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The term structure of interest rates in a pure exchange economy with heterogeneous investors

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Abstract

This paper presents an equilibrium model of the term structure of interest rates when investors have heterogeneous preferences. The basic model considers a pure exchange economy of two classes of investors with different (but constant) relative risk aversion and gives closed-form solutions to bond prices. I use the model to examine the effect of preference heterogeneity on the behavior of bond yields. The model is also extended to cases of more than two classes of investors.

Key words: Asset pricing; Interest rates; Term structure; Preferences heterogeneity
JEL classification: G12; E43; D51; D91

1. Introduction

Existing models of the equilibrium term structure of interest rates are often based on the representative agent framework with specific parametric assumptions about the preferences of the representative agent. For example, the well-known model of Cox, Ingersoll, and Ross (1985a) assumes that the representative agent has logarithmic preferences and faces a production opportunity

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with an expected return that follows a 'square-root' process (see also Longstaff and Schwartz, 1992; Sun, 1992). When the financial market is complete (in the sense of Harrison and Kreps, 1979), a representative agent can be constructed whose marginal utility under the given process of aggregate consumption determines the equilibrium security prices (see, e.g., Constantinides, 1982). However, the preferences of the representative agent are in general quite complicated even when the preferences of individual investors are simple (see, e.g., Dumas, 1989; Rubinstein, 1974, considers special cases of investor preferences when the representative agent's preference exhibits simple forms). Instead of being assumed, they should be derived from the primitives of the economy, such as the individual preferences, as part of the equilibrium analysis. Thus, even though strong assumptions about the representative agent's preferences can lead to simple bond pricing formulas, they are often too restrictive to reflect the effect of any investor heterogeneity on the behavior of bond prices. Furthermore, starting from the representative agent without explicitly modeling the interaction among individual investors leaves out any implications for quantities (such as the amount investors borrow and lend) and how they are related to bond prices and interest rates. In addition, linking bond prices and disaggregated variables such as the amount of borrowing and the distribution of consumption and wealth among investors makes the model more appealing empirically. Note that representative agent models only relate bond prices to underlying state variables that may not be directly observable. By explicitly modeling individual investors, bond prices can be related to disaggregated variables that are directly observable. These variables can then be used as instruments for the underlying state variables in any empirical implementation of the model.

This paper considers a simple pure exchange economy with two classes of investors who have time-additive, state-independent, constant relative risk-aversion preferences with risk-aversion coefficients a_1 and a_2, respectively, where $a_1 > a_2$. Equilibrium bond prices and yields are solved in closed form. The main goal of the paper is to examine how the heterogeneity in preferences can affect the behavior of the term structure of interest rates. In particular, the equilibrium term structure of interest rates with both classes of investors present is compared with the term structure with only one class of investors.

In general, the yield curve with both classes of investors behaves differently from the two yield curves each with only one of the two classes of the investors present in the economy. In the simple case in which aggregate consumption follows a geometric Brownian motion, its growth will be independently and identically distributed over time. In worlds populated by only one class of investors, aggregate consumption is simply their aggregate consumption. Since their utility function is isoelastic, the growth of their marginal utility will then be i.i.d. over time also. Thus interest rates will be constant over time and the term structure will be flat, independent of the level of aggregate consumption (see, e.g., Stapleton and Subrahmanyam, 1990; McCulloch, 1993). When both classes of investors are present, however, the growth of investors' marginal utility will be endogenous and non-i.i.d. over time. The a_1 investors (with low elasticity of intertemporal substitution) prefer lower consumption growth than the a_2 investors. In equilibrium, the consumption of the a_1 investors will be less sensitive to changes in aggregate consumption than the consumption of the a_2 investors. Furthermore, the a_1 investors' share of aggregate consumption is higher (lower) than that of the a_2 investors when the level of aggregate consumption is high (low). The consumption growth of individual investors thus depends on the level of aggregate consumption. Consequently, the instantaneous interest rate and the shape of the yield curve change over time as aggregate consumption changes.

Borrowing and lending between the two classes of investors in financing their optimal consumption plans tend to increase the volatility of short-term yields. In particular, short-term yields can move outside the range bounded by the values they would take in worlds populated by only one class of investors. On the other hand, the long-term yields with both classes of investors present are closely related to the bounds given by the two yield curves with only one of the two classes of investors. The long-term yield always approaches the lower bound as the maturity increases. With mild growth in the economy, the preferences of the a_1 investors dominate the long-term yields even though the a_2 investors may eventually own the whole economy (independent of the current wealth distribution between the two classes of investors). This result implies that investors with small relative wealth can have large effects on bond yields. Note that the more risk-averse investors are more averse to low levels of future consumption. Long-term bonds are more attractive to them as hedging instruments against future downturns of the economy. Consequently, the a_1 investors can exert stronger influence on the equilibrium prices of long-term bonds when the probability of future downturns is not too small, i.e., the growth of the economy is not too high.

A close cousin of the current model is Dumas (1989). He considers the equilibrium of a production economy with two investors, one of whom has logarithmic preferences and the other power preferences. Since the growth of the economy is endogenously determined in a production economy, Dumas has to conjecture the existence of equilibrium and resorts to numerical solutions in his analysis. The pure exchange economy considered here allows closed form solutions to the equilibrium so that the behavior of bond yields can be analyzed. The current model is also related to international growth models with heterogeneous agents (see, e.g., Solow, 1956; Cass, 1965; Koopmans, 1965; Becker, 1978; Lucas and Stokey, 1984). These models often assume certainty and are less interesting for studying the term structure of interest rates.

The paper is organized as follows. Section 2 defines the basic model in which there are two classes of investors and the aggregate endowment follows a simple geometric Brownian motion. The equilibrium of the economy is derived in Section 3. Section 4 calculates equilibrium bond prices and analyzes the effect of preference heterogeneity on the behavior of bond yields. Extensions of the basic model to allow more than two classes of investors and more general processes of the aggregate endowment are discussed in Section 5. Some further comments are given in Section 6. All proofs are in the Appendix.

2. The basic model

Consider a pure exchange economy of a single perishable consumption good (the numeraire). The economy is endowed with a flow of the consumption good. The rate of endowment flow is Y_t at t for $t \in [0, T]$ and follows a geometric Brownian motion:

$$\mathrm{d}Y_t = \mu Y_t \mathrm{d}t + \sigma Y_t \mathrm{d}w_t, \qquad t \in [0, T], \tag{1}$$

where $Y_0 > 0$, $\mu \geqslant 0$, and $\sigma > 0$ are constants, and w_t is a standard Wiener process. (Throughout the paper, equalities or inequalities involving random variables are in the sense of almost surely with respect to the underlying probability measure.) The process Y_t has a natural boundary at zero which is attractive when $\mu < \frac{1}{2}\sigma^2$ but always unattainable (see Karlin and Taylor, 1981). This implies that Y_t is strictly positive with probability one. Y_t as defined by (1) has the following solution:

$$Y_t = Y_0 \exp\left\{(\mu - \tfrac{1}{2}\sigma^2)t + \sigma \int_0^t \mathrm{d}w_s\right\}. \tag{2}$$

Conditional on Y_t, $Y_{t+\tau}$ is log-normally distributed. Define $g_t(\tau) \equiv Y_{t+\tau}/Y_t$ as the (gross) growth rate of aggregate consumption; $\mathrm{E}[\log g_t(\tau)] = (\mu - \frac{1}{2}\sigma^2)\tau$ and $\mathrm{var}[\log g_t(\tau)] = \sigma^2\tau$.

There exists a market where shares of the aggregate endowment (the 'stock') are traded. Holding one share of the stock from $t = 0$ to $t = T$ yields the payoff (i.e., the dividend) at rate $\{Y_t,\ t \in [0, T]\}$. In addition, there exists a 'money market' in which a locally risk-free security can be traded (i.e., investors can borrow from or lend to each other without default). For $t \in [0, T]$, let S_t be the price of the stock (ex-dividend) and r_t the instantaneous interest rate.

Investors in the economy can trade competitively in the securities market and consume the proceeds. Let c_t be an investor's consumption rate at t, α_t his holdings of the risk-free asset, and θ_t his holdings of the stock. The consumption and trading strategies $\{c_t,\ (\alpha_t, \theta_t)\}$ are adopted processes satisfying the standard integrability conditions:

$$\int_0^T c_t^2 \mathrm{d}t < \infty\,, \quad \int_0^T |\alpha_t r_t \mathrm{d}t + \theta_t(Y_t \mathrm{d}t + \mathrm{d}S_t)| < \infty\,, \quad \int_0^T \theta_t^2 \mathrm{d}[S_t] < \infty\,, \tag{3}$$

where $[S_t]$ denotes the quadratic variation process of S_t. [See, e.g., Karatzas and Shreve (1988) for a discussion on the quadratic variation process of a given process.] The investor's wealth process defined by $W_t \equiv \alpha_t + \theta_t S_t$ must be positive with probability one and must conform to the stochastic differential equation

$$\mathrm{d}W_t = \alpha_t r_t \mathrm{d}t + \theta_t(Y_t \mathrm{d}t + \mathrm{d}S_t) - c_t \mathrm{d}t\,.$$

The restriction of positive wealth is to rule out arbitrage opportunities (following Dybvig and Huang, 1988). Let Θ denote the set of trading strategies that satisfy the above conditions.

There are two classes of identical investors in the economy, denoted 1 and 2. Both classes of investors are initially endowed with only shares of the stock. Let $\alpha_{i,0_-}$ and $\theta_{i,0_-}$ be the initial shares of the risk-free security and stock of class i investors. Then $\alpha_{i,0_-} = 0$, $\theta_{i,0_-} > 0$, $i = 1, 2$, and $\sum_{i=1}^{2} \theta_{i,0_-} = 1$. Note that $(\alpha_{i,0}, \theta_{i,0})$, which denotes the optimal holdings of class i investors, is in general different from their endowment $(0, \theta_{i,0_-})$. A class i investor, $i = 1, 2$, chooses his consumption/trading strategy $\{c_i, (\alpha_i, \theta_i)\}$ to maximize his lifetime expected utility

$$\mathrm{E}_t\left[\int_t^T \mathrm{e}^{-\rho(s-t)}\,\frac{c_{i,s}^{1-a_i} - 1}{1 - a_i}\mathrm{d}s\right], \quad a_i > 0\,, \tag{4}$$

where $\rho > 0$ is the time discount parameter and is the same across investors. Since investors within each class are identical, we do not distinguish them and simply denote them respectively as investor i, $i = 1, 2$. Both classes of investors have constant relative risk aversion. Sections 3 and 4 further assume that $a_1 = 1$ and $a_1 = \frac{1}{2}$ to obtain simple solutions (the utility function with $a = 1$ is obtained by taking the limit $\lim_{a \to 1}(c^{1-a} - 1)/(1 - a) = \log c$). Thus, class 1 investors have a logarithmic utility function and class 2 investors have a square-root utility function, and class 1 investors are more risk-averse than class 2 investors (in terms of relative risk aversion). Section 5 relaxes these restrictive assumptions.

In addition, the parameter values are subject to the following growth condition:

$$\rho > \tfrac{1}{2}\max\left[0, \mu - \tfrac{1}{4}\sigma^2\right]. \tag{5}$$

This growth condition guarantees that investors' expected utilities are uniformly bounded for all $T \in [0, \infty)$ given the aggregate consumption process in (1), allowing taking the limit $T \to \infty$ in future discussions.

Before considering the equilibrium of the economy as defined above, a few comments on the economy are in order. For simplicity in exposition, Y_t is restricted to be a univariate diffusion process with linear drift and diffusion coefficient. Section 5 considers extensions to the multivariate case. Extensions to more general forms for the drift and the diffusion coefficient, including path dependence, are also possible.

In specifying the securities market, the only traded securities are the stock and the locally risk-free security. As will be shown later, given the current process of Y_t, the stock and the risk-free security are sufficient to dynamically complete the securities market in the sense of Harrison and Kreps (1979). Arbitrary consumption plans (satisfying certain integrability conditions as specified later) can be financed by continuously trading in the stock and the risk-free security. Allowing additional securities will not affect the nature of the equilibrium. Thus, the market equilibrium is derived with the securities market consisting of only the stock and the risk-free security. Simple arbitrage arguments can then be used to price other securities if they exist.

A principal assumption is that there are only two classes of investors in the economy and that they behave competitively in the market. Since investors within each class have the same isoelastic preferences, each class can be represented by a single representative investor who has the same preferences as the individual investors and the total endowment of the class (see, e.g., Rubinstein, 1974). In effect, the economy is populated with only the two representative investors, who behave competitively. In the remainder of the paper, the two representative investors are treated as two individual investors without referring to the class of investors they represent and denoted investor 1 and 2, respectively.

3. Market equilibrium

This section considers the market equilibrium of the economy defined above. I first derive a solution to the market equilibrium and then discuss the general nature of the equilibrium and the pricing implications.

3.1. Deriving the equilibrium

The definition of a market equilibrium follows Radner (1972):

Definition 1. A market equilibrium of the economy is the pair of price process $\{S, r\}$ *and consumption-trading strategies* $\{c_i, (\alpha_i, \theta_i);\ i = 1, 2\}$ *such that* $\{c_i, (\alpha_i, \theta_i)\}$, $i = 1, 2$, *maximizes investor i's expected utility*:

$$\sup_{\{c_i,(\alpha_i,\theta_i)\}} \mathrm{E}_t\left[\int_t^T \mathrm{e}^{-\rho(s-t)}\,\frac{c_{i,s}^{1-a_i}-1}{1-a_i}\,\mathrm{d}s\right], \qquad t\in[0, T]\,, \tag{6}$$

subject to

$$\mathrm{d}W_{i,t} = \alpha_{i,t} r_t \mathrm{d}t + \theta_{i,t}(Y_t \mathrm{d}t + \mathrm{d}S_t) - c_{i,t}\mathrm{d}t\,,$$

and markets clear:

$$\sum_{i=1}^{2} \theta_{i,t} = 1, \quad \sum_{i=1}^{2} \alpha_{i,t} = 0\,. \tag{7}$$

Here, $W_{i,0} = \theta_{i,0\_} S_0$.

