informs
http://pubsonline.informs.org/journal/mnsc/

MANAGEMENT SCIENCE
Vol. 65, No. 5, May 2019, pp. 2015–2040
ISSN 0025-1909 (print), ISSN 1526-5501 (online)

Dynamic Portfolio Execution

Gerry Tsoukalas,[a] Jiang Wang,[b] Kay Giesecke[c]

[a] The Wharton School, University of Pennsylvania, Philadelphia, Pennsylvania 19104; [b] MIT Sloan School of Management, Cambridge, Massachusetts 02142 and China Academy of Financial Research and National Bureau of Economic Research; [c] Management Science and Engineering, Stanford University, Stanford, California 94305

Contact: gtsouk@wharton.upenn.edu, http://orcid.org/0000-0003-2011-3646 (GT); wangj@mit.edu (JW); giesecke@stanford.edu (KG)

Received: July 29, 2012
Revised: February 20, 2013; March 15, 2014; November 2, 2014; April 30, 2016; November 3, 2016; April 29, 2017
Accepted: May 22, 2017
Published Online in Articles in Advance: October 27, 2017

https://doi.org/10.1287/mnsc.2017.2865

Abstract. We analyze the optimal execution problem of a portfolio manager trading multiple assets. In addition to the liquidity and risk of each individual asset, we consider cross asset interactions in these two dimensions, which substantially enriches the nature of the problem. Focusing on the market microstructure, we develop a tractable order book model to capture liquidity supply/demand dynamics in a multiasset setting, which allows us to formulate and solve the optimal portfolio execution problem. We find that cross asset risk and liquidity considerations are of critical importance in constructing the optimal execution policy. We show that even when the goal is to trade a single asset, its optimal execution may involve transitory trades in other assets. In general, optimally managing the risk of the portfolio during the execution process affects the time synchronization of trading in different assets. Moreover, links in the liquidity across assets lead to complex patterns in the optimal execution policy. In particular, we highlight cases where aggregate costs can be reduced by temporarily "overshooting" one's target portfolio.

History: Accepted by Jerome Detemple, finance.
Funding: G. Tsoukalas and K. Giesecke are grateful to Jeff Blokker and the Mericos Foundation for a grant that supported this work.

Keywords: portfolio management • market microstructure • limit order book • liquidity risk • market impact • programming • dynamic • quadratic

1. Introduction

This paper formulates and solves the optimal execution problem of a portfolio manager trading multiple assets with correlated risks and cross price impact. The execution process, even for a single asset, exhibits several main challenges: the *at-the-money* liquidity available is finite and the act of trading can influence current and future prices. For instance, a large buy order can push prices higher, making subsequent purchases more expensive. Similarly, a sell order can push prices lower, implying that subsequent sales generate less revenue. The connection between trading and price is known as *price impact* and its consequence on investment returns can be substantial.[1] The desire to minimize the overall price impact prompts the manager to split larger orders into smaller ones and execute them over time, in order to source more liquidity. However, trading over longer periods leads to more price uncertainty, increasing risk from the gap between remaining and targeted position. These considerations jointly influence the optimal execution strategy.

The execution of a portfolio generates two additional challenges: First, how to balance liquidity considerations with risks from multiple assets? In particular, reducing costs may require trading assets with different liquidity characteristics at different paces, whereas reducing risk may require more synchronized trading across assets. Second, how should cross asset liquidity be managed? To the extent that liquidity can be connected across assets, properly coordinating trades can help improve execution.

Controlling price impact is a challenging problem because it requires modeling how markets react to one's discrete actions. In practice, this requires a significant investment in information technology and human capital, which can be prohibitive. Therefore, many firms choose to outsource their execution needs or use black-box algorithms from specialized third parties, such as banks with sophisticated electronic trading desks. Moreover, this execution services industry has been growing rapidly over the past decade. Not surprisingly, there is a vast literature studying optimal execution. Most of the existing work focuses on a specific type of execution objective, namely, the problem of optimal liquidation for a single risky asset.

One strand of literature seeks to develop parsimonious functional forms of price impact, grounded in empirical observations, such as Bertsimas and Lo (1998) and Almgren and Chriss (2000).[2] The other focuses on the market microstructure foundations of price impact. Recent protechnology regulations have continued to fuel the widespread adoption of electronic communication networks driven by limit order books. The order books aggregate and publish available orders submitted by market participants, which represents the instantaneous supply/demand of liquidity available in the market. Consequently, more recent papers incorporate this aspect into the analysis. In particular, Obizhaeva and Wang (2013) propose a market microstructure framework in which price impact can be understood as the dynamic responses in the supply and demand of liquidity.[3] One advantage of this approach is that the optimal strategies obtained are robust to different order book profiles. This literature highlights the fact that supply/demand dynamics are crucial.

The key question we seek to address in this paper is how managers can maximize their expected wealth from execution, or more generally their expected utility, when trading portfolios composed of dynamically interacting assets. As much interest as the single-asset case has generated, the multiasset problem has been less studied, perhaps because "the portfolio setting clearly is considerably more complex than the single-stock case" (Bertsimas et al. 1999, p. 41). Our motivation to pursue the multiasset problem is based on the following observation: Even when the execution object is about a single asset, in the general multiasset setting, it is optimal to consider transitory trades in other assets. There are at least two reasons. First, other assets provide natural opportunities for risk reduction through diversification/hedging. Second, price impact across assets may provide additional benefits in reducing execution costs by trading in other assets. Thus, to limit trading to the target asset is in general suboptimal. Of course, when the execution involves a portfolio, we would need to consider both effects from correlation in risk and supply/demand evolution, respectively.[4]

To tackle the problem, we develop a multiasset order book model with correlated risks and coupled supply/demand dynamics. Here, an order executed in one direction (buy or sell) will affect both the currently available inventory of limit orders and also future incoming orders on either side. This is in line with the empirical results in Biais et al. (1995) who find that "downward (upward) shifts in both bid and ask quotes occur after large sales (purchases)." Therefore, there is a priori no reason to rule out the possibility that double-sided (buy and sell) strategies may be optimal even if the original objective is unidirectional (e.g., in the standard liquidation problem). However, allowing for arbitrary dynamics leads to modeling difficulties. In particular, there is no reason to assume that the supply and demand sides of the order books are identical, implying that the manager's buy and sell orders need to be treated separately. To this end, we need to introduce inequality constraints on the optimization variables, which can render the optimization computationally challenging.

To solve the problem, we show that in our setting the optimal policy is path independent, under some restrictions on the asset price processes (namely, that they are random walks). This allows us to solve a equivalent static reformulation of the problem, which is cast as a tractable quadratic program (QP) over the manager's inputs, without sacrificing optimality.

Our model implies that managers can utilize cross asset interactions to significantly reduce risk-adjusted execution costs. The resultant optimal policies involve advanced strategies, such as conducting a series of buy and sell trades in multiple assets. In other words, we find that managers can benefit by *overtrading* during the execution phase. This result may a priori seem counterintuitive. Indeed, we demonstrate that one can lower risk-adjusted trading costs by trading "more." We show that this is the case because a unique tradeoff arises in the multiasset setting. While consuming greater liquidity generally leads to higher charges, one can also take advantage of asset correlation and cross impact to reduce risk via offsetting trades.

We show that multiasset strategies turn out to be optimal for simple unidirectional execution objectives. Even in the trivial case where the objective is to either buy or sell units in a single asset, we find that the manager can benefit by simultaneously trading back and forth in other securities. Previous work has focused on modeling the available buy-side or sell-side liquidity independently of each other. Our results suggest that these two cannot generally be decoupled when accounting for cross asset interactions. Furthermore, the associated strategies are often nontrivial. For instance, when liquidating (constructing) a portfolio, one can reduce execution risk by simultaneously selling (purchasing) shares in positively correlated assets. Our model explains why this type of trade provides an effective hedge against subsequent price volatility.

Extending the analysis to portfolios with heterogeneous liquidity across assets (e.g., portfolios composed of small- and large-cap stocks, ETFs and underlying basket securities, and stocks and options), we find that the presence of illiquid assets in the portfolio drastically affects the optimal policies of liquid assets. In particular, it can be optimal to temporarily overshoot targeted positions in some of the most liquid assets in order to improve execution efficiency at the portfolio level. However, the different trading strategies associated with each asset type could leave managers overexposed to illiquidity at certain times during the execution phase. This *synchronization risk* can be addressed by introducing constraints on the asset weights that synchronize the portfolio trades, while maintaining efficient execution. The constrained optimal policies obtained combine aspects of the optimal stand alone policies of both liquid and illiquid assets.

Our analysis has implications for other important problems in portfolio management. The DP and/or QP formulations can be integrated into existing portfolio optimization problems that treat transaction costs as a central theme. For example, the portfolio selection problem with transaction costs is one of the most central problems in portfolio management (see Brown and Smith 2011 for recent advances). Our model provides an understanding of the origin of these costs and of their propagation dynamics in the portfolio setting. The insights we develop can thus allow portfolio managers to better assess the applicability of some common cost assumptions in this strand of literature (such as assuming cost convexity and diagonal impact matrices, and prohibiting counterdirectional trading).[5]

There is prior work on the multiasset liquidation problem. Bertsimas et al. (1999) develop an approximation algorithm for a risk-neutral agent, which solves the multiasset portfolio problem, while efficiently handling inequality constraints. Almgren and Chriss (2000) briefly discuss the portfolio problem with a mean-variance objective in their appendix and obtain a solution for the simplified case without cross impact. Engle and Ferstenberg (2007) solve a joint composition and execution mean-variance problem with no cross impact using the model from Almgren and Chriss (2000). They find that cross asset trading can become optimal even without cross impact effects. Brown et al. (2010) treat a multiasset two-period liquidation problem with distress risk, focusing on the trade-offs between liquid and illiquid assets. In contrast to these papers, we analyze the more general multiobjective execution problem focusing on the market microstructure origins of price impact. This allows us to characterize the optimal policies as a function of intuitive order book parameters, such as inventory levels, replenishment rates and bid-ask transaction costs. These parameters could be calibrated to tick by tick high-frequency data.[6]

The remainder of the paper is structured as follows: Section 2 details the multiasset liquidity model. Section 3 formulates and solves the manager's dynamic optimization problem. Section 4 focuses on numerical applications and economic insights. Section 5 treats mixed liquidity portfolios. Section 6 concludes. The appendix contains proofs and some additional results.

2. The Liquidity Model

In this section, we develop a model specifying how the manager's trades affect the supply/demand and price processes of all assets. We start with the investment space and admissible trading strategies in Section 2.1. Each buy or sell order submitted to the exchange will be executed against available inventory in the limit order books. Section 2.2 explains the distribution of orders in the order book. Section 2.3 describes the replenishment process: Following each executed trade, new limit orders arrive, reverting prices and collapsing the bid-ask spread toward a defined steady state. This liquidity mean-reversion property provides an incentive for the manager to split his original order over time. Doing so, he can take advantage of more favorable limit orders arriving at future periods. However, delaying trading also introduces more price uncertainty. We formulate and eventually solve this essential trade-off between risk and liquidity.

2.1. Investment Space and Admissible Strategies

We adopt the following notation convention: vectors/matrices are in bold and scalars in standard font. Time t is discrete, with N equally spaced intervals. The manager has a finite execution window, $[0,1]$, where the length of the horizon is normalized to 1 without loss of generality. Thus, there are $N+1$, equally spaced, discrete trading times, indexed by $n \in \{0,\ldots,N\}$, with period length $\tau = 1/N$.

Uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A filtration $(\mathcal{F}_n)_{n\in\{0,\ldots,N\}}$ models the flow of information. The stochastic process generating the information flow is specified in Assumption 2.

We consider a portfolio of m assets indexed by $i \in \{1,\ldots,m\} = I$. Regardless of the manager's objective, we assume that he has the option of purchasing or selling/shorting units in any of the assets during any of the discrete times, as long as he satisfies his boundary conditions at the horizon N. Let $x^+_{i,n} \geq 0$ and $x^-_{i,n} \geq 0$ be his order sizes for buy and sell orders, respectively, in asset i at time n. These will constitute the decision variables over the trading horizon. We also define the following corresponding buy and sell vectors:

$$\text{buy at } n\text{: } \mathbf{x}^+_n = \begin{bmatrix} x^+_{1,n} \\ \vdots \\ x^+_{m,n} \end{bmatrix}, \quad \text{sell at } n\text{: } \mathbf{x}^-_n = \begin{bmatrix} x^-_{1,n} \\ \vdots \\ x^-_{m,n} \end{bmatrix},$$

$$\text{aggregate: } \mathbf{x}_n = \begin{bmatrix} \mathbf{x}^+_n \\ \mathbf{x}^-_n \end{bmatrix}.$$

Next, we define part of the execution objective by formulating the boundary conditions. Let $z_{i,n}$ be a state variable representing the net amount of shares left to be purchased (or sold, if negative) in asset i at time n, before the incoming order at n. Following the vector conventions defined above, the manager's total net trades in each asset must sum to $\mathbf{z}_0$, that is,

$$\sum_{n=0}^{N} (\mathbf{x}^+_n - \mathbf{x}^-_n) = \sum_{n=0}^{N} \boldsymbol{\delta}'\mathbf{x}_n = \mathbf{z}_0,$$

where $\boldsymbol{\delta} = [\mathbf{I}; -\mathbf{I}]$ and $\mathbf{I}$ the identity matrix of size m. Thus, $\boldsymbol{\delta}'\mathbf{x}_n = \mathbf{x}^+_n - \mathbf{x}^-_n$. Following these definitions, it is easy to show that the dynamics for the state vector $\mathbf{z}_n$ can be written as

$$\mathbf{z}_n = \mathbf{z}_{n-1} - \boldsymbol{\delta}'\mathbf{x}_{n-1} \quad \text{and} \quad \mathbf{z}_N = \boldsymbol{\delta}'\mathbf{x}_N. \tag{1}$$

The manager's trades must be adapted to the information filtration. The set S of admissible trading strategies is specified in the definition below.

Definition 1 (Admissible Execution Strategies). The set S of all admissible trading strategies for $n \in \{0, \ldots, N\}$ takes the form

$$S = \left\{ \mathbf{x}_n \in \mathbb{R}_+^{2m} \,\middle|\, \mathcal{F}_n\text{-adapted}; \sum_{n=0}^{N} \boldsymbol{\delta}'\mathbf{x}_n = \mathbf{z}_0 \right\}. \tag{2}$$

The set of strategies in Definition 1 is broad in the sense that no restrictions (e.g., shorting or budget constraints) are imposed during the trading window, as long as the boundary constraints are satisfied by N.

Having established the preliminary notations, the next step is to model the manager's price impact. In other words, we need to describe how his actions affect asset prices over time. The next section is dedicated to developing an adequate liquidity model, which will allow us to formulate the manager's dynamic optimization problem.

2.2. Order Book

In a limit order book market (Parlour and Seppi 2008), the supply/demand of each asset is described by the order book. The basic building blocks of limit order markets consist of three order types: *Limit orders* are placed by market participants who commit their intent to buy (bids) or sell (asks) a certain volume at a specified worst-case (or limit) price. They represent the current visible and available inventory of orders in the market. *Market orders* are immediate orders placed by market participants who want to buy or sell a specific size at the current best prices available in the market. They are executed against existing supply or demand in the limit order book. *Cancelation orders* remove unfilled orders from the book.

To preserve tractability, we follow the existing literature in assuming that the manager is a liquidity taker, that is, he submits market orders executed against available inventory in the book on a single exchange.[7] Although prices and quantities are discrete, we adopt a continuous model of the order book which is entirely described by its density functions: $q^a_{i,n}(p)$ for the ask side and $q^b_{i,n}(p)$ for the bid side. The density functions map available units (q) to limit order prices (p) and thus describe the distribution of available inventory in the order book overall price levels, at any given point in time.

To illustrate, Figure 1 displays a partial snapshot of (a) the oil futures limit order book as of November 8, 2011, at 11:10 A.M., and as a comparison, an equivalent continuous-model (b) and a simplified continuous model (c). The continuous model along with a simplifying assumption on the order book density functions (i.e., Assumption 1) keeps the problem tractable and focused on the multiasset aspect of the model.

Following Obizhaeva and Wang (2013), we assume that all assets in the portfolio have block-shaped order books with infinite depth and time-invariant steady-state densities.

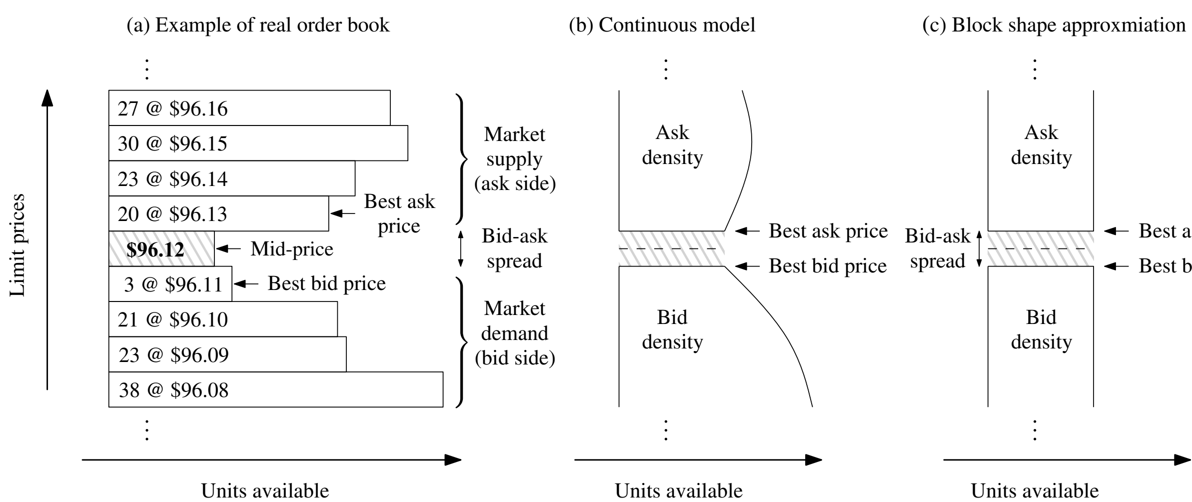
Assumption 1 (Order Book Shapes). *Letting q^a_i, q^b_i be constants, and denoting by $a_{i,n}$, $b_{i,n}$ the best available ask and bid prices in each order book at n, right before the trade arrives at n,*[8] *we have*

$$q^a_{i,n}(p) = q^a_i 1_{\{p \geq a_{i,n}\}} \quad \text{and} \quad q^b_{i,n}(p) = q^b_i 1_{\{p \leq b_{i,n}\}}, \quad i \in \{1, \ldots, m\}. \tag{3}$$

Figure 1(c) provides an illustration of this assumption.[9]

In addition to the shape of order book, we also need to specify the location of $a_{i,n}$ and $b_{i,n}$ and their evolution

Figure 1. Partial Snapshot of (a) an Order Book: One-Month Oil Futures Contracts as of November 8, 2011, at 11:10 A.M. (to Be Read as "Units Available" @ "Limit Price"), (b) an Equivalent Continuous Model Utilizing Density Functions, and (c) the Shape of the Order Book Following Assumption 1



over time. Two components are driving each asset's best bid and ask prices: its fundamental value and the price impact of trading. We will focus on the first component and return to the second later.

In absence of trading, the best bid and ask prices should be determined by the assets' fundamental values. We will assume these are given by a vector of random walks $\mathbf{u}_n$.

Assumption 2 (Random-Walk Fundamental Values). *Let $\boldsymbol{\epsilon}_n \sim N(\mathbf{0}, \tau\boldsymbol{\Sigma})$ be a vector of normal random variables with covariance $\tau\boldsymbol{\Sigma}$, and $\forall\, n \in \{1, \dots, N\}$, $\mathbb{E}[\epsilon_{i,n}\epsilon_{i,n-1}] = 0$ and $\mathbb{E}[\epsilon_{i,n}\epsilon_{j,n}] = \tau\sigma_{ij}$. We have*

$$\mathbf{u}_n = \mathbf{u}_{n-1} + \boldsymbol{\epsilon}_n, \quad \mathbf{u}_0 > 0, \tag{4}$$

with $\mathbb{E}[u_{i,n} \mid \mathcal{F}_{n-1}] = u_{i,n-1}$.[10]

The possibility of relaxing Assumption 2 is discussed in Section 3.3. Thus, we can express the best bid and ask prices, in the absence of the manager's trades, as follows:

$$a_{i,n} = u_{i,n} + \tfrac{1}{2}s_i, \quad b_{i,n} = u_{i,n} - \tfrac{1}{2}s_i, \quad \forall\, i, n. \tag{5}$$

Here, s_i gives the bid-ask spread of asset i in steady-state.

2.3. Order Book Dynamics

Next, we need to describe the evolution of $a_{i,n}$ and $b_{i,n}$ when the manager trades in the market, which affects the supply/demand dynamics of the order books. For this purpose we extend the single-asset, one-sided, order book model in Obizhaeva and Wang (2013) in two directions. First, we develop a single-asset, two-sided, order book model with coupled bid and ask sides (i.e., a trade in one direction will affect both sides of the order book) and bid-ask transaction costs. Second, we extend to allow multiple assets. We start with the two-asset case and show that interactions between assets justify the need for a dynamic two-sided order book model. We then provide the general multiasset case (m assets).

2.3.1. Single Asset. We break down the price-impact process into two phases: In phase 1, the manager submits an order which is executed against available inventory of orders, creating an immediate change in the limit order book. The order book updates itself and displaces the asset's mid-price accordingly, creating both a temporary price impact (TPI) and a permanent price impact (PPI). In phase 2, new limit orders arrive in the books, gradually absorbing the temporary price impact and collapsing the bid-ask spread toward its new steady state. We then describe how these dynamics could be affected in a two-sided model.

Consider a market order arriving at time n to buy $x = x^+_{i,n} > 0$ units in an arbitrary asset i.[11] Figure 2 shows possible dynamics that i can face after getting hit by the order.[12] At time $n-1$, we illustrate i in its steady state (see Figure 2(a)). At the next period in time n (see Figure 2(b)), the incoming order is executed against available inventory on the ask side of i's order book, starting from the best available price and rolling up i's supply curve toward less-favorable prices. This instantaneously drives i's best ask price from $a_{i,n}$ to $a^*_{i,n}$, where the superscript denotes the moment immediately following an executed order. This results in a displacement of $a^*_{i,n}(x) - a_{i,n}$. Given a density shape $q^a_i(p)$ the amount of units executed over a small increment in price is simply $dx = q^a_i(p)dp$. An executed buy order of size x therefore shifts the best ask price according to

$$\int_{a_{i,n}}^{a^*_{i,n}(x)} q^a_i(p)\,dp = x. \tag{6}$$

Combining the above expression with Assumption 1 we have the following Lemma.

Lemma 1 (Impact of Trading on the Order Book). *An incoming market order to buy (sell) x^+_{in} ($x^-_{i,n}$) shares at time n will instantaneously displace the ask (bid) price of asset i according to*

$$a^*_{i,n} = a_{i,n} + \frac{x^+_{i,n}}{q^a_i} \quad \textit{and} \quad b^*_{i,n} = b_{i,n} - \frac{x^-_{i,n}}{q^b_i}. \tag{7}$$

Clearly, the corresponding displacements in the best bid/ask limit order prices are linear in the order size: $a^*_{i,n} - a_{i,n} = x^+_{i,n}/q^a_i$ and $b^*_{i,n} - b_{i,n} = -x^-_{i,n}/q^b_i$.

The immediate cost the manager incurs in this phase can then simply be calculated by integrating the price over the total amount of units executed:

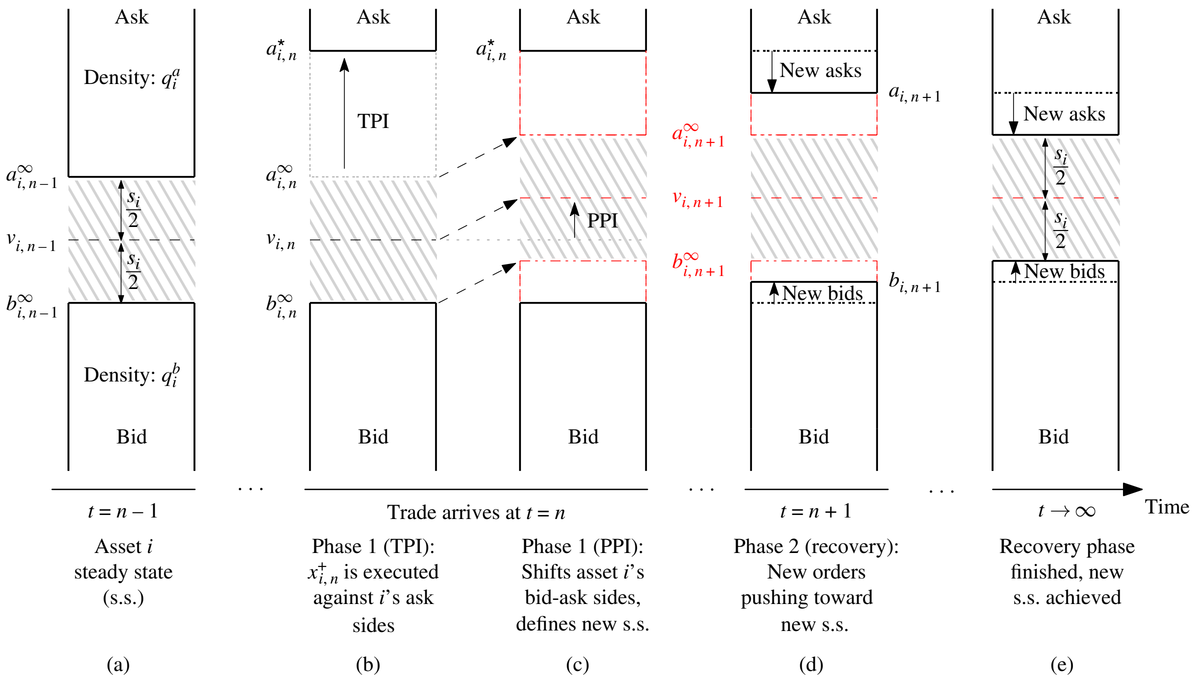
$$\int_0^x a^*_{i,n}(u)\,du.$$

Next, as shown in Figure 2(c), we assume that the current and future supply/demand will adjust accordingly. In particular, we assume that trading gives rise to a permanent impact on prices, which is proportional to the cumulative trade size.[13] To capture the permanent price impact, we introduce what we will call the "steady-state" mid-price $v_{i,n}$, ($i = 1, \dots, m$), before the trade arrives at n, which is given by

$$\begin{aligned} v_{i,n} &= v_{i,n-1} + \lambda_{ii}(x^+_{i,n-1} - x^-_{i,n-1}) + \epsilon_{i,n} \\ &= u_{i,n} + \lambda_{ii}\sum_{k=0}^{n-1}(x^+_{i,k} - x^-_{i,k}), \end{aligned} \tag{8}$$

where the second term gives the permanent price impact of trades up to and including the previous period $(n-1)$, and λ_{ii} is the permanent price impact for each unit of trading in asset i itself. Hence, if the manager doesn't submit any trades after n, the best ask/bid prices of asset i will eventually converge to $v_{i,n+1} + \frac{1}{2}s_i$ and $v_{i,n+1} - \frac{1}{2}s_i$, respectively. For convenience, we introduce the steady-state best ask/bid prices.

Figure 2. (Color online) Evolution of Asset i's Order Book, After Being Hit by a Single Buy Order of Size x_{in}^+ at Time n



Assumption 3 (Steady-State Prices). *Asset i's best ask and bid prices have steady-state levels, before the trade arrives at n, which are given by*

$$a_{i,n}^\infty = v_{i,n} + \tfrac{1}{2}s_i, \quad b_{i,n}^\infty = v_{i,n} - \tfrac{1}{2}s_i, \tag{9}$$

where the steady-state mid-price is given by Equation (8).

The best available ask and bid prices may generally differ from their steady-state levels.

After the order is executed at n, the replenishment process (phase 2) begins (see Figure 2(d) for an illustration). During this phase, supply/demand is replenished as new limit orders arrive to refill the order books. Replenishment is spread over time, and the order book may remain in transitory state over an extended period of time. In the absence of any new market orders after n, the new limit orders will gradually push the best bid/ask prices toward their new steady states $a_{i,n+1}^\infty$ and $b_{i,n+1}^\infty$. The rate at which this happens depends on the dislocation size, the inherent properties of the asset and the behavior of market participants.

We follow Obizhaeva and Wang (2013) in describing the order book replenishment process. For convenience, we define the order book displacement functions to keep track of the difference between the best ask and bid prices and their steady-state levels, that is,

$$d_{i,n}^a = a_{i,n} - a_{i,n}^\infty, \quad d_{i,n}^b = b_{i,n}^\infty - b_{i,n}. \tag{10}$$

The order book replenishment process is given as follows.

Assumption 4 (Order Book Replenishment). *The limit order demand and supply are replenished exponentially, with constant decay parameters ρ_i^a and ρ_i^b, for the ask and bid prices, respectively. Specifically, over period τ, the order book displacements are given by*

$$d_{i,n+1}^a = e^{-\rho_i^a\tau}\left(d_{i,n}^a + \left[\frac{x_{i,n}^+}{q_i^a} - \lambda_{ii}(x_{i,n}^+ - x_{i,n}^-)\right]\right), \tag{11a}$$

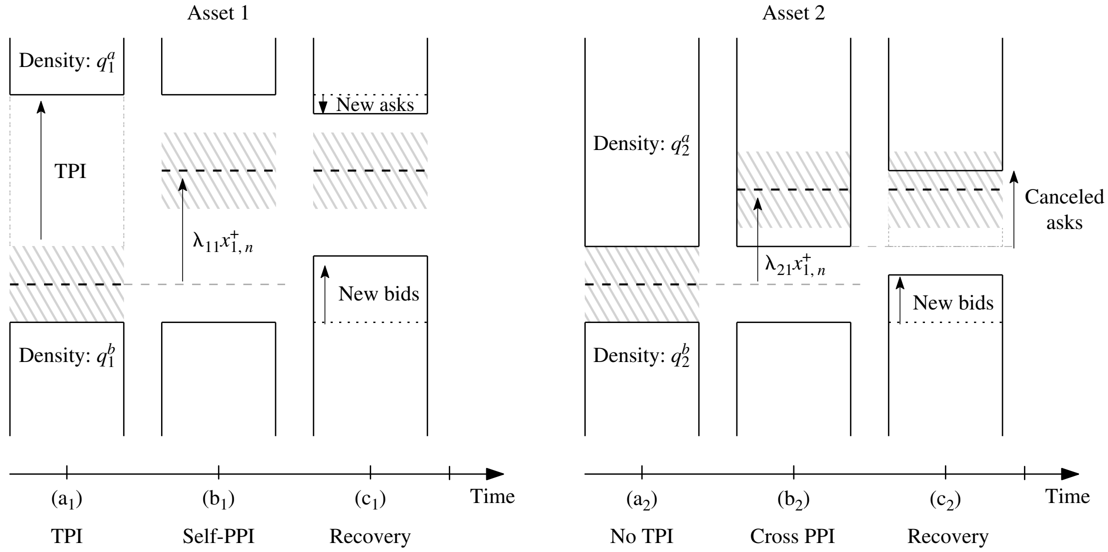
$$d_{i,n+1}^b = e^{-\rho_i^b\tau}\left(d_{i,n}^b + \left[\frac{x_{i,n}^-}{q_i^b} + \lambda_{ii}(x_{i,n}^+ - x_{i,n}^-)\right]\right). \tag{11b}$$

Clearly, as ρ_i^a and $\rho_i^b \to \infty$, the asset is highly liquid the displacements are null, and the order books are replenished instantaneously after each trade. As ρ_i^a and $\rho_i^b \to 0$, the asset is highly illiquid no new limit orders arrive, and the displacements are permanent (i.e., they do not decay over time).[14]

From the order book replenishment process described in Equation (11) and the steady-state bid/ask prices in Equation (9), the dynamics of the best bid and ask prices at any time are simply given by Equation (10).

2.3.2. Two Assets. Adding a second asset to the problem introduces several new features. We need to take into account the correlation between the stochastic processes driving the mid-prices but also the cross impact that a trade in one asset can have on the supply/demand curves of the other. These two features are distinct. Correlation is exogenous, whereas cross impact is a direct result of the manager's action. Although the former is straightforward, we provide an example of the latter in Figure 3.

Figure 3. Dynamics of a Two-Asset Portfolio in Transient Regime (Nonsteady State) After Getting Hit by an Incoming Buy Order in Asset 1: $\{x^+_{1,n} > 0, x^+_{2,n} = 0\}$



Note. Executing the order leads to a PPI on asset 2 given by $\lambda_{21}x^+_{1,n}$ and to a subsequent response in its supply/demand curves.

Consider a portfolio comprised of two assets, and an incoming order to buy $x^+_{1,n}$ shares in the first asset—the second asset being "inactive." Let $\lambda_{21} > 0$ be the cross impact parameter of asset 1 on asset 2. We illustrate how the buy order affects the mid-price of the inactive asset via the term $\lambda_{21}x^+_{1,n}$, as shown in Figure 3(b_2). Given the resultant price change, the portfolio value could be significantly affected. Furthermore, the cross impact will have a secondary effect on the supply/demand curves of the inactive asset. As is shown in Figure 3(c_2), the change in the second asset's mid-price defines a new steady state, initiating a response in the bid/ask books. Specifically, new buy orders arrive to replenish demand, while existing ask orders are canceled as prices converge toward the new steady states. Thus, if any orders are later submitted in the inactive asset, these would be executed at prices which could diverge from the initial state. This effect is further exacerbated as the number of assets in the portfolio increases, since a trade in one could affect the prices of all others. Section 4 provides a numerical study.

Analytically, for both assets, $i = 1, 2$, the steady-state mid-prices and best bid/ask prices are still given by Equations (8) and (9), with only the following modification required on the steady-state mid-prices to incorporate the effect of cross asset price impact.

Assumption 5 (Cross Asset Price Impact). *When there is trading in both assets, the steady-state mid-price remains linear in the trade size and is given by*

$$v_{i,n} = u_{i,n} + \sum_{j=1,2} \lambda_{ij} \sum_{k=1}^{n} (x^+_{j,k-1} - x^-_{j,k-1}), \quad i = 1, 2. \tag{12}$$

The order book replenishment dynamics for both assets are still given by Equation (11) with only a slight modification required to adjust the permanent price impact term for both ask and bid sides:

$$d^k_{i,n+1} = e^{-\rho^k_i \tau}\left(d^k_{i,n} + \left[\frac{x^{\pm}_{i,n}}{q^k_i} \mp \sum_{j=1,2} \lambda_{ij}(x^+_{j,n} - x^-_{j,n})\right]\right), \quad k = a, b.$$

2.3.3. Multiple Assets. Once the two-asset case is understood, the generalization to the m-asset case is straightforward. For ease in exposition, we adopt simple vector notations. Let $\mathbf{a}_n$ and $\mathbf{b}_n$ denote the vector of ask and bid prices for the m assets in period n, $n = 0, 1, \ldots, T$, $\mathbf{u}_n$ the vector of their fundamental values, $\mathbf{s}$ the vector of steady-state bid-ask spread, $\mathbf{z}_n$ the vector of positions in the m assets, $\mathbf{d}^a_n$ and $\mathbf{d}^b_n$ the vectors of order book displacements from steady-state, and $\boldsymbol{\Lambda}$ the matrix of permanent price impact coefficients:

$$\mathbf{a}_n = \begin{bmatrix} a_{1,n} \\ \vdots \\ a_{m,n} \end{bmatrix}, \ \mathbf{b}_n = \begin{bmatrix} b_{1,n} \\ \vdots \\ b_{m,n} \end{bmatrix}, \ \mathbf{u}_n = \begin{bmatrix} u_{1,n} \\ \vdots \\ u_{m,n} \end{bmatrix}, \ \mathbf{s}_0 = \begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix}, \tag{13a}$$

$$\mathbf{z}_n = \begin{bmatrix} z_{1,n} \\ \vdots \\ z_{m,n} \end{bmatrix}, \mathbf{d}^a_n = \begin{bmatrix} d^a_{1,n} \\ \vdots \\ d^a_{m,n} \end{bmatrix}, \mathbf{d}^b_n = \begin{bmatrix} d^b_{1,n} \\ \vdots \\ d^b_{m,n} \end{bmatrix}, \boldsymbol{\Lambda} = \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1m} \\ \vdots & \ddots & \vdots \\ \lambda_{m1} & \cdots & \lambda_{mm} \end{bmatrix}. \tag{13b}$$

Furthermore, let $\mathbf{I}_m$ be the identity matrix of order m and

$$\mathbf{e}^{-\rho^k \tau} = \begin{bmatrix} e^{-\rho^k_1 \tau} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{-\rho^k_m \tau} \end{bmatrix}, \ \mathbf{Q}^k = \begin{bmatrix} \frac{1}{2q^k_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{2q^k_m} \end{bmatrix}, \quad k = a, b \tag{14a}$$

$$\boldsymbol{\delta}_a = [\mathbf{I}_m; \mathbf{0}], \quad \boldsymbol{\delta}_b = [\mathbf{0}; -\mathbf{I}_m], \tag{14b}$$

$$\boldsymbol{\kappa}^a = 2\mathbf{Q}^a \boldsymbol{\delta}'_a - \boldsymbol{\Lambda}\boldsymbol{\delta}', \quad \boldsymbol{\kappa}^b = 2\mathbf{Q}^b \boldsymbol{\delta}'_b + \boldsymbol{\Lambda}\boldsymbol{\delta}'. \tag{14c}$$

Then, $\boldsymbol{\delta}'_a \mathbf{x}_{n-1} = \mathbf{x}^+_{n-1}$ and $\boldsymbol{\delta}'_b \mathbf{x}_{n-1} = \mathbf{x}^-_{n-1}$. The assets' best ask and bid prices are given as follows.

Lemma 2 (Bid/Ask Price Processes). *Following Assumptions 1–5 and Lemma 1, the best bid/ask prices available in the order books at time n, are given by*

$$\mathbf{a}_n = \mathbf{u}_n + \tfrac{1}{2}\mathbf{s} + \boldsymbol{\Lambda}(\mathbf{z}_0 - \mathbf{z}_n) + \mathbf{d}^a_n, \tag{15a}$$

$$\mathbf{b}_n = \mathbf{u}_n - \tfrac{1}{2}\mathbf{s} + \boldsymbol{\Lambda}(\mathbf{z}_0 - \mathbf{z}_n) - \mathbf{d}^b_n. \tag{15b}$$

where the state vector $\mathbf{z}_n$ is defined in (1) and the vectors $\mathbf{d}^a_n$ and $\mathbf{d}^b_n$, which keep track of the replenishment process for the ask and bid sides, are given by

$$\mathbf{d}^a_n = \mathbf{e}^{-\rho^a \tau}(\mathbf{d}^a_{n-1} + \boldsymbol{\kappa}^a \mathbf{x}_{n-1}), \tag{16a}$$

$$\mathbf{d}^b_n = \mathbf{e}^{-\rho^b \tau}(\mathbf{d}^b_{n-1} + \boldsymbol{\kappa}^b \mathbf{x}_{n-1}). \tag{16b}$$

3. Optimal Execution Problem

3.1. Dynamic Program

Having detailed the liquidity model in Section 2, the next step is to derive the manager's execution costs, as a function of his trading strategy. Using Lemma 2, we can calculate the total costs and revenues resulting from an order $\mathbf{x}_n$ submitted at time n.

Lemma 3 (Costs and Revenues). *An incoming order to execute $\mathbf{x}_n$ shares at time n will have associated total costs (c_n) and revenues (r_n), given by*

$$c_n = \mathbf{x}^{+\prime}_n(\mathbf{a}_n + \mathbf{Q}^a \mathbf{x}^+_n), \quad r_n = \mathbf{x}^{-\prime}_n(\mathbf{b}_n - \mathbf{Q}^b \mathbf{x}^-_n).$$

Let π_n be the manager's reward function at n which can be written as the difference between his total revenues (from his selling orders) and his total costs (from his purchasing orders). It follows that

$$\pi_n = r_n - c_n = \mathbf{x}^{-\prime}_n(\mathbf{b}_n - \mathbf{Q}^b \mathbf{x}^-_n) - \mathbf{x}^{+\prime}_n(\mathbf{a}_n + \mathbf{Q}^a \mathbf{x}^+_n). \tag{17}$$

The manager's total terminal wealth is thus

$$W_N = \sum_{n=0}^{N} \pi_n,$$

which can recursively be written as

$$\begin{aligned} W_n &= W_{n-1} + \pi_n \\ &= W_{n-1} + \mathbf{x}^{-\prime}_n(\mathbf{b}_n - \mathbf{Q}^b \mathbf{x}^-_n) - \mathbf{x}^{+\prime}_n(\mathbf{a}_n + \mathbf{Q}^a \mathbf{x}^+_n). \end{aligned} \tag{18}$$

For convenience, it is helpful to restate the wealth function in a more compact form. This requires some additional notation. Let the $(3m+1) \times 1$ vector $\mathbf{y}_n$ represent the aggregate state of the system at time n, excluding the random walk:

$$\mathbf{y}_n = \begin{bmatrix} 1 \\ \mathbf{z}_n \\ \mathbf{d}_n \end{bmatrix}, \quad \text{with } \mathbf{d}_n = \begin{bmatrix} \mathbf{d}^a_n \\ \mathbf{d}^b_n \end{bmatrix}.$$

Using this notation, the state dynamics from (1) and the order books (16) can be aggregated as follows:

$$\mathbf{y}_{n+1} = \mathbf{A}\mathbf{y}_n + \mathbf{B}\mathbf{x}_n,$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}_{m+1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{e}^{-\rho^a \tau} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{e}^{-\rho^b \tau} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ -\boldsymbol{\delta}' \\ \mathbf{e}^{-\rho^a \tau}\boldsymbol{\kappa}^a \\ \mathbf{e}^{-\rho^b \tau}\boldsymbol{\kappa}^b \end{bmatrix}, \tag{19}$$

where $\mathbf{I}_{m+1}$ is the identity matrix of size $m+1$, and $\mathbf{e}^{-\rho^{a,b}\tau}$ and $\boldsymbol{\kappa}^{a,b}$ are given in (14). Using this compact notation and the bid and ask expressions in (15a) and (15b), the wealth dynamics in (18) (after some algebra) can be simplified to the following:

$$\begin{aligned} W_{n+1} = W_n &- (\mathbf{u}'_{n+1}\boldsymbol{\delta}' + \mathbf{z}'_0 \boldsymbol{\Lambda}\boldsymbol{\delta}' + \mathbf{y}'_{n+1}\mathbf{N})\mathbf{x}_{n+1} \\ &- \mathbf{x}'_{n+1}\mathbf{Q}\mathbf{x}_{n+1}, \end{aligned} \tag{20}$$

where $\mathbf{Q}$ and $\mathbf{N}$ are constant matrices that contain the problem parameters:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}^a & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}^b \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} \frac{1}{2}\mathbf{s}'\boldsymbol{\iota}' \\ -\boldsymbol{\Lambda}\boldsymbol{\delta}' \\ \mathbf{I}_{2m} \end{bmatrix}, \quad \text{with } \boldsymbol{\iota} = \begin{bmatrix} \mathbf{I}_m \\ \mathbf{I}_m \end{bmatrix}.$$

Having defined the wealth function in (20) at each time step, we can formulate the manager's DP. To capture the trade-off between liquidity and risk, we will assume an exponential utility function with risk-aversion coefficient α, over the manager's total terminal wealth. This choice is motivated by several factors: First, it allows us to focus exclusively on the utility derived from execution, regardless of the manager's initial wealth—a well-known property of constant absolute risk aversion (CARA) utility functions. Second, in our framework, the exponential objective is equivalent to a mean-variance objective—a common modeling choice in the existing portfolio management and price impact literature. Lastly, this form leads to a tractable optimization problem which can be solved in polynomial time (see Section 3.2). Then, letting $J_n(\cdot)$ be the value function at time n, the manager's dynamic program $\forall\, n \in \{0, \ldots, N\}$, is given by

$$J_n = \underset{\mathbf{x}_n \geq 0}{\text{maximize}}\ \mathbb{E}_n[J_{n+1}] \tag{21}$$

$$\begin{aligned} \text{s.t. } & \boldsymbol{\delta}'\mathbf{x}_N = \mathbf{z}_N \quad \text{(terminal trade)}, \\ & J_{N+1} = -e^{-\alpha W_N}, \quad \text{(terminal value function)}, \\ \text{with state dynamics:} & \\ & W_{n+1} = W_n - (\mathbf{u}'_{n+1}\boldsymbol{\delta}' + \mathbf{z}'_0\boldsymbol{\Lambda}\boldsymbol{\delta}' + \mathbf{y}'_{n+1}\mathbf{N})\mathbf{x}_{n+1} - \mathbf{x}'_{n+1}\mathbf{Q}\mathbf{x}_{n+1}, \\ & \mathbf{y}_{n+1} = \mathbf{A}\mathbf{y}_n + \mathbf{B}\mathbf{x}_n, \\ & \mathbf{u}_{n+1} = \mathbf{u}_n + \boldsymbol{\epsilon}_{n+1}. \end{aligned}$$

Here, $\mathbb{E}_n$ denotes the conditional expectation given $\mathcal{F}_n$. The initial conditions are W_{0^-} (initial wealth), $\mathbf{u}_0, \mathbf{y}_0$ (specified by the user), in which without loss, we assume $\mathbf{d}_0 = \mathbf{0}$ (i.e., we assume the order books are initially in their steady states). To ensure the dynamic problem above is well defined we impose sufficient concavity conditions on the terminal value function W_N.

To this end, we first reformulate W_N to express it simply as a function of the trades $\mathbf{x}_0, \ldots, \mathbf{x}_N$. We do this in two steps: First, notice from the recursive equation (20), the terminal wealth function can be written as

$$W_N = W_{0^-} - \mathbf{u}'\bar{\boldsymbol{\delta}}'\mathbf{x} - \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_0 - \mathbf{y}'\bar{\mathbf{N}}\mathbf{x} - \mathbf{x}'\bar{\mathbf{Q}}\mathbf{x}, \tag{22}$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_0 \\ \vdots \\ \mathbf{x}_N \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \mathbf{u}_0 \\ \vdots \\ \mathbf{u}_N \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}_0 \\ \vdots \\ \mathbf{y}_N \end{bmatrix}, \quad \bar{\boldsymbol{\delta}} = \begin{bmatrix} \boldsymbol{\delta} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \boldsymbol{\delta} \end{bmatrix},$$
$$\bar{\mathbf{N}} = \begin{bmatrix} \mathbf{N} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{N} \end{bmatrix}, \quad \bar{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{Q} \end{bmatrix}, \tag{23}$$

and the term $\mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_0$ is a constant which comes from $\sum_n \mathbf{z}_0'\boldsymbol{\Lambda}\boldsymbol{\delta}'\mathbf{x}_n$ with $\sum_n \boldsymbol{\delta}'\mathbf{x}_n = \mathbf{z}_0$. Second, notice that we can use the linear state dynamics of $\mathbf{y}_n$ (which tie it to $\mathbf{y}_{n-1}, \mathbf{x}_{n-1}$) to eliminate intermediate states by substitution, that is,

$$\mathbf{y} = \bar{\mathbf{A}}\mathbf{y}_0 + \bar{\mathbf{B}}\mathbf{x}, \quad \bar{\mathbf{A}} = [\mathbf{I} \;\; \mathbf{A} \;\; \cdots \;\; \mathbf{A}^N]^T,$$
$$\bar{\mathbf{B}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{AB} & \mathbf{B} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{A}^{N-1}\mathbf{B} & \mathbf{A}^{N-2}\mathbf{B} & \cdots & \mathbf{B} & \mathbf{0} \end{bmatrix}. \tag{24}$$

Substituting this back into the wealth function, we obtain a *stochastic quadratic function* of $\mathbf{x}$:

$$\begin{aligned} W_N &= W_{0^-} - \mathbf{u}'\bar{\boldsymbol{\delta}}'\mathbf{x} - \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_0 - (\bar{\mathbf{A}}\mathbf{y}_0 + \bar{\mathbf{B}}\mathbf{x})'\bar{\mathbf{N}}\mathbf{x} - \mathbf{x}'\bar{\mathbf{Q}}\mathbf{x} \\ &= W_{0^-} - \mathbf{u}'\bar{\boldsymbol{\delta}}'\mathbf{x} - \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_0 - (\mathbf{y}_0'\bar{\mathbf{A}}'\bar{\mathbf{N}})\mathbf{x} - \mathbf{x}'(\bar{\mathbf{B}}'\bar{\mathbf{N}} + \bar{\mathbf{Q}})\mathbf{x}. \end{aligned} \tag{25}$$

Let $\mathbf{D}$ be the symmetric matrix characterizing the quadratic form $\mathbf{x}'(\bar{\mathbf{B}}'\bar{\mathbf{N}} + \bar{\mathbf{Q}})\mathbf{x}$. We have

$$\mathbf{D} \stackrel{\text{def}}{=} \tfrac{1}{2}\big((\bar{\mathbf{B}}'\bar{\mathbf{N}} + \bar{\mathbf{Q}}) + (\bar{\mathbf{B}}'\bar{\mathbf{N}} + \bar{\mathbf{Q}})'\big) = \tfrac{1}{2}(\bar{\mathbf{B}}'\bar{\mathbf{N}} + \bar{\mathbf{N}}'\bar{\mathbf{B}}) + \bar{\mathbf{Q}}. \tag{26}$$

We then have the following result.

Lemma 4 (Concavity Condition). *Suppose* $\mathbf{D}$, *defined in* (26), *is positive semidefinite* (*positive definite*), *then the terminal wealth function* W_N *is* (*strictly*) *jointly concave in* $\mathbf{x}_0, \ldots, \mathbf{x}_N$. *It follows that the dynamic problem* (21) *is* (*strictly*) *concave.*

The proof is provided in Appendix A.3. Here and below, all results are subject to the concavity condition stated above.

In Appendix A.4, we show that the optimal policy which solves the problem (21) is path independent, that is, it does not depend on any $\mathbf{u}_n$. This statement is formalized in Proposition 1.

Proposition 1 (Deterministic Property of Optimal Policy). *The optimal trading policy* $\mathbf{x}_0^*, \ldots, \mathbf{x}_N^*$ *which solves the problem* (21), *is path independent, that is, it does not depend on the random walk* $\mathbf{u}_0 \ldots, \mathbf{u}_N$.

The proof is provided in Appendix A.4 and relies on three critical modeling features: the exponential utility adopted, the random walk in Assumption 2, and a "separability" property of the wealth function W_n, in which the random walk appears as an additive term. Proposition 1 does not describe how an optimal policy can be obtained, yet it suffices to establish an equivalence between dynamic and static problem formulations given in Section 3.2. Nonetheless before proceeding, we can provide more detail about the shape of the optimal DP policy.

Proposition 2 (Shape of Optimal Policy and Value Function). *The optimal trade at any time* n, $\mathbf{x}_n^*$, *is a continuous, piecewise affine function of the deterministic state variables of the problem,* $\mathbf{z}_n$ *and* $\mathbf{d}_n$ *contained in* $\mathbf{y}_n$. *The optimal policy can be written as*

$$\mathbf{x}_n^*(\mathbf{y}_n) = \mathbf{K}_n\mathbf{y}_n, \quad n \in \{0, \ldots, N\}, \tag{27}$$

where $\mathbf{K}_n = \mathbf{K}_n(\mathbf{y}_n)$ *is piecewise constant in* $\mathbf{y}_n$. *Furthermore, the value function takes the form*:

$$\begin{aligned} &J_n(W_{n-1}, \mathbf{y}_n, \mathbf{u}_n) \\ &\quad = -\exp\big[-\alpha(W_{n-1} - \mathbf{u}_n'\mathbf{z}_n - \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_n - \mathbf{y}_n'\hat{\mathbf{M}}_n\mathbf{y}_n)\big], \\ &\qquad\qquad n \in \{0, \ldots, N\}, \end{aligned} \tag{28}$$

where $\hat{\mathbf{M}}_n = \hat{\mathbf{M}}_n(\mathbf{y}_n)$ *is piecewise constant in* $\mathbf{y}_n$.

The proof is provided in Appendix A.4.

3.2. Equivalent Static Quadratic Program

Though Proposition 2 provides the structure of the optimal policy, directly solving the DP can be computationally tedious because of the positivity constraints on the controls, which lead to piecewise-defined value functions. On the other hand, the path-independence property of the optimal policy described in Proposition 1 allows us to reformulate the problem (21) into a static one without loss of optimality. This type of reformulation is often referred to as a "batch approach" in the predictive control literature (e.g., Borrelli et al. 2017, section 8.2). The batch approach can be more convenient when inequalities are present on the controls as it effectively bypasses the need to compute the value function (see discussion in Borrelli et al. 2017, section 8.4). This reformulation leads to the following result.

Proposition 3 (Quadratic Program). *The dynamic portfolio execution problem* (21) *is equivalent to the following* *static quadratic program which minimizes risk-adjusted execution shortfall*:

$$\begin{aligned} \underset{\mathbf{x}\geq 0}{\text{minimize}} \quad & \tfrac{1}{2}\mathbf{x}'\bar{\mathbf{D}}\mathbf{x} + (\mathbf{y}_0'\bar{\mathbf{A}}'\bar{\mathbf{N}})\mathbf{x} + \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_0 \\ \text{s.t.} \quad & \bar{\mathbf{I}}\bar{\delta}'\mathbf{x} = \mathbf{z}_0 \quad (\textit{total shares executed}) \end{aligned} \tag{29}$$

where $\bar{\mathbf{I}} = [\mathbf{I}_m, \ldots, \mathbf{I}_m]$ *is a collection of* $N+1$ *identity matrices, each of size* m, *and* $\bar{\mathbf{D}} = 2\mathbf{D} + \alpha\bar{\delta}\boldsymbol{\Sigma}_{\mathbf{u}}\bar{\delta}'$, *with* $\boldsymbol{\Sigma}_{\mathbf{u}}$ *being the covariance matrix of* $\mathbf{u}$.

The proof is provided in Appendix A.5.

The sufficient condition in Lemma 4 requires that $\bar{\mathbf{D}}$ is positive (semi)definite, which is also the sufficient convexity conditions for the QP.[15] For all the numerical examples in the paper, we verified positive definiteness of $\bar{\mathbf{D}}$.

3.3. Discussion

We compare the static optimal policy described in Proposition 1 to other types of policies found in the literature: Bertsimas et al. (1999) develop a static approximation algorithm, allowing the manager to reoptimize his objective at every period, and show that their solution is close to optimal. Basak and Chabakauri (2010) compare static precommitment strategies with "adaptive" strategies in the context of the portfolio composition problem and argue that the manager can be better off by precommitting in certain cases. In contrast, Lorenz and Almgren (2012) develop an adaptive execution model and show that the gain in trading flexibility can indeed be valuable for the manager.

In our framework, a static solution is optimal without exogenously enforcing precommitment—a result that is sensitive to the random walk assumption, but which significantly simplifies the problem. Intuitively, this result states that the generated filtration provides no useful information for the optimal policy in our framework. This implies that the manager has nothing to gain by utilizing path dependent trading strategies in the CARA framework, under the random walk assumption. Alfonsi et al. (2008) develop a comparable static solution methodology in the context of an optimal liquidation problem for a single asset and a risk-neutral investor. Similarly, Huberman and Stanzl (2005) find a comparable static solution in their framework with a mean-variance objective.

Our formulation can be extended to include additional deterministic linear or quadratic constraints one may want to impose on the set of feasible strategies. This feature is of consequence to practitioners. For instance, in many large-scale portfolio execution programs, managers may want to exercise particular control over certain assets. We provide an example in Section 5.2. Further, the model can easily incorporate agency trading constraints which some execution desks may face when trading on behalf of their clients. For example, an execution desk liquidating an agency position may not be allowed to trade counterdirectionally and conduct any purchasing orders. This constraint could be captured in our model by setting $\mathbf{x}^+ = 0$. A more detailed discussion on agency trading can be found in Moallemi and Sağlam (2017).

Our formulation can also handle deterministic time-dependent parameters (relaxing the Assumptions 1, 3, and 4). Time dependence can be critical in many situations, for instance, when markets are in turmoil and liquidity variations are expected to occur in the future (see Brown et al. 2010 for a detailed treatment with uncertain liquidity shocks). In our framework, expected liquidity variations during the execution window could be integrated into the model by adjusting the values of the density q, the replenishment rate ρ and the steady-state bid-ask spread s, at the desired periods. Similarly, one could capture expected intraday fluctuations in volume of trade (thus accounting for the well-known intraday "smile" effect). Details are provided in Appendix A.1.

The liquidity model in Section 2 can capture various forms of transaction costs observed in the market, including fixed, proportional and quadratic costs. The proportional (linear) trading costs are captured by the constant bid-ask spread s_i. The quadratic trading costs are captured by the linear price impact assumed in the liquidity model. The fixed trading costs are not directly modeled but reflected implicitly in our setting. In particular, we assume a finite number of trading periods in part to reflect the fixed cost in trading. Presumably, the number of trading periods N is connected to the fixed cost. Although in our model N is taken as given, we can easily endogenize it as an optimal choice in the presence of fixed trading costs at say c_0. Clearly, larger N would decrease execution costs by allowing the manager more flexibility in spreading trades. But it would also increase total fixed costs, which would be Nc_0. An optimal choice of N will result from this trade-off. See, for example, He and Mamaysky (2005) for a more detailed discussion on this issue.

4. Optimal Execution Policy

This section presents several case studies which illustrate the solution to problem (29). We highlight cases where advanced execution strategies are optimal. These strategies constructively utilize order book cross elasticities to improve execution efficiency. In what follows, we set the steady-state bid-ask spread to zero to simplify the exposition.[16] Furthermore, we only consider the problem of liquidating assets. The asset purchasing problem is fully equivalent (by interchanging "buy" and "sell" labels). The model can also treat mixed buy and sell objectives without any modifications.

4.1. Base Case (No Correlation in Asset Fundamentals, No Cross Impact in Liquidity)

Our base case consists of a portfolio with two identical assets, but with no correlation in their risks ($\gamma = 0$) or cross impact ($\lambda_{12} = \lambda_{21} = 0$) in their liquidity. The manager needs to liquidate his position in the first asset, but has no initial and final position, or predefined objective in the second. We refer to the first asset as the *active* asset (with boundary conditions $z_{1,0} \neq 0$ and $z_{1,N} = 0$), whereas the second is *inactive* (with boundary conditions $z_{2,0} = z_{2,N} = 0$). Consider a long position in the active asset, consisting of $z_{1,0} = 100$ shares that need to be liquidated over $N = 100$ periods (i.e., $z_{1,100} = 0$). The horizon $T = 1$ day. The mid-price is $v_{1,0} = \$1$ at time 0, implying a preliquidation market value of \$100.[17]

Figure 4 displays the manager's optimal execution policy (OEP), in the form of his net position over time, comparing the risk-neutral (RN) case to the risk-averse (RA) case. Unsurprisingly, in the absence of correlation and cross impact between the two assets, it is never optimal to trade the inactive asset (dashed line). Doing so, would increase overall execution costs without any risk reduction. It is useful to provide some intuition on the resultant OEP of the active asset (solid line).

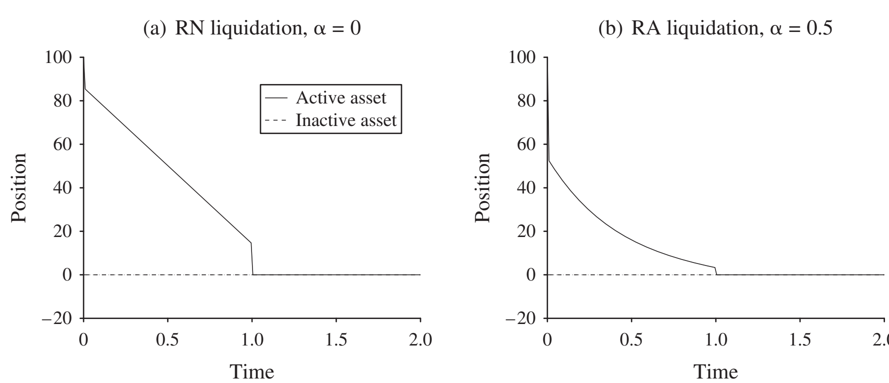
In the RN case (Figure 4(a)), the OEP consists of placing two large orders at times 0 and N, and splitting the rest of the order evenly across time. The slope of the execution curve represents the manager's trading rate. The steeper the slope, the faster he is executing shares. The slope is related to the order book replenishment process. The faster the order book inventory gets replenished after each executed order, the more sell orders the manager can submit per unit time. The liquidity spikes on the boundaries are related to the replenishment and boundary conditions of the order book. Assuming that the order books are initially "full," it is natural that the first order should be large. In essence, one can obtain "cheaper" liquidity at the start. Similarly, the last order also should be large because one cannot constructively utilize order book dynamics after the execution horizon N. These spikes fade as the inventory recovery rate increases and disappear at the limit when liquidity is infinite and inventories are instantaneously replenished after each executed order, $\rho \to \infty$ (we omit the plot).

In the RA case (Figure 4(b)), the manager consumes greater liquidity early on in the liquidation process. This dampens the adverse impact of future price uncertainty, reducing execution risk, but at a cost. To understand this result, it is helpful to consider extreme values of α. When $\alpha \to \infty$, the manager is only sensitive to the variance of his costs and the solution is trivial: execute everything at time zero (we omit the plot). This strategy is effectively risk free, as it guarantees zero standard deviation in execution costs. But it also understandably the worst-case scenario from a cost perspective.

The impact the OEP has on the expected ask and bid prices of the active asset is shown in Figure 5. The ask and bid prices are initially equal to \$1 before the first sell order is placed. The liquidation process pushes the ask and bid prices of the active asset down over time. As there is no correlation or cross impact between assets, the price of the inactive asset remains unaffected at \$1 (we omit the plot). At any fixed time t, the gap between the ask and bid prices defines the instantaneous bid-ask spread, the dynamics of which depend on two opposing forces: on the one hand, each executed order widens the bid-ask spread (as the manager is consuming liquidity in the order books). On the other hand, new limit orders arrive over time collapsing the bid-ask spread back toward its steady state. The mid-point of the bid-ask spread is the instantaneous mid-price (this is not shown on the plot).

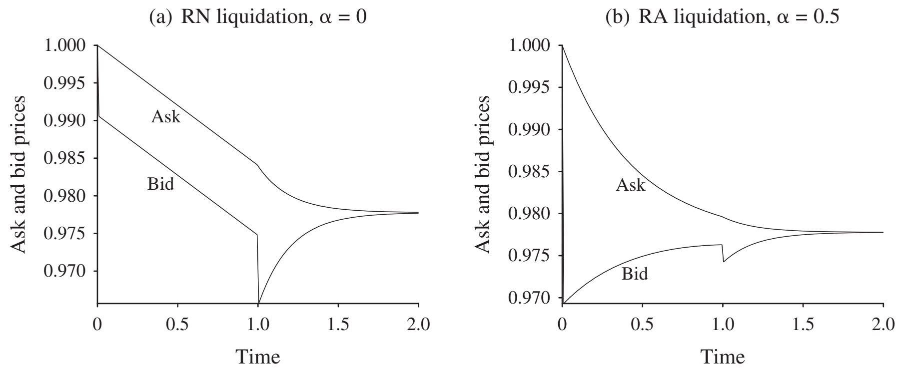
The trade sizes and utility implications of the aforementioned strategies are reported in cases 1 and 5, of Table 1. The table shows traded volume (in shares), the expected execution costs, the execution risk (stated in terms of the standard deviation of the execution costs), the manager's execution certainty equivalent, and a

Figure 4. Optimal Execution Policies in the Base Case



Notes. Correlation ($\gamma = 0$) and cross impact ($\lambda_{ij} = 0$) are turned off, implying that it is never optimal to trade in the inactive asset. Other parameter values include $\sigma_1 = \sigma_2 = 0.05$, $q_1 = q_2 = 1{,}500$, $\rho_1 = \rho_2 = 5$, and $\lambda_{11} = \lambda_{22} = 1/(3q_1)$.

Figure 5. Dynamics of the Expected Ask and Bid Prices of the Active Asset, Responding to the Manager's Execution Policy



Notes. Both prices are initially equal to \$1 at time 0. Correlation ($\gamma = 0$) and cross impact ($\lambda_{ij} = 0$) are turned off, implying that it is never optimal to trade in the inactive asset. Prices are plotted beyond $T = 1$ to illustrate the convergence process toward a new steady state.

measure of execution efficiency which we refer to as the execution Sharpe ratio. The latter is defined as the ratio of the cost savings achieved over the most costly, risk-free, execution strategy ($\alpha \to \infty$), divided by the standard deviation of those costs. The higher the execution Sharpe ratio, the more efficient the execution.

4.2. Effect of Correlation in Risk

Next, we build on the base case by introducing correlation between the two assets, while maintaining cross impact at zero (see Table 1, case 2).

Figure 6 compares the impact of correlation ($\gamma = 0.7$) between the RN and RA cases. Unsurprisingly, if the manager is RN ($\alpha = 0$, Figure 6(a)), there are no trades in the inactive asset. On the other hand, risk aversion ($\alpha = 0.5$, Figure 6(b)), combined with correlation, leads to a complex strategy in the inactive asset. In particular, it becomes optimal to (1) go short the inactive asset at time zero, (2) hold the short position for some time, and (3) start covering the short at variable rates toward the end of the execution window, to satisfy the boundary conditions. The resultant asset price dynamics for the inactive asset are shown in Figure 7, whereas the number of shares traded with this strategy are reported in case 2 of Table 1.

The results in Table 1 show that on the one hand, the RA strategy trades off higher execution costs for risk reduction, compared to the RN case. In other words, to reduce execution risk, one generally has to be willing to incur higher expected execution costs. On the other hand, the RA strategy has a higher execution Sharpe ratio compared to the RN strategy, implying more efficient execution.

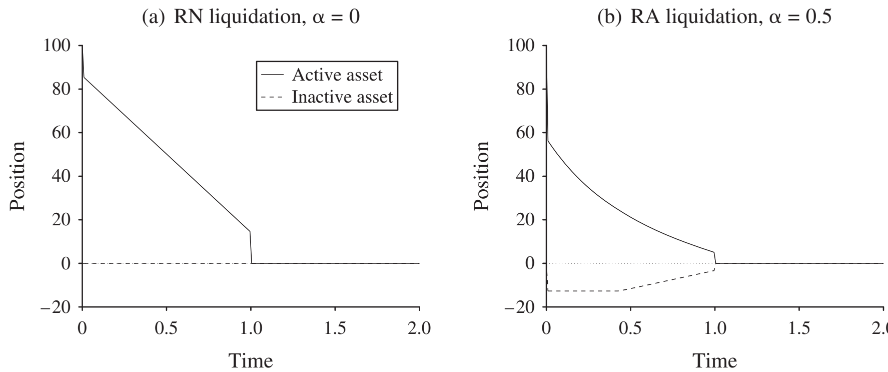
More importantly, a comparison between cases 1 and 2 in Table 1 suggests that shorting the positively correlated inactive asset during the execution process

Table 1. Shares Executed in Active and Inactive Assets, Expected Execution Costs, Standard Deviation of Costs, Certainty Equivalent, and Execution Sharpe Ratio, Defined as the Cost Savings Achieved over the Most Inefficient Case 5, Divided by the Standard Deviation of the Costs

	Active asset		Inactive asset					
Case	1st trade	Total volume	1st trade	Total volume	Exp. costs (\$)	Std. dev. (\$)	Cert. eq. (\$)	Exec. sharpe ratio (−)
1: No correl., no cross impact								
Figure 4(a)	14.7	100	0	—	1.75	2.70	1.75	0.59
Figure 4(b)	47.6	100	0	—	2.17	1.20	2.54	0.97
2: Effect of correlation								
Figure 6(a)	14.7	100	—	—	1.75	2.70	1.75	0.59
Figure 6(b)	43.8	100	12.7	25.4	2.12	1.12	2.43	1.1
3: Effect of cross impact								
Figure 8(a)	14.7	100	—	—	1.75	2.70	1.75	0.59
Figure 8(b)	49.1	100	6.4	12.8	2.15	1.19	2.51	0.99
Figure 8(c)	14.1	100	9.6	33.3	1.45	2.73	1.45	0.69
4: Effect of correl. and cross impact								
Figure 10	46.4	100	18.0	36.0	2.05	1.02	2.31	1.25
5: Execute everything at time 0 (not plotted)	100	100	—	—	3.33	—	3.33	—

Notes. The higher the Sharpe ratio, the more efficient the execution is. Costs are provided in (\$) terms. As a comparison, the portfolio preliquidation market value is \$100.

Figure 6. OEPs with Correlation ($\gamma = 0.7$)



Notes. In (a) the lack of RA implies no trades in the inactive asset. Introducing RA in (b) triggers trades in the inactive asset. Cross impact ($\lambda_{ij} = 0$) is turned off in both cases. Other parameter values include $\sigma_1 = \sigma_2 = 0.05$, $q_1 = q_2 = 1{,}500$, $\rho_1 = \rho_2 = 5$, and $\lambda_{11} = \lambda_{22} = 1/(3q_1)$.

(see Figure 6(b)) allows the manager to reduce both his total execution costs (by \$0.05) and execution risk (by \$0.08), compared to the single-asset strategy in Figure 4(b). The risk reduction is because the short position acts as a hedge, dampening future price volatility. Note that despite the assumed positive correlation, the trades at time 0 involve selling shares in both assets simultaneously. This may seem counterintuitive given that positive correlation is generally associated with offsetting trades (buy and sell) in the classical portfolio choice analysis. To understand why, consider the following scenario: assume the price of the active asset randomly decreases in the future, implying that subsequent sell orders generate less revenue for the manager. In this case, his short position in the positively correlated inactive asset will also accrue in value, thus compensating him or her for the decreased revenues. An analogous argument holds for the opposite case of a random price increase.

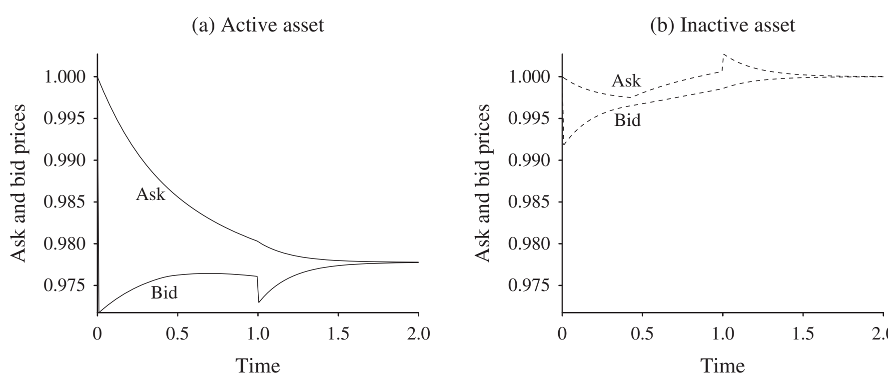
Beyond a reduction in risk, we emphasize that expected costs are also reduced over the single-asset case, despite that one is trading more shares and incurring additional price impact in the inactive asset. To understand why, consider the risk-reduction/cost-reduction trade-off mentioned previously. In the RA case, one can decrease execution costs at the expense of higher risk, and vice versa. Trading the inactive asset as a hedge leads to more efficient risk reduction compared to the unhedged strategy. In turn, this implies that one does not have to give up as much “upside” in execution costs, to achieve a desired risk level.

Figure 7(b) shows the evolution of the bid and ask prices of the inactive asset, resulting from the OEP portrayed in Figure 6(b). The figure clearly demonstrates why one cannot generally model bid and ask sides independently of one another, when considering cross asset effects. As one is required to sell and subsequently purchase back shares in the same asset, it is necessary to keep track of the price impact that each order has on both sides of the book, over time.

4.3. Effect of Cross Impact in Liquidity

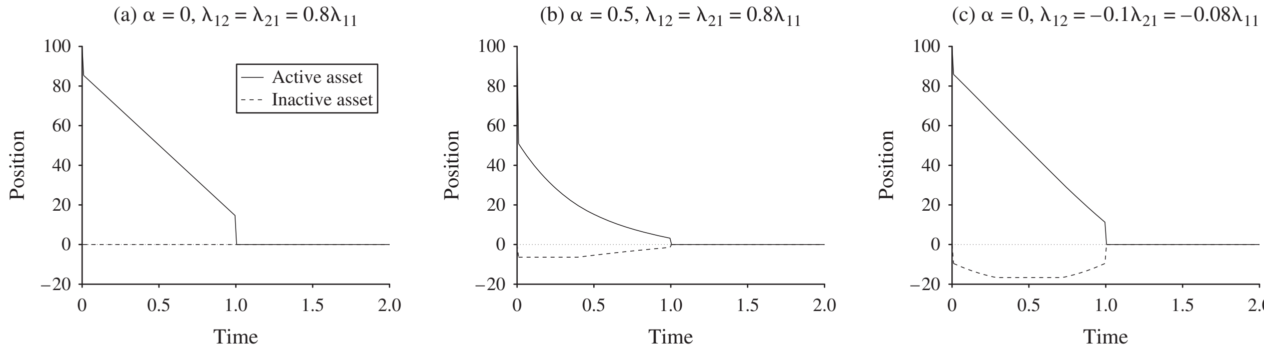
Here, we remain with the previous liquidation scenario, removing correlation between the two assets and focusing instead on the effect of cross impact. In contrast to correlation which is assumed exogenous, cross impact accounts for the impact an order in one asset has on the price and order book supply/demand

Figure 7. Dynamics of the Expected Ask and Bid Prices of Both Assets in the Case with RA ($\alpha = 0.5$) and Correlation ($\gamma = 0.7$), as Depicted in Figure 6(b)



Note. Cross impact ($\lambda_{ij} = 0$) is turned off.

Figure 8. OEPs with Cross Impact



Notes. In (a) the lack of RA implies no trades in the inactive asset. Introducing RA in (b) triggers trades in the inactive asset. In (c), asymmetric cross impact triggers trades in the inactive asset, even in the RN case. Correlation ($\gamma = 0$) is turned off. Other parameter values include $\sigma_1 = \sigma_2 = 0.05$, $q_1 = q_2 = 1{,}500$, $\rho_1 = \rho_2 = 5$, and $\lambda_{11} = \lambda_{22} = 1/(3q_1)$.

dynamics of the other (see Figure 3 for an illustration). Moreover, this impact does not need to be symmetric between the two assets: an order in stock A may affect stock B in one way, whereas changing the order and trading in stock B first, ceteris paribus, may affect stock A differently. Figure 8 illustrates this idea by comparing a case with symmetric cross impact ($\lambda_{12} = \lambda_{21}$ in Figure 8(b)) to a case with asymmetric cross impact ($\lambda_{12} = -\lambda_{21}$ in Figure 8(c)).

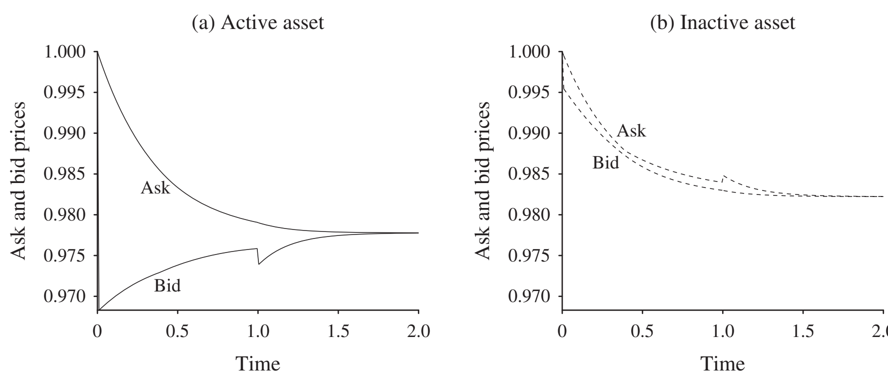
4.3.1. Effect of Cross Impact on the Liquidation Strategy. *Symmetric cross impact* without RA (Figure 8(a)) does not result in any trades in the inactive asset and the costs over the base case remain unchanged. Adding RA (Figure 8(b)) triggers trades in the inactive asset, comparable to the ones observed in the case with correlation. Therefore, if RA is considered, symmetric cross impact and correlation can have similar implications for the manager's OEP. The resultant trades are reported in case 3 of Table 1, whereas the resultant price dynamics are reported in Figure 9.

Figure 8(c) presents a case that clearly differentiates correlation from cross impact. We consider *asymmetric cross impact* between two assets and show that even a RN manager could be better off by trading in both the active and the inactive asset. This is in stark contrast to the previous example of correlations which become irrelevant for a RN manager.

4.3.2. Effect of Cross Impact on Execution Utility. The results in case 3 of Table 1 suggest that the effect of symmetric cross impact on execution costs and risk reduction is comparable to that of correlation. Trading the inactive asset during the liquidation (Figure 8(b)) allows the manager to slightly reduce his total execution costs (by \$0.02) and his execution risk (by \$0.01) over the optimal single-asset trading strategy (Figure 4(b)). In contrast, asymmetric cross impact allows the manager to achieve the greatest cost reduction of all cases (although, this comes with increased execution risk).

The introduction of symmetric cross impact has implications for the price dynamics of the inactive asset that are not observed when only considering correlation. Figure 9(b) shows the cross impact the active asset has on the inactive one (λ_{21}). The liquidation of the

Figure 9. Dynamics of the Expected Ask and Bid Prices of Both Assets in the Case with RA ($\alpha = 0.5$) and Cross Impact, as Depicted in Figure 8(b)

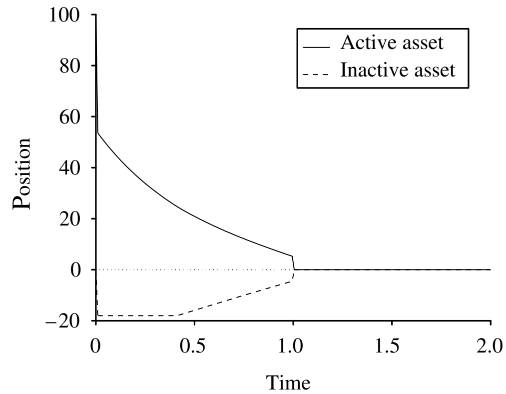


Notes. Correlation is turned off. The manager's trades in the active asset affect the bid and ask side prices of the inactive asset, and vice versa.

active asset pushes the price of the inactive asset down over time. This effect may seem favorable to the manager. He could short the inactive asset at time 0 and buy it back later at a lower price. However, putting in place the initial short position in the inactive asset also adversely affects the price of the active asset via the cross impact term (λ_{12}). This implies that the manager would be liquidating his active position at lower prices. This effectively restricts the manager's ability to take advantage of the favorable price dynamics expected in the inactive asset. This trade-off is fully internalized in the OEP.

4.3.3. Asymmetric Cross Impact and Arbitrage. Huberman and Stanzl (2004, section 5) illustrate an example of asymmetric cross impact that can lead to price manipulation and arbitrage. The authors derive sufficient no-arbitrage conditions in the multiasset setting: (1) cross impact symmetry between assets and (2) lack of temporary impact costs. As the authors state, these conditions are sufficient, but they are not necessary. Case 3 of Table 1 shows that asymmetric cross impact does not necessarily lead to arbitrage opportunities when considering positive temporary impact costs. Although some cost benefits can be achieved under these scenarios over the RN base case, net execution costs remain positive at \$1.45. To understand this result, observe that every executed order has both a permanent and temporary impact component, and although asymmetric cross impact can understandably reduce costs on the permanent component, the manager is also consistently incurring costs from the temporary component during trade (i.e., he is "rolling" up or down the supply/demand curves getting executed at increasingly costly limit price levels). This trade-off between temporary and permanent price impacts is fully internalized in the OEP. Thus, similar to Proposition 3 in Huberman and Stanzl (2004), absence of arbitrage will hold if temporary impact costs are sufficiently larger than permanent impact costs. Formally, positive definiteness of $\bar{\mathbf{D}}$ implies no arbitrage.

Figure 10. OEP with RA ($\alpha = 0.5$), Cross Impact ($\lambda_{12} = \lambda_{21} = 0.8\lambda_{11}$), and Correlation ($\gamma = 0.7$)



Note. Other parameter values include $\sigma_1 = \sigma_2 = 0.05$, $q_1 = q_2 = 1{,}500$, $\rho_1 = \rho_2 = 5$, and $\lambda_{11} = \lambda_{22} = 1/(3q_1)$.

4.4. Joint Effect of Correlation in Risk and Cross Impact in Liquidity

Here, we consider both correlation and cross impact simultaneously. The results reported in Figure 10 and in case 4 of Table 1, suggest that the cost benefits obtained exhibit positive convexity. In other words, correlation and cros -impact work constructively, providing benefits that are greater than the sum of the individual contributions each of them brings independently.

The results in case 4 of Table 1 suggest that expected costs can be reduced to \$2.05, whereas risk can be reduced to \$1.02, the lowest of all cases. The execution Sharpe ratio obtained is the greatest of all cases, at 1.25. To achieve these benefits, the OEP requires trading a significant volume in the inactive asset, equal to approximately 1/3 of the total volume of the active asset. The price dynamics of the inactive asset observed in Figure 11 combine the cross impact and correlation effects we described previously. In this case, the manager is generally selling "high" and buying "low" in the inactive asset, while also benefiting from a reduced initial order size in the active asset, and limit order mean-reversion dynamics.

Figure 11. Dynamics of the Expected Ask and Bid Prices of Both Assets in the Scenario Depicted in Figure 10

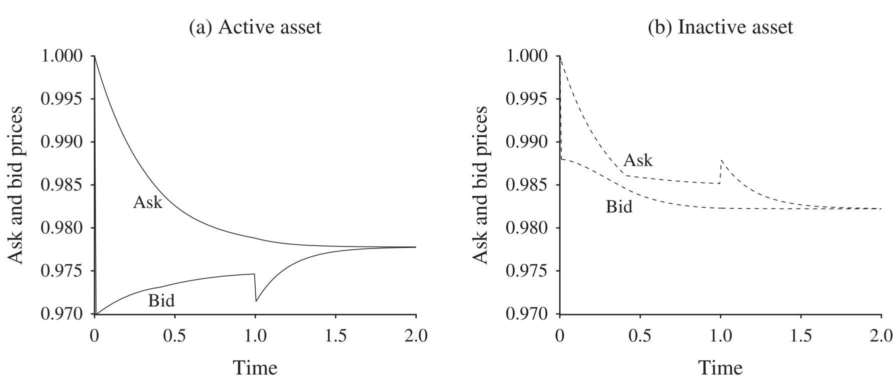
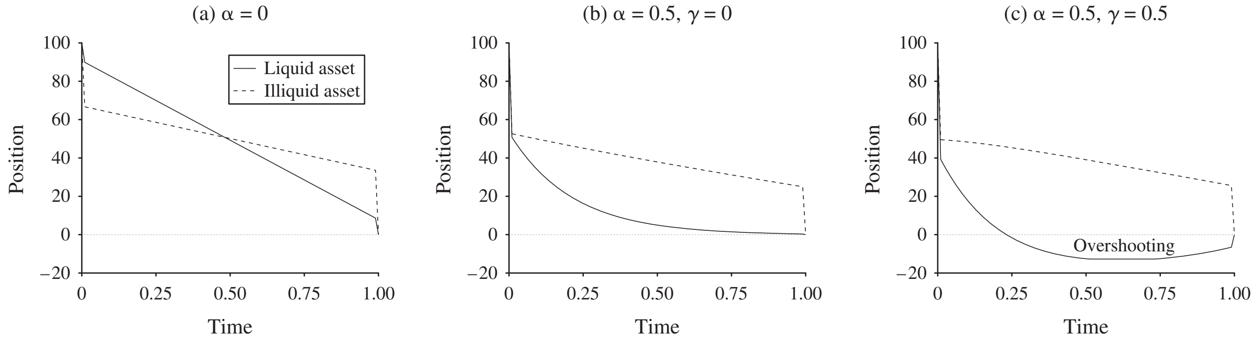


Figure 12. OEPs in the Case of a Portfolio with Mixed Liquidity



Note. Parameter values include $v_{1,0} = v_{2,0} = 1$, $\sigma_1 = \sigma_2 = 0.05$, $\lambda_{11} = \lambda_{22} = 1/(3q_1)$, and $\lambda_{12} = \lambda_{21} = 0$.

5. Mixed Liquidity Portfolios

This section illustrates additional results that are of consequence to practitioners.

5.1. Portfolio Overshooting

Execution objectives are typically richer than the ones illustrated in the previous section. Portfolio managers often need to liquidate or acquire positions in multiple assets with different risk and liquidity characteristics. This section illustrates the optimal liquidation of a portfolio comprising two assets with different liquidity levels. The first asset is considered liquid, with limit order replenishment rate $\rho_1 = 10$ and limit order density $q_1 = 3{,}000$, whereas the second is (comparatively) illiquid, with rate $\rho_2 = 1$ and density $q_2 = 300$.[18]

Figure 12 shows the OEPs obtained for different RA and correlation assumptions. The results suggest that liquid assets generally will be more smoothly executed throughout the horizon, whereas illiquid assets tend to corner solutions (i.e., it is optimal to execute two larger trades at times 0 and 1). The intuition here is simple: illiquid assets have order books with low replenishment rates leading to asset prices with low mean-reversion. The lack of replenishment implies that one cannot take advantage of order book dynamics in any meaningful way and thus, the optimal solutions tend to be trivial. On the other hand, liquid assets with high replenishment have more interesting dynamics that can be utilized toward the execution problem, leading to richer optimal strategies.

When the two assets are correlated (see Figure 12(c)), further advanced strategies become optimal. We obtain two-sided buy and sell strategies, despite the simple unidirectional liquidation objective. The position in the liquid asset becomes negative near time 0.25, implying overshooting. The excess shares sold are gradually purchased back in order to meet the boundary conditions as the horizon approaches. From a hedging perspective, the transient short position in the liquid asset dampens future price uncertainty and reduces execution risk. We emphasize that although overshooting was also observed in the example of Section 4.2 (because any trades in the inactive asset could be considered as overshooting trades), here, this effect is entirely driven by the assumed liquidity differences between the two assets.

5.2. Synchronization Risk

The results in the previous section suggest that liquid and illiquid assets will be executed at different speeds. The manager could therefore be left over/under-exposed to individual assets during the execution process, that is, he could be facing *synchronization risk*. To highlight this more clearly, we plot the weight of each asset (expressed as the ratio of net shares held in each asset over total shares held in the portfolio) over time, in Figure 13, for the same cases presented in Figure 12.

Figure 13. Asset Weights (Expressed as a % of Total Shares Held) of the Mixed Liquidity Portfolios Depicted in Figure 12

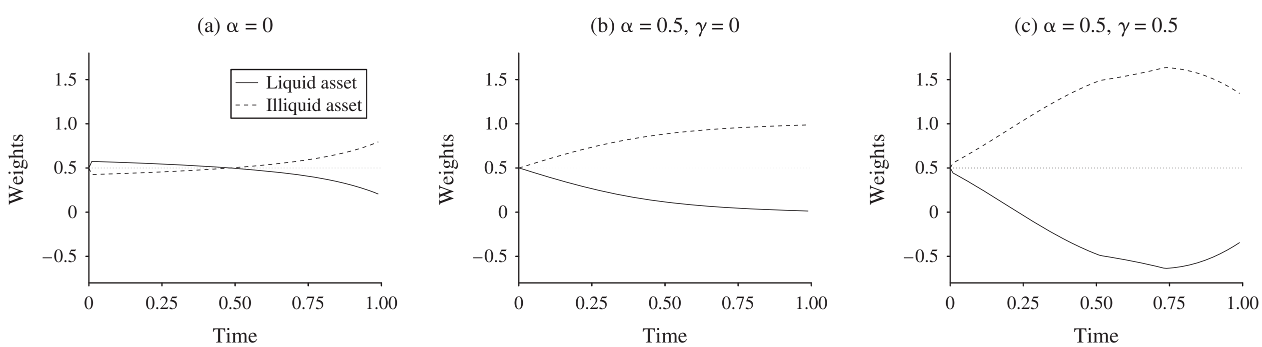


Figure 14. OEPs in Mixed Liquidity Portfolios With (Subfigures (b) and (c)) and Without (Subfigure (a)) Synchronization Constraints

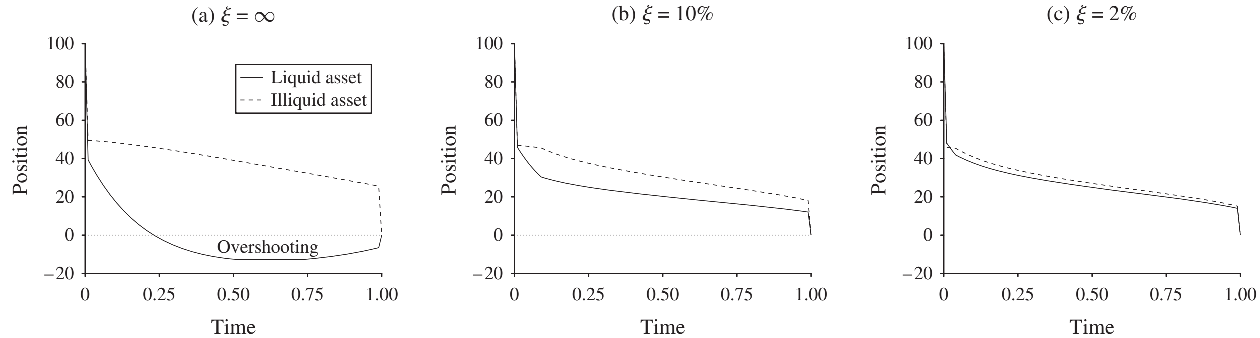
