode is 0 if $v \geq 0$, $+\infty$ otherwise
  co-content is 0 if $i \leq 0$ and $-\infty$ otherwise
- content of transformer is 0 if $\widehat{v} = A\widetilde{v}$, ∞ otherwise
  co-content is 0 if $\widetilde{i} = -A^T\widehat{\imath}$, $-\infty$ otherwise

primal problem: voltages minimize total content

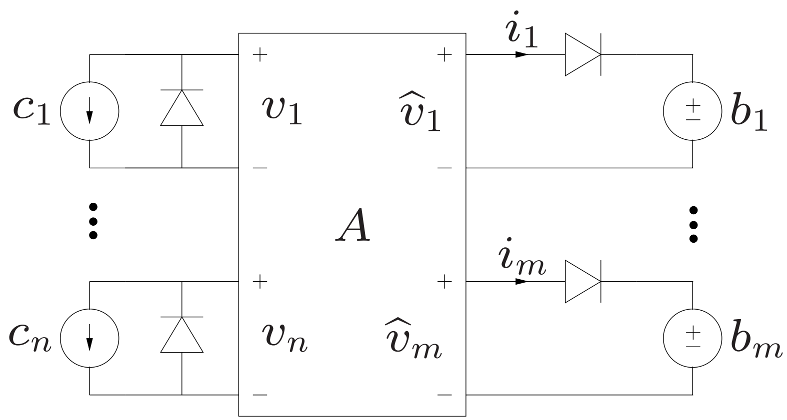
dual problem: currents maximize total co-content

example

primal problem

$$\begin{array}{ll} \text{minimize} & c^T v \\ \text{subject to} & Av \leq b \\ & v \geq 0 \end{array}$$

circuit equivalent:



dual problem:

$$\begin{array}{ll} \text{maximize} & -b^T i \\ \text{subject to} & A^T i + c \geq 0 \\ & i \geq 0 \end{array}$$

Two-person zero-sum games (matrix games)

described by a **payoff matrix**

$$A \in \mathbf{R}^{m \times n}$$

- player 1 chooses a number in $\{1, \ldots, m\}$ (corresponding to m possible actions or strategies)
- player 2 chooses a number in $\{1, \ldots, n\}$
- players make their choice simultaneously and independently
- if P1's choice is i and P2's choice is j, then P1 pays a_{ij} to P2 (negative a_{ij} means P2 pays $-a_{ij}$ to P1)

Mixed (randomized) strategies

players make random choices according to some probability distribution

- player 1 chooses randomly according to distribution $x \in \mathbf{R}^m$:

$$\mathbf{1}^T x = 1, \qquad x \geq 0$$

  (x_i is probability of choosing i)

- player 2 chooses randomly (and independently from 1) according to distribution $y \in \mathbf{R}^n$:

$$\mathbf{1}^T y = 1, \qquad y \geq 0$$

  (y_j is probability of choosing j)

expected payoff from player 1 to 2, if they use mixed stragies x and y:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j a_{ij} = x^T A y$$

Optimal mixed strategies

optimal strategy for player 1:

$$\begin{array}{ll} \text{minimize}_x & \max_{\mathbf{1}^T y=1,\, y\geq 0} x^T A y \\ \text{subject to} & \mathbf{1}^T x = 1, \quad x \geq 0 \end{array}$$

note:

$$\max_{\mathbf{1}^T y=1,\, y\geq 0} x^T A y = \max_{j=1,\ldots,n} (A^T x)_j$$

optimal strategy $x^\star$ can be computed by solving an LP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A^T x \leq t\mathbf{1} \\ & \mathbf{1}^T x = 1, \quad x \geq 0 \end{array} \qquad (1)$$

(variables x, t)

optimal strategy for player 2:

$$\begin{array}{ll} \text{maximize}_y & \min_{\mathbf{1}^T x = 1,\, x \succeq 0} x^T A y \\ \text{subject to} & \mathbf{1}^T y = 1, \quad y \succeq 0 \end{array}$$

note:

$$\min_{\mathbf{1}^T x = 1,\, x \succeq 0} x^T A y = \min_{i=1,\ldots,m} (Ay)_i$$

optimal strategy $y^\star$ can be computed by solving an LP

$$\begin{array}{ll} \text{maximize} & w \\ \text{subject to} & Ay \succeq w\mathbf{1} \\ & \mathbf{1}^T y = 1, \quad y \succeq 0 \end{array} \qquad (2)$$

(variables y, w)

The minimax theorem

for all mixed strategies x, y,

$$x^{\star T} A y \leq x^{\star T} A y^{\star} \leq x^T A y^{\star}$$

proof: the LPs (1) and (2) are duals, so they have the same optimal value

example

$$A = \begin{bmatrix} 4 & 2 & 0 & -3 \\ -2 & -4 & -3 & 3 \\ -2 & -3 & 4 & 1 \end{bmatrix}$$

optimal strategies

$$x^{\star} = (0.37, 0.33, 0.3), \qquad y^{\star} = (0.4, 0, 0.13, 0.47)$$

expected payoff: $x^{\star T} A y^{\star} = 0.2$

Lecture 10
The simplex method

- extreme points
- adjacent extreme points
- one iteration of the simplex method
- degeneracy
- initialization
- numerical implementation

Idea of the simplex method

move from one extreme point to an adjacent extreme point with lower cost until an optimal extreme point is reached

- invented in 1947 (George Dantzig)
- usually developed and implemented for LPs in standard form

questions

1. how do we characterize extreme points? (answered in lecture 3)
2. how do we move from an extreme point to an adjacent one?
3. how do we select an adjacent extreme point with a lower cost?
4. how do we find an initial extreme point?

Extreme points

to check whether x is an extreme point of a polyhedron defined by

$$a_i^T x \leq b_i, \quad i = 1, \ldots, m$$

- check that $Ax \leq b$
- define

$$A_I = \begin{bmatrix} a_{i_1}^T \\ a_{i_2}^T \\ \vdots \\ a_{i_K}^T \end{bmatrix}, \qquad I = \{i_1, \ldots, i_K\}$$

  where I is the set of active constraints at x:

$$a_k^T x = b_k, \quad k \in I, \qquad a_k^T x < b_k, \quad k \notin I$$

- x is an extreme point if and only if $\mathbf{rank}(A_I) = n$

Degeneracy

an extreme point is **nondegenerate** if exactly n constraints are active at x

- A_I is square and nonsingular ($K = n$)
- $x = A_I^{-1} b_I$, where $b_I = (b_{i_1}, b_{i_2}, \ldots, b_{i_n})$

an extreme point is **degenerate** if more than n constraints are active at x

- extremality is a geometric property (depends on $\mathcal{P}$)
- degeneracy/nondegeneracy depend on the representation of $\mathcal{P}$ (*i.e.*, A and b)

Assumptions

we will develop the simplex algorithm for an LP in inequality form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

with $A \in \mathbf{R}^{m \times n}$

we assume throughout the lecture that $\mathbf{rank}(A) = n$

- if $\mathbf{rank}(A) < n$, we can reduce the number of varables
- implies that the polyhedron has at least one extreme point (page 3–25)
- implies that if the LP is solvable, it has an optimal extreme point (page 3–28)

until page 10–20 we assume that all the extreme points are nondegenerate

Adjacent extreme points

extreme points are **adjacent** if they have $n-1$ common active constraints

moving to an adjacent extreme point

given extreme point x with active index set I and an index $k \in I$, find an extreme point $\hat{x}$ that has the active constraints $I \setminus \{k\}$ in common with x

1. solve the n equations

$$a_i^T \Delta x = 0, \quad i \in I \setminus \{k\}, \qquad a_k^T \Delta x = -1$$

2. if $A\Delta x \le 0$, then $\{\hat{x} + \alpha \Delta x \mid \alpha \ge 0\}$ is a feasible half-line:

$$A(x + \alpha \Delta x) \le b \quad \forall \alpha \ge 0$$

3. otherwise, $\hat{x} = x + \alpha \Delta x$, where

$$\alpha = \min_{i : a_i^T \Delta x > 0} \frac{b_i - a_i^T x}{a_i^T \Delta x}$$

comments

- step 1: equations are solvable because A_I is nonsingular
- step 3: $\alpha > 0$ because $a_i^T \Delta x > 0$ means $i \notin I$, hence $a_i^T x < b_i$ (for nondegenerate x)
- new active set is $\hat{I} = I \setminus \{k\} \cup \{j\}$ where

$$j = \underset{i : a_i^T \Delta x > 0}{\operatorname{argmin}} \frac{b_i - a_i^T x}{a_i^T \Delta x}$$

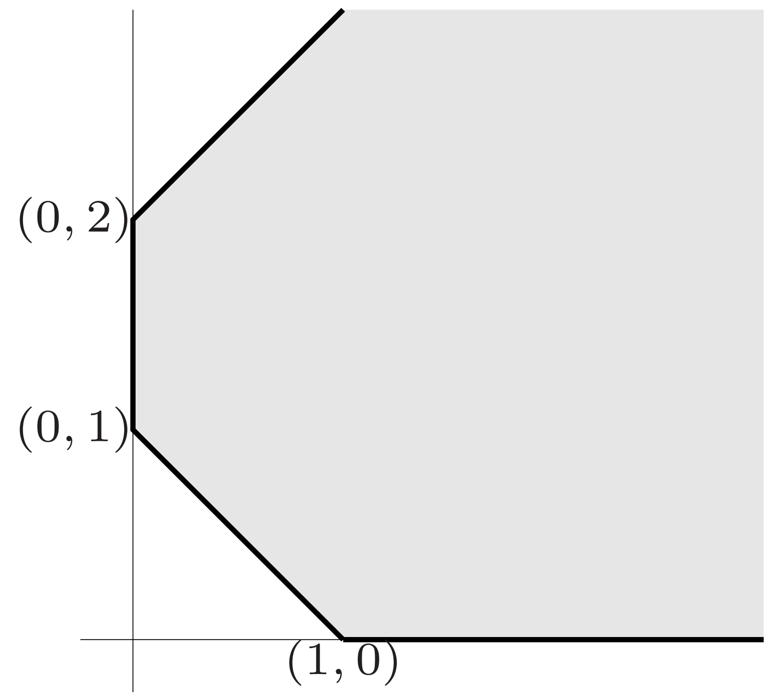
- $A_{\hat{I}}$ is nonsingular because

$$a_i^T \Delta x = 0, \quad i \in I \setminus \{k\}, \qquad a_j^T \Delta x > 0$$

implies that a_j is linearly independent of the vectors a_i, $i \in I \setminus \{k\}$

Example

$$\begin{bmatrix} 0 & -1 \\ -1 & -1 \\ -1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$



extreme points

x	$b - Ax$	I
$(1, 0)$	$(0, 0, 1, 3)$	$\{1, 2\}$
$(0, 1)$	$(1, 0, 0, 1)$	$\{2, 3\}$
$(0, 2)$	$(2, 1, 0, 0)$	$\{3, 4\}$

compute extreme points adjacent to $x = (1, 0)$

1. try to remove $k = 1$ from active set $I = \{1, 2\}$

   - compute Δx

$$\begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \Delta x = (-1, 1)$$

   - minimum ratio test: $A\Delta x = (-1, 0, 1, 2)$

$$\alpha = \min\{\frac{b_3 - a_3^T x}{a_3^T \Delta x}, \frac{b_4 - a_4^T x}{a_4^T \Delta x}\} = \min\{1/1, 3/2\} = 1$$

   new extreme point: $\hat{x} = (0, 1)$ with active set $\hat{I} = \{2, 3\}$

2. try to remove $k = 2$ from active set $I = \{1, 2\}$

- compute Δx

$$\begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad \Longrightarrow \qquad \Delta x = (1, 0)$$

- $A\Delta x = (0, -1, -1, -1)$:

$$\{(1, 0) + \alpha(1, 0) \mid \alpha \geq 0\}$$

is an unbounded edge of the feasible set

Finding an adjacent extreme point with lower cost

given extreme point x with active index set I

1. define $z \in \mathbf{R}^m$ with
$$A_I^T z_I + c = 0, \qquad z_j = 0, \quad j \notin I$$

2. if $z \geq 0$, then x, z are primal and dual optimal

3. otherwise select k with $z_k < 0$ and determine Δx as on page 10–6:
$$\begin{aligned} c^T(x + \alpha \Delta x) &= c^T x - \alpha z_I^T A_I \Delta x \\ &= c^T x + \alpha z_k \end{aligned}$$

   cost decreases in the direction Δx

One iteration of the simplex method

given an extreme point x with active set I

1. compute $z \in \mathbf{R}^m$ with

$$A_I^T z_I + c = 0, \qquad z_j = 0, \quad j \notin I$$

   if $z \geq 0$, terminate (x is optimal)

2. choose k such that $z_k < 0$, compute $\Delta x \in \mathbf{R}^n$ with

$$a_i^T \Delta x = 0, \quad i \in I \setminus \{k\}, \qquad a_k^T \Delta x = -1$$

   if $A\Delta x \leq 0$, terminate ($p^\star = -\infty$)

3. set $I := I \setminus \{k\} \cup \{j\}$, $x := x + \alpha \Delta x$ where

$$j = \operatorname*{argmin}_{i:a_i^T\Delta x > 0} \frac{b_i - a_i^T x}{a_i^T \Delta x}, \qquad \alpha = \frac{b_j - a_j^T x}{a_j^T \Delta x}$$

Pivot selection and convergence

step 2: which k do we choose if z_k has several negative components?

many variants, *e.g.*,

- choose most negative z_k
- choose maximum decrease in cost αz_k
- choose smallest k

all three variants work (if all extreme points are nondegenerate)

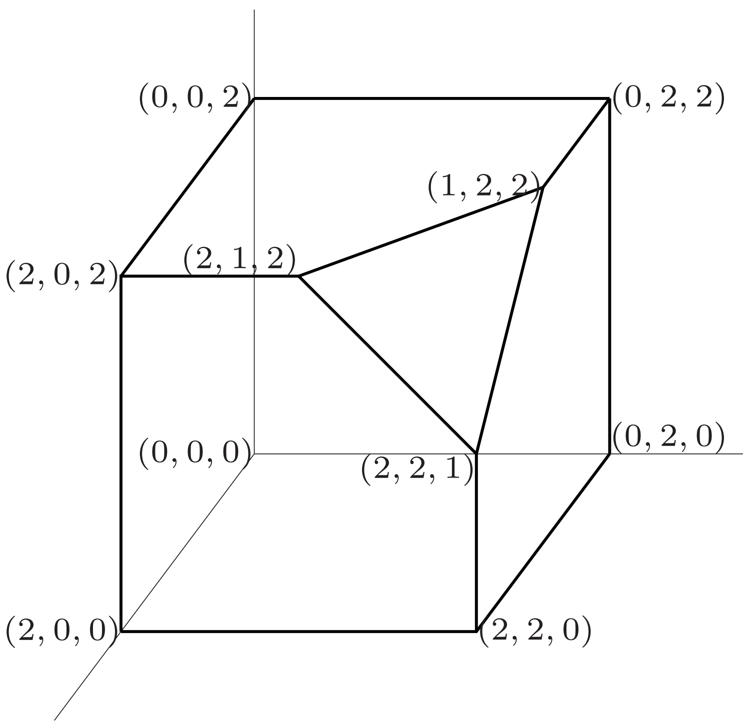
step 3: j is unique and $\alpha > 0$ (if all extreme points are nondegenerate)

convergence follows from:

- number of extreme points is finite
- cost strictly decreases at each step

Example

$$\text{min. } x_1 + x_2 - x_3 \qquad \text{s.t.} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 2 \\ 2 \\ 5 \end{bmatrix}$$



- optimal point is $x = (0, 0, 2)$
- start simplex method at $x = (2, 2, 0)$

iteration 1: $x = (2, 2, 0)$, $\quad b - Ax = (2, 2, 0, 0, 0, 2, 1)$, $\quad I = \{3, 4, 5\}$

1. compute z:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_3 \\ z_4 \\ z_5 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad z = (0, 0, -1, -1, -1, 0, 0)$$

   not optimal; remove $k = 3$ from active set

2. compute Δx

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \Delta x = (0, 0, 1)$$

3. minimum ratio test: $A\Delta x = (0, 0, -1, 0, 0, 1, 1)$

$$\alpha = \operatorname{argmin}\{2/1, 1/1\} = 1, \qquad j = 7$$

iteration 2: $x = (2, 2, 1)$, $\quad b - Ax = (2, 2, 1, 0, 0, 1, 0)$, $\quad I = \{4, 5, 7\}$

1. compute z:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_4 \\ z_5 \\ z_7 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad z = (0, 0, 0, -2, -2, 0, 1)$$

   not optimal; remove $k = 5$ from active set

2. compute Δx

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \Delta x = (0, -1, 1)$$

3. minimum ratio test: $A\Delta x = (0, 1, -1, 0, -1, 1, 0)$

$$\alpha = \operatorname{argmin}\{2/1, 1/1\} = 1, \qquad j = 6$$

iteration 3: $x = (2, 1, 2), \quad b - Ax = (2, 1, 2, 0, 1, 0, 0), \quad I = \{4, 6, 7\}$

1. compute z:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_4 \\ z_6 \\ z_7 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad z = (0, 0, 0, 0, 0, 2, -1)$$

not optimal; remove $k = 7$ from active set

2. compute Δx

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad \Delta x = (0, -1, 0)$$

3. minimum ratio test: $A\Delta x = (0, 1, 0, 0, -1, 0, -1)$

$$\alpha = \mathrm{argmin}\{1/1\} = 1, \qquad j = 2$$

iteration 4: $x = (2, 0, 2)$, $\quad b - Ax = (2, 0, 2, 0, 2, 0, 1)$, $\quad I = \{2, 4, 6\}$

1. compute z:

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_2 \\ z_4 \\ z_6 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad z = (0, 1, 0, -1, 0, 1, 0)$$

   not optimal; remove $k = 4$ from active set

2. compute Δx

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \Delta x = (-1, 0, 0)$$

3. minimum ratio test: $A\Delta x = (1, 0, 0, -1, 0, 0, -1)$

$$\alpha = \operatorname{argmin}\{2/1\} = 2, \qquad j = 1$$

iteration 5: $x = (0, 0, 2)$, $\quad b - Ax = (0, 0, 2, 2, 2, 0, 3)$, $\quad I = \{1, 2, 6\}$

1. compute z:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_6 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \Longrightarrow \quad z = (1, 1, 0, 0, 0, 1, 0)$$

optimal

Degeneracy

- if x is degenerate, A_I has rank n but is not square
- if next point is degenerate, we have a tie in the argmin of step 3

solution

- define I to be a subset of n linearly independent active constraints
- A_I is square; steps 1 and 2 work as in the nondegenerate case
- in step 3, break ties arbitrarily

does it work?

- in step 3 we can have $\alpha = 0$ (*i.e.*, x does not change)
- maybe this does not hurt, as long as I keeps changing

Example

$$\begin{array}{ll} \text{minimize} & -3x_1 + 5x_2 - x_3 + 2x_4 \\ \text{subject to} & \begin{bmatrix} 1 & -2 & -2 & 3 \\ 2 & -3 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

- $x = (0, 0, 0, 0)$ is a degenerate extreme point with

$$b - Ax = (0, 0, 1, 0, 0, 0, 0)$$

- start simplex with $I = \{4, 5, 6, 7\}$

iteration 1: $I = \{4, 5, 6, 7\}$

1. $z = (0, 0, 0, -3, 5, -1, 2)$: remove 4 from active set
2. $\Delta x = (1, 0, 0, 0)$
3. $A\Delta x = (1, 2, 0, -1, 0, 0, 0)$: $\alpha = 0$, add 1 to active set

iteration 2: $I = \{1, 5, 6, 7\}$

1. $z = (3, 0, 0, 0, -1, -7, 11)$: remove 5 from active set
2. $\Delta x = (2, 1, 0, 0)$
3. $A\Delta x = (0, 1, 0, -2, -1, 0, 0)$: $\alpha = 0$, add 2 to active set

iteration 3: $I = \{1, 2, 6, 7\}$

1. $z = (1, 1, 0, 0, 0, -4, 6)$: remove 6 from active set
2. $\Delta x = (-4, -3, 1, 0)$
3. $A\Delta x = (0, 0, 1, 4, 3, -1, 0)$: $\alpha = 0$, add 4 to active set

iteration 4: $I = \{1, 2, 4, 7\}$

1. $z = (-2, 3, 0, 1, 0, 0, -1)$: remove 7 from active set
2. $\Delta x = (0, -1/4, 7/4, 1)$
3. $A\Delta x = (0, 0, 7/4, 0, 1/4, -7/4, -1)$: $\alpha = 0$, add 5 to active set

iteration 5: $I = \{1, 2, 4, 5\}$

1. $z = (-1, 1, 0, -2, 4, 0, 0)$: remove 1 from active set
2. $\Delta x = (0, 0, -1, -1)$
3. $A\Delta x = (-1, 0, -1, 0, 0, 1, 1)$: $\alpha = 0$, add 6 to active set

iteration 6: $I = \{2, 4, 5, 6\}$

1. $z = (0, -2, 0, -7, 11, 1, 0)$: remove 2 from active set
2. $\Delta x = (0, 0, 0, -1)$
3. $A\Delta x = (-3, -1, 0, 0, 0, 0, 1)$: $\alpha = 0$, add 7 to active set

iteration 7: $I = \{4, 5, 6, 7\}$, the initial active set

Bland's pivoting rule

no cycling will occur if we follow the following rule

- in step 2, always choose the smallest k for which $z_k < 0$
- if there is a tie in step 3, always choose the smallest j

proof (by contradiction) suppose there is a cycle $i.e.$, for some $q > p$

$$x^{(p)} = x^{(p+1)} = \cdots = x^{(q)}, \qquad I^{(p)} \neq I^{(p+1)} \neq \cdots \neq I^{(q)} = I^{(p)}$$

where $x^{(s)}$ ($I^{(s)}$, $z^{(s)}$, $\Delta x^{(s)}$) is the value of x (I, z, Δx) at iteration s

we also define

- k_s: index removed from I in iteration s; j_s: index added in iteration s
- $\bar{k} = \max_{p \leq s \leq q-1} k_s$
- r: the iteration ($p \leq r \leq q - 1$) in which $\bar{k}$ is removed ($\bar{k} = k_r$)
- t: the iteration ($r < t \leq q$) in which $\bar{k}$ is added back again ($\bar{k} = j_t$)

at iteration r we remove index $\bar{k}$ from $I^{(r)}$; therefore

- $z_{\bar{k}}^{(r)} < 0$
- $z_i^{(r)} \geq 0$ for $i \in I^{(r)}$, $i < \bar{k}$ (otherwise we should have removed i)
- $z_i^{(r)} = 0$ for $i \notin I^{(r)}$ (by definition of $z^{(r)}$)

at iteration t we add index $\bar{k}$ to $I^{(t)}$; therefore

- $a_{\bar{k}}^T \Delta x^{(t)} > 0$
- $a_i^T \Delta x^{(t)} \leq 0$ for $i \in I^{(r)}$, $i < \bar{k}$

  (otherwise we should have added i, since $b_i - a_i^T x = 0$ for all $i \in I^{(r)}$)
- $a_i^T \Delta x^{(t)} = 0$, for $i \in I^{(r)}$, $i > \bar{k}$

  (if $i > \bar{k}$ and $i \in I^{(r)}$ then it is never removed, so $i \in I^{(t)} \setminus \{k_t\}$)

conclusion: ${z^{(r)}}^T A \Delta x^{(t)} < 0$

a contradiction, because $-{z^{(r)}}^T A \Delta x^{(t)} = c^T \Delta x^{(t)} \leq 0$

Example

LP of page 10–21, same starting point but applying Bland's rule

iteration 1: $I = \{4, 5, 6, 7\}$

1. $z = (0, 0, 0, -3, 5, -1, 2)$: remove 4 from active set
2. $\Delta x = (1, 0, 0, 0)$
3. $A\Delta x = (1, 2, 0, -1, 0, 0, 0)$: $\alpha = 0$, add 1 to active set

iteration 2: $I = \{1, 5, 6, 7\}$

1. $z = (3, 0, 0, 0, -1, -7, 11)$: remove 5 from active set
2. $\Delta x = (2, 1, 0, 0)$
3. $A\Delta x = (0, 1, 0, -2, -1, 0, 0)$: $\alpha = 0$, add 2 to active set

iteration 3: $I = \{1, 2, 6, 7\}$

1. $z = (1, 1, 0, 0, 0, -4, 6)$: remove 6 from active set
2. $\Delta x = (-4, -3, 1, 0)$
3. $A\Delta x = (0, 0, 1, 4, 3, -1, 0)$: $\alpha = 0$, add 4 to active set

iteration 4: $I = \{1, 2, 4, 7\}$

1. $z = (-2, 3, 0, 1, 0, 0, -1)$: remove 1 from active set
2. $\Delta x = (0, -1/4, 3/4, 1)$
3. $A\Delta x = (-1, 0, 3/4, 0, 1/4, -3/4, 0)$: $\alpha = 0$, add 5 to active set

iteration 5: $I = \{2, 4, 5, 7\}$

1. $z = (0, -1, 0, -5, 8, 0, 1)$: remove 2 from active set
2. $\Delta x = (0, 0, 1, 0)$
3. $A\Delta x = (-2, -1, 1, 0, 0, -1, 0)$: $\alpha = 1$, add 3 to active set

new $x = (0, 0, 1, 0)$, $b - Ax = (2, 1, 0, 0, 0, 1, 0)$

iteration 6: $I = \{3, 4, 5, 7\}$

1. $z = (0, 0, 1, -3, 5, 0, 2)$: remove 4 from active set
2. $\Delta x = (1, 0, 0, 0)$
3. $A\Delta x = (1, 2, 0, -1, 0, 0, 0)$: $\alpha = 1/2$, add 2 to active set

new $x = (1/2, 0, 1, 0)$, $b - Ax = (3/2, 0, 0, 1/2, 0, 1, 0)$

iteration 7: $I = \{1, 3, 5, 7\}$

1. $z = (3, 0, 7, 0, -1, 0, 11)$: remove 5 from active set
2. $\Delta x = (2, 1, 0, 0)$
3. $A\Delta x = (0, 1, 0, -2, -1, 0, 0)$: $\alpha = 0$, add 2 to active set

iteration 8: $I = \{1, 2, 3, 7\}$

1. $z = (1, 1, 4, 0, 0, 0, 6)$: optimal

Initialization via phase I

linear program with variable bounds

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b, \quad x \geq 0 \end{array}$$

general; can split free x_k as $x_k = x_k^+ - x_k^-$, $x_k \geq 0$, $x_k^- \geq 0$

phase I problem

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax \leq (1-t)b, \quad x \geq 0, \quad 0 \leq t \leq 1 \end{array}$$

- $x = 0$, $t = 1$ is an extreme point of phase I LP
- can compute an optimal extreme point $x^\star$, $t^\star$ of phase I LP via simplex
- if $t^\star > 0$, original problem is infeasible
- if $t^\star = 0$, then $x^\star$ is an extreme point of original problem

Numerical implementation

- most expensive step: solution of two sets of linear equations

$$A_I^T z_I = -c, \qquad A_I \Delta x = (e_k)_I$$

  where e_k is kth unit vector

- one row of A_I changes at each iteration

efficient implementation: propagate LU factorization of A_I

- given the factorization, can solve the equations in $O(n^2)$ operations
- updating LU factorization after changing a row costs $O(n^2)$ operations

total cost is $O(n^2)$ per iteration ($\ll O(n^2)$ if A is sparse)

Complexity of the simplex method

in practice: very efficient (#iterations grows linearly with m, n)

worst-case:

- for most pivoting rules, there exist examples where the number of iterations grows exponentially with n and m
- it is an open question whether there exists a pivoting rule for which the number of iterations is bounded by a polynomial of n and m

Lecture 11
The barrier method

- brief history of interior-point methods
- Newton's method for smooth unconstrained minimization
- logarithmic barrier function
- central points, the central path
- the barrier method

The ellipsoid method

- 1972: ellipsoid method for (nonlinear) convex nondifferentiable optimization (Nemirovsky, Yudin, Shor)
- 1979: Khachiyan proves that the ellipsoid method applied to LP has polynomial worst-case complexity

- much slower in practice than simplex
- very different approach from simplex method; extends gracefully to nonlinear convex problems
- solved important open theoretical problem (polynomial-time algorithm for LP)

Interior-point methods

early methods (1950s–1960s)

- methods for solving convex optimization problems via sequence of smooth unconstrained problems
- logarithmic barrier method (Frisch), sequential unconstrained minimization (Fiacco & McCormick), affine scaling method (Dikin), method of centers (Huard & Lieu)
- no worst-case complexity theory; (often) worked well in practice
- fell out of favor in 1970s

new methods (1984–)

- 1984 Karmarkar: new polynomial-time method for LP (projective algorithm)
- later recognized as closely related to earlier interior-point methods
- many variations since 1984; widely believed to be faster than simplex for very large problems (over 10,000 variables/constraints)

Gradient and Hessian

differentiable function $f : \mathbf{R}^n \to \mathbf{R}$

gradient and Hessian (evaluated at x):

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}, \qquad \nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$$

2nd order Taylor series expansion around x:

$$f(y) \simeq f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(x)(y - x)$$

Positive semidefinite matrices

a *quadratic form* is a function $f : \mathbf{R}^n \to \mathbf{R}$ with

$$f(x) = x^T A x = \sum_{i,j=1}^{n} A_{ij} x_i x_j$$

may as well assume $A = A^T$ since $x^T A x = x^T((A + A^T)/2)x$

$A = A^T$ is *positive semidefinite* if

$$x^T A x \geq 0 \text{ for all } x$$

$A = A^T$ is *positive definite* if

$$x^T A x > 0 \text{ for all } x \neq 0$$

Convex differentiable functions

$f : \mathbf{R}^n \to \mathbf{R}$ is *convex* if for all x and y

$$0 \leq \lambda \leq 1 \quad \Longrightarrow \quad f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

$f : \mathbf{R}^n \to \mathbf{R}$ is *strictly convex* if for all x and y

$$0 < \lambda < 1 \quad \Longrightarrow \quad f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y)$$

- for differentiable f:

$$\nabla^2 f(x) \text{ positive semidefinite} \quad \Longleftrightarrow \quad f \text{ convex}$$

$$\nabla^2 f(x) \text{ positive definite} \quad \Longrightarrow \quad f \text{ strictly convex}$$

- for convex differentiable f:

$$\nabla f(x) = 0 \quad \Longleftrightarrow \quad x = \operatorname{argmin} f$$

f strictly convex $\Rightarrow \operatorname{argmin} f$ is unique (if it exists)

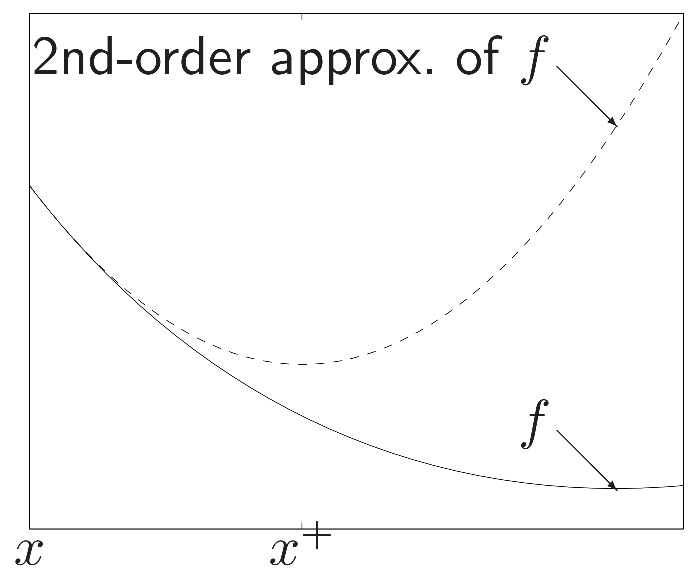
Pure Newton method

algorithm for minimizing convex differentiable f:

$$x^+ = x - \nabla^2 f(x)^{-1} \nabla f(x)$$

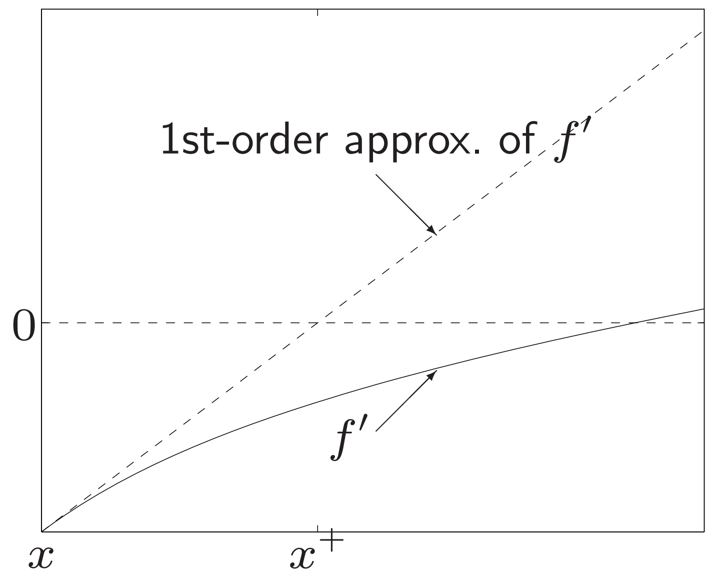
- x^+ minimizes 2nd order expansion of $f(y)$ at x:

$$f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(x)(y - x)$$



- x^+ solves linearized optimality condition:

$$0 = \nabla f(x) + \nabla^2 f(x)(y - x)$$

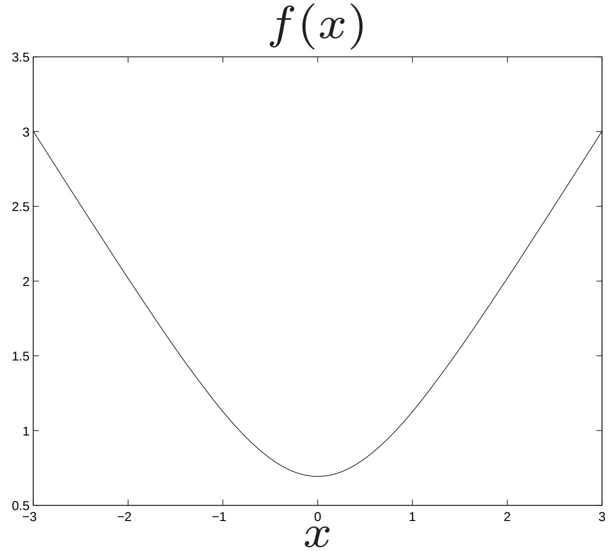


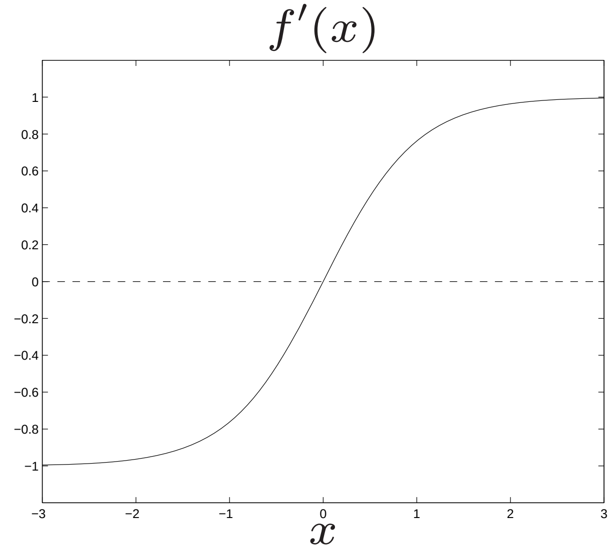
intepretations suggest method works **very well** near optimum

Global behavior

pure Newton method can diverge

example: $f(x) = \log(e^x + e^{-x})$, start at $x^{(0)} = 1.1$





k	$x^{(k)}$	$f(x^{(k)}) - f^\star$
1	$-1.129 \cdot 10^0$	$5.120 \cdot 10^{-1}$
2	$1.234 \cdot 10^0$	$5.349 \cdot 10^{-1}$
3	$-1.695 \cdot 10^0$	$6.223 \cdot 10^{-1}$
4	$5.715 \cdot 10^0$	$1.035 \cdot 10^0$
5	$-2.302 \cdot 10^4$	$2.302 \cdot 10^4$

Newton method with exact line search

given starting point x
repeat
1. *Compute Newton direction*
   $v = -\nabla^2 f(x)^{-1} \nabla f(x)$
2. *Line search.* Choose a step size t
   $t = \operatorname{argmin}_{t>0} f(x + tv)$
3. *Update.* $x := x + tv$

until stopping criterion is satisfied

- globally convergent
- very fast local convergence

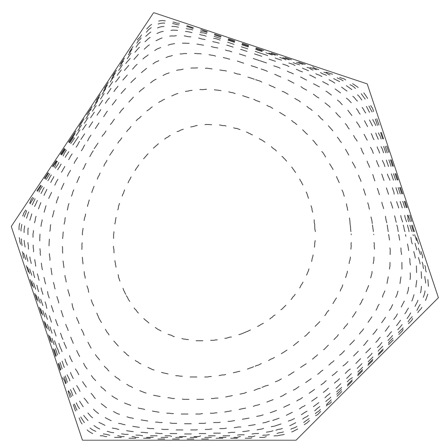
(more later)

Logarithmic barrier function

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

assume strictly feasible: $\{x \mid Ax < b\} \neq \emptyset$

define **logarithmic barrier** $\phi(x) = \begin{cases} \displaystyle\sum_{i=1}^{m} -\log(b_i - a_i^T x) & Ax < b \\ +\infty & \text{otherwise} \end{cases}$



$\phi \to \infty$ as x approaches boundary of $\{x \mid Ax < b\}$

Derivatives of barrier function

$$
\begin{aligned}
\nabla \phi(x) &= \sum_{i=0}^{m} \frac{1}{b_i - a_i^T x} a_i = A^T d \\
\nabla^2 \phi(x) &= \sum_{i=1}^{m} \frac{1}{(b_i - a_i^T x)^2} a_i a_i^T = A^T \mathbf{diag}(d)^2 A
\end{aligned}
$$

where $d = (1/(b_1 - a_1^T x), \ldots, 1/(b_m - a_m^T x))$

- ϕ is smooth on $C = \{x \mid Ax < b\}$
- ϕ is convex on C: for all $y \in \mathbf{R}^n$,

$$
y^T \nabla^2 \phi(x) y = y^T A \, \mathbf{diag}(d)^2 A y = \| \mathbf{diag}(d) A y \|^2 \geq 0
$$

- strictly convex if $\mathbf{rank}\, A = n$

The analytic center

$\operatorname{argmin} \phi$ (if it exists) is called **analytic center** of inequalities

optimality conditions:

$$\nabla \phi(x) = \sum_{i=1}^{m} \frac{1}{b_i - a_i^T x} a_i = 0$$

- exists if and only if $C = \{x \mid Ax < b\}$ is bounded
- unique if A has rank n
- different descriptions of the same polyhedron may have different analytic centers ($e.g.$, adding redundant inequalities moves analytic center)
- efficiently computed via Newton's method (given strictly feasible starting point)

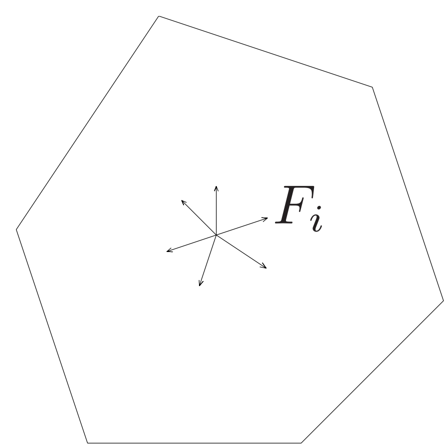
Force field interpretation

- associate with constraint $a_i^T x \le b_i$, at point x, the force

$$F_i = \frac{-a_i}{b_i - a_i^T x}$$

  - F_i points away from constraint plane
  - $\|F_i\| = 1/\text{dist}(x, \text{constraint plane})$

- ϕ is potential of $1/r$ force field associated with each constraint plane



forces balance at analytic center

Central path

$$x^*(t) = \underset{x}{\operatorname{argmin}}(tc^T x + \phi(x)) \text{ for } t > 0$$

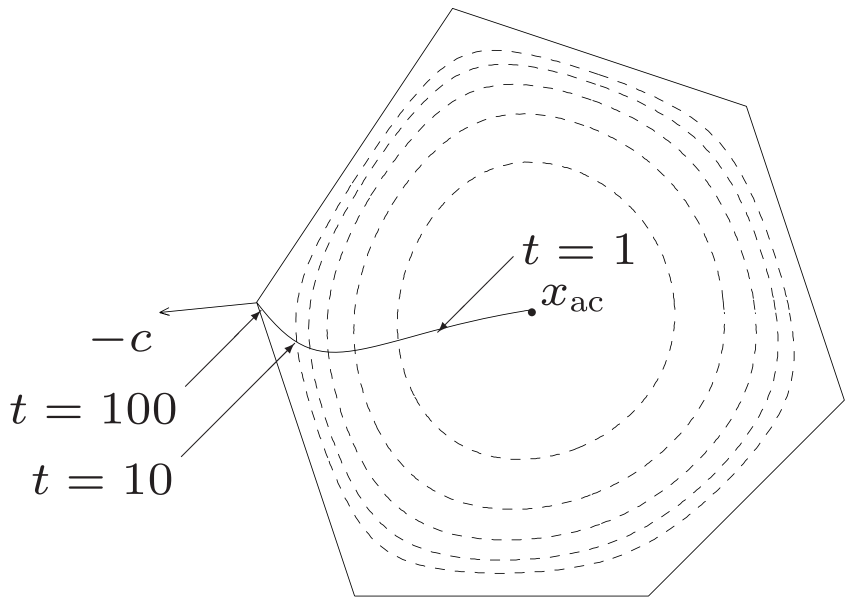
(we assume minimizer exists and is unique)

- curve $x^*(t)$ for $t \geq 0$ called **central path**
- can compute $x^*(t)$ by solving smooth unconstrained minimization problem (given a strictly feasible starting point)
- t gives relative weight of objective and barrier
- barrier 'traps' $x^*(t)$ in strictly feasible set
- intuition suggests $x^*(t)$ converges to optimal as $t \to \infty$

$x^*(t)$ characterized by

$$tc + \sum_{i=1}^{m} \frac{1}{b_i - a_i^T x^*(t)} a_i = 0$$

example



Force field interpretation

imagine a particle in C, subject to forces

ith constraint generates **constraint force field**

$$F_i(x) = -\frac{1}{b_i - a_i^T x} a_i$$

- ϕ is *potential* associated with constraint forces
- constraint forces push particle away from boundary of feasible set

superimpose **objective force field** $F_0(x) = -tc$

- pulls particle toward small $c^T x$
- t scales objective force

at $x^*(t)$, constraint forces balance objective force; as t increases, particle is pulled towards optimal point, trapped in C by barrier potential

Central points and duality

recall $x^* = x^*(t)$ satisfies

$$c + \sum_{i=1}^{m} z_i a_i = 0, \qquad z_i = \frac{1}{t(b_i - a_i^T x^*)} > 0$$

$i.e.$, z is dual feasible and

$$p^\star \geq -b^T z = c^T x^* + \sum_i z_i (a_i^T x^* - b_i) = c^T x^* - m/t$$

summary: a point on central path yields dual feasible point and lower bound:

$$c^T x^*(t) \geq p^\star \geq c^T x^*(t) - m/t$$

(which proves $x^*(t)$ becomes optimal as $t \to \infty$)

Central path and complementary slackness

optimality conditions: x optimal $\Longleftrightarrow$ $Ax \leq b$ and $\exists z$ s.t.

$$z \geq 0, \qquad A^T z + c = 0, \qquad z_i(b_i - a_i^T x) = 0$$

centrality conditions: x is on central path $\Longleftrightarrow$ $Ax < b$ and $\exists z$, $t > 0$ s.t.

$$z \geq 0, \qquad A^T z + c = 0, \qquad z_i(b_i - a_i^T x) = 1/t$$

- for t large, $x^*(t)$ 'almost' satisfies complementary slackness
- central path is continuous deformation of complementary slackness condition

Unconstrained minimization method

given strictly feasible x, desired accuracy $\epsilon > 0$

1. $t := m/\epsilon$
2. compute $x^*(t)$ starting from x
3. $x := x^*(t)$

- computes ϵ-suboptimal point on central path (and dual feasible z)
- solve constrained problem via Newton's method
- works, but can be slow

Barrier method

given strictly feasible x, $t > 0$, tolerance $\epsilon > 0$
repeat
1. compute $x^*(t)$ starting from x, using Newton's method
2. $x := x^*(t)$
3. if $m/t \leq \epsilon$, return(x)
4. increase t

- also known as SUMT (Sequential Unconstrained Minimization Technique)
- generates sequence of points on central path
- simple updating rule for t: $t^+ = \mu t$ (typical values $\mu \approx 10 \ \sim \ 100$)

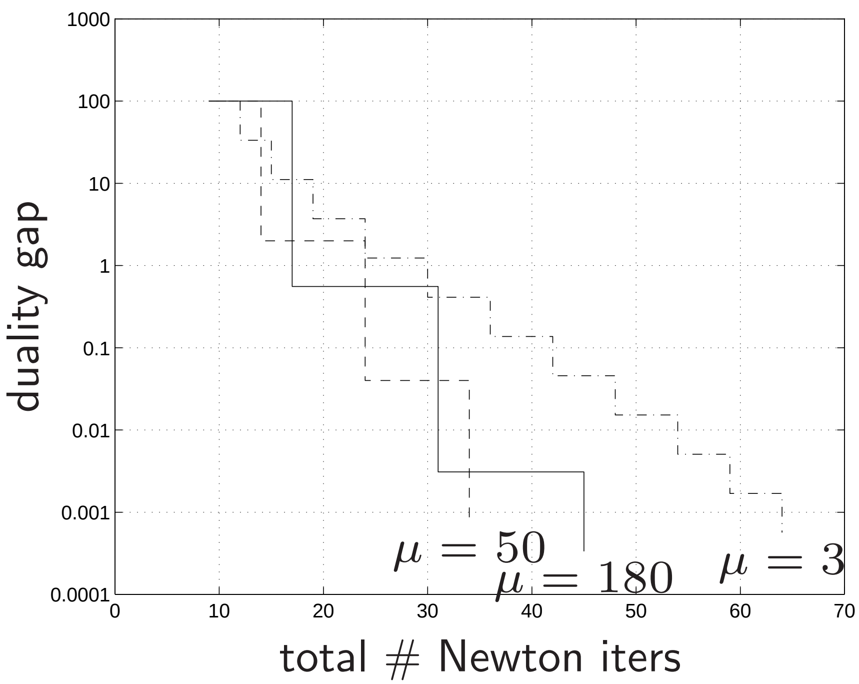
steps 1–4 above called **outer iteration**; step 1 involves **inner iterations** ($e.g.$, Newton steps)

tradeoff: small $\mu \Longrightarrow$ few inner iters to compute $x^{(k+1)}$ from $x^{(k)}$, but more outer iters

Example

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

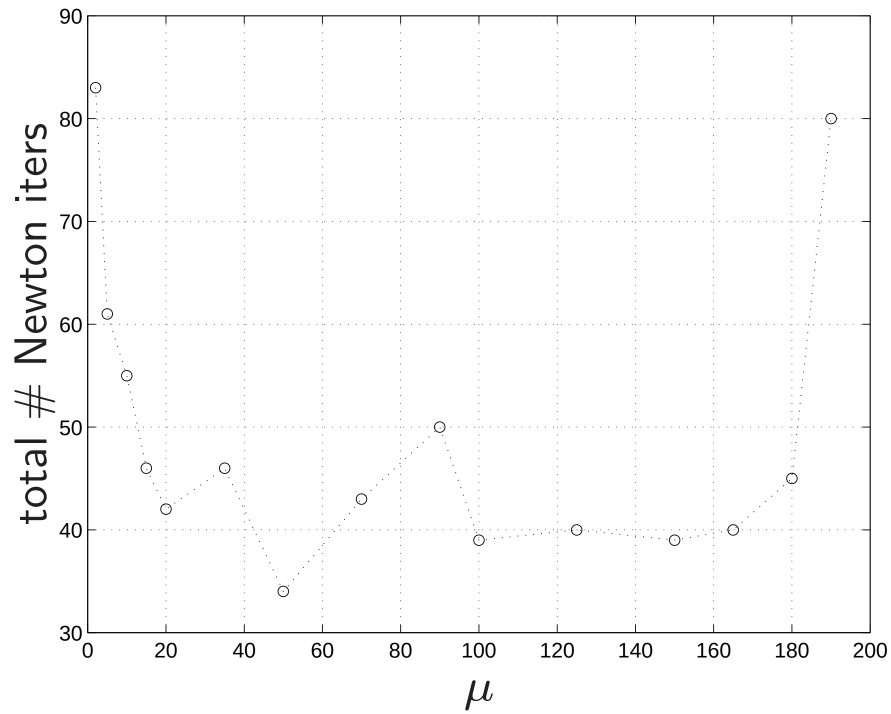
$A \in \mathbf{R}^{100\times 50}$, Newton with exact line search



- width of 'steps' shows #Nt. iters per outer iter; height of 'steps' shows reduction in dual. gap ($1/\mu$)
- gap reduced by 10^5 in few tens of Newton iters
- gap decreases geometrically
- can see trade-off in choice of μ

example continued . . .

trade-off in choice of μ: #Newton iters required to reduce duality gap by 10^6



works very well for wide range of μ

Phase I

to compute strictly feasible point (or determine none exists) set up auxiliary problem:

$$\begin{array}{ll} \text{minimize} & w \\ \text{subject to} & Ax \preceq b + w\mathbf{1} \end{array}$$

- easy to find strictly feasible point (hence barrier method can be used)
- can use stopping criterion with target value 0

if we include constraint on $c^T x$,

$$\begin{array}{ll} \text{minimize} & w \\ \text{subject to} & Ax \preceq b + w\mathbf{1} \\ & c^T x \leq M \end{array}$$

phase I method yields point on central path of original problem

many other methods for finding initial primal (& dual) strictly feasible points

Lecture 12
Convergence analysis of the barrier method

- complexity analysis of the barrier method
  - convergence analysis of Newton's method
  - choice of update parameter μ
  - bound on the total number of Newton iterations
- initialization

Complexity analysis

we'll analyze the method of page 11–21 with

- update $t^+ = \mu t$
- starting point $x^*(t^{(0)})$ on the central path

main result: #Newton iters is bounded by

$$O(\sqrt{m}\log(\epsilon^{(0)}/\epsilon)) \qquad \text{(where } \epsilon^{(0)} = m/t^{(0)}\text{)}$$

caveats:

- methods with good worst-case complexity don't necessarily work better in practice
- we're not interested in the numerical values for the bound—only in the exponent of m and n
- doesn't include initialization
- insights obtained from analysis are more valuable than the bound itself

Outline

1. convergence analysis of Newton's method for

$$\varphi(x) = tc^T x - \sum_{i=1}^{m} \log(b_i - a_i^T x)$$

   (will give us a bound on the number of Newton steps per outer iteration)

2. effect of μ on total number of Newton iterations to compute $x^*(\mu t)$ from $x^*(t)$

3. combine 1 and 2 to obtain the total number of Newton steps, starting at $x^*(t^{(0)})$

The Newton decrement

Newton step at x:

$$\begin{aligned} v &= -\nabla^2\varphi(x)^{-1}\nabla\varphi(x) \\ &= -(A^T \mathbf{diag}(d)^2 A)^{-1}(tc + A^T d) \end{aligned}$$

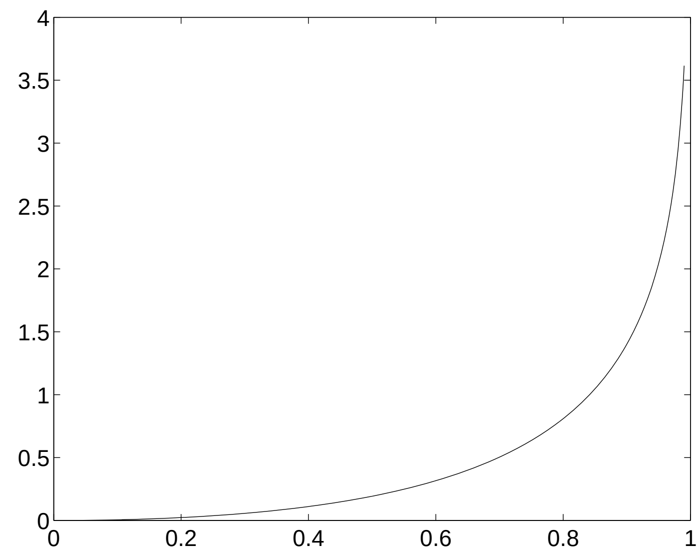
where $d = (1/(b_1 - a_1^T x), \ldots, 1/(b_m - a_m^T x))$

Newton decrement at x:

$$\begin{aligned} \lambda(x) &= \sqrt{\nabla\varphi(x)^T \nabla^2\varphi(x)^{-1}\nabla\varphi(x)} \\ &= \sqrt{v^T \nabla^2\varphi(x) v} \\ &= \left(\sum_{i=1}^m \left(\frac{a_i^T v}{b_i - a_i^T x}\right)^2\right)^{1/2} \\ &= \|\mathbf{diag}(d) A v\| \end{aligned}$$

theorem. if $\lambda = \lambda(x) < 1$, then φ is bounded below and

$$\varphi(x) \leq \varphi(x^*(t)) - \lambda - \log(1 - \lambda)$$



- if $\lambda \leq 0.81$, then $\varphi(x) \leq \varphi(x^*(t)) + \lambda$
- useful as stopping criterion for Newton's method

proof: w.l.o.g. assume $b - Ax = \mathbf{1}$; let $x^* = x^*(t)$, $z = \mathbf{1} + Av$

$$\lambda = \|Av\| < 1 \quad \Longrightarrow \quad z = \mathbf{1} + Av \geq 0$$

$$\nabla^2\varphi(x)v = A^TAv = -\nabla\varphi(x) = -tc - A^T\mathbf{1} \quad \Longrightarrow \quad A^Tz = -tc$$

$$\begin{aligned}
tc^Tx^* - \sum_{i=1}^m \log(b_i - a_i^Tx^*) &= -z^TAx^* - \sum_{i=1}^m \log(b_i - a_i^Tx^*) \\
&\geq -z^TAx^* + \sum_{i=1}^m \log z_i - z^T(b - Ax^*) + m \\
&= -(\mathbf{1} + Ax)^Tz + \sum_{i=1}^m \log z_i + m \\
&= tc^Tx + \sum_i (-a_i^Tv + \log(1 + a_i^Tv)) \\
&\geq tc^Tx + \lambda + \log(1 - \lambda)
\end{aligned}$$

inequalities follow from:

1. $\log y \leq -\log z + zy - 1$ for $y, z > 0$

2. $\sum_{i=1}^{m}(y_i - \log(1 + y_i)) \leq -\|y\| - \log(1 - \|y\|)$ if $\|y\| < 1$

Local convergence analysis

$$x^+ = x - \nabla^2 \varphi(x)^{-1} \nabla \varphi(x)$$

theorem: if $\lambda < 1$, then $Ax^+ < b$ and $\lambda^+ \leq \lambda^2$

(λ is Newton decrement at x; λ^+ is Newton decrement at x^+)

- gives bound on number of iterations: suppose we start at $x^{(0)}$ with $\lambda^{(0)} \leq 0.5$, then $\varphi(x) - \varphi(x^*(t)) < \delta$ after fewer than

$$\log_2 \log_2 (1/\delta) \text{ iterations}$$

- called region of **quadratic convergence**
- practical rule of thumb: 5–6 iterations

proof.

1. $\lambda^2 = \sum_{i=1}^m (a_i^T v)^2/(b_i - a_i^T x)^2 < 1$ implies $a_i^T(x+v) < b_i$

2. assume $b - Ax^+ = \mathbf{1}$; let $w = \mathbf{1} - d - \mathbf{diag}(d)^2 Av$

$$
\begin{aligned}
(\lambda^+)^2 = \|Av^+\|^2 &= \|Av^+\|^2 - 2(Av^+)^T(w + Av^+) &\quad (1)\\
&\le \|w + Av^+ - Av^+\|^2 \\
&= \sum_{i=1}^m (1-d_i)^4 &\quad (2)\\
&= \sum_{i=1}^m (d_i a_i^T v)^4 \\
&\le \|\mathbf{diag}(d)Av\|^4 = \lambda^4
\end{aligned}
$$

(1) uses $A^T w = tc + A^T\mathbf{1}$, $A^T A v^+ = -tc - A^T\mathbf{1}$
(2) uses $Av = Ax^+ - b - Ax + b = -\mathbf{1} + \mathbf{diag}(d)^{-1}\mathbf{1}$,
therefore $d_i a_i^T v = 1 - d_i$ and $w_i = (1-d_i)^2$

Global analysis of Newton's method

damped Newton algorithm: $x^+ = x + sv$, $v = -\nabla^2\varphi(x)^{-1}\nabla\varphi(x)$

step size to the boundary: $s = \alpha^{-1}$ where

$$\alpha = \max\left\{ \frac{a_i^T v}{b_i - a_i^T x} \;\middle|\; a_i^T v > 0 \right\} \qquad (\alpha = 0 \text{ if } Av \le 0)$$

theorem. for $s = 1/(1+\alpha)$,

$$\varphi(x + sv) \le \varphi(x) - (\lambda - \log(1+\lambda))$$

- very simple expression for step size
- same bound if s is determined by an exact line search

if $\lambda \ge 0.5$,

$$\varphi(x + (1+\alpha)^{-1}v) \le \varphi(x) - 0.09$$

(hence, convergence)

proof. define $f(s) = \varphi(x + sv)$ for $0 \leq s < 1/\alpha$

$$f'(s) = v^T \nabla\varphi(x + sv), \qquad f''(s) = v^T \nabla^2\varphi(x + sv)^T v$$

for Newton direction v: $f'(0) = -f''(0) = -\lambda^2$

by integrating the upper bound

$$f''(s) = \sum_{i=1}^{m} \left(\frac{a_i^T v}{b_i - a_i^T x - s a_i^T v} \right)^2 \leq \frac{f''(0)}{(1 - s\alpha)^2}$$

twice, we obtain

$$f(s) \leq f(0) + s f'(0) - \frac{f''(0)}{\alpha^2}(s\alpha + \log(1 - s\alpha))$$

upper bound is minimized by $s = -f'(0)/(f''(0) - \alpha f'(0)) = 1/(1 + \alpha)$

$$\begin{aligned} f(s) &\leq f(0) - \frac{f''(0)}{\alpha^2}(\alpha - \log(1 + \alpha)) \\ &\leq f(0) - (\lambda - \log(1 + \lambda)) \qquad \text{(since } \alpha \leq \lambda\text{)} \end{aligned}$$

Summary

given x with $Ax < b$, tolerance $\delta \in (0, 0.5)$

repeat

1. *Compute Newton step at x:* $v = -\nabla^2\varphi(x)^{-1}\nabla\varphi(x)$
2. *Compute Newton decrement:* $\lambda = (v^T\nabla^2\varphi(x)v)^{1/2}$
3. If $\lambda \leq \delta$, return(x)
4. *Update x:* If $\lambda \geq 0.5$,

   $x := x + (1+\alpha)^{-1}v$ where $\alpha = \max\{0, \max_i a_i^T v/(b_i - a_i^T x)\}$

   else, $x := x + v$

upper bound on #iterations, starting at x:

$$\log_2\log_2(1/\delta) + 11\,(\varphi(x) - \varphi(x^*(t)))$$

usually very pessimistic; good measure in practice:

$$\beta_0 + \beta_1\,(\varphi(x) - \varphi(x^*(t)))$$

with empirically determined β_i ($\beta_0 \leq 5$, $\beta_1 \ll 11$)

#Newton steps per outer iteration

#Newton steps to minimize $\varphi(x) = t^+ c^T x - \sum_{i=1}^m \log(b_i - a_i^T x)$

theorem. if $z > 0$, $A^T z + c = 0$, then

$$\varphi(x^*(t^+)) \geq -t^+ b^T z + \sum_{i=1}^m \log z_i + m(1 + \log t^+)$$

in particular, for $t^+ = \mu t$, $z_i = 1/t(b_i - a_i^T x^*(t))$:

$$\varphi(x^*(t^+)) \geq \varphi(x^*(t)) - m(\mu - 1 - \log \mu)$$

yields estimates for #Newton steps to minimize φ starting at $x^*(t)$:

$$\beta_0 + \beta_1 m(\mu - 1 - \log \mu)$$

- is an upper bound for $\beta_0 = \log_2 \log_2(1/\delta)$, $\beta_1 = 11$
- is a good measure in practice for empirically determined β_0, β_1

proof. if $z > 0$, $A^T z + c = 0$, then

$$\begin{aligned} \varphi(x) &= t^+ c^T x - \sum_{i=1}^m \log(b_i - a_i^T x) \\ &\geq t^+ c^T x + \sum_{i=1}^m \log z_i - t^+ z^T (b - Ax) + m(1 + \log t^+) \\ &= -t^+ b^T z + \sum_{i=1}^m \log z_i + m(1 + \log t^+) \end{aligned}$$

for $z_i = 1/(t(b_i - a_i^T x^*(t))$, $t^+ = \mu t$, this yields

$$\varphi(x^\star(t^+)) \geq \varphi(x^\star(t)) - m(\mu - 1 - \log \mu)$$

Bound on total #Newton iters

suppose we start on central path with $t = t^{(0)}$

number of outer iterations:

$$\#\text{outer iters} = \left\lceil \frac{\log(\epsilon^{(0)}/\epsilon)}{\log \mu} \right\rceil$$

- $\epsilon^{(0)} = m/t^{(0)}$: initial duality gap
- $\epsilon^{(0)}/\epsilon$: reduction in duality gap

upper bound on total #Newton steps:

$$\left\lceil \frac{\log(\epsilon^{(0)}/\epsilon)}{\log \mu} \right\rceil \left(\beta_0 + \beta_1 m(\mu - 1 - \log \mu)\right)$$

- $\beta_0 = \log_2 \log_2(1/\delta)$, $\beta_1 = 11$
- can use empirical values for β_i to estimate average-case behavior

Strategies for choosing μ

- μ independent of m:

$$\#\text{Newton steps per outer iter} \leq O(m)$$

$$\text{total } \#\text{Newton steps} \leq O(m \log(\epsilon^{(0)}/\epsilon)))$$

- $\mu = 1 + \gamma/\sqrt{m}$ with γ independent of m

$$\#\text{Newton steps per outer iter} \leq O(1)$$

$$\text{total } \#\text{Newton steps} \leq O(\sqrt{m} \log(\epsilon^{(0)}/\epsilon)))$$

  follows from:
  - $m(\mu - 1 - \log \mu) \leq \gamma^2/2$, because $x - x^2/2 \leq \log(1 + x)$ for $x > 0$
  - $\log(1 + \gamma/\sqrt{m}) \geq \log(1 + \gamma)/\sqrt{m}$ for $m \geq 1$

Choice of initial t

rule of thumb: given estimate $\widehat{p}$ of $p^\star$, choose

$$m/t \approx c^T x - \widehat{p}$$

(since m/t is duality gap)

via complexity theory (c.f. page 12–12) given dual feasible z, #Newton steps in first iteration is bounded by an affine function of

$$\begin{aligned} & t(c^T x + b^T z) + \phi(x) - \sum_{i=1}^m \log z_i - m(1 + logt) \\ = \quad & t(c^T x + b^T z) - m \log t + \text{const.} \end{aligned}$$

choose t to minimize bound; yields $m/t = c^T x + b^T z$

there are many other ways to choose t

Lecture 13
Primal-dual interior-point methods

- Mehrotra's predictor-corrector method
- computing the search directions

Central path and complementary slackness

$$
\begin{aligned}
s + Ax - b &= 0 \\
A^T z + c &= 0 \\
z_i s_i &= 1/t, \quad i = 1, \ldots, m \\
z \geq 0, &\quad s \geq 0
\end{aligned}
$$

- continuous deformation of optimality conditions
- defines central path: solution is $x = x^*(t)$, $s = b - Ax^*(t)$, $z_i = 1/ts_i$
- $m + n$ linear and m nonlinear equations in the variables $s \in \mathbf{R}^m$, $x \in \mathbf{R}^n$, $z \in \mathbf{R}^m$

Interpretation of barrier method

apply Newton's method to

$$s + Ax - b = 0, \qquad A^T z + c = 0, \qquad z_i - 1/(ts_i) = 0, \quad i = 1, \ldots, m$$

i.e., linearize around current x, z, s:

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ X & 0 & X^{-1}/t \end{bmatrix} \begin{bmatrix} \Delta z \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -(Ax + s - b) \\ -(A^T z + c) \\ \mathbf{1}/t - Xz \end{bmatrix}$$

where $X = \mathbf{diag}(s)$

solution (for $s + Ax - b = 0$, $A^T z + c = 0$):

- determine Δx from $A^T X^{-2} A \Delta x = -tc - A^T X^{-1} \mathbf{1}$ *i.e.*, Δx is the Newton direction used in barrier method
- substitute to obtain Δs, Δz

Primal-dual path-following methods

- modifications to the barrier method:
  - different linearization of central path
  - update both x and z after each Newton step
  - allow infeasible iterates
  - very aggressive step size selection (99% or 99.9% of step to the boundary)
  - update t after each Newton step (hence distinction between outer & inner iteration disappears)
  - linear or polynomial approximation to the central path
- limited theory, fewer convergence results
- work better in practice (faster and more reliable)

Primal-dual linearization

apply Newton's method to

$$s + Ax - b = 0$$
$$A^T z + c = 0$$
$$z_i s_i - 1/t = 0, \quad i = 1, \ldots, m$$

i.e., linearize around s, x, z:

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ X & 0 & Z \end{bmatrix} \begin{bmatrix} \Delta z \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -(Ax + s - b) \\ -(A^T z + c) \\ \mathbf{1}/t - Xz \end{bmatrix}$$

where $X = \mathbf{diag}(s)$, $Z = \mathbf{diag}(z)$

- iterates can be infeasible: $b - Ax \neq s$, $A^T z + c \neq 0$
- we assume $s > 0$, $z > 0$

computing Δx, Δz, Δs

1. compute Δx from

$$A^T X^{-1} Z A \Delta x = A^T z - A^T X^{-1} \mathbf{1}/t - r_z - A^T X^{-1} Z r_x$$

   where $r_x = Ax + s - b$, $r_z = A^T z + c$

2. $\Delta s = -r_x - A\Delta x$

3. $\Delta z = X^{-1}\mathbf{1}/t - z - X^{-1} Z \Delta s$

the most expensive step is step 1

Affine scaling direction

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ X & 0 & Z \end{bmatrix} \begin{bmatrix} \Delta z^{\mathrm{aff}} \\ \Delta x^{\mathrm{aff}} \\ \Delta s^{\mathrm{aff}} \end{bmatrix} = \begin{bmatrix} -(Ax + s - b) \\ -(A^T z + c) \\ -Xz \end{bmatrix}$$

where $X = \mathbf{diag}(s)$, $Z = \mathbf{diag}(z)$

- limit of Newton direction for $t \to \infty$
- Newton step for

$$s + Ax - b = 0$$

$$A^T z + c = 0$$

$$z_i s_i = 0, \quad i = 1, \ldots, m$$

  i.e., the primal-dual optimality conditions

Centering direction

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ X & 0 & Z \end{bmatrix} \begin{bmatrix} \Delta z^{\mathrm{cent}} \\ \Delta x^{\mathrm{cent}} \\ \Delta s^{\mathrm{cent}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{1} \end{bmatrix}$$

where $X = \mathbf{diag}(s)$, $Z = \mathbf{diag}(z)$

- limit of Newton direction for $t \to 0$
- search direction is weighted sum of centering direction and affine scaling direction

$$\begin{aligned} \Delta x &= (1/t)\Delta x^{\mathrm{cent}} + \Delta x^{\mathrm{aff}} \\ \Delta z &= (1/t)\Delta z^{\mathrm{cent}} + \Delta z^{\mathrm{aff}} \\ \Delta s &= (1/t)\Delta s^{\mathrm{cent}} + \Delta s^{\mathrm{aff}} \end{aligned}$$

- in practice:
  - compute affine scaling direction first
  - choose t
  - compute centering direction and add to affine scaling direction

Heuristic for selecting t

- compute affine scaling direction
- compute primal and dual step lengths to the boundary along the affine scaling direction

$$\begin{aligned}\alpha_x &= \max\{\alpha \in [0,1] \mid s + \alpha \Delta s^{\mathrm{aff}} \geq 0\} \\ \alpha_z &= \max\{\alpha \in [0,1] \mid z + \alpha \Delta z^{\mathrm{aff}} \geq 0\}\end{aligned}$$

- compute
$$\sigma = \left(\frac{(s + \alpha_x \Delta s^{\mathrm{aff}})^T (z + \alpha_z \Delta z^{\mathrm{aff}})}{s^T z}\right)^3$$
small σ means affine scaling directions are good search directions (significant reduction in $s^T z$)
- use $t = m/(\sigma s^T z)$ *i.e.*, search direction will be the Newton direction towards the central point with duality gap $\sigma s^T z$

a *heuristic*, based on extensive experiments

Mehrotra's corrector step

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ X & 0 & Z \end{bmatrix} \begin{bmatrix} \Delta z^{\mathrm{cor}} \\ \Delta x^{\mathrm{cor}} \\ \Delta s^{\mathrm{cor}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\Delta X^{\mathrm{aff}} \Delta z^{\mathrm{aff}} \end{bmatrix}$$

- higher-order correction to the affine scaling direction:

$$(s_i + \Delta s_i^{\mathrm{aff}} + \Delta s_i^{\mathrm{cor}})(z_i + \Delta z_i^{\mathrm{aff}} + \Delta z_i^{\mathrm{cor}}) \approx 0$$

- computation can be combined with centering step, *i.e.*, use

$$\Delta x = \Delta x^{\mathrm{cc}} + \Delta x^{\mathrm{aff}}, \qquad \Delta z = \Delta z^{\mathrm{cc}} + \Delta z^{\mathrm{aff}}, \qquad \Delta s = \Delta s^{\mathrm{cc}} + \Delta s^{\mathrm{aff}}$$

where

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ X & 0 & Z \end{bmatrix} \begin{bmatrix} \Delta z^{\mathrm{cc}} \\ \Delta x^{\mathrm{cc}} \\ \Delta s^{\mathrm{cc}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{1}/t - \Delta X^{\mathrm{aff}} \Delta z^{\mathrm{aff}} \end{bmatrix}$$

Step size selection

- determine step to the boundary

$$\begin{aligned}\alpha_x &= \max\{\alpha \geq 0 \mid s + \alpha\Delta s \geq 0\} \\ \alpha_z &= \max\{\alpha \geq 0 \mid z + \alpha\Delta z \geq 0\}\end{aligned}$$

- update x, s, z

$$x := x + \min\{1, 0.99\alpha_x\}\Delta x$$

$$s := s + \min\{1, 0.99\alpha_x\}\Delta s$$

$$z := z + \min\{1, 0.99\alpha_z\}\Delta z$$

Mehrotra's predictor-corrector method

choose starting points x, z, s with $s > 0$, $z > 0$

1. evaluate stopping criteria

- primal feasibility: $\|Ax + s - b\| \leq \epsilon_1(1 + \|b\|)$
- dual feasibility: $\|A^T z + c\| \leq \epsilon_2(1 + \|c\|)$
- maximum absolute error: $c^T x + b^T z \leq \epsilon_3$
- maximum relative error:

$$
\begin{array}{ll}
c^T x + b^T z \leq \epsilon_4 |b^T z| & \text{if } -b^T z > 0 \\
c^T x + b^T z \leq \epsilon_4 |c^T x| & \text{if } c^T x < 0
\end{array}
$$

2. compute affine scaling direction ($X = \mathbf{diag}(s)$, $Z = \mathbf{diag}(z)$)

$$
\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ X & 0 & Z \end{bmatrix}
\begin{bmatrix} \Delta z^{\mathrm{aff}} \\ \Delta x^{\mathrm{aff}} \\ \Delta s^{\mathrm{aff}} \end{bmatrix}
=
\begin{bmatrix} -(Ax + s - b) \\ -(A^T z + c) \\ -Xz \end{bmatrix}
$$

3. compute steps to the boundary

$$\begin{aligned} \alpha_x &= \max\{\alpha \in [0,1] \mid s + \alpha \Delta s^{\rm aff} \geq 0\} \\ \alpha_z &= \max\{\alpha \in [0,1] \mid z + \alpha \Delta z^{\rm aff} \geq 0\} \end{aligned}$$

4. compute centering-corrector steps

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ X & 0 & Z \end{bmatrix} \begin{bmatrix} \Delta z^{\rm cc} \\ \Delta x^{\rm cc} \\ \Delta s^{\rm cc} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sigma \dfrac{s^T z}{m} \mathbf{1} - \Delta X^{\rm aff} \Delta z^{\rm aff} \end{bmatrix}$$

where $\Delta X^{\rm aff} = \mathbf{diag}(\Delta s^{\rm aff})$, and

$$\sigma = \left(\frac{(s + \alpha_x \Delta s^{\rm aff})^T (z + \alpha_z \Delta z^{\rm aff})}{s^T z} \right)^3$$

5. compute search directions

$$\Delta x = \Delta x^{\mathrm{aff}} + \Delta x^{\mathrm{cc}}, \qquad \Delta s = \Delta s^{\mathrm{aff}} + \Delta s^{\mathrm{cc}}, \qquad \Delta z = \Delta z^{\mathrm{aff}} + \Delta z^{\mathrm{cc}}$$

6. determine step sizes and update

$$\begin{aligned} \alpha_x &= \max\{\alpha \geq 0 \mid s + \alpha \Delta s \geq 0\} \\ \alpha_z &= \max\{\alpha \geq 0 \mid z + \alpha \Delta z \geq 0\} \end{aligned}$$

$$\begin{aligned} x &:= x + \min\{1, 0.99\alpha_x\}\Delta x \\ s &:= s + \min\{1, 0.99\alpha_x\}\Delta s \\ z &:= z + \min\{1, 0.99\alpha_z\}\Delta z \end{aligned}$$

go to step 1

Computing the search direction

most expensive part of one iteration: solve two sets of equations

$$A^T X^{-1} Z A \Delta x^{\mathrm{aff}} = r_1, \qquad A^T X^{-1} Z A \Delta x^{\mathrm{cc}} = r_2$$

for some r_1, r_2

two methods

- sparse Cholesky factorization: used in all general-purpose solvers
- conjugate gradients: used for extremely large LPs, or LPs with special structure

Cholesky factorization

if $B = B^T \in \mathbf{R}^{n \times n}$ is positive definite, then it can be written as

$$B = LL^T$$

L lower triangular with $l_{ii} > 0$

- L is called the Cholesky factor of B
- costs $O(n^3)$ if B is dense

application: solve $Bx = d$ with $B = LL^T$

- solve $Ly = d$ (forward substitution)
- solve $L^T x = y$ (backward substitution)

Sparse Cholesky factorization

solve $Bx = d$ with B positive definite and sparse

1. *reordering* of rows and columns of B to increase sparsity of L
2. *symbolic factorization*: based on sparsity pattern of B, determine sparsity pattern of L
3. *numerical factorization*: determine L
4. *forward and backward substitution*: compute x

only steps 3,4 depend on the numerical values of B; only step 4 depends on the right hand side; most expensive steps: 2,3

in Mehrotra's method with sparse LP: $B = A^T X^{-1} Z A$

- do steps 1,2 once, at the beginning of the algorithm ($A^T X^{-1} Z A$ has same sparsity pattern as $A^T A$)
- do step 3 once per iteration, step 4 twice

Conjugate gradients

solve $Bx = d$ with $B = B^T \in \mathbf{R}^{n \times n}$ positive definite

- iterative method
- requires n evaluations of Bx (in theory)
- faster if evaluation of Bx is cheap ($e.g.$, B is sparse, Toeplitz, ...)
- much cheaper in memory than Cholesky factorization
- less accurate and robust (requires preconditioning)

in Mehrotra's method:

$$B = A^T X^{-1} Z A$$

evaluations Bx are cheap if evaluations Ax and $A^T y$ are cheap ($e.g.$, A is sparse)

Lecture 14
Self-dual formulations

- initialization and infeasibility detection
- skew-symmetric LPs
- homogeneous self-dual formulation
- self-dual formulation

Complete solution of an LP

given a pair of primal and dual LPs

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax + s = b \\ & s \geq 0 \end{array} \qquad\qquad \begin{array}{ll} \text{maximize} & -b^T z \\ \text{subject to} & A^T z + c = 0 \\ & z \geq 0, \end{array}$$

classify problem as solvable, primal infeasible, or dual infeasible

- if solvable, find optimal x, s, z

$$Ax+s=b, \qquad A^Tz+c=0, \qquad c^Tx+b^Tz=0, \qquad s\geq 0, \qquad z\geq 0$$

- if primal infeasible, find certificate z: $A^Tz=0$, $z\geq 0$, $b^Tz<0$
- if dual infeasible, find certificate x: $Ax\leq 0$, $c^Tx<0$

Methods for initialization and infeasibility detection

- **phase I – phase II**

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax \leq b + t\mathbf{1}, \quad t \geq 0 \end{array}$$

  disadvantage: phase I is as expensive as phase II

- **'big M' method**

$$\begin{array}{ll} \text{minimize} & c^T x + Mt \\ \text{subject to} & Ax \leq b + t\mathbf{1}, \quad t \geq 0 \end{array}$$

  for some large M
  disadvantage: large M causes numerical problems

- **infeasible-start methods** (lecture 13)
  disadvantage: do not return certificate of (primal or dual) infeasibility

- **self-dual embeddings:** this lecture

Self-dual LP

primal LP (variables u, v)

$$\begin{array}{ll} \text{minimize} & f^T u + g^T v \\ \text{subject to} & Cu + Dv \leq f \\ & -D^T u + Ev = g \\ & u \geq 0 \end{array}$$

with $C = -C^T$, $E = -E^T$

dual LP (variables $\tilde{u}$, $\tilde{v}$)

$$\begin{array}{ll} \text{maximize} & -f^T \tilde{u} - g^T \tilde{v} \\ \text{subject to} & C\tilde{u} + D\tilde{v} \leq f \\ & -D^T \tilde{u} + E\tilde{v} = g \\ & \tilde{u} \geq 0 \end{array}$$

- primal LP $=$ dual LP
- we assume the problem is feasible: hence $p^\star = d^\star = -p^\star = 0$

Optimality conditions for self-dual LP

primal & dual feasibility, complementary slackness:

$$\begin{bmatrix} C & D \\ -D^T & E \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} w \\ 0 \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$
$$\begin{bmatrix} C & D \\ -D^T & E \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} + \begin{bmatrix} \tilde{w} \\ 0 \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

$$u \geq 0, \qquad w \geq 0, \qquad \tilde{u} \geq 0, \qquad \tilde{w} \geq 0, \qquad w^T \tilde{u} + u^T \tilde{w} = 0$$

- **observation 1**: if u, v, w are primal optimal, then $\tilde{u} = u$, $\tilde{v} = v$, $\tilde{w} = w$ are dual optimal; hence optimal u, v, w must satisfy

$$\begin{bmatrix} C & D \\ -D^T & E \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} w \\ 0 \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

$$u \geq 0, \qquad w \geq 0, \qquad u^T w = 0$$

- **observation 2**: there exists a strictly complementary optimal pair (u, v, w), $(\tilde{u}, \tilde{v}, \tilde{w})$ (true for any LP with finite optimal value; see hw); can show that u, w are strictly complementary:

$$
\begin{array}{rcll}
w_i = 0 & \Longrightarrow & \tilde{u}_i > 0 & \text{(by strict complementarity of } w \text{ and } \tilde{u}) \\
& \Longrightarrow & \tilde{w}_i = 0 & \text{(because } \tilde{w}^T\tilde{u} = 0) \\
& \Longrightarrow & u_i > 0 & \text{(by strict complementarity of } u \text{ and } \tilde{w}) \\
w_i > 0 & \Longrightarrow & u_i = 0 & \text{(because } u^T w = 0)
\end{array}
$$

conclusion: a feasible self-dual LP has optimal u, v, w for which

$$
\begin{bmatrix} C & D \\ -D^T & E \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} w \\ 0 \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}
$$

$$
u \geq 0, \qquad w \geq 0, \qquad u^T w = 0
$$

$$
u + w > 0
$$

Homogeneous self-dual embedding of LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax + s = b \\ & s \geq 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & -b^T z \\ \text{subject to} & A^T z + c = 0 \\ & z \geq 0, \end{array}$$

homogeneous self-dual (HSD) formulation:

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & \begin{bmatrix} 0 & b^T & c^T \\ -b & 0 & A \\ -c & -A^T & 0 \end{bmatrix} \begin{bmatrix} \tau \\ z \\ x \end{bmatrix} + \begin{bmatrix} \lambda \\ s \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ & \tau \geq 0, \quad z \geq 0, \quad \lambda \geq 0, \quad s \geq 0 \end{array}$$

- homogeneous (rhs zero) and self-dual
- all feasible points are optimal
- feasible, but not strictly feasible

LP solution from HSD formulation

let $\tau^\star$, $z^\star$, $x^\star$, $\lambda^\star$, $s^\star$ be optimal for HSD and strictly complementary:

$$\tau^\star\lambda^\star = {z^\star}^T s^\star = 0, \qquad \tau^\star + \lambda^\star > 0, \qquad z^\star + s^\star > 0$$

two cases:

1. $\tau^\star > 0$, $\lambda^\star = 0$: primal and dual LP are solvable, with optimal solution

$$x = x^\star/\tau^\star, \qquad s = s^\star/\tau^\star, \qquad z = z^\star/\tau^\star$$

follows from HSD constraints, divided by $\tau^\star$:

$$Ax + s = b, \qquad A^T z + c = 0, \qquad c^T x + b^T z = 0, \qquad s \geq 0, \qquad z \geq 0$$

2. $\tau^\star = 0$, $\lambda^\star > 0$:

$$c^T x^\star + b^T z^\star < 0$$

- if $c^T x^\star < 0$, dual problem is infeasible

$$Ax^\star \leq 0, \qquad c^T x^\star < 0$$

  $x^\star$ is a certificate of dual infeasibility

- if $b^T z^\star < 0$, primal problem is infeasible

$$A^T z^\star = 0, \qquad b^T z^\star < 0$$

  $z^\star$ is a certificate of primal infeasibility

Extended self-dual embedding of LP

choose x_0, $z_0 > 0$, $s_0 > 0$, and define

$$r_{\mathrm{pri}} = b - Ax_0 - s_0, \qquad r_{\mathrm{du}} = A^T z_0 + c, \qquad r = -(c^T x_0 + b^T z_0 + 1)$$

self-dual (SD) formulation

$$\begin{array}{ll}
\text{min.} & (z_0^T s_0 + 1)\theta \\
\text{s.t.} & \left[\begin{array}{cccc} 0 & b^T & c^T & r \\ -b & 0 & A & r_{\mathrm{pri}} \\ -c & -A^T & 0 & r_{\mathrm{du}} \\ -r & -r_{\mathrm{pri}}^T & -r_{\mathrm{du}}^T & 0 \end{array}\right] \left[\begin{array}{c} \tau \\ z \\ x \\ \theta \end{array}\right] + \left[\begin{array}{c} \lambda \\ s \\ 0 \\ 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ z_0^T s_0 + 1 \end{array}\right] \\
& \tau \geq 0, \qquad z \geq 0, \qquad \lambda \geq 0, \qquad s \geq 0
\end{array}$$

- self-dual, not homogeneous
- strictly feasible: take $x = x_0$, $z = z_0$, $s = s_0$, $\tau = \theta = \lambda = 1$

- at optimum:

$$
\begin{aligned}
0 &= \begin{bmatrix} \tau \\ z \end{bmatrix}^T \begin{bmatrix} \lambda \\ s \end{bmatrix} \\
&= -\begin{bmatrix} \tau \\ z \end{bmatrix}^T \left(\begin{bmatrix} 0 & b^T \\ -b & 0 \end{bmatrix} \begin{bmatrix} \tau \\ z \end{bmatrix} + \begin{bmatrix} c^T & r \\ A & r_{\rm pri} \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} \right) \\
&= 0 - \begin{bmatrix} x \\ \theta \end{bmatrix}^T \begin{bmatrix} c & A^T \\ r & r_{\rm pri}^T \end{bmatrix} \begin{bmatrix} \tau \\ z \end{bmatrix} \\
&= \theta(1 + z_0^T s_0) - \begin{bmatrix} x \\ \theta \end{bmatrix}^T \begin{bmatrix} 0 & r_{\rm du} \\ -r_{\rm du}^T & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} \\
&= \theta(1 + z_0^T s_0)
\end{aligned}
$$

hence $\theta = 0$

LP solution from SD formulation

let $\tau^\star$, $z^\star$, $x^\star$, $\theta^\star = 0$, $\lambda^\star$, $s^\star$ be optimal for SD form and strictly complementary:

$$\tau^\star \lambda^\star = {z^\star}^T s^\star = 0, \qquad \tau^\star + \lambda^\star > 0, \qquad z^\star + s^\star > 0$$

two cases:

1. $\tau^\star > 0$, $\lambda^\star = 0$: primal and dual LP are solvable, with optimal solution

$$x = x^\star/\tau^\star, \qquad s = s^\star/\tau^\star, \qquad z = z^\star/\tau^\star$$

2. $\tau^\star = 0$, $\lambda^\star > 0$:
   - $c^T x^\star < 0$: dual problem is infeasible
   - $b^T z^\star < 0$: primal problem is infeasible

Conclusion

- status of the LP can be determined unambiguously from strictly complementary solution of HSD or SD formulation
- can apply any algorithm (barrier, primal-dual, feasible, infeasible) to solve SD form
- can apply any infeasible-start algorithm to solve HSD form
- HSD and SD formulations are twice the size of the original LP; however by exploiting (skew-)symmetry in the equations, one can compute the search directions at roughly the same cost as for the original LP

Lecture 15
Network optimization

- network flows
- extreme flows
- minimum cost network flow problem
- applications

Networks

network (directed graph): m nodes connected by n directed arcs

- arcs are ordered pairs (i, j)
- we assume there is at most one arc from node i to node j
- we assume there are no self-loops (arcs (i, i))

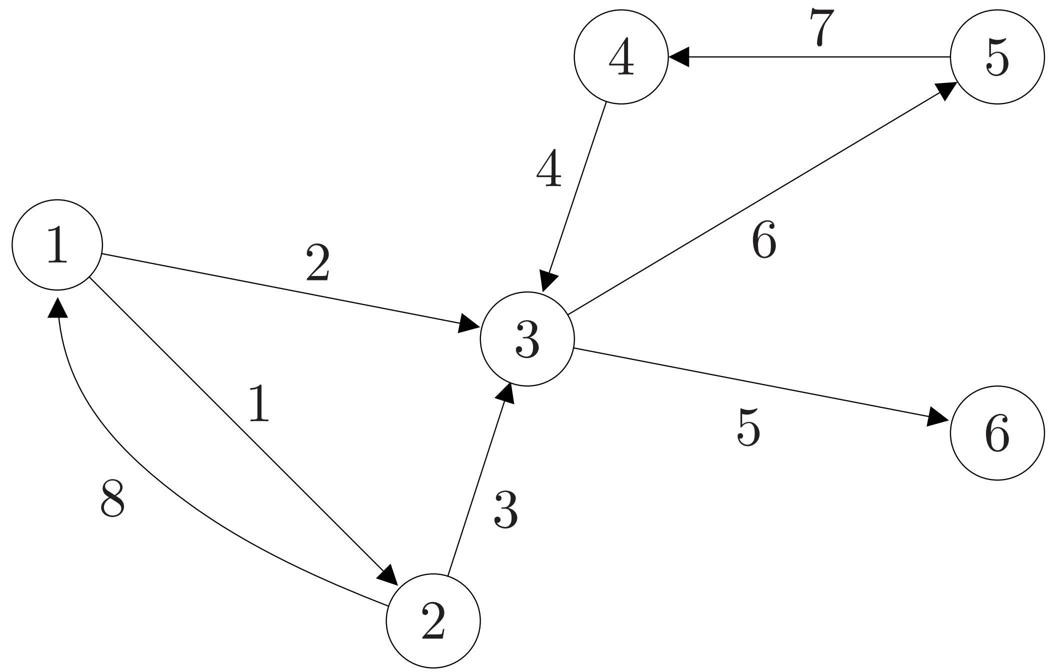
arc-node incidence matrix $A \in \mathbf{R}^{m \times n}$:

$$A_{ij} = \begin{cases} 1 & \text{arc } j \text{ starts at node } i \\ -1 & \text{arc } j \text{ ends at node } i \\ 0 & \text{otherwise} \end{cases}$$

column sums of A are zero: $\mathbf{1}^T A = 0$

reduced arc-node incidence matrix $\tilde{A} \in \mathbf{R}^{(m-1) \times n}$: the matrix formed by the first $m - 1$ rows of A

example ($m = 6$, $n = 8$)



$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

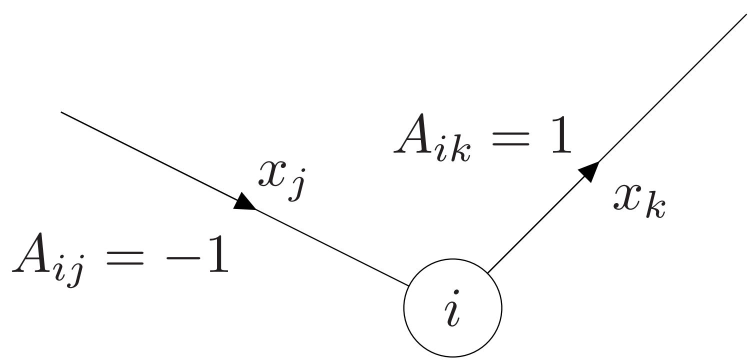
Network flow

flow vector $x \in \mathbf{R}^n$

- x_j: flow (of material, traffic, charge, information, . . .) through arc j
- positive if in direction of arc; negative otherwise

total flow leaving node i:

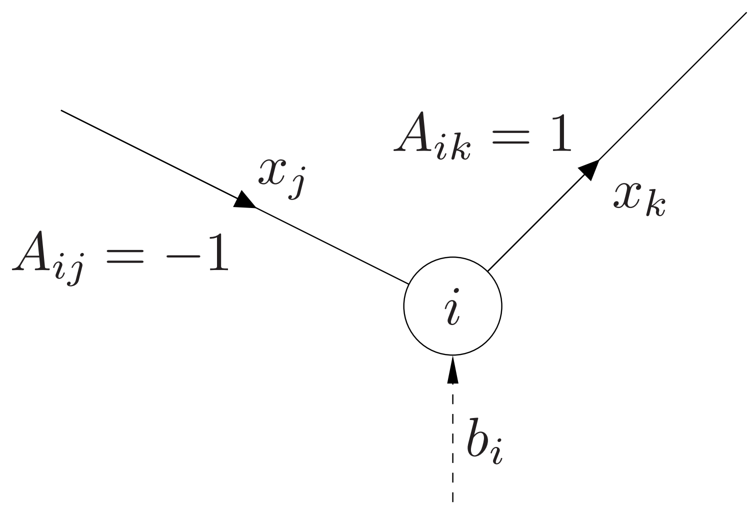
$$\sum_{j=1}^{n} A_{ij} x_j = (Ax)_i$$



External supply

supply vector $b \in \mathbf{R}^m$

- b_i: external supply at node i
- negative b_i represents external demand from the network
- must satisfy $\mathbf{1}^T b = 0$ (total supply = total demand)



balance equations: $Ax = b$

reduced balance equations: $\tilde{A}x = (b_1, \ldots, b_{m-1})$

Minimum cost network flow problem

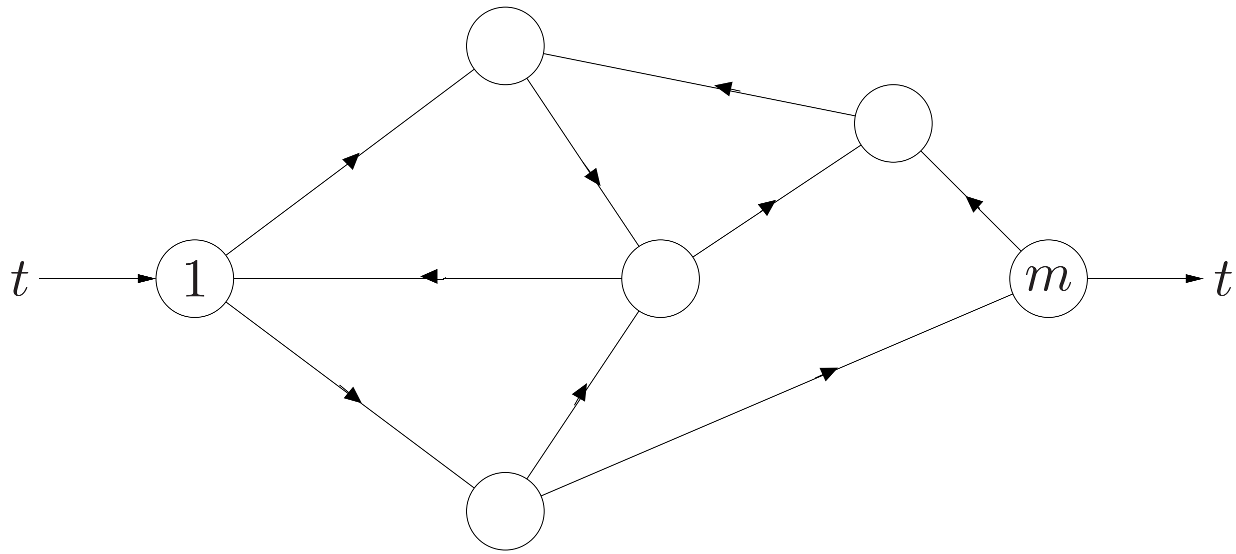
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & l \preceq x \preceq u \end{array}$$

- c_i is unit cost of flow through arc i
- l_j and u_j are limits on flow through arc j (typically, $l_j \leq 0$, $u_j \geq 0$)
- we assume $l_j < u_j$, but allow $l_j = -\infty$ and $u_j = \infty$

includes many network problems as special cases

Max-flow problem

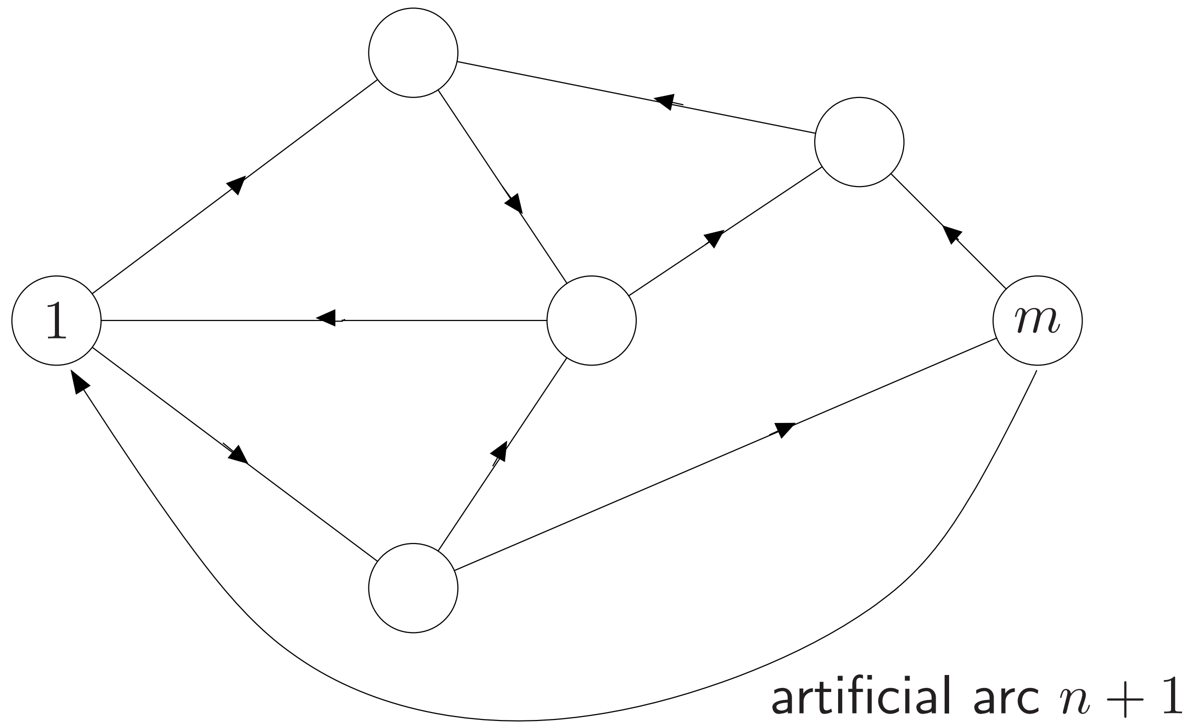
maximize flow between node 1 (source) and node m (sink)



$$\begin{array}{ll} \text{maximize} & t \\ \text{subject to} & Ax = te \\ & l \leq x \leq u \end{array}$$

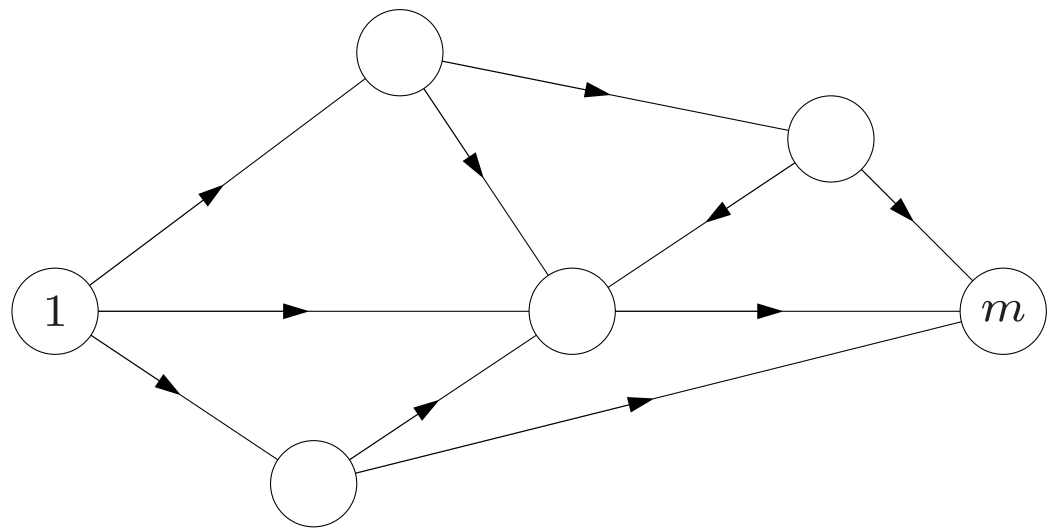
where $e = (1, 0, \ldots, 0, -1)$

interpretation as minimum cost flow problem



$$\begin{array}{ll} \text{minimize} & -t \\ \text{subject to} & \left[\begin{array}{cc} A & -e \end{array}\right] \left[\begin{array}{c} x \\ t \end{array}\right] = 0 \\ & l \leq x \leq u \end{array}$$

Project scheduling



- arcs represent n tasks to be completed in a period of length T
- t_k is duration of task k; must satisfy $\alpha_k \le t_k \le \beta_k$
- cost of completing task k in time t_k is $c_k(\beta_k - t_k)$
- nodes represent precedence relations: if arc k ends at node i and arc j starts at node i, then task k must be completed before task j can start

LP formulation

$$\begin{array}{ll} \text{minimize} & c^T(\beta - t) \\ \text{subject to} & t_k + y_i \leq y_j \quad \text{for all arcs } k = (i,j) \\ & y_m - y_1 \leq T \\ & \alpha \leq t \leq \beta \end{array}$$

- variables $t_1, \ldots, t_n, y_1, \ldots, y_m$
- $y_i - y_1$ is an upper bound on the total duration of tasks preceding node i

in matrix form

$$\begin{array}{ll} \text{minimize} & c^T(\beta - t) \\ \text{subject to} & t + A^T y \leq 0 \\ & y_m - y_1 \leq T \\ & \alpha \leq t \leq \beta \end{array}$$

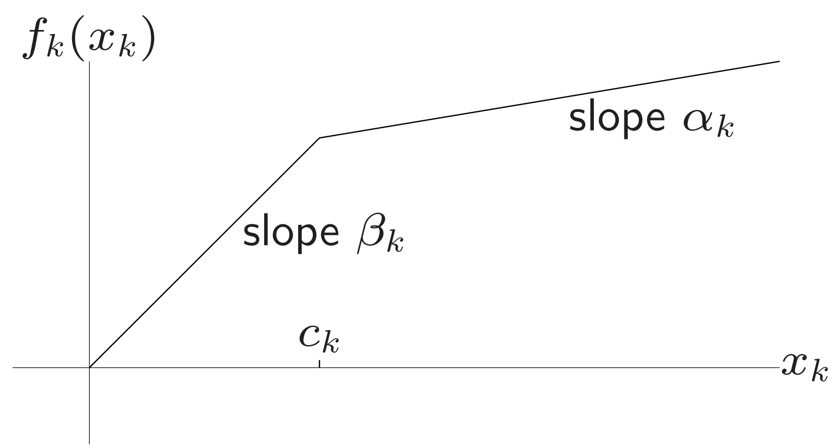
dual problem (after a simplification)

$$\begin{array}{ll} \text{maximize} & -T\lambda + \alpha^T x + (\beta - \alpha)^T s \\ \text{subject to} & Ax = \lambda e \\ & x \geq 0, \quad s \leq x, \quad s \leq c, \quad \lambda \geq 0 \end{array}$$

variables λ, x, s; $e = (1, 0, \ldots, 0, -1)$

interpretation: minimum cost network flow problem with nonlinear cost

$$\begin{array}{ll} \text{maximize} & -T\lambda + \sum_k f_k(x_k) \\ \text{subject to} & Ax = \lambda e \\ & x \geq 0, \quad \lambda \geq 0 \end{array}$$



Paths and cycles

- **path** from node s to node t: sequence of arcs $P_1, \ldots, P_N$

$$P_k = (i_{k-1}, i_k) \quad \text{or} \quad P_k = (i_k, i_{k-1}), \qquad i_0 = s, \qquad i_N = t$$

example (page 15–3): arcs 1, 3, 4, 7 form a path from node 1 to node 5

- **directed path** sequence of arcs $P_1, \ldots, P_N$

$$P_k = (i_{k-1}, i_k) \qquad i_0 = s, \qquad i_N = t$$

example: arcs 1, 3, 6 form a directed path from node 1 to node 5

- **(directed) cycle:** (directed) path from a node to itself

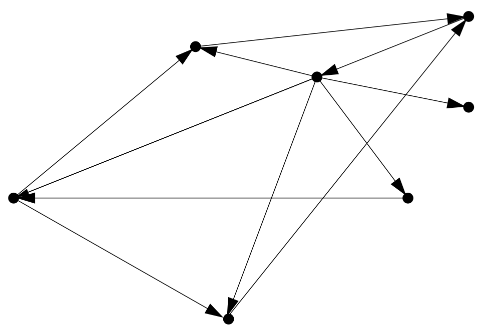
example: arcs 1, 2, 3 form a cycle; arcs 4, 6, 7 form a directed cycle

Acyclic networks and trees

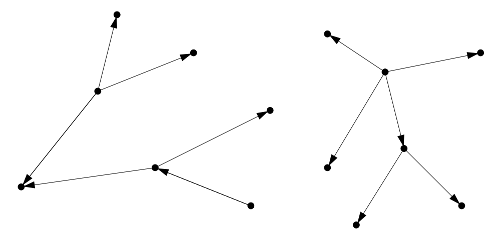
connected network: there exists a path between every pair of nodes

acyclic network: does not contain cycles

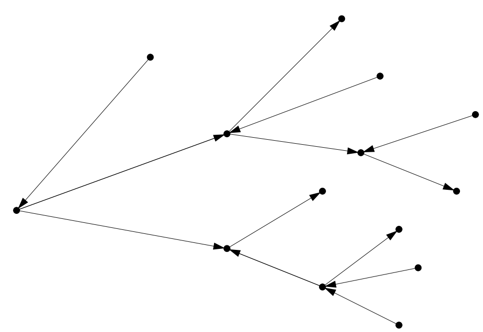
tree: connected acyclic network



connected, not acyclic



acyclic, not connected



tree

Topology and rank of incidence matrix

- network is **connected** if and only if

$$\mathbf{rank}\, A = \mathbf{rank}\, \tilde{A} = m - 1$$

  $Ax = b$ is solvable for all b with $\mathbf{1}^T b = 0$

- network is **acyclic** if and only if

$$\mathbf{rank}\, A = \mathbf{rank}\, \tilde{A} = n$$

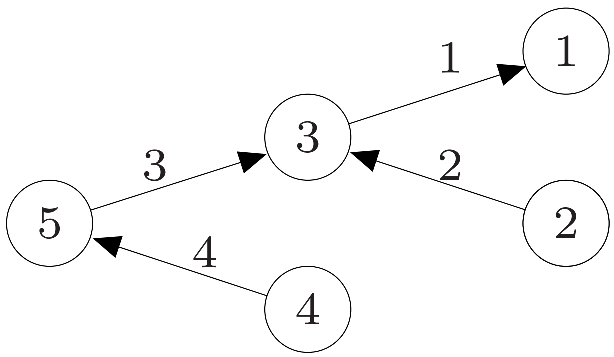
  if $Ax = b$ is solvable, its solution is unique

- network is a **tree** if and only if

$$\mathbf{rank}(A) = \mathbf{rank}\, \tilde{A} = n = m - 1$$

  $Ax = b$ has a unique solution for all b with $\mathbf{1}^T b = 0$

Solving balance equations for tree networks



$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

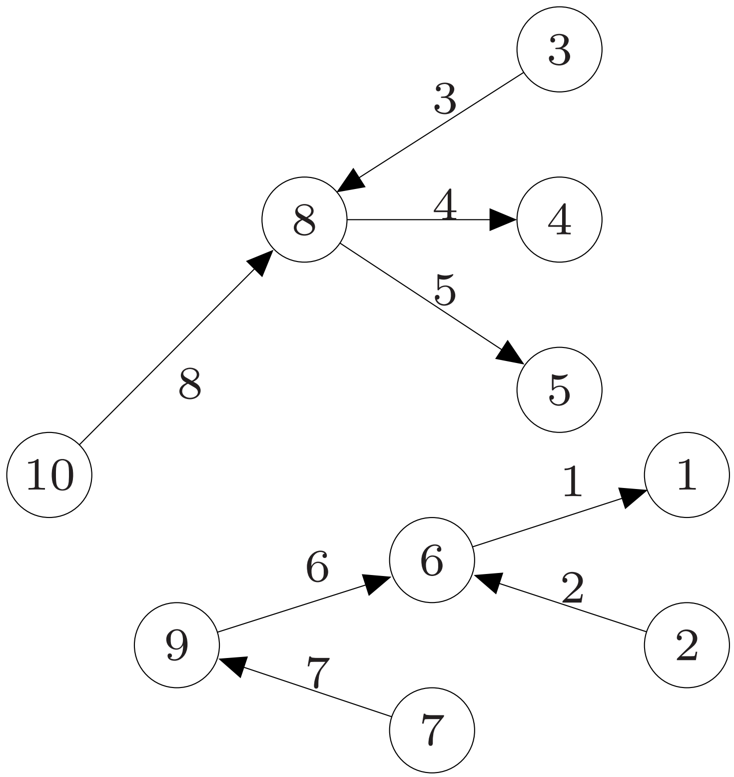
$$x_1 = -b_1, \qquad x_2 = b_2, \qquad x_3 = -b_1 - b_2 - b_3, \qquad x_4 = b_4$$

in general, choose node m as 'root' node and take

$$x_j = \pm \sum_{\text{nodes } i \text{ downstream of arc } j} b_i$$

important consequence: $x \in \mathbf{Z}^n$ if $b \in \mathbf{Z}^m$

Solving balance equations for acyclic networks



$$
\begin{aligned}
x_1 &= -b_1 \\
x_2 &= b_2 \\
x_3 &= b_3 \\
x_4 &= -b_4 \\
x_5 &= -b_5 \\
x_6 &= -b_1 - b_2 - b_6 \\
x_7 &= b_7 \\
x_8 &= -b_3 - b_4 - b_5 - b_8
\end{aligned}
$$

- can solve using only additions/subtractions
- $x \in \mathbf{Z}^n$ if $b \in \mathbf{Z}^m$

Integrality of extreme flows

$\mathcal{P}$ is polyhedron of feasible flows

$$Ax = b, \qquad l \leq x \leq u$$

we will show that the extreme points of $\mathcal{P}$ are integer vectors if

- the external supplies b_i are integer
- the flow limits l_i, u_i are integer (or $\pm\infty$)

proof. suppose x is an extreme flow with

$$l_j < x_j < u_j, \quad j = 1, \ldots, K, \qquad x_j = \begin{cases} l_j & j = K+1, \ldots, L \\ u_j & j = L+1, \ldots, n \end{cases}$$

we prove that x_1,. . . , x_K are integers

1. apply rank test of page 3–19 to the inequalities

$$l \leq x \leq u, \qquad Ax \leq b, \qquad -Ax \leq -b$$

   rank test:

$$\mathbf{rank}\left(\begin{bmatrix} 0 & -I & 0 \\ 0 & 0 & I \\ B_0 & B_- & B_+ \\ -B_0 & -B_- & -B_+ \end{bmatrix}\right) = n$$

   where $A = \begin{bmatrix} B_0 & B_- & B_+ \end{bmatrix}$, $B_0 \in \mathbf{R}^{m \times K}$, etc.

   conclusion: $\mathbf{rank}\, B_0 = K$ (subnetwork with arcs 1, . . . , K is acyclic)

2. $y = (x_1, \ldots, x_K)$ satisfies

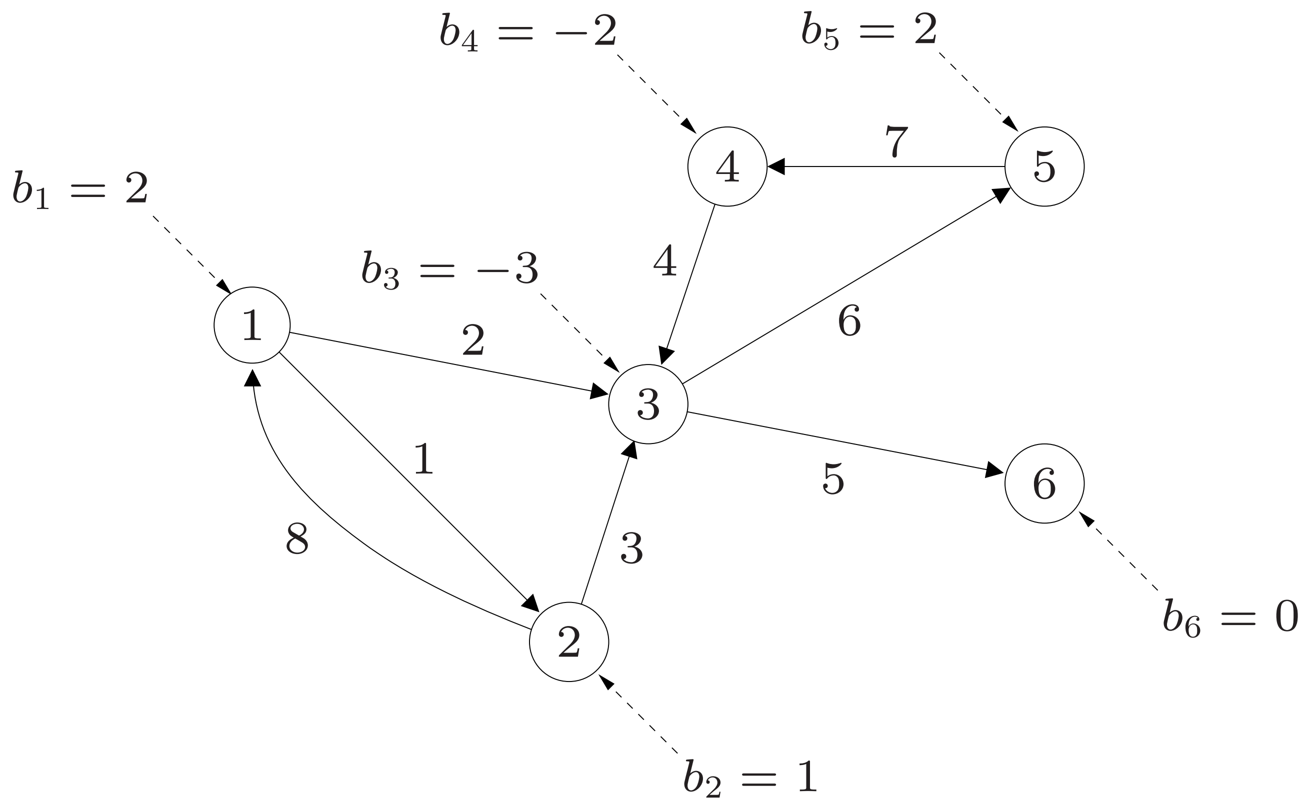
$$B_0 y = b - \left[\begin{array}{cc} B_- & B_+ \end{array}\right] \left[\begin{array}{c} x_{K+1} \\ \vdots \\ x_n \end{array}\right] \qquad (1)$$

interpretation: balance equations of an *acyclic* subnetwork with incidence matrix B_0, flow vector y, and *integer* external supplies

$$b - \left[\begin{array}{cc} B_- & B_+ \end{array}\right] \left[\begin{array}{c} x_{K+1} \\ \vdots \\ x_n \end{array}\right]$$

conclusion (from page 15–16): y is an integer vector

example ($l_i = 0$, $u_i = \infty$ for all arcs)



$x = (0, 2, 1, 0, 0, 0, 2, 0)$ is an extreme flow:

- it is feasible
- subgraph with arcs 2, 3, 7 is acyclic

Shortest path problem

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T x \\ \text{subject to} & Ax = (-1, 0, \ldots, 0, 1) \\ & 0 \leq x \leq \mathbf{1} \end{array}$$

- extreme optimal solutions satisfy $x_i \in \{0, 1\}$
- arcs with $x_i = 1$ form a shortest (forward) path between nodes 1 and m
- extends to arcs with non-unit lengths
- can be solved very efficiently via specialized algorithms

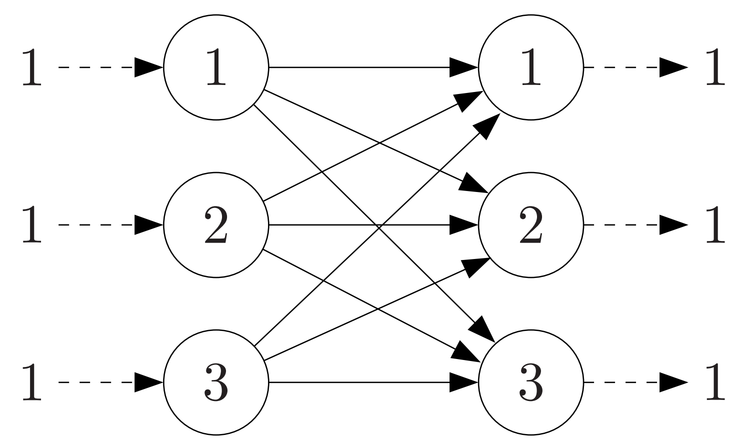
Assignment problem

- match N people to N tasks
- each person assigned to one task; each task assigned to one person
- cost of matching person i to task j is a_{ij}

minimum cost flow formulation

$$\begin{array}{ll} \text{min.} & \sum_{i,j=1}^{N} a_{ij}x_{ij} \\ \text{s.t.} & \sum_{i=1}^{N} x_{ij} = 1, \quad j = 1, \ldots, N \\ & \sum_{j=1}^{N} x_{ij} = 1, \quad i = 1, \ldots, N \\ & 0 \leq x_{ij} \leq 1, \quad i, j = 1, \ldots, N \end{array}$$

example ($N = 3$)



integrality: extreme optimal solution satisfies $x_{ij} \in \{0, 1\}$

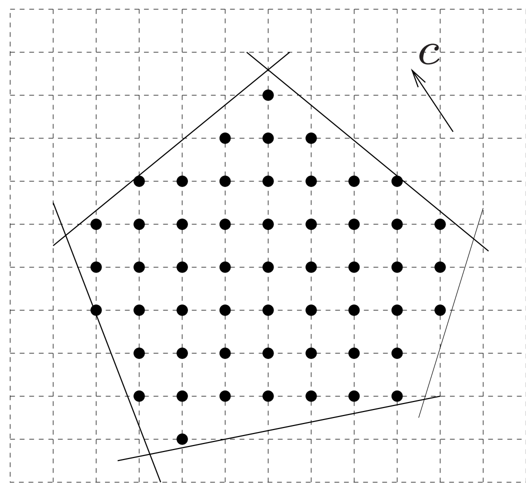
Lecture 16
Integer linear programming

- integer linear programming, 0-1 linear programming
- a few basic facts
- branch-and-bound

Definition

integer linear program (ILP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b, \quad Gx = d \\ & x \in \mathbf{Z}^n \end{array}$$



mixed integer linear program: only some of the variables are integer

0-1 (Boolean) linear program variables take values 0 or 1

Example: facility location problem

- n potential facility locations, m clients
- c_i, $i = 1, \ldots, n$: cost of opening a facility at location i
- d_{ij}, $i = 1 \ldots, m$, $j = 1, \ldots, n$: cost of serving client i from location j

determine optimal location:

$$\begin{array}{ll} \text{minimize} & \sum_{j=1}^n c_j y_j + \sum_{i=1}^m \sum_{j=1}^n d_{ij} x_{ij} \\ \text{subject to} & \sum_{j=1}^n x_{ij} = 1, \quad i = 1, \ldots, m \\ & x_{ij} \leq y_j, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n \\ & x_{ij}, y_j \in \{0, 1\} \end{array}$$

- $y_j = 1$ if location j is selected
- $x_{ij} = 1$ if location j serves client i

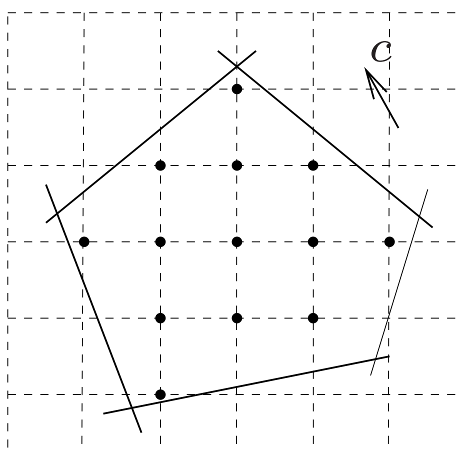
a 0-1 LP

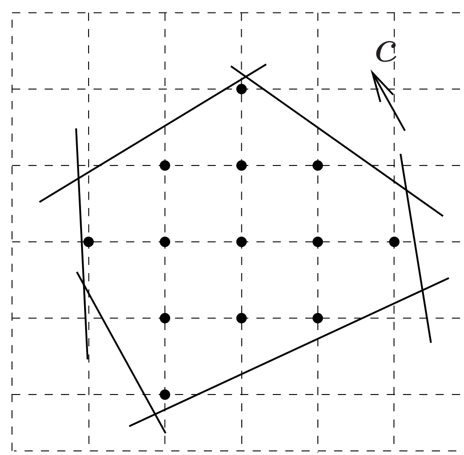
Linear programming relaxation

the LP obtained by deleting the constraints $x \in \mathbf{Z}^n$ (or $x \in \{0,1\}^n$) is called the LP *relaxation*

- provides a lower bound on the optimal value of the integer LP
- if the solution of the relaxation has integer components, then it also solves the integer LP

equivalent ILP formulations of the same problem can have different relaxations



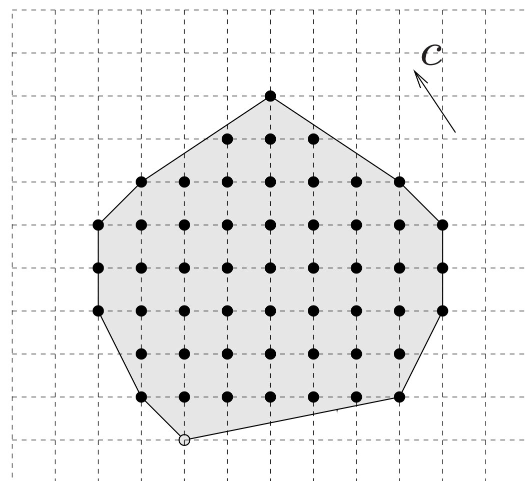


Strong formulations

the *convex hull* of the feasible set $\mathcal{S}$ of an ILP is:

$$\mathbf{conv}\,\mathcal{S} = \left\{ \sum_{i=1}^{K} \lambda_i x^i \;\middle|\; x^i \in \mathcal{S}, \lambda_i \geq 0, \sum_i \lambda_i = 1 \right\}$$

(the smallest polyhedron containing $\mathcal{S}$)



for any c, the solution of the ILP also solves the relaxation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathbf{conv}\,\mathcal{S} \end{array}$$

Branch-and-bound algorithm

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P} \end{array}$$

where $\mathcal{P}$ is a finite set

general idea:

- decompose in smaller problems

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P}_i \end{array}$$

  where $\mathcal{P}_i \subset \mathcal{P}$, $i = 1, \ldots, K$

- to solve subproblem: decompose recursively in smaller problems

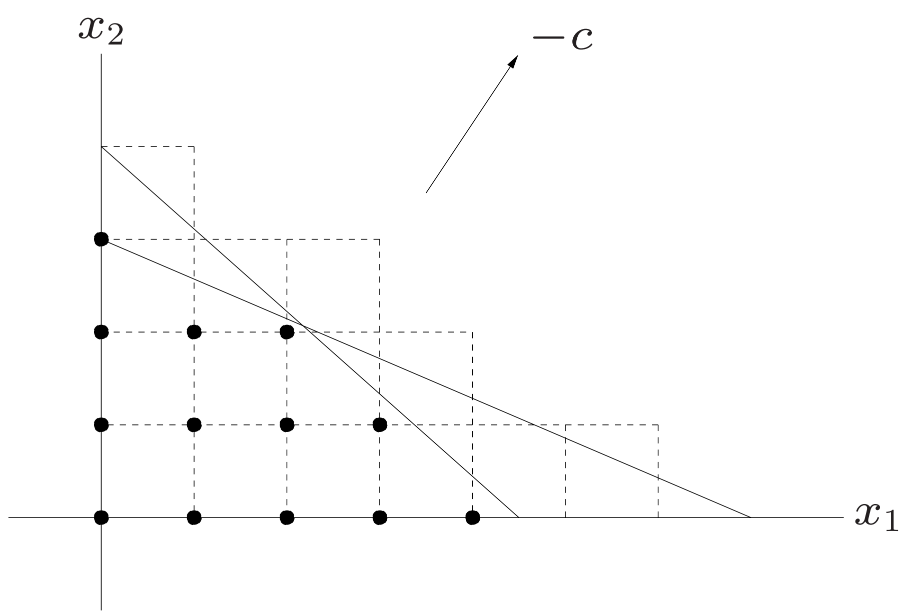
- use lower bounds from LP relaxation to identify subproblems that don't lead to a solution

example

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P} \end{array}$$

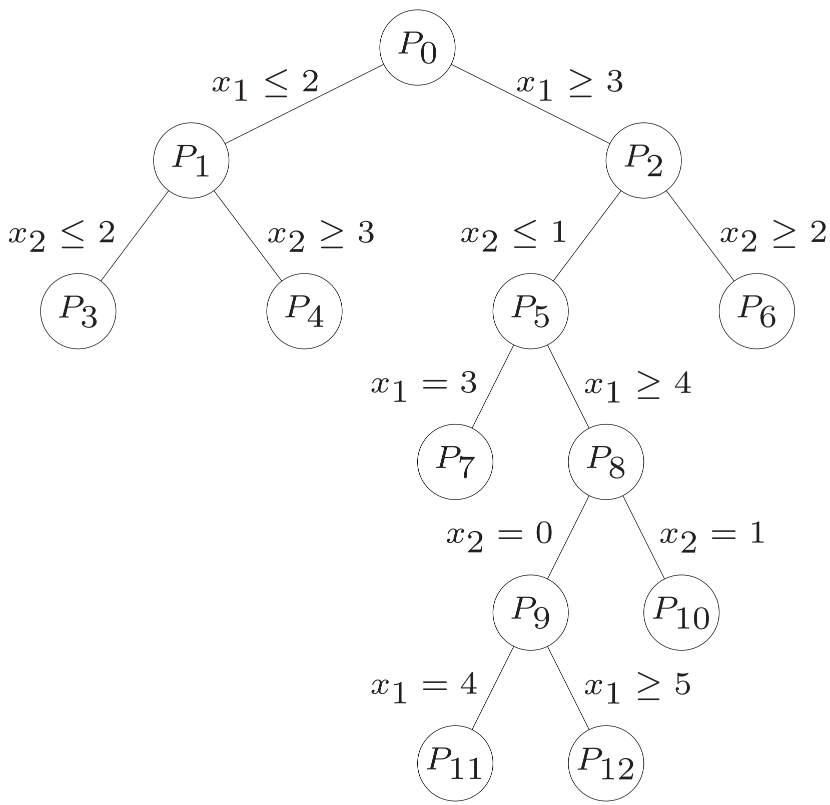
where $c = (-2, -3)$, and

$$\mathcal{P} = \left\{ x \in \mathbf{Z}_+^2 \;\middle|\; \frac{2}{9}x_1 + \frac{1}{4}x_2 \leq 1, \quad \frac{1}{7}x_1 + \frac{1}{3}x_2 \leq 1 \right\}$$



optimal point: $(2, 2)$

tree of subproblems and results of LP relaxations:



	$x^\star$	$p^\star$
P_0	$(2.17, 2.07)$	-10.56
P_1	$(2.00, 2.14)$	-10.43
P_2	$(3.00, 1.33)$	-10.00
P_3	$(2.00, 2.00)$	-10.00
P_4	$(0.00, 3.00)$	-9.00
P_5	$(3.38, 1.00)$	-9.75
P_6		$+\infty$
P_7	$(3.00, 1.00)$	-9.00
P_8	$(4.00, 0.44)$	-9.33
P_9	$(4.50, 0.00)$	-9.00
P_{10}		$+\infty$
P_{11}	$(4.00, 0.00)$	-8.00
P_{12}		$+\infty$

conclusions from subproblems:

- P_2: the optimal value of

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P}, \quad x_1 \geq 3 \end{array}$$

  is greater than or equal to -10.00

- P_3: the solution of

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P}, \quad x_1 \leq 2, \quad x_2 \leq 2 \end{array}$$

  is $(2, 2)$

- P_6: the problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P}, \quad x_1 \leq 3, \quad x_2 \geq 2 \end{array}$$

  is infeasible

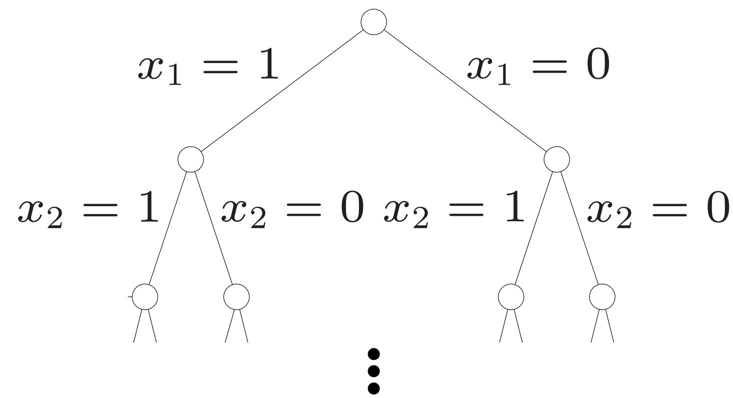
suppose we enumerate the subproblems in the order

$$P_0, \quad P_1, \quad P_2, \quad P_3, \quad \ldots$$

then after solving subproblem P_4 we can conclude that $(2, 2)$ is optimal

branch-and-bound for 0-1 linear program

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax \leq b, \quad x \in \{0,1\}^n
\end{array}
$$



can solve by enumerating all 2^n possible x; every node represents a problem

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax \leq b \\
& x_i = 0, \quad i \in I_1, \qquad x_i = 1, \quad i \in I_2 \\
& x_i \in \{0,1\}, \quad i \in I_3
\end{array}
$$

where I_1, I_2, I_3 partition $\{1,\ldots,n\}$

branch-and-bound method

set $U = +\infty$, mark all nodes in the tree as active

1. select an active node k, and solve the corresponding LP relaxation

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax \leq b \\
& x_i = 0, \ i \in I_1^k \\
& x_i = 1, \ i \in I_2^k \\
& 0 \leq x_i \leq 1, \ i \in I_3^k
\end{array}
$$

   let $\hat{x}$ be the solution of the relaxation

2. if $c^T \hat{x} \geq U$, mark all nodes in the subtree with root k as inactive
3. if all components of $\hat{x}$ are 0 or 1, mark all nodes in the subtree with root k as inactive; if moreover $c^T \hat{x} < U$, then set $U := c^T \hat{x}$ and save $\hat{x}$ as the best feasible point found so far
4. otherwise, mark node k as inactive
5. go to step 1

Lecture 17
Conclusions

- topics we didn't cover
- choosing an algorithm
- EE236B

Topics we didn't cover

network flow problems

- LPs defined in terms of graphs, *e.g.*, assignment, shortest path, transportation problems
- huge problems solvable via specialized methods

see 232E

integer linear programming

- examples and applications
- other methods (cutting plane, dynamic programming, . . .)

Choice of method

interior-point methods vs. simplex

- both work very well
- interior-point methods believed to be (usually) faster than simplex for problems with more than 10,000 variables/constraints

general-purpose vs. custom software

- several widely available and efficient general-purpose packages
- unsophisticated custom software that exploits specific structure can be faster than general-purpose solvers; some examples:
  - column generation via simplex method
  - ℓ_1-minimization via interior-point methods
  - interior-point method using conjugate gradients

some interesting URLs

- `http://plato.la.asu.edu/guide.html`
  (decision tree for optimization software)

- `http://www.mcs.anl.gov/home/otc/Guide`
  (NEOS guide of optimization software)

- `http://gams.cam.nist.gov`
  (guide to available mathematical software)

EE236B (winter quarter)

benefits of expressing a problem as an LP:

- algorithms will find the global optimum
- very large instances are readily solved

both advantages extend to nonlinear convex problems

- duality theory, interior-point algorithms extend gracefully from LP to nonlinear convex problems
- nonlinear convex optimization covers a much wider range of applications
- recognizing convex problems is more difficult than recognizing LPs