1 Lecture One

Main ideas: vector spaces, norms, inner product, analysis review, an existence result

Finite dimensional optimization breaths on three branches of mathematics: analysis, geometry, and (linear) algebra. We will start with some algebra, add geometry, and then review the key analytic notions that are used subsequently in the course.

It seems that in engineering, we can not really escape from the notion of a vector space (an algebraic structure).

1. For a set $\mathbf{F}$ to be a field, it has to be closed under two binary operations (addition and multiplication), both operations must be associative, commutative, and have distinct identity elements in $\mathbf{F}$; additive inverses exist and multiplicative inverses exist except for the additive identity, and multiplication operation must be distributive over the addition. We will mainly work with the field of real numbers $\mathbf{R}$, and occasionally with the field of complex numbers $\mathbf{C}$.

2. A vector space over $\mathbf{F}$ (scalars) is a set of objects, called vectors, which is closed under a binary operation (addition) that is associative, commutative, and has an identity element; moreover, for all $\alpha, \beta \in \mathbf{F}$ and all $x, y \in \mathbf{V}$, $\alpha(x+y) = \alpha x + \alpha y, (\alpha+\beta)x = \alpha x + \beta x, \alpha(\beta x) = (\alpha\beta)x$ and $ex = x$, where $e \in \mathbf{F}$ is the multiplicative identity.

3. Let $S \subseteq \mathbf{V}$. The span of S is the set all possible linear combinations of the vectors in S; S is called a linear independent set (or a set with are linearly independent vectors) if none of its elements can be expressed as a linear combination of the others (otherwise it is called linearly dependent). A basis for $\mathbf{V}$ is a linearly independent set whose span is $\mathbf{V}$. If $\mathbf{V}$ admits a finite basis, it is called finite dimensional.

   We now add some geometry ...

4. Let $\mathbf{V}$ be a vector space over $\mathbf{R}$. A function

   $$\langle .,. \rangle : \mathbf{V} \times \mathbf{V} \to \mathbf{R}$$

   is an inner product if it is symmetric, self-positive ($\langle x, x \rangle = 0$ if and only if $x = 0$), and it is additive (individually, in each of its arguments) and homogeneous (with respect to the scalar multiplication). One has

   $$|\langle x, y \rangle| \leq \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2}, \tag{1.1}$$

   for all $x, y \in \mathbf{V}$ (Cauchy-Schwarz).

   In this course, we will mainly work with **finite dimensional vector spaces over R with an inner product defined on them**- Euclidean spaces. An arbitrary Euclidean space will be denoted by $\mathbf{E}$.

5. Let $\mathbf{V}$ be a vector space over $\mathbf{R}$. A function

   $$||.|| : \mathbf{V} \to \mathbf{R}$$

   is a norm if it is positive (except when its argument is the zero vector), positive homogeneous (with respect to the scalars),

   $$||\alpha\, x|| = |\alpha|\, ||x||, \quad \alpha \in \mathbf{R},$$

   and satisfies the triangular inequality

   $$||x+y|| \leq ||x|| + ||y||.$$

   A norm is really a topological notion, but we can induce a topology with the aid of geometry ...

6. The inner product induces a (canonical) norm:
$$\|x\| := \langle x, x\rangle^{1/2}.$$
7. The unit ball is the set $B = \{x \in \mathbf{E} \,|\, \|x\| \leq 1\}$.