Eq. (7) gives the market clearing of the securities market. The market clearing of the goods market is guaranteed by Walras' law. Combining the two market clearing conditions in (7), we have $W_{1,t} + W_{2,t} = S_t$.

The equilibrium is derived in three steps. The first step is to solve the Pareto-optimal allocations of the economy. The next step is to show that each Pareto-optimal allocation can be supported by an Arrow–Debreu equilibrium where investors can trade arbitrary future payoff streams at the initial date and achieve the given allocation. The final step is to construct the dynamic implementation of the Arrow–Debreu equilibrium (see Duffie and Huang, 1985) where investors continuously trade the stock and the risk-free security at prices given by the pricing functional in the Arrow–Debreu equilibrium and achieve the same allocation in equilibrium. This then gives the market equilibrium of the economy.[1]

When both investors have positive initial wealth, an allocation $\{c_1, c_2\}$ is Pareto-optimal if and only if there is a *constant* $\lambda \in (0, 1)$ such that $\{c_1, c_2\}$ solves the problem

$$\sup_{\{c_1,c_2\}} \mathrm{E}_0 \left\{ \int_0^T \mathrm{e}^{-\rho t} [\lambda \log c_{1,t} + 2(1-\lambda)\sqrt{c_{2,t}}\,]\,\mathrm{d}t \right\}, \tag{8}$$

subject to

$$c_{1,t} + c_{2,t} \leqslant Y_t, \qquad t \in [0,\ T]\,.$$

Here, λ is the weight of the investor with logarithm utility in the welfare function to be maximized. Note that in an exchange economy, there is no intertemporal transformation of resources. The intertemporal resource constraint in (8) is simply the collection of resource constraints for each date and each state. Furthermore, the investors' preferences are time-additive and state-separable, and so is the welfare function. Thus maximizing the expected intertemporal welfare function in (8) is equivalent to maximizing the welfare function period by

[1]Many authors have studied the existence of market equilibrium in quite general settings in continuous time (see, e.g., Duffie and Zame, 1989; Mas-Colell and Zame, 1991; Karatzas, Lehoczky, and Shreve, 1990). However, the model defined above does not directly fit into their framework. In particular, the aggregate endowment process as specified by (1) is not bounded away from zero which is often assumed in the literature.

period and state by state subject to the corresponding resource constraint. For each period and each state, the maximization problem takes the following form:

$$\sup_{c_1 + c_2 \leqslant Y} \leqslant e^{-\rho t}[\lambda \log c_1 + 2(1-\lambda)\sqrt{c_2}]\,. \tag{9}$$

Its solution gives the optimal sharing rule between the two investors.

Lemma 1. Given $\lambda \in (0, 1)$, *the optimal sharing rule between the two investors is*

$$\hat{c}_1(Y, \lambda) = \frac{1}{2}\left(\frac{\lambda}{1-\lambda}\right)^2 \left[\sqrt{1 + 4\left(\frac{1-\lambda}{\lambda}\right)^2 Y} - 1\right],$$

$$\hat{c}_2(Y, \lambda) = Y_t - \hat{c}_1(Y, \lambda)_2\,, \tag{10}$$

Also, $\lambda \hat{c}_1(Y, \lambda)^{-1} = (1-\lambda)\hat{c}_2(Y, \lambda)^{-1/2}$.

The optimal sharing rule as a function of Y is nonlinear and depends only on λ. Furthermore, in a Pareto optimum the marginal utilities of the two investors are linearly related. Given λ, the Pareto-optimal allocation is simply $\hat{c}_{i,t} = \hat{c}_i(Y_t, \lambda)$, $i = 1, 2$, $\forall\ Y_t$, and $t \in [0,\ T]$. For future convenience, define a representative agent by his utility function at t over aggregate consumption Y_t as follows: $u_\lambda(Y_t, t) \equiv e^{-\rho t} u_\lambda(Y_t)$ and

$$u_\lambda(Y_t) \equiv [\lambda \log \hat{c}_1(Y_t, \lambda) + 2(1-\lambda)\sqrt{\hat{c}_2(Y_t, \lambda)}]\,.$$

For simplicity in notation, u_λ represents both the time discounted and the undiscounted utility function of the representative agent. Let $b \equiv 4(1-\lambda)^2/\lambda^2$. The marginal utility of the representative agent over aggregate consumption is

$$m_t \equiv \frac{\partial u_\lambda(Y_t, t)}{\partial Y_t} = \left(\frac{b}{2+\sqrt{b}}\right)\frac{e^{-\rho t}}{\sqrt{1 + bY_t} - 1}, \qquad t \in [0,\ T]\,. \tag{11}$$

It is easy to see that the relative marginal utility of the representative agent (between any two states) is the same as the relative marginal utilities of the two individual investors.

For any Pareto-optimal allocation, an Arrow–Debreu equilibrium can be derived that supports the allocation. In an Arrow–Debreu equilibrium, investors can trade arbitrary payoff streams at the initial date. The equilibrium is defined as the pricing function $\{\phi_{0,s},\ s \in [0,\ T]\}$, such that the price of an arbitrary payoff stream $\{X_s,\ s \in [0,\ T]\}$ at $t = 0$ is given by the linear functional

$\Phi_0(X) = \mathrm{E}_0[\int_0^T \phi_{0,s} X_s \mathrm{d}s]$, and the market clears. The specific form of investor preferences assumed here leads to the following lemma:

Lemma 2. *Given $\lambda \in (0, 1)$ and the corresponding optimal allocation $(\hat{c}_1, \hat{c}_2)$, there exists an Arrow–Debreu equilibrium that leads to the same allocation, with the pricing function given by $\phi_{0,s} = m_s/m_0$, $s \in [0, T]$.*[2]

Clearly, the pricing function $\phi_{0,s}$ is positive. The value of the pricing function for any state at s is simply the ratio between the marginal utility of the representative agent in that state and his marginal utility at time zero. In general, $\phi_{0,s}$ can depend on s and the whole time path of Y_s up to s, which gives the complete description of the underlying state of the economy at s. In the current setting, however, due to the time-additive and state-separable preferences of the investors, $\phi_{0,s}$ only depends on s, Y_s, Y_0, and λ. Thus $\phi_{0,s} = \phi(Y_s, s; Y_0; \lambda)$. Although $\phi_{0,s}$ gives the pricing function at the initial date, the pricing function at any future time t is simply $\phi_{t,s} = m_s/m_t$, where $t, s \in [0, T]$ and $s \geqslant t$. Clearly, $\phi_{t,s} = \phi(Y_s, s; Y_t, t; \lambda)$. The Arrow–Debreu price of payoff $\{X_s, s \in [0, T]\}$ at t is then $\Phi_t(X) = \mathrm{E}_t[\int_t^T \phi_{t,s} X_s \mathrm{d}s]$.

The literature often assumes that the pricing function ϕ_t is bounded above and away from zero (see, e.g., Duffie and Huang, 1985; Duffie, 1986; Huang, 1987; Duffie and Zame, 1989). These conditions are not satisfied here, implying that securities with payoffs satisfying simple integrability conditions such as $\mathrm{E}_0[\int_0^T X_s^2 \mathrm{d}s] < \infty$ do not always have finite prices. This is not surprising when the state prices are unbounded. Securities that have nontrivial payoffs in states with high state prices will certainly have high prices at time zero. The remainder of the paper will be restricted to securities that have finite Arrow–Debreu prices.

Turning to the market equilibrium as defined at the beginning of this section, Duffie and Huang (1985) have shown, in a quite general setting, that for any Arrow–Debreu equilibrium a corresponding market equilibrium can be constructed as its dynamic implementation to achieve the same allocation. Unfortunately, the current model does not meet some of the regularity conditions required by their results. However, by slightly modifying their approach the dynamic equilibrium can be derived as follows. Given an Arrow–Debreu equilibrium as specified in Lemma 2, the stock prices and interest rates are first calculated using the Arrow–Debreu pricing function. Budget-feasible trading strategies can then be found for each individual investor to finance his consumption plan given in the Arrow–Debreu equilibrium. Finally, the above

[2] In a setting more general than the current one, Araujo and Monteiro (1989) have shown that the Second Welfare Theorem holds (see also Duffie and Zame, 1989; Mas-Colell and Zame, 1991).

consumption/trading strategy for each investor is shown to be optimal since any trading strategy that gives higher expected utility is not budget-feasible (see the Appendix for a formal proof).

Lemma 3. Given an Arrow–Debreu equilibrium as defined in Lemma 2, there exists a dynamic implementation in which prices of traded securities are given by

$$S_t = \mathrm{E}_t\left[\int_t^T \left(\frac{m_s}{m_t}\right) Y_s \,\mathrm{d}s\right], \quad r_t = -\frac{\mathrm{E}_t[\mathrm{d}m_t]}{m_t \mathrm{d}t}, \qquad t \in [0,\ T]. \tag{12}$$

Investors optimally choose the consumption plan $(\hat{c}_1, \hat{c}_2)$ financed respectively by budget-feasible trading strategies, and the securities market clears.

Given the definition of m_t and the process for Y_t, S_t and r_t can be expressed as functions of Y_t, t, and λ. Thus, we can write $S_t = S(Y_t, t; \lambda)$ and $r_t = r(Y_t, t; \lambda)$.

Combining Lemmas 1–3 gives the solution to the market equilibrium as summarized in the following theorem:

Theorem 1. For the economy defined in Section 1, there exists a market equilibrium in which (i) the equilibrium prices of traded securities are given by (12); (ii) investors' optimal consumption strategies are

$$\hat{c}_{1,t} = \frac{2}{b}[\sqrt{1 + bY_t} - 1], \qquad \hat{c}_{2,t} = Y_t - \hat{c}_{1,t}, \tag{13}$$

which are financed, respectively, by the following trading strategies:

$$\hat{\alpha}_{1,t} = \frac{2}{b\rho}[1 - \mathrm{e}^{-\rho(T-t)}](\sqrt{1 + bY_t} - 1) - \hat{\theta}_{1,t} S_t, \quad \hat{\theta}_{1,t} = \frac{b}{\rho}\frac{1 - \mathrm{e}^{-\rho(T-t)}}{S_Y \sqrt{1 + bY_t}},$$

$$\hat{\alpha}_{2,t} = -\hat{\alpha}_{1,t}, \quad \hat{\theta}_{2,t} = 1 - \hat{\theta}_{1,t},$$

where $S_Y = \partial S/\partial Y$; and (iii) b is determined by

$$\theta_{1,0\text{-}} \mathrm{E}_0\left[\int_0^T \mathrm{e}^{-\rho\tau}(\sqrt{1 + bY_0 g_t(\tau)} + 1)\mathrm{d}\tau\right] = \frac{2}{\rho}(1 - \mathrm{e}^{-\rho T}). \tag{14}$$

Furthermore, $S_Y > 0$.

Note that multiplying both sides of (14) by $1/(bm_0)$ gives $W_{1,0} = \theta_{1,0\text{-}} S_0$ for the left-hand side, which is investor 1's initial wealth, and $\mathrm{E}_0[\int_0^T \phi_{0,t}\hat{c}_{1,t}\mathrm{d}t]$ for the right-hand side, which is the cost of his optimal consumption plan. Thus (14) is simply investor 1's budget constraint, which uniquely determines b (or λ) in terms of the initial condition of the economy, $\theta_{1,0\text{-}}$ and Y_0. Since

$W_{1,0} + W_{2,0} = S_0 = S(Y_0, 0; \lambda)$, λ can also be expressed in terms of the two investors' initial wealth.

3.2. Properties of the equilibrium

Given that the uncertainty of the economy is completely characterized by the process of aggregate consumption Y_t which is a univariate diffusion, the stock and the (locally) risk-free security allow the market to be dynamically complete. Any consumption patterns (that have finite Arrow–Debreu prices) can be financed by continuous trading in these two securities. Thus investors are able to achieve Pareto-optimal allocations in the market equilibrium. Introducing other securities will not change the equilibrium allocations. Furthermore, any other securities can be synthesized by trading only in the stock and the risk-free security. Their prices should equal the cost of the synthesizing strategy. As seen in Section 5, when Y_t follows more general processes more securities will be needed to complete the market.

In deriving the equilibrium, λ, the relative weight of the two investors in the welfare function, fully characterizes the Pareto-optimal allocations and the supporting equilibria. Eq. (14) uniquely determines λ (or b) in terms of $\theta_{1,0_-}$ and Y_0. It is easy to show that λ is an increasing function of $\theta_{1,0_-}$ (holding Y_0 constant) and an increasing function of Y_0 (holding $\theta_{1,0_-}$ constant). When $\theta_{1,0_-} \to 1$, $\lambda \to 1$, $\hat{c}_1(Y, \lambda) \to Y$, and $\hat{c}_2(Y, \lambda) \to 0$. This is the limiting case when the economy is populated only by investor 1. When $\theta_{1,0_-} \to 0$, $\lambda \to 0$, $\hat{c}_1(Y, \lambda) \to 0$, and $\hat{c}_2(Y, \lambda) \to Y$. This is the limiting case when the economy is populated only by investor 2. It is also interesting to consider the allocation of consumption when the initial aggregate endowment is very low or very high, i.e., when $Y_0 \to 0$ or $Y \to \infty$ (holding $\theta_{1,0_-}$ constant). It is easy to show that when $Y_0 \to \infty$, $\lambda \to 1$, $\hat{c}_1(Y, \lambda) \to Y$, and $\hat{c}_2(Y, \lambda) \to 0$. When $Y_0 \to 0$, $\lambda \to 0$, $\hat{c}_1(Y, \lambda) \to 0$, and $\hat{c}_2(Y, \lambda) \to Y$. This suggests that λ does not simply represent the relative wealth of the two investors, even though it can be expressed as a function of the wealth of the two investors. For example, even when $\theta_{1,0_-} \gg \theta_{2,0_-} > 0$ (thus $W_{1,0}/W_{2,0} \gg 1$), λ can be very small if Y_0 is very large. In other words, λ not only depends on the initial relative wealth of the two investors but also on the level of total initial wealth.

It is important to note that λ depends only on the initial conditions of the economy, and remains constant afterwards. Given the initial condition of the economy (i.e., Y_0 and $\theta_{1,0_-}$), Y_t completely determines the state of the economy at t. As the economy evolves, the state of the economy, security prices, investors' wealth, and their security holdings do change. But the sharing rule does not. The intuition behind this result is simple. In the current setting, the securities market is dynamically complete. In equilibrium, investors follow optimal trading strategies to achieve consumption distributions such that the relative marginal utilities (for any two states) are equal for all investors. (Otherwise, gains could be made for the investors by deviating from their optimal trading strategies.) For example, if investor 1's marginal utility is more sensitive to changes in the level of consumption than that of investor 2 at the current level of consumption, investor 1 will then optimally hold a portfolio that yields lower (higher) returns than the portfolio of investor 2 when aggregate consumption increases (decreases). Consequently, their marginal utility remains proportional independent of future changes in aggregate consumption. This implies that in all states, the two investors' marginal utilities are linearly related with a constant proportionality. This condition then gives the sharing rule between the two investors, which does not change over time. If one recalculates the equilibrium at a later date, the same λ will be obtained. As the aggregate endowment changes, investors' wealth also changes. But λ as a function of both investors' wealth remains constant.

Even though the sharing rule between the two investors does not change over time, the actual consumption of the two investors does change as the aggregate consumption Y_t changes. For example, as Y_t increases, investor 1's percentage share in aggregate consumption decreases and investor 2's share increases. When Y_t drifts to zero, investor 1's percentage share in aggregate consumption drifts to one. On the other hand, when Y_t drifts to infinity, investor 1's percentage share in aggregate consumption drifts to zero. This result is quite intuitive given the investors' preferences. At low (high) levels of consumption, investor 1's marginal utility is higher (lower) than investor 2's marginal utility. In equilibrium, investor 1 maintains higher (lower) level of consumption than investor 2 when the aggregate consumption is low (high). As investors' consumption changes, security prices also change.

Investors' optimal consumption policies are financed by their corresponding trading strategies. [For more general discussions on optimal trading policies, see, e.g., Merton (1969, 1990) and Cox and Huang (1989).] Theorem 1 has the following corollary:

Corollary 1. For $\lambda \in (0, 1)$, $\alpha_{1,t} > 0$ and $\alpha_{2,t} < 0$. When $Y_t \to 0$, $\theta_{1,t}S_t/W_{1,t} \to 1$, $W_{1,t}/S_t \to 1$, and $W_{2,t}/S_t \to 0$. When $Y_t \to \infty$, $\theta_{1,t}S_t/W_{1,t} \to \frac{1}{2}$, $W_{1,t}/S_t \to 0$, and $W_{2,t}/S_t \to 1$.

Thus, investor 1 is the lender and investor 2 is the borrower. This is not surprising given that investor 1 is more risk-averse than investor 2. Furthermore, investor 1 shifts his portfolio toward the stock (the risk-free security) when the stock price drops (arises) while investor 2 does the opposite. Also, investor 1's relative wealth approaches one and zero as the level of aggregate consumption shifts to zero and infinity, respectively. This implies that investor 1 follows a strategy that pays off in bad states of the economy since his marginal utility is higher than that of investor 2 at low levels of consumption.

3.3. Security prices in equilibrium

In the market equilibrium, it is also possible to price securities that can be replicated by dynamic trading strategies at finite costs. If a security has payoff $\{X_s, s \in [t, T]\}$ $[t \geqslant 0]$, its price is

$$P_t = \mathrm{E}_t\left[\int_t^T \left(\frac{m_s}{m_t}\right) X_s \mathrm{d}s\right] = \mathrm{E}_t\left[\int_t^T \mathrm{e}^{-\rho(s-t)} \frac{\sqrt{1+bY_t}-1}{\sqrt{1+bY_s}-1} X_s \mathrm{d}s\right]. \tag{15}$$

If X_t only depends on Y_t and t, its price P_t as a function of Y and t satisfies the stochastic equation:

$$\mathrm{d}P = \mu_P P \mathrm{d}t + \sigma_P P \mathrm{d}w\,,$$

where $\mu_P = [(\partial P/\partial t) + \mu Y(\partial P/\partial Y) + \frac{1}{2}\sigma^2 Y^2(\partial^2 P/\partial Y^2)]/P$ and $\sigma_P = \sigma Y P_Y/P$. Here, it is assumed that $X = f(Y, t)$ is twice differentiable with respect to Y. From (15), we obtain the following partial differential equation for P:

$$\frac{\partial P}{\partial t} + \mu Y \frac{\partial P}{\partial Y} + \tfrac{1}{2}\sigma^2 Y^2 \frac{\partial^2 P}{\partial Y^2} - rP + X = \pi\sigma Y \frac{\partial P}{\partial Y}, \tag{16}$$

where r is given by (12) and

$$\pi \equiv -\sigma \frac{Y}{m}\frac{\partial m}{\partial Y} = \frac{b\sigma Y}{2\sqrt{1+bY}(\sqrt{1+bY}-1)}. \tag{17}$$

Given X_t and proper boundary conditions, the solution to (16) gives the equilibrium price of the security. [For a general framework of intertemporal asset pricing based on investor optimality conditions, see Merton (1973, 1990) and Cox, Ingersoll, and Ross (1985b).]

The variable π can be interpreted as the market price of risk. Rewrite (16) as

$$\frac{\mu_P + X/P - r}{\sigma_P} = \pi\,.$$

The left-hand side is simply the Sharpe measure of the security, which is the expected excess return on the security normalized by its standard deviation. Given that there is only one source of risk in the current situation, the Sharpe measure is the same for all risky securities and we can call it the market price of risk. Since m_t is simply the marginal utility of the representative agent at time t, (17) can be rewritten as

$$\pi_t = \left[-\frac{Y_t u_\lambda''(Y_t)}{u_\lambda'(Y_t)}\right]\sigma \equiv a_\lambda(Y_t)\sigma\,, \tag{18}$$

where $a_\lambda(Y)$ is the relative risk aversion of the representative agent at consumption level Y. Thus, the market price of risk is proportional to the uncertainty in consumption growth σ and the proportionality constant is just the relative risk aversion of the representative agent. As Y_t changes over time, the risk aversion of the representative agent also changes and so does the market price of risk. It can be shown that $a_\lambda(Y_t)$ monotonically decreases with Y_t and lies in the interval $(a_2, a_1) = (\frac{1}{2}, 1)$. It approaches a_1 and a_2 as Y_t approaches zero and infinity, respectively. Note that the market price of risk when only investor i, $i = 1, 2$, is present is simply $\pi^{(i)} = a_i\sigma$ which is constant. Thus, the market price of risk when both investors are present lies between $\pi^{(2)}$ and $\pi^{(1)}$ and varies over time.

Given the growth condition (5), the stock price and bond prices are well defined at the limit $T \to \infty$. As a matter of fact, the limiting economy and its equilibrium are well-defined. Certain technical modifications are needed in analyzing the infinite-horizon counterpart of a finite-horizon economy (see Huang and Pagès 1990, for more detailed discussions). However, these modifications are quite straightforward in the current setting. For simplicity in exposition, in the remainder of this paper, we will consider the limiting economy and its equilibrium when $T \to \infty$. In this case, the economy has an infinite horizon, hence the state of the economy at time t only depends on the level of aggregate consumption given the initial condition of the economy, not on t itself.

4. Bond prices and yields

Turning to the equilibrium term structure of interest rates, let $B_t(\tau)$ be the price of a pure discount bond at t that matures at $t + \tau$ where $t, \tau > 0$. Its payoff process is $X_s^B = \delta(s - t - \tau)$ where $\delta(\cdot)$ is the Dirac δ-function. Substituting X_s^B into the pricing equation (15) gives the following expression for the price at t:

$$B_t(\tau) = \mathrm{e}^{-\rho\tau}\,\mathrm{E}_t\left[\frac{\sqrt{1 + bY_t} - 1}{\sqrt{1 + bY_t g_t(\tau)} - 1}\right]. \tag{19}$$

The yield to maturity $y_t(\tau)$ is defined by $y_t(\tau) \equiv -(1/\tau)\log B_t(\tau)$. How the bond yield changes with maturity gives the term structure of interest rates. Since the state of the economy at t depends on Y_t, the bond prices and the term structure will also depend on Y_t. As Y_t changes over time, the term structure also changes.

4.1. Limiting cases

Before considering the bond prices and the term structure of interest rates when both investors are present, it is useful to first examine the limiting cases when only one of the two investors is present in the economy (i.e., when $\theta_{1,0_-} \to 1$ or 0). The model in the two limiting cases is similar to Cox, Ross, and Ingersoll (1985a), except that the specific process of aggregate consumption is different.[3] The resulting interest rate process is identical to the one analyzed by Vasicek (1977) in a partial equilibrium context and Stapleton and Subrahmanyam (1990) and McCulloch (1993) in a general equilibrium context.

Let $B_t^{(i)}(\tau)$ be the price of a pure discount bond at t with maturity τ when only investor i is present in the economy. $B_t^{(i)}(\tau)$ can be calculated from (19) by properly taking the limits: $b \to 0$ for $i = 1$ and $b \to \infty$ for $i = 2$, respectively.[4] Then

$$B_t^{(i)}(\tau) = \mathrm{e}^{-\rho\tau}\,\mathrm{E}_t\left[g_t(\tau)^{-a_i}\right], \qquad i = 1, 2.$$

In both of these two limiting cases, the bond prices do not depend on the current level of aggregate endowment. They depend only on the expectations of future growth rates. This result is well-known (see, for example, Cox, Ingersoll, and Ross, 1985a; Dumas, 1989; Stapleton and Subrahmanyam, 1990). From the distributional assumptions about the growth rates, the following expressions are obtained for bond prices in the two limiting cases:

Lemma 4. Given the process of Y_t,

$$B_t^{(i)}(\tau) = \mathrm{e}^{-r^{(i)}\tau},$$

where

$$r^{(i)} = \rho + a_i\left(\mu - \frac{1 + a_i}{2}\sigma^2\right)$$

and $i = 1, 2$.

It is clear that in the two limiting cases the interest rate is constant over time and the term structure is flat, i.e., $y_t^{(i)}(\tau) = r^{(i)}$, $\forall t$. Even though the aggregate consumption Y_t varies over time, the yield curve stays constant.

4.2. Bond prices with two investors

Now consider the bond prices and yields when both investors are present in the economy. For simplicity in exposition, let $b = 1$ in the pricing equation (15) from now on. This implies that the following weights are assigned to the two investors in the welfare function: $\lambda = \frac{2}{3}$ and $1 - \lambda = \frac{1}{3}$. As discussed in

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[3]Cox, Ingersoll, and Ross (1985a) consider a production economy in which the aggregate consumption is endogenously determined by the representative investor's optimal consumption–trading strategies. Sun (1992) shows that an exchange economy can be constructed which is analogous to the production economy. The endowment process in the exchange economy is taken to be the same as the optimal consumption process in the Cox, Ingersoll, and Ross model. The pricing implications are the same for the two economies.

[4]In obtaining the bond prices in the two limiting cases, the limit $b \to 0$ or $b \to \infty$ is taken under the integration. The order of taking the limit and integration is irrelevant here. This can be easily shown by applying standard convergence results.

Section 3.1, this choice of λ involves certain choices of the initial condition of the economy. The qualitative behavior of bond prices and yields does not depend on this particular choice of the initial condition. Extending the analysis to the general case of $\lambda \in (0, 1)$ is trivial. As a matter of fact, there is no loss of generality by setting $b = 1$ here when both investors are present. Note that there is a one-to-one correspondence between the economy with $b \in (0, 1)$ and initial aggregate endowment Y_0 and the economy with $b' = 1$ and $Y'_0 = bY_0$. Given the initial conditions of the economy, the bond prices are completely determined by the current level of aggregate consumption.

The equilibrium bond prices can be calculated by computing the conditional expectation in (19). The results are summarized in the following theorem:

Theorem 2. When both investors are present in the economy, the equilibrium prices are given by

$$B_t(\tau) = \mathrm{e}^{-\rho\tau}(\sqrt{1+Y_t}-1)\,[I_{1,t}(\tau) + I_{2,t}(\tau)]\,, \tag{20}$$

where

$$I_{1,t}(\tau) = \frac{1}{Y_t}\mathrm{E}_t\,[g_t(\tau)^{-1}]\,, \qquad I_{2,t}(\tau) = \mathrm{E}_t\left[\frac{\sqrt{1+Y_t g_t(\tau)}}{Y_t g_t(\tau)}\right].$$

Furthermore, let $\delta_t(\tau) = [\mu - \frac{1}{2}\sigma^2\tau + \log Y_t]/(\sigma^2\tau)$ *and* $\xi_{n,t}(\tau) = [n - 1 + \delta_t(\tau)]\sigma\sqrt{\tau}$. *Then*

$$I_{1,t}(\tau) = \mathrm{e}^{-\xi_{1/2,t}(\tau)\sigma\sqrt{\tau}}\,,$$

$$I_{2,t}(\tau) = \mathrm{e}^{\frac{1}{2}\delta_t(\tau)^2\sigma^2\tau}\sum_{n=0}^{\infty}\alpha_n\{\mathrm{e}^{\frac{1}{2}\xi_{n,t}(\tau)^2}\Phi[-\xi_{n,t}(\tau)] + \mathrm{e}^{\frac{1}{2}\xi_{1/2-n,t}(\tau)^2}\Phi[\xi_{1/2-n,t}(\tau)]\}\,,$$

where $\alpha_0 = 1$, $\alpha_n = (-1)^{n-1}(2n-3)!!/(2n)!!$ *for* $n \geqslant 1$, *and* $\Phi(x) \equiv (1/\sqrt{2\pi})\int_{-\infty}^{x}\mathrm{e}^{-x'^2/2}\mathrm{d}x'$ *is the cumulative normal distribution function. Here,* $n!! = 1$ *for* $n \leqslant 0$ *and* $n!! = n(n-2)!!$ *for* $n > 0$.

Although the bond prices are expressed in the form of infinite summation, their numerical values are easy to calculate.

It can be shown that for a given maturity, when $Y_t \to 0$, $I_{1,t}(\tau) \gg I_{2,t}(\tau) - I_{1,t}(\tau)$. Then $I_{1,t}(\tau) + I_{2,t}(\tau) \to 2I_{1,t}(\tau)$ and $B_t(\tau) \to B^{(1)}(\tau)$. On the other hand, when $Y_t \to \infty$, $I_{1,t}(\tau) \ll I_{2,t}(\tau)$. Then $I_{1,t}(\tau) + I_{2,t}(\tau) \to I_{2,t}(\tau) \to [Y_t g_t(\tau)]^{-1/2}$ and $B_t(\tau) \to B^{(2)}(\tau)$.

The equilibrium bond prices can be used to derive the equilibrium yield curve. Two yields are of particular interest. One is the instantaneous interest rate r_t, which is the limiting yield as maturity goes to zero: $r_t \equiv \lim_{\tau\to 0} y_t(\tau)$. The other is the long yield which is defined as the limiting yield as maturity goes to infinity: $y_t(\infty) \equiv \lim_{\tau\to\infty} y_t(\tau)$. They give, respectively, the two ends of the yield curve.

4.3. Instantaneous interest rate

We first consider the instantaneous interest rate r_t. Theorem 1 and applying Itô's lemma to $m_t = m(t, Y_t)$ as given in (11) [with $b = 1$], give the following result:

Theorem 3. When both investors are present in the economy, the instantaneous interest rate is given by

$$r_t = \rho + \frac{\mu Y_t}{2\sqrt{1+Y_t}(\sqrt{1+Y_t}-1)} - \frac{\sigma^2 Y_t^2(3\sqrt{1+Y_t}-1)}{8(1+Y_t)^{\frac{3}{2}}(\sqrt{1+Y_t}-1)^2}. \tag{21}$$

Given the value of λ determined by the initial conditions of the economy (i.e., $\theta_{1,0-}$ and Y_0), r_t depends only on the current level of aggregate consumption Y_t (independent of the path taken to arrive at Y_t).

Before analyzing the dynamics of r_t, let us examine the range within which the interest rate moves. Using a similar two-investor economy but one which has production, Dumas (1989) conjectures that the instantaneous interest rate r_t should always lie within the range bounded by $r^{(1)}$ and $r^{(2)}$, the values it would take in worlds populated by investor 1 only and investor 2 only, respectively. In the pure exchange economy considered here, this is generally not the case. Note that $r^{(1)} = \rho + \mu - \sigma^2$ and $r^{(2)} = \rho + \frac{1}{2}\mu - \frac{3}{8}\sigma^2$. If $\mu = \frac{5}{4}\sigma^2$, then $r^{(1)} = r^{(2)} = \rho + \frac{1}{4}\sigma^2$. It is easy to show that in this case, $r_t < r^{(1)} = r^{(2)}$ for $Y_t \in (0, \infty)$. r_t reaches a unique local minimum of $\rho + \frac{25}{108}\sigma^2$ in the interval at $Y_t = \frac{5}{4}$. Thus, in the current model the interest rate with both investors present can move outside the range bounded by $r^{(1)}$ and $r^{(2)}$.

In order to understand this behavior of interest rates, recall that the interest rates in equilibrium should make investors indifferent between consuming now or later. The lower the investors' expected marginal utilities are in the next instant (relative to the current value), the higher the equilibrium interest rate should be. In other words, the equilibrium interest rate is negatively related to the expected growth of investors' marginal utility as shown in (16). Consider an investor in the economy with utility function $e^{-\rho t}u(c_t)$ and optimal consumption process c_t. From (12),

$$r_t = -\frac{E_t[de^{-\rho t}u'(c_t)]}{e^{-\rho t}u'(c_t)dt} = \rho - \frac{c_t u''(c_t)}{u'(c_t)}\mu_{c,t} - \frac{1}{2}\frac{c_t^2 u'''(c_t)}{u'(c_t)}\sigma_{c,t}^2,$$

where $\mu_{c,t} \equiv E_t[dc_t]/c_t dt$ and $\sigma_{c,t}^2 \equiv E_t[dc_t^2]/(c_t^2 dt)$ are, respectively, the expected value and the variance of the investor's consumption growth. Thus the interest rate is related to both the expected value and the variance of instantaneous consumption growth in equilibrium. High expected consumption growth implies low expected marginal utility in the future. The equilibrium interest rate then must be high. In other words, r_t increases with the expected consumption growth. The proportionality coefficient $-c_t u''(c_t)/u'(c_t)$ is the inverse of the elasticity of intertemporal substitution, which is also the relative risk-aversion coefficient $a(c_t)$ given the time-separable preferences. High variance in consumption growth, on the other hand, implies high expected marginal utility in the future by Jensen's inequality, assuming that $u'''(c_t) > 0$, i.e., the marginal utility function is convex. The equilibrium interest rate then must be low. In other words, r_t decreases with the variance of consumption growth and the proportionality constant is $c_t u'''(c_t)/u'(c_t)$. Since $c_t u'''(c_t)/u'(c_t) = a(c_t)\ [1 + a(c_t)] - c_t a'(c_t)$, the above expression can be rewritten as

$$r_t = \rho + a(c_t)\,\mu_{c,t} - \tfrac{1}{2}\{a(c_t)[1 + a(c_t)] - c_t a'(c_t)\}\,\sigma_{c,t}^2 .$$

$a(c_t)$ in general depends on the consumption level, although for power utility functions, it is constant and $a'(c_t) = 0$.

When the economy is populated only by an investor with constant relative risk aversion a, his consumption will be the aggregate consumption, hence $\mu_{c,t} = \mu$ and $\sigma_{c,t}^2 = \sigma^2$. The interest rate will be $r(a) = \rho + a\mu - [(1 + a)/2]\sigma^2$, which for $a = 1$ and $\frac{1}{2}$ simplify to $r^{(1)}$ and $r^{(2)}$, respectively. It is important to note that $r(a)$ is not monotonic in a. As a increases, the elasticity of intertemporal substitution decreases which tends to increase the equilibrium interest rate. On the other hand, the risk aversion increases which tends to decrease the equilibrium interest rate. For $a_1 = 1 > a_2 = \frac{1}{2}$, $r^{(1)} \geqslant r^{(2)}$ when $\mu \geqslant \frac{5}{4}\sigma^2$ and $r^{(1)} < r^{(2)}$ when $\mu < \frac{5}{4}\sigma^2$.

When the economy is populated with both investors, each investor's consumption and marginal utility are endogenously determined. For example, when $\mu < \frac{5}{4}\sigma^2$, $r^{(1)} < r^{(2)}$. If the current level of aggregate consumption is close to zero, the interest rate is then close to $r^{(1)}$. As Y_t increases, investor 1 shifts his portfolio towards the risk-free security. His expected consumption growth decreases and so does its variance. The decrease in expected consumption growth tends to decrease the interest rate, while the decrease in the variance of consumption growth tends to increase the interest rate. If the effect of expected consumption growth dominates, the interest rate will then be less than $r^{(1)}$ which is outside the range $[r^{(1)}, r^{(2)}]$.

To further analyze this situation, consider the representative agent. The consumption of the representative agent is simply the aggregate endowment which is exogenously specified. His relative risk-aversion coefficient $a(Y_t)$ is given in (18) (with $\lambda = \frac{2}{3}$), which is also the inverse of his elasticity of intertemporal substitution. $a(Y_t)$ now varies with the consumption level. In particular, $a(Y_t)$ monotonically decreases with Y_t, $a(0) = a_1 = 1$, $a(\infty) = a_2 = \frac{1}{2}$, and $a_1 < a(Y_t) < a_2$. The interest rate given by

$$r_t = \rho + a(Y_t)\,\mu - \tfrac{1}{2}a(Y_t)\left[1 + a(Y_t) - \frac{Y_t a'(Y_t)}{a(Y_t)}\right]\sigma^2$$

is, however, nonmonotonic in Y_t. Note that when $\mu > 0$ and $\sigma^2 = 0$, r_t monotonically increases with Y_t and $r^{(2)} = \rho + \frac{1}{2}\mu < r_t < \rho + \mu = r^{(1)}$. When $\mu = 0$ and $\sigma^2 > 0$, r_t monotonically decreases with Y_t and $r^{(1)} = \rho - \sigma^2 < r_t < \rho - \frac{3}{8}\sigma^2 = r^{(2)}$. In the general case when $\mu > 0$ and $\sigma^2 > 0$, it is possible to have $a(Y_t) \in (\frac{1}{2}, 1)$ and $r_t < \min[r^{(1)}, r^{(2)}]$. Thus in this model, as Y_t changes over time, the equilibrium interest rate can move outside the range bounded by $r^{(1)}$ and $r^{(2)}$.

The difference in the behavior of the interest rate between this model and Dumas' may be due to the difference between an exchange economy and a production economy. This difference is best seen by considering the two limiting cases under certainty when $\sigma = 0$. In a production economy, the interest rate is simply μ, independent of the preferences, because with production, the consumption process is endogenous. Under linear production technology, the equilibrium interest rate must equal to the intertemporal rate of transformation (as given by the production technology) which is μ. In the exchange economy, the consumption path is exogenously specified. Given the consumption process, the equilibrium interest rate is $\rho + a_i\mu$, which does depend on investors' preferences. With positive growth (i.e., $\mu > 0$), the interest rate increases with a_i in this case. When there is uncertainty, the interest rate also depends (negatively) on the risk in future consumption. If σ^2 is large the interest rate decreases with a_i due to the effect of risk aversion as discussed above. Thus, as a_i changes, the effect on the interest rate may be negative in the production economy of Dumas (1989) while in the exchange economy considered here it is ambiguous.

Consider now the dynamics of instantaneous interest rate. In order to simplify the analysis, define a new variable $\omega_t \equiv \hat{c}_{1,t}/Y_t$, which represents investor 1's share of aggregate consumption in equilibrium. Lemma 1, with $\lambda = \frac{2}{3}$, implies that there exists the following one-to-one mapping between ω_t and Y_t:

$$\omega_t = \frac{2(\sqrt{1+Y_t} - 1)}{Y_t} \quad \text{or} \quad Y_t = \frac{4(1-\omega_t)}{\omega_t^2}. \tag{22}$$

It maps $\omega_t \in (0, 1)$ onto $Y_t \in (0, \infty)$ and Y_t is monotonically decreasing with ω_t. $\omega_t \to 1$ as $Y_t \to 0$ and $\omega_t \to 0$ as $Y_t \to \infty$. The state variable of the economy can then be ω_t instead of Y_t. Expressed in ω_t, the equilibrium interest rate is

$$r_t = \rho + \frac{\mu}{2-\omega_t} - \sigma^2 \frac{3-2\omega_t}{(2-\omega_t)^3}. \tag{23}$$

Hence, r_t depends on the growth rate of aggregate consumption as well as the consumption distribution across investors. Clearly, r_t approaches $r^{(2)}$ as $\omega_t \to 0$ while it approaches $r^{(1)}$ as $\omega_t \to 1$. Under certain parameter constraints, the interest rate is bounded below by a positive constant. For example, for $\rho > \sigma^2 - \frac{1}{2}\mu$, $r_t \geqslant \rho + \frac{1}{2}\mu - \sigma^2 > 0$. Fig. 1 plots r_t as a function of ω_t for

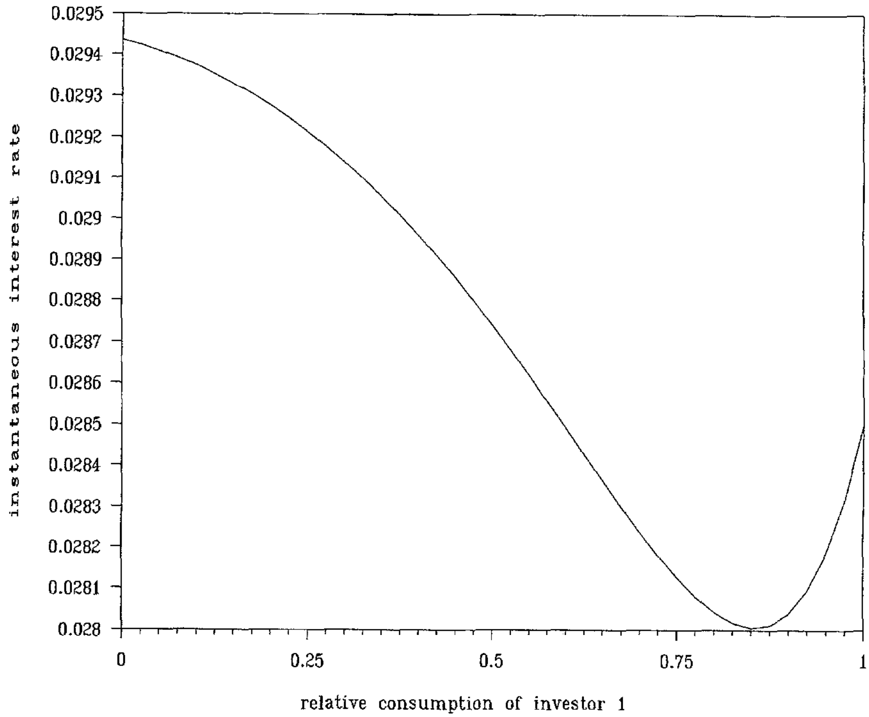

Fig. 1. Instantaneous interest rate r_t plotted as a function of ω_t, the consumption of investor 1 relative to the aggregate consumption. The parameters are set at the following values: investors' time discount coefficient $\rho = 0.02$, expected rate of aggregate consumption growth $\mu = 0.05$, instantaneous standard deviation of aggregate consumption growth $\sigma = 0.20$. The instantaneous interest rate with only investor 1 present is $r^{(1)} = 0.0285$ and the instantaneous interest rate with only investor 2 present is $r^{(2)} = 0.0294$.

a specific set of parameter values. Note that r_t reaches an interior minimum which is smaller than both $r^{(1)}$ and $r^{(2)}$ as discussed earlier.

Given the process of Y_t, the dynamics of ω_t can be easily obtained by applying Itô's lemma to (22):

$$d\omega_t = \mu_\omega(\omega_t)\,dt - \sigma_\omega(\omega_t)\,dw_t\,, \tag{24}$$

where

$$\mu_\omega(\omega) = -\frac{\omega(1-\omega)}{2-\omega}\left\{\mu - \sigma^2\left[1 - \frac{\omega}{(2-\omega)^2}\right]\right\}, \qquad \sigma_\omega(\omega) = \sigma\frac{\omega(1-\omega)}{2-\omega}\,.$$

Similarly, the dynamics of r_t are obtained from (23):

$$dr_t = \mu_r(\omega_t)\,dt + \sigma_r(\omega_t)\,dw_t\,, \tag{25}$$

where

$$\mu_r(\omega) = \left[-\frac{\mu}{(2-\omega)^2} + \frac{\sigma^2(5-4\omega)}{(2-\omega)^4}\right]\mu_\omega + \left[\frac{\mu}{(2-\omega)^3} + \frac{6\sigma^2(1-\omega)}{(2-\omega)^5}\right]\sigma_\omega^2,$$

$$\sigma_r(\omega) = \left[-\frac{\mu}{(2-\omega)^2} + \frac{\sigma^2(5-4\omega)}{(2-\omega)^4}\right]\sigma_\omega .$$

Note that $\sigma_r(0) = \sigma_r(1) = 0$. Thus, in the two limiting cases in which there is only one investor present, the interest rate is constant and its volatility is zero. When the two investors coexist, however, the interest rate volatility is nonzero unless r_t is at its local minimum value. Thus, preference heterogeneity among investors can increase interest rate variability. Fig. 2 plots the instantaneous drift and volatility of interest rate $\sigma_{r,t}^2$ as a function of ω_t. Note that in the case of a local minimum of r_t as a function of ω_t for $\omega_t \in [0, 1]$, the interest rate volatility drops to zero at its local minimum as it should when it follows a diffusion process.

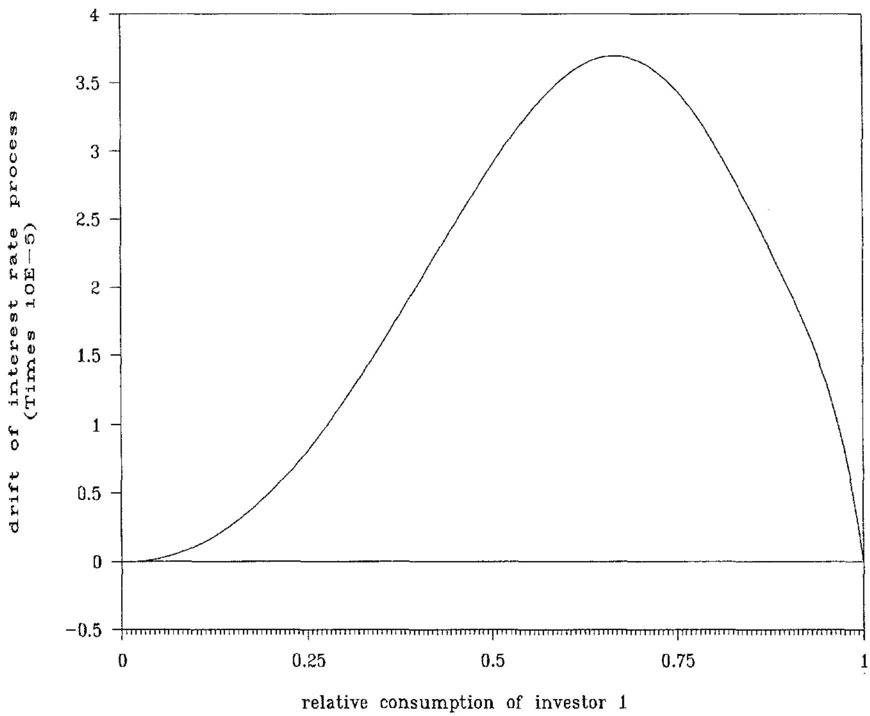

Fig. 2a. The instantaneous drift of the interest rate process $\mu_{r,t}$ plotted as a function of ω_t, the consumption of investor 1 relative to the aggregate consumption. The parameters are set at the following values: investors' time discount coefficient $\rho = 0.02$, expected rate of aggregate consumption growth $\mu = 0.05$, instantaneous standard deviation of aggregate consumption growth $\sigma = 0.20$.

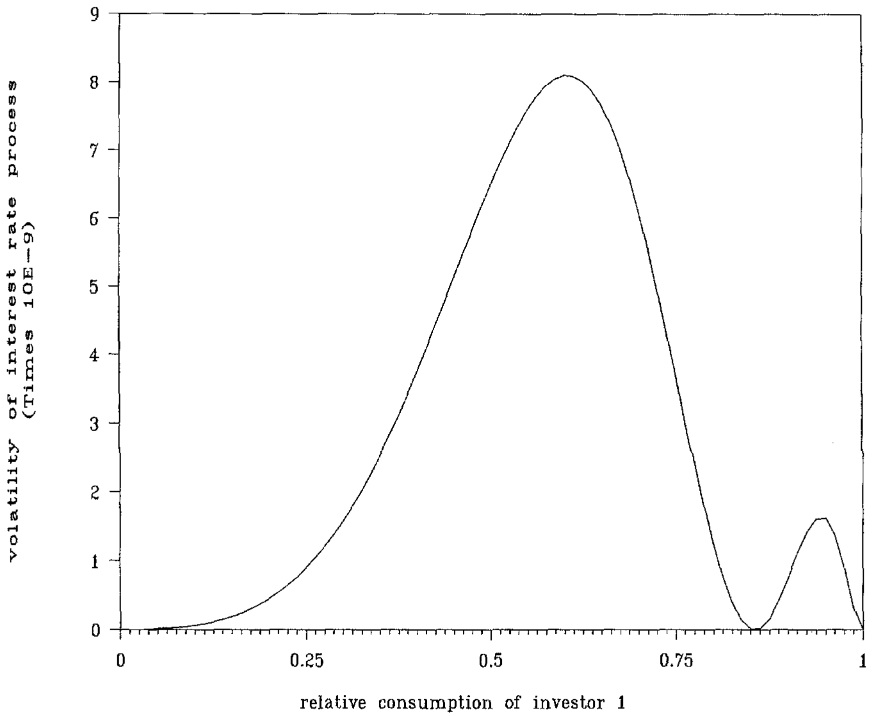

Fig. 2b. The instantaneous variance of the interest rate process $\sigma_{r,t}^2$ plotted as a function of ω_t, the consumption of investor 1 relative to the aggregate consumption. The parameters are set at the following values: investors' time discount coefficient $\rho = 0.02$, expected rate of aggregate consumption growth $\mu = 0.05$, instantaneous standard deviaiton of aggregate consumption growth $\sigma = 0.20$.

4.4. Long yield

Consider now the long yield $y_t(\infty)$. From Theorem 2, we have the following lemma:

Lemma 5. Given the aggregate endowment process (1) and the equilibrium bond prices (19), $y_t(\infty)$ is a constant independent of the current value of Y_t.

As discussed earlier, when Y_t becomes large (small), the investor with lower (higher) risk aversion dominates the economy in relative wealth and consumption. It is easy to show that for $\mu > \sigma^2/2$, $Y_{t+\tau}$ will be greater than any given positive constant with probability one as $\tau \to \infty$, i.e., $Y_{t+\tau} \to \infty$ as $\tau \to \infty$. Thus investor 2 will eventually own the whole economy. This seems to imply that long-term bond yields should be determined mainly by the preferences of investor 2. In other words, $y(\infty) = y^{(2)}(\infty)$ which is the long-term yield when the economy is only populated with investor 2. (The subscript t has been dropped given that the long yields are constant.) This, however, is not true as shown by the following theorem:

Theorem 4. When both investors are present in the economy, i.e., $\lambda \in (0, 1)$,

$$y(\infty) = \min\left[y^{(1)}(\infty),\ y^{(2)}(\infty)\right], \tag{26}$$

where $y^{(1)}(\infty) = r^{(1)}$ and $y^{(2)}(\infty) = r^{(2)}$ are given in Lemma 4.

When $\mu > \frac{5}{4}\sigma^2$, $y^{(1)}(\infty) > y^{(2)}(\infty)$. Then, $y(\infty) = y^{(2)}(\infty)$. When $\mu < \frac{5}{4}\sigma^2$, $y^{(1)}(\infty) < y^{(2)}(\infty)$ and $y(\infty) = y^{(1)}(\infty)$. It is important to note that the critical value is $\frac{5}{4}\sigma^2$, not $\sigma^2/2$. Thus under mild long-run growth (i.e., $\mu < 5\sigma^2/4$), the current long yields are still determined by the preferences of investor 1 even though his relative wealth will be negligible in the future.

This seemingly counterintuitive result arises for the following reason. Even though *in expectation* investor 2 may eventually dominate the economy (in terms of his wealth and consumption), there are still possible future states of the economy in which investor 1 actually dominates. The probability of those states may be small, but the marginal utilities for consumption in these states can be high. Thus, these states can be very important in determining today's asset prices despite their small probability of occurrence. Note that a long-term bond pays one unit of consumption at maturity independent of the state of the economy at that time. Thus it provides an instrument to hedge against future downturns of the economy. Of course, the probability of a severe downturn in the future (leading to low consumption levels) decreases with the expected long-run growth of the economy. Under mild growth, the probability of such a downturn is nontrivial. At low levels of consumption, the marginal utility of investor 1 is much higher than that of investor 2. Thus, a long-term bond as a hedging instrument is more attractive to investor 1 than to investor 2. Consequently, investor 1 exerts a stronger influence on its equilibrium price. The longer the bond's maturity, the higher the expected wealth of investor 2 (relative to that of investor 1) at the maturity date and the less attractive it is to investor 2. Thus its price will more disproportionally reflect investor 1's preferences. Further notice that $y(\infty)$ is independent of the wealth distribution today. This implies that investors with only a small proportion of the total market wealth can have a large effect in determining asset prices.

The limiting result in Theorem 4 is obtained by letting maturity goes to infinity, given the state of the economy. The convergence of $y_t(\tau)$ to $y(\infty)$ as $\tau \to \infty$ is, however, not necessarily uniform. A similar notion of long-term bond yield is considered in Dybvig, Ingersoll, and Ross (1995).

4.5. The yield curve

For arbitrary maturity between zero and infinity, the bond yield can be calculated from Theorem 2. Fig. 3 plots the bond yields for a wide range of maturities. The parameters are set at the same values as in Figs. 1 and 2: $\rho = 0.02$, $\mu = 0.05$, and $\sigma = 0.20$. It then follows that $y^{(1)}(\tau) = r^{(1)} = 0.0285$ and $y^{(2)}(\tau) = r^{(2)} = 0.0294$. (The subscript t has been dropped for the two limiting yield curves since they are constant over time.) In this case, $y^{(2)}(\tau) > y^{(1)}(\tau)$ and $y(\infty) = \min[y^{(1)}(\infty), y^{(1)}(\infty)] = y^{(1)}(\infty) = 0.285$.

It is seen that at any time $t \in (0, \infty)$, the yield curve with both investors present can be downward-sloping, upward-sloping, or nonmonotonic, depending

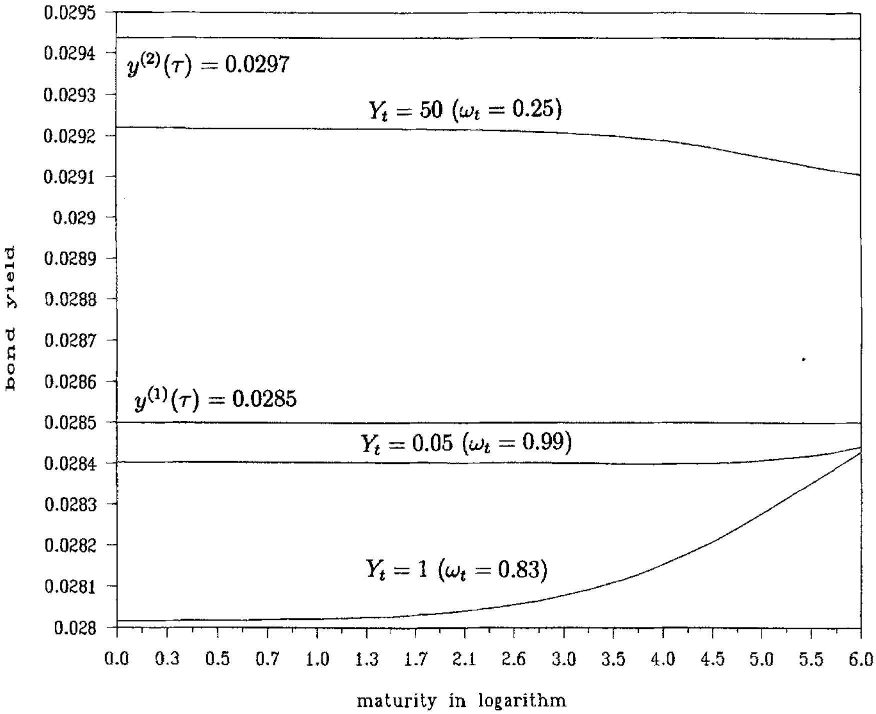

Fig. 3. The bond yield $y_t(\tau)$ plotted against logarithm of maturity $\log \tau$ at different levels of current aggregate consumption Y_t or equivalently the consumption of investor 1 relative to the aggregate consumption ω_t. The level of Y_t (or ω_t) is chosen to be $Y_t = 0.05$, 1, 50 (or $\omega_t = 0.99$, 0.83, 0.25), respectively. The parameters are set at the following values: investors' time discount coefficient $\rho = 0.02$, expected rate of aggregate consumption growth $\mu = 0.05$, instantaneous standard deviation of aggregate consumption growth $\sigma = 0.20$. The bond yield with only investor 1 present is $y^{(1)}(\tau) = 0.0285$ and the bond yield with only investor 2 present is $y^{(2)}(\tau) = 0.0294$, both independent of maturity.

on the current level of aggregate endowment Y_t. The yield curve has $\min[y^{(1)}(\infty),\ y^{(2)}(\infty)]$ as its asymptotic limit as the maturity increases. As Y_t changes over time, the shape of the yield curve also changes. For $Y_t = 50$ (i.e., $\omega_t = 0.25$), the yield curve is downward-sloping. It lies inside the range bounded by the two limiting yield curves, $[y^{(1)}(\tau),\ y^{(2)}(\tau)]$. The yields at short maturities are close to $y^{(2)}(\tau)$ while the yields at long maturities decrease and approach $y^{(1)}(\infty)$. For $Y_t = 1$ (i.e., $\omega_t = 0.83$), the yield curve is upward-sloping. For the range of maturities shown in the figure, it lies outside the range $[y^{(1)}(\tau),\ y^{(2)}(\tau)]$ and is lower than $y^{(1)}(\tau) = 0.28$. As maturity increases, the yield increases and approaches $y^{(1)}(\infty)$. For $Y_t = 0.05$ (i.e., $\omega_t = 0.99$), the yield curve is nonmonotonic, first decreasing and then increasing as maturity increases, and lies outside the range bounded by $y^{(1)}$ and $y^{(2)}$.

Similar behavior of the yield curve is found for other parameter values. For certain parameter values and aggregate consumption level, the yield curve can exhibit a humped shape, upward-sloping at short maturities and downward-sloping at long maturities.

5. Extensions and discussions

The previous sections present a parsimonious model of the term structure of interest rates with heterogeneous investors. For simplicity in exposition, only the case of two investors (with respectively the logarithm and square-root utility function and the same time discount parameter) and the simple endowment process is considered. This section considers some extensions of the basic model.

5.1. More general preference heterogeneity

In the basic model, there are only two investors, one with logarithmic utility function and the other with square-root utility function. The equilibrium is tractable because closed-form solutions can be obtained for the optimal sharing rules given the specific preferences of the investors. Within the class of isoelastic utility functions, there are other combinations of the two risk-aversion coefficients, a_1 and a_2, that also allow closed-form solutions to the optimal sharing rule. For example, when $a_1 > a_2 > 0$ and $a_1 = na_2$ where $n = 2, 3, 4$, the situation is similar to the basic model and a closed-form solution can be obtained for the optimal sharing rule and the equilibrium.

Another extension of the model is to consider more than two investors. Again, consider the situation when closed-form solutions can be obtained for the optimal sharing rule. The following three-investor economy provides such an example. All investors have isoelastic utility functions with the following exponents: $a_1 = 2$, $a_2 = 1$, and $a_3 = \frac{1}{2}$. More generally, for $a_1 > a_2 > a_3 > 0$, closed-form solutions to the optimal sharing rule can be obtained if a_1/a_2 and a_1/a_3

belong to the set $\{2, 3, 4\}$. The case of four investors when $a_1 > a_2 > a_3 > a_4 > 0$ and a_1/a_2, a_1/a_3, and a_1/a_4 belong to the set $\{2, 3, 4\}$ also yield closed-form solutions. In the more general case with more than four investors within the class of power utility functions, it is more difficult to find closed-form solutions to the optimal sharing rule under general wealth distributions.

The following theorem summarizes the above discussion:

Theorem 5. Suppose that the economy consists of I investors with power utility functions of the form (4). Let a_i be the relative risk aversion of investor i, $i = 1, \dots, I$, and $0 < a_I \leqslant \cdots \leqslant a_2 \leqslant a_1$. The optimal sharing rule has a closed-form solution if

$$\left\{\frac{a_1}{a_1}, \frac{a_1}{a_2}, \dots, \frac{a_1}{a_I}\right\} \subseteq \{1, 2, 3, 4\}\,.$$

Furthermore, the utility function of the representative agent defined by

$$u_\lambda(Y) \equiv \sup_{\sum_{i=1}^{I} c_i = y} \sum_{i=1}^{I} \lambda_i \frac{c^{1-a_i} - 1}{1 - a_i}, \quad \text{where} \quad \lambda_i \geqslant 0 \quad \text{and} \quad \sum_{i=1}^{I} \lambda_i = 1\,,$$

exhibits relative risk aversion that is bounded by a_I and a_1, i.e., $a_I \leqslant -\,Yu''_\lambda(Y)/u'_\lambda(Y) \leqslant a_1$, where $Y \in (0, \infty)$.[5]

The previous discussions on the behavior of bond yields in the two-investor case can easily be extended to the multiple-investor case here. The qualitative results are similar.

5.2. More general endowment processes

In the basic model, the special case of geometric Brownian motion is considered for the aggregate endowment in the two-investor economy. The simple process was chosen in order to illustrate the effect of heterogeneity in investor preferences on asset prices, in particular, bond prices and the term structure of interest rates. Since the aggregate endowment Y_t is the single variable that drives the economy, all asset prices have one explanatory factor. Price changes of bonds with different maturities are perfectly correlated. This section provides some generalizations of the previous endowment process in order to relax its restrictive nature. The resulting term structures will depend on multiple factors.

[5]The last part of the theorem was first suggested to me by Chi-fu Huang. Bruce Grundy later brought to my attention the work of Benninga and Mayshar (1993) of which this is a special case.

For simplicity in notation, define $y_t \equiv \log Y_t$. The endowment process is as follows:

$$\mathrm{d}y_t = [\alpha - \beta y_t + z_t]\mathrm{d}t + \sigma \mathrm{d}w_t, \quad y_{t=0} = y_0, \tag{27}$$

$$\mathrm{d}z_t = -z_t \mathrm{d}t + \sigma_z \mathrm{d}w_{z,t}, \qquad z_{t=0} = z_0, \quad t \in [0, \infty). \tag{28}$$

w_t and $w_{z,t}$ are two independent standard Wiener processes, and α, β, σ, and σ_z are nonnegative constants. Although for simplicity in exposition w_t and $w_{z,t}$ are assumed to be independent, this assumption can be easily relaxed. z_t is assumed to be a standard Ornstein–Uhlenbeck process. (There is no loss of generality by making the negative coefficient of the linear drift to be -1.) The linear system $[y_t, z_t]$ contains several interesting special cases. For example, $\beta = 0$ and $z_t \equiv 0$ (i.e., when $z_0 = 0$ and $\sigma_z = 0$) reduce to the simple case considered in previous sections where y_t follows a simple Brownian motion with constant drift α ($\alpha = \mu - \sigma^2/2$). When $\beta > 0$ and $z_t \equiv 0$, y_t follows an Ornstein–Uhlenbeck process which is stationary. When $\beta = 0$, y_t has a drift linear in z_t. Since z_t is stationary and has an unconditional mean of zero, α gives the long-run growth of y_t and z_t the transitory growth of y_t. The aggregate consumption process in this case is quite similar to the case considered in Cox, Ingersoll, and Ross (1985a) and Sun (1992).

Lemma 6. The solution to the linear system (27)–(28) is

$$y_{t+\tau} = y_t + \mu_t(\tau) + \sigma \int_t^{t+\tau} \mathrm{e}^{-\beta(t+\tau-s)} \mathrm{d}w_s + \frac{\sigma_z}{1-\beta} \int_t^{t+\tau} [\mathrm{e}^{-\beta(t+\tau-s)} - \mathrm{e}^{-(t+\tau-s)}] \, \mathrm{d}w_{z,s},$$

$$z_{t+\tau} = \mathrm{e}^{-\tau} z_t + \sigma_z \int_t^{t+\tau} \mathrm{e}^{-(t+\tau-s)} \mathrm{d}w_{z,s},$$

where

$$\mu_t(\tau) \equiv \left(\frac{\alpha}{\beta} - y_t\right)(1 - \mathrm{e}^{-\beta\tau}) + \frac{1}{1-\beta}(\mathrm{e}^{-\beta\tau} - \mathrm{e}^{-\tau}).$$

Let $g_t(\tau) = Y_{t+\tau}/Y_t = \mathrm{e}^{y_{t+\tau} - y_t}$ *and*

$$v(\tau) \equiv \frac{\sigma^2}{2\beta}(1 - \mathrm{e}^{-2\beta\tau}) + \frac{\sigma_z^2}{(1-\beta)^2}\left\{\frac{1 - \mathrm{e}^{-2\beta\tau}}{2\beta} + \frac{1 - \mathrm{e}^{-2\tau}}{2} - \frac{2[1 - \mathrm{e}^{-(1+\beta)\tau}]}{1+\beta}\right\}.$$

Conditional on $[y_t, z_t]$, $\log g_t(\tau) = y_{t+\tau} - y_t$ *is normally distributed and* $\mathrm{E}_t[\log g_t(\tau)] = \mu_t(\tau)$, $\mathrm{var}_t[\log g_t(\tau)] = v(\tau)$.

Now consider the equilibrium and bond prices under the current endowment process. Given that the state of the economy is characterized by $[y_t, z_t]$, more traded securities are needed in addition to the stock and the (locally) risk-free security in order to make the financial market dynamically complete. Without further specification, assume that enough securities are traded so that the market is complete. Following the same steps as in Section 3 gives the same sharing rules between the two investors and the same pricing equations in terms of Y_t and $g_t(\tau)$. Bond prices are then calculated by applying the pricing equation (15). The results are summarized in the following lemma and theorem.

When only one of the two investors is present in the economy, the resulting term structures of interest rates are reminiscent of those in the Cox, Ingersoll, and Ross (1985a) model.

Lemma 7. Given the aggregate endowment process as specified in Lemma 6, the bond prices and yields in the two limiting cases are

$$B_t^{(i)}(\tau) = \mathrm{e}^{-\rho\tau - a_i\mu_t(\tau) + \frac{1}{2}a_i^2 v(\tau)}, \qquad y_t^{(i)}(\tau) = \rho + a_i\mu_t(\tau) - \frac{a_i^2}{2}v(\tau),$$

where $\mu_t(\tau)$ and $v(\tau)$ are given in Lemma 6. The corresponding instantaneous interest rates and the long yields are

$$r_t^{(i)} = \rho + a_i[\alpha - \beta y(t) + z_t] - \frac{a_i^2}{2}\sigma^2,$$

$$y^{(i)}(\infty) = \begin{cases} \rho, & \beta > 0, \\ \rho + a_i\alpha - \dfrac{a_i^2}{2}(\sigma^2 + \sigma_z^2), & \beta \leqslant 0, \end{cases}$$

where $i = 1, 2$.

When $\beta = 0$, the result is the special case that is very similar to the Cox, Ingersoll, and Ross model except that the interest rate follows the linear Ornstein–Uhlenbeck process here while in their model it follows the square-root process. In this case, the growth rate of the economy depends only on z_t, not on the size of the economy y_t. The current model obtains the single-factor structure for bond prices when investor preferences are homogeneous. Similar to the Cox, Ingersoll, and Ross model, the term structure can exhibit rich patterns even under this one-factor structure.

When both investors are present in the economy, the following theorem holds:

Theorem 6. Let $\xi_{n,t}(\tau) \equiv \{n - 1 + [\mu_t(\tau) + y_t]/v(\tau)\}\sqrt{v(\tau)}$. Given the endowment process (27)–(28), the bond prices with both investors present are given as follows:

$$B_t(\tau) = \mathrm{e}^{-\rho\tau}(\sqrt{1 + Y_t} - 1)[I_{1,t}(\tau) + I_{2,t}(\tau)],$$

where

$$I_{1,t}(\tau) = \mathrm{e}^{-\xi_{1/2,t}(\tau)\sqrt{v(\tau)}},$$

$$I_{2,t}(\tau) = \mathrm{e}^{-[\mu_t(\tau)+y_t]^2/2v(\tau)} \sum_{n=0}^{\infty} \alpha_n \{\mathrm{e}^{-\frac{1}{2}\xi_{n,t}^2(\tau)} \Phi[-\xi_{n,t}(\tau)]$$

$$+ \mathrm{e}^{\frac{1}{2}\xi^2_{1/2-n,t}(\tau)} \Phi[\xi_{1/2-n,t}(\tau)]\},$$

and α_n, $n = 0, 1, \ldots, \Phi(\cdot)$, *are defined in Theorem 2. Furthermore, the instantaneous interest rate is given by*

$$r_t = \rho + \frac{(\alpha - \beta y_t + z_t)Y_t}{2\sqrt{1+Y_t}(\sqrt{1+Y_t}-1)} - \frac{\sigma^2 Y_t[Y_t^2 - 2(\sqrt{1+Y_t}-1)^2]}{8(1+Y_t)^{\frac{3}{2}}(\sqrt{1+Y_t}-1)^3}.$$

The long yield $y_t(\infty)$ *is a constant independent of* y_t *and* z_t, *and is given by*

$$y(\infty) = \min[y^{(1)}(\infty), y^{(2)}(\infty)],$$

where $y^{(2)}(\infty)$ *and* $y^{(2)}(\infty)$ *are given in Lemma 7.*

The endowment process defined by (27)–(28) is Gaussian. It leads to normal distributions for the growth rate of the economy over any finite periods, permitting the calculation of bond prices. However, it is easy to see from the calculations that more general (non-Gaussian) processes can be considered for the aggregate endowment. One example is the following square-root process:

$$\mathrm{d}y_t = (\alpha - \beta y_t)\mathrm{d}t + \sigma\sqrt{y_t}\,\mathrm{d}w_t, \qquad y_0 \geqslant 0, \quad t \in [0, \infty), \tag{29}$$

where α ($\geqslant 0$) and β are constants. This process is reminiscent of the process of aggregate endowment assumed in the Cox, Ingersoll, and Ross model. It is easy to show that under (29),

$$r_t^{(i)} = \rho + a_i\alpha - a_i(\beta + a_i^2)\sigma^2 y_t.$$

Since $y_t \geqslant 0$, $r_t^{(i)} \geqslant 0$ if $k \equiv -\beta - a_i^2 > 0$. Defining $\hat{r}_t^{(i)} = r_t^{(i)} - (\rho + a_i\alpha)$,

$$\mathrm{d}\hat{r}_t^{(i)} = \left[\alpha + \frac{|\beta|}{k}\hat{r}_t^{(i)}\right]\mathrm{d}t + (\sigma/\sqrt{k})\sqrt{\hat{r}_t^{(i)}}\mathrm{d}w_t. \tag{30}$$

$\hat{r}_t^{(i)}$ will be nonnegative (if it starts from nonnegative values). Cox, Ingersoll, and Ross provide a thorough examination on the properties of process (29) or (30).

When both investors are present, $m_t = \mathrm{e}^{-\rho t}/(\sqrt{1+\mathrm{e}^{y_t}}-1)$. Simply apply Theorem 1 to derive the equilibrium interest rate. Since the probability density of $y_{t+\tau}$ conditional on y_t (with $\tau \geqslant 0$) can be calculated in closed form, the bond prices can be calculated in the same way as in Theorem 2 (these calculations are omitted here).

5.3. Stationarity

One unattractive feature of the current model is its long-run behavior. With positive growth, the economy will eventually be dominated by the less risk-averse investor. The steady state distribution of bond yields will then simply be the one when only the less risk-averse investor is present. In other words, the importance of investor heterogeneity will eventually disappear. Although stationary distributions of bond yields can be obtained in which the effect of heterogeneity remains important, it requires the stationarity of the endowment process. This feature is particularly undesirable for the empirical implementation of the model given the positive growth observed in the data.

One way to allow positive growth of the aggregate endowment and to maintain the importance of investor heterogeneity in the steady state (in terms of the distribution of bond yields) is to relax the assumption that the time discount parameters of the two investors are the same and to modify the aggregate endowment process. Let ρ_i, $i = 1, 2$, be the time discount parameter of investor i. It is easy to show that the optimal sharing rule now is

$$\hat{c}_{1,t}(Y_t,\lambda) = \frac{2}{b(t)}[\sqrt{1 + b(t)Y_t} - 1], \qquad \hat{c}_{2,t}(Y_t, \lambda) = Y_t - \hat{c}_{1,t}(Y_t, \lambda),$$

where $b(t) = b_0 e^{-2(\rho_2 - \rho_1)t}$ and $b_0 = 4(1-\lambda)^2/\lambda^2$. The marginal utility of the representative agent m_t now has the form

$$m_t = \frac{e^{-(2\rho_2 - \rho_1)t}}{\sqrt{1 + b_0 e^{-2(\rho_2 - \rho_1)t} Y_t} - 1}.$$

In order to maintain the importance of investor heterogeneity in the steady state, $e^{-2(\rho_2 - \rho_1)t} Y_t$ must be stationary. Instead of assuming a geometric Brownian motion for Y_t, Y_t can be trend-stationary, i.e., $Y_t \equiv e^{(\mu - \frac{1}{2}\sigma^2)t + y_t}$ and y_t follows a stationary process such as an Ornstein–Uhlenbeck process as discussed in Section 5.2. A further assumption is that $\mu - \sigma^2/2 = 2(\rho_2 - \rho_1)$. Thus, for $\rho_2 - \rho_1 > 0$, positive growth can be allowed in the model and the steady state of the equilibrium does not degenerate to the case with only one investor.[6]

It should be pointed out that this is a knife-edge case. When $\rho_2 - \rho_1 > 0$, only the growth at $\mu = \sigma^2/2 + 2(\rho_2 - \rho_1)$ gives the desirable behavior in the long run. One remedy to this situation is to consider preferences that are time-nonseparable. For example, one can endogenize the time discount parameter by making it depend on past consumption. (High levels of past consumption lead to large

[6] I thank George Constantinides for pointing out a problem with the stationary distributions of consumption distribution and state prices when Y_t follows Geometric Brownian motion.

values of the time discount parameter.) The development of a detailed model of this type will not be further pursued here. For models with time nonadditive preferences under certainty, see, e.g., Koopmans (1962), Uzawa (1965), and Lucas and Stokey (1984).

6. Further comments

Section 4 only considers the prices of pure discount bonds. Eq. (20) can be used to price any security given its payoff stream. In particular, the equilibrium stock price derived in Theorem 1 is given by the following expectation:

$$S_t = S(Y_t) = (\sqrt{1+Y_t} - 1)\left[\frac{1}{\rho} + \int_0^\infty e^{-\rho\tau} E_t \sqrt{1 + Y_t g_t(\tau)}\, d\tau\right].$$

Clearly, the current stock price is only a function of the current level of aggregate consumption (given the initial conditions of the economy). In the two limiting cases, the corresponding stock prices are, respectively, $S_t^{(1)} = S^{(1)}(Y_t) = Y_t/\rho$ and $S_t^{(2)} = S^{(2)}(Y_t) = Y_t/(r^{(2)} - \mu + \frac{1}{2}\sigma^2)$. It is easy to show that $S(Y) \to S^{(1)}(Y)$ when $Y \to 0$ and $S(Y) \to S^{(2)}(Y)$ when $Y \to \infty$. Note that from (16), $S(Y)$ satisfies the following ordinary differential equation:

$$\tfrac{1}{2}\sigma^2 S''(Y) + [\mu Y - \sigma\pi(Y)]S'(Y) - r(Y)\, S(Y) + Y = 0\,.$$

Thus the stock price is given by the solution to this equation with the above boundary conditions (see Wang, 1994, for a more detailed discussion).

The current price of a European call option on a pure discount bond, $c(B, t; K, \tau)$, is simply $E_t[(m_{t+\tau}/m_t)c(B, t+\tau; K, 0)]$, where K is the strike price of the option, τ the maturity of the option, $B_t(\tau)$ the price of the discount bond with same maturity, and $c(B, t+\tau; K, 0)$ the terminal payoff of the option. With some algebra, the conditional expectation can be explicitly calculated.

The basic model is presented in a continuous-time setting. This is purely for mathematical convenience. The model can also be presented in a discrete-time setting and most of the results remain the same. As pointed out by Sun (1992) in the case of the Cox, Ingersoll, and Ross (1985a) model, the discrete-time representation may be easier to estimate empirically, especially in the presence of general nominal shocks. Similar arguments can be made here, although a detailed discussion on this issue is outside the scope of this paper.

Appendix

This appendix provides proofs to some of the results in the text. The proofs are only for the basic model. Extensions to more general endowment processes are straightforward.

For convenience, let the continuous-time economy be defined on the finite time span $[0, T]$. The uncertainty and the information structure are represented by a filtered, complete probability space $(\Omega, \mathscr{F}, \mathbf{F}, P)$ on which a one-dimensional Brownian motion w_t, $t \in [0, T]$, is defined. The filtration $\mathbf{F} = [\mathscr{F}_t, t \in [0, T]\}$ is the augmentation under P of the filtration generated by w. For a reference of the terminology used here, see, e.g., Duffie (1992).

The consumption space C_+ is defined as the set of positive, adapted consumption rate process that satisfy (3). The securities market consists of the (locally) risk-free security which pays a sure interest r_t and the stock which pays dividend at rate Y_t and is traded at (ex-dividend) price S_t. Y_t is given by (2). The trading strategy (α, θ) is a two-dimensional predictable process adapted to $\mathscr{F}_t$ where α_t denotes holdings of the risk-free security and θ_t denotes holdings of the stock. A trading strategy is admissible if it satisfies condition (4) and $W_t \equiv \alpha_t + \theta_t S_t \geqslant 0$, $t \in [0, T]$. Let Θ denote the set of admissible trading strategies. A consumption/trading strategy is budget-feasible if

$$W_T = W_0 + \int_0^T \{\alpha_t r_t \mathrm{d}t + \theta_t(Y_t \mathrm{d}t + \mathrm{d}S_t) - c_t \mathrm{d}t\}\,.$$

Investors' preferences are given in (4) which are continuous, smooth, and strictly concave. Investors' initial endowments are in shares of the stock, $(0, \theta_{i,0_-})$, $i = 1, 2$, and $\sum_i \theta_{i,0_-} = 1$.

Proof of Lemma 2. Let $u_i(c) = \mathrm{E}_0[\int_0^T \mathrm{e}^{-\rho t}(c_t^{1-a_i} - 1)/(1 - a_i)\mathrm{d}t]$, $\partial u_i(c) = \mathrm{e}^{-\rho t} c^{-a_i}$, and $\phi \cdot c = \mathrm{E}_0[\int_0^T \phi_t c_t \mathrm{d}t]$. It must be shown that $\phi_{0,t} = \mathrm{e}^{-\rho t}(\sqrt{1 + bY_0} - 1)/(\sqrt{1 + bY_t} - 1)$ is the pricing function that supports an Arrow–Debreu equilibrium given an optimum $(\hat{c}_1, \hat{c}_2)$. Clearly, $\phi > 0$. Furthermore, $\forall c \in C_+$ such that $u_i(c) > u_i(\hat{c}_i)$ $[i = 1, 2]$, $\partial u_i(\hat{c}_i) \cdot (c - \hat{c}_i) > 0$ since u_i is strictly concave. Note that $\phi_{0,t} = (\sqrt{1 + bY_0} - 1)\partial u_i(\hat{c}_{i,t})$. Thus, $\phi \cdot c > \phi \cdot \hat{c}_i$. This completes the proof.

Proof of Lemma 3. To show that the price processes given in (12) characterize an equilibrium requires showing that (a) $(\hat{c}_i, (\hat{\alpha}_i, \hat{\theta}_i))$ is a budget-feasible consumption/trading strategy for each investor i and (b) any other admissible trading strategies that yield higher expected utility are not budget-feasible. For (a), first note that $(\hat{\alpha}_1, \hat{\theta}_1)$ and $(\alpha_2, \hat{\theta}_2)$ given in Theorem 1 are admissible ($W_{1,t}, W_{2,t} > 0$ and the corresponding gain processes are integrable). Next, to show that $(\hat{c}_i, (\hat{\alpha}_i, \hat{\theta}_i))$, $i = 1, 2$, is budget-feasible, I use the standard equivalent martingale approach. Let Q be a measure on the space $(\Omega, \mathscr{F})$ defined by its Radon–Nikodym derivative with respect to P: $\mathrm{d}Q/\mathrm{d}P = \phi(T, \omega)$. Clearly, Q and P are equivalent measures. It can be shown that the gain process of any admissible trading strategy, defined as $\alpha r_t \mathrm{d}t + \theta_t[Y_t \mathrm{d}t + \mathrm{d}S_t]$, is a martingale under Q. For $W_T = 0$, the budget-feasibility of a consumption/trading policy $(c, (\alpha, \theta))$ under Q becomes

$$W_0 = \mathrm{E}_0^*\left[\int_0^T c_t \mathrm{d}t\right],$$

where E^* denotes the expectation under Q. It is easy to verify that this is true for $(\hat{c}_i, (\hat{\alpha}_i, \hat{\theta}_i))$, $i = 1, 2$. For (b), note that for any consumption strategy c financed by (α, θ), $u_i(c) > u_i(\hat{c}_i)$ implies that $\phi \cdot c > \phi \cdot \hat{c}_i$. Thus, $\mathrm{E}_0^*[\int_0^T c_t \mathrm{d}t] > \mathrm{E}_0^*[\int_0^T \hat{c}_{i,t} \mathrm{d}t] = W_{i,0}$, i.e., c is not budget-feasible. This completes the proof.

Proof of Theorem 2. Since $g_t(\tau)$ is log-normally distributed, $g_t(\tau) = \mathrm{e}^{\mu(\tau) + \sigma(\tau)\tilde{\varepsilon}}$ where $\mu(\tau) \equiv (\mu - \frac{1}{2}\sigma^2)\tau$, $\sigma(\tau) \equiv \sigma\sqrt{\tau}$, and $\tilde{\varepsilon} \sim \mathcal{N}(0, 1)$. The calculation of $I_{1,t}(\tau)$ is trivial. For $I_{2,t}(\tau)$,

$$\begin{aligned}
I_{2,t}(\tau) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-[\delta_t(\tau) + \varepsilon/\sigma(\tau)]\sigma^2\tau} \sqrt{1 + \mathrm{e}^{[\delta_t(\tau) + \varepsilon/\sigma(\tau)]\sigma^2\tau}}\, \mathrm{e}^{-\frac{1}{2}\varepsilon^2} \mathrm{d}\varepsilon \\
&= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-\delta_t(\tau)\sigma(\tau)} \sum_{n=0}^{\infty} a_n \mathrm{e}^{(n-1)[\delta_t(\tau) + \varepsilon/\sigma(\tau)]\sigma^2\tau} \mathrm{e}^{-\frac{1}{2}\varepsilon^2} \mathrm{d}\varepsilon \right. \\
&\quad \left. + \int_{-\delta_t(\tau)\sigma(\tau)}^{\infty} \sum_{n=0}^{\infty} a_n \mathrm{e}^{-(n+\frac{1}{2})[\delta_t(\tau) + \varepsilon/\sigma(\tau)]\sigma^2\tau} \mathrm{e}^{-\frac{1}{2}\varepsilon^2} \mathrm{d}\varepsilon \right\} \\
&= \sum_{n=0}^{\infty} a_n \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\delta_t(\tau)\sigma(\tau)} \mathrm{e}^{(n-1)[\delta_t(\tau) + \varepsilon/\sigma(\tau)]\sigma^2\tau} \mathrm{e}^{-\frac{1}{2}\varepsilon^2} \mathrm{d}\varepsilon \right. \\
&\quad \left. + \frac{1}{\sqrt{2\pi}} \int_{-\delta_t(\tau)\sigma(\tau)}^{\infty} \mathrm{e}^{-(n+\frac{1}{2})[\delta_t(\tau) + \varepsilon/\sigma(\tau)]\sigma^2\tau} \mathrm{e}^{-\frac{1}{2}\varepsilon^2} \mathrm{d}\varepsilon \right\},
\end{aligned}$$

where $\delta_t(\tau) \equiv [(\mu - \frac{1}{2}\sigma^2)\tau + \log Y_t]/(\sigma^2\tau)$ and a_n, $n = 0, \ldots$, are the coefficients of the Taylor expansion of $\sqrt{1 + x}$. Let $\xi_{n,t}(\tau) \equiv \sigma(\tau)[n - 1 + \delta_t(\tau)]$. It is easy to show that

$$\begin{aligned}
&\int_{-\infty}^{-\delta_t(\tau)\sigma(\tau)} \mathrm{e}^{(n-1)[\delta_t(\tau) + \varepsilon/\sigma(\tau)]\sigma^2\tau} \mathrm{e}^{-\frac{1}{2}\varepsilon^2} \frac{\mathrm{d}\varepsilon}{\sqrt{2\pi}} \\
&\qquad = \mathrm{e}^{-\frac{1}{2}\delta_t(\tau)^2\sigma^2\tau + \frac{1}{2}\xi_{n,t}(\tau)^2} \Phi[-\xi_n(t, \tau)], \\
&\int_{-\delta(t,\tau)\sigma(\tau)}^{\infty} \mathrm{e}^{-(n+\frac{1}{2})[\delta_t(\tau) + \varepsilon/\sigma(\tau)]\sigma^2\tau} \mathrm{e}^{-\frac{1}{2}\varepsilon^2} \frac{\mathrm{d}\varepsilon}{\sqrt{2\pi}} \\
&\qquad = \mathrm{e}^{-\frac{1}{2}\delta_t(\tau)^2\sigma^2\tau + \frac{1}{2}\xi_{1/2-n,t}(\tau)^2} \Phi[\xi_{1/2-n,t}(\tau)],
\end{aligned}$$

where $\Phi(z) = (1/\sqrt{2\pi})\int_{-\infty}^{z} \mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x$. This yields the results in Theorem 2.

Proof of Theorem 4. When $\tau \gg 1$, $\mu(\tau) \gg 1$ and $\sigma^2(\tau) \gg 1$. Rewrite $I_{2,t}(\tau)$ as

$$I_{2,t}(\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mu(\tau)-\sigma(\tau)\varepsilon-\frac{1}{2}\varepsilon^2} \sqrt{1 + Y(t)\mathrm{e}^{\mu(\tau)+\sigma(\tau)\varepsilon}}\mathrm{d}\varepsilon$$

$$= \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\mu(\tau)+\frac{1}{2}\sigma^2(\tau)} \int_{-\infty}^{\infty} \sqrt{\mathrm{e}^{-\varepsilon'^2} + \mathrm{e}^{\mu(\tau)-\frac{3}{4}\sigma^2(\tau)-[\varepsilon'-\frac{1}{2}\sigma(\tau)]^2}}\,\mathrm{d}\varepsilon',$$

where the last equation is obtained by the change of variable $\varepsilon' = \varepsilon + \sigma(\tau)$. Observing that when $\mathrm{d} \gg 1$,

$$0 \leqslant \int_{-\infty}^{\infty} \frac{[\sqrt{\mathrm{e}^{-x^2}} + \sqrt{c\mathrm{e}^{-(x-d)^2}}]^2 - [\sqrt{\mathrm{e}^{-x^2} + c\mathrm{e}^{-(x-d)^2}}]^2}{\sqrt{\mathrm{e}^{-x^2}} + \sqrt{c\mathrm{e}^{-(x-d)^2}} + \sqrt{\mathrm{e}^{-x^2} + c\mathrm{e}^{-(x-d)^2}}}\mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \frac{2\sqrt{c\mathrm{e}^{-x^2}\mathrm{e}^{-(x-d)^2}}}{\sqrt{\mathrm{e}^{-x^2}} + \sqrt{c\mathrm{e}^{-(x-d)^2}} + \sqrt{\mathrm{e}^{-x^2} + c\mathrm{e}^{-(x-d)^2}}}\mathrm{d}x \ll 1\,.$$

This then implies that for $\mathrm{d} \gg 1$,

$$\int_{-\infty}^{\infty} \sqrt{\mathrm{e}^{-} + c\mathrm{e}^{-(x-d)^2}}\mathrm{d}x \approx \int_{-\infty}^{\infty} [\sqrt{\mathrm{e}^{-x^2}} + \sqrt{c\mathrm{e}^{-(x-d)^2}}]\,\mathrm{d}x\,.$$

Applying this approximation to the $I_2(t, \tau)$ when $\tau \gg 1$,

$$I_{2,t}(\tau) \approx \max[\mathrm{e}^{-\mu(\tau)+\frac{1}{2}\sigma^2(\tau)}, \sqrt{Y(t)}\,\mathrm{e}^{-\frac{1}{2}\mu(\tau)-\frac{3}{8}\sigma^2(\tau)}]\,.$$

Theorem 4 follows then immediately.

Proof of Theorem 5. The optimal sharing rule is obtained by solving the following optimization problem:

$$\sup_{s\{c_i\}} \sum_{i}^{I} \lambda_i \frac{c^{1-a_i} - 1}{1 - a_i},$$

subject to

$$c_i \geqslant 0, \quad \forall i = 1, \dots, I, \quad \text{and} \quad \sum_{i=1}^{I} c_i = Y\,,$$

where $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_I > 0$, $i = 2, \dots, I$. Given the strict concavity of the objective function and the linear constraints, there is a unique solution to the maximization problem. The corresponding first-order conditions are

$$c_i^{-a_i} = \frac{\lambda_1}{\lambda_i} c_1^{-a_i}, \qquad i = 2, \dots, I.$$

Then,

$$c_i = \left(\frac{\lambda_i}{\lambda_1}\right)^{1/a_i} c_1^{a_1/a_i}, \qquad i = 2, \dots, I .$$

From the resource constraint,

$$\sum_{i=1}^{I} \left(\frac{\lambda_i}{\lambda_1}\right)^{1/a_i} c_1^{a_1/a_i} = Y .$$

When $\{a_1/a_1, a_1/a_2, \dots, a_1/a_I\} \subseteq \{1, 2, 3, 4\}$, the above equation is a fourth-order polynomial equation and has closed form solutions. The unique solution that guarantees $0 < c_1 < Y$ gives the optimal sharing rule.

References

Araujo, A. and Paulo Monteiro, 1989, Equilibrium without uniform conditions, Journal of Economic Thoery 48, 416–427.

Becker, Robert A., 1978, The existence of perfect foresight competitive equilibrium in a simple dynamic model of capital accumulations with heterogeneous households, Working paper (Indiana University, Bloomington, IN).

Benninga, Simon and Joram Mayshar, 1993, Dynamic wealth distribution, trade and asset pricing, Working paper (Rodney L. White Center for Financial Research, University of Pennsylvania, Philadelphia, PA).

Cass, David, 1965, Optimum growth in an aggregate model of capital accumulation, Review of Economic Studies 32, 233–240.

Constantinides, George M., 1982, Intertemporal asset pricing with heterogeneous consumers and without demand aggregation, Journal of Business 55, 253–267.

Cox, John, Jonathan E. Ingersoll, and Stephen Ross, 1985a, A theory of term structure of interest rates, Econometrica 53, 385–408.

Cox, John, Jonathan Ingersoll, and Stephen Ross, 1985b, An intertemporal general equilibrium model of asset prices, Econometrica 53, 363–384.

Duffie, Darrell, 1986, Stochastic equilibria: Existence, spanning number, and the 'no expected financial gain from trade' hypothesis, Econometrica 54, 1161–1184.

Duffie, Darrell, 1992, Dynamic asset pricing theory (Princeton University Press, Princeton, NJ).

Duffie, Darrell and Chi-fu Huang, 1985, Implementing Arrow–Debreu equilibria by continuous trading of few long-lived securities, Econometrica 53, 1337–1356.

Duffie, Darrell and William Zame, 1989, The consumption-based capital asset pricing model, Econometrica 57, 1279–1297.

Dumas, Bernard, 1989, Two-person dynamic equilibrium in the capital market, Review of Financial Studies 2, 157–188.

Dybvig, Philip H. and Chi-fu Huang, 1988, Nonnegative wealth, absence of arbitrage and feasible consumption plans, Review of Financial Studies 1, 377–401.

Dybvig, Philip H., Jonathan Ingersoll, and Stephen Ross, 1995, Long forward and zero-coupon rates can never fall, Journal of Business, forthcoming.

Harrison, J. Michael and David Kreps, 1979, Martingales and arbitrage in multiperiod securities markets, Journal of Economic Theory 20, 381–408.

Huang, Chi-fu, 1987, An intertemporal general equilibrium asset pricing model: The case of diffusion information, Econometrica 55, 117–142.

Huang, Chi-fu and Henri Pagès, 1992, Optimal consumption and portfolio policies with an infinite horizon: Existence and convergence, Annals of Applied Probability 2, 36–64.
Karatzas, Ionnis and Steven E. Shreve, 1988, Brownian motion and stochastic calculus (Springer-Verlag, New York, NY).
Karatzas, Ionnis, John P. Lehoczky, and Steven E. Shreve, 1990, Existence and uniqueness of multi-agent equilibrium in a stochastic, dynamic consumption/investment model, Mathematics of Operations Research 15, 80–128.
Karlin, Samuel and Howard M. Taylor, 1981, A second course in stochastic processes (Academic Press, New York, NY).
Koopmans, Tjalling C., 1962, Stationary utility and impatience, Econometrica 28, 287–309.
Koopmans, Tjalling C., 1965, On the concept of optimal economic growth, in: The econometric approach to development planning (Rand McNally, Chicago, IL).
Longstaff, Francis and Eduardo Schwartz, 1992, Interest rate volatility and the term structure: A two-factor general equilibrium model, Journal of Finance 47, 1259–1282.
Lucas, Robert E., Jr., 1978, Asset prices in an exchange economy, Econometrica 40, 1429–1444.
Lucas, Robert E., Jr. and Nancy L. Stokey, 1984, Optimal growth with many consumers, Journal of Economic Theory 32, 139–171.
Mas-Colell, Andreu and William Zame, 1991, Equilibrium theory in infinite dimensional spaces, in: W. Hildenbrand and H. Sonnenschein, Handbook of mathematical economics, Vol. 4 (North-Holland, Amsterdam) 1835–1898.
McCulloch, J. Huston, 1993, A reexamination of traditional hypothesis about the term structure: A comment, Journal of Finance 48, 779–790.
Merton, Robert C., 1969, Lifetime portfolio selection under uncertainty: The continuous time case, Review of Economics and Statistics 51, 247–257.
Merton, Robert C., 1973, An intertemporal capital asset pricing model, Econometrica 41, 867–887.
Merton, Robert C., 1990, Continuous time finance (Blackwell, Oxford).
Radner, Ray, 1972, Existence of equilibrium of plans, prices, and price expectations in a sequence of markets, Econometrica 40, 289–303.
Rubinstein, Mark, 1974, An aggregation theorem for security markets, Journal of Financial Economics 3, 201–224.
Solow, Robert M., 1956, A contribution to the theory of economic growth, Quarterly Journal of Economics 70, 65–94.
Stapleton, Richard C. and M.G. Subrahmanyam, 1990, Risk aversion and the intertemporal behavior of asset prices, Review of Financial Studies 3, 677–694.
Sun, Tong-sheng, 1992, Real and nominal interest rates: A discrete-time model and its continuous-time limit, Review of Financial Studies 5, 581–612.
Uzawa, Hirofumi, 1968, Time preference, the consumption function, and optimum asset holdings, in: J.N. Wolfe, ed., Value, capital and growth: Papers in honor of Sir John Hicks (University of Edinburgh Press, Edinburgh 485–504.
Vasicek, Oldrich, 1977, An equilibrium characterization of the term structure, Journal of Financial Economics 5, 177–188.
Wang, Jiang, 1994, The term structure of interest rates in a pure exchange economy with heterogeneous investors, Working paper (MIT, Cambridge, MA).