   We find a good use for some basic set theoretic operations ...
8. For $\Lambda \subseteq \mathbf{R}$ and $C \subseteq \mathbf{E}$
$$\Lambda C := \{\lambda x \,|\, \lambda \in \Lambda,\ x \in C\}.$$
9. The sum of the two sets $C, D \subseteq \mathbf{E}$ is defined by
$$C + D := \{x + y \,|\, x \in C,\, y \in D\}; \quad C - D := C + (-D).$$

   From analysis, we need to understand openness, closed-ness, compactness, and lacking or having an interior, etc.:
10. A point x is in the interior of the set $D \subseteq \mathbf{E}$ ($\mathbf{int}\ D$) if there is a real $\delta > 0$ such that $x + \delta B \subseteq D$.
11. x is the limit of a sequence of points $x^1, x^2, \ldots, x^n$ in $\mathbf{E}$, written as
$$x^j \to x \quad \text{as} \quad j \to \infty, \quad \text{if} \quad \|x^j - x\| \to 0 \quad \text{as} \quad j \to \infty.$$
12. The closure of D , cl D, is the set of limits of sequences of points in D. The boundary of D, bd D is cl $D \backslash \mathbf{int}\ D$.
13. $D \subseteq \mathbf{E}$ is open if $D = \mathbf{int}\ D$ and closed if $D = \mathrm{cl}\ D$.
14. $D \subseteq \mathbf{E}$ is bounded if there is a real k such that
$$D \subseteq kB.$$
15. The set $D \subseteq \mathbf{E}$ is compact if it is closed and bounded.
16. One of the most important results in classical analysis is the following:

   Theorem 1.1 (Bolzano-Weierstrass) *Bounded sequences in* $\mathbf{E}$ *have convergent subsequences.*
17. Let $D \subseteq \mathbf{E}$ and $f : D \to \mathbf{R}$. The function f is continuous (on D) if
$$f(x^j) \to f(x) \quad \text{when} \quad x^j \to x.$$
18. The set of real numbers is ordered.
19. Given a set $\Gamma \subseteq \mathbf{R}$, the infimum of Γ ($\inf \Gamma$) is the greatest lower bound on Γ; the least upper bound on Γ is denoted by $\sup \Gamma$ (supremum).
20. To make sure that inf and sup always exists (what the inf of $\mathbf{R}$?) we append $-\infty$ and $+\infty$ to $\mathbf{R}$; we write $\mathbf{R} \cup \{+\infty\}$ if necessary. One has, by convention, $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.
21. Let $D \subseteq \mathbf{E}$ and $f : D \to \mathbf{R}$. Note that $f(D) \subseteq \mathbf{R}$. The global minimizer of f in D is a point $\bar{x}$ where f attains its infimum, i.e., $\bar{x} \in D$ and
$$\inf_D f = \inf\{f(x) \,|\, x \in D\} = f(\bar{x}).$$

   In this case $\bar{x}$ is *optimal solution* of the optimization problem $\inf_D f$.

22. Recall that the level sets of a function $f : D \to \mathbf{R}$ are, for each $\alpha \in \mathbf{R}$,

$$L_f(\alpha) := \{x \in D \mid f(x) \leq \alpha\}.$$

Another important result in analysis, well, it is really in *optimization*, is the following existence result ...

23. **Theorem 1.2 (Weierstrass)** *Suppose that the set $D \subseteq \mathbf{E}$ is nonempty and closed and that all the level sets of the continuous function $f : D \to \mathbf{R}$ are bounded. Then f has a global minimizer (in D).*

Proof: Since D is nonempty, $\inf_D f < +\infty$. Consider a decreasing sequence $\alpha^i \to \inf_D f$ $(i = 1, 2, \ldots)$. Now construct a sequence of vectors $x^i \in \mathbf{E}$ such that $x^i \in L_f(\alpha^i)$; note that

$$f(x^i) \to \inf_D f.$$

This sequence is bounded: for all $i \geq 1$, $x^i \in L_f(\alpha^i) \subseteq L_f(\alpha^1)$. Thus it has a convergent subsequence (Theorem 1.1), i.e., there exists a increasing sequence $j_1, j_2, \ldots$, and x^* such that $x^{j_1}, x^{j_2}, \ldots \to x^*$. The set D is closed thus $x^* \in D$. Moreover,

$$f(x^{j_1}), f(x^{j_2}), \ldots \to f(x^*).$$

Thereby $\inf_D f = f(x^*)$. $\square$

Note that the proof does not give us a clue on actually finding this global minimizer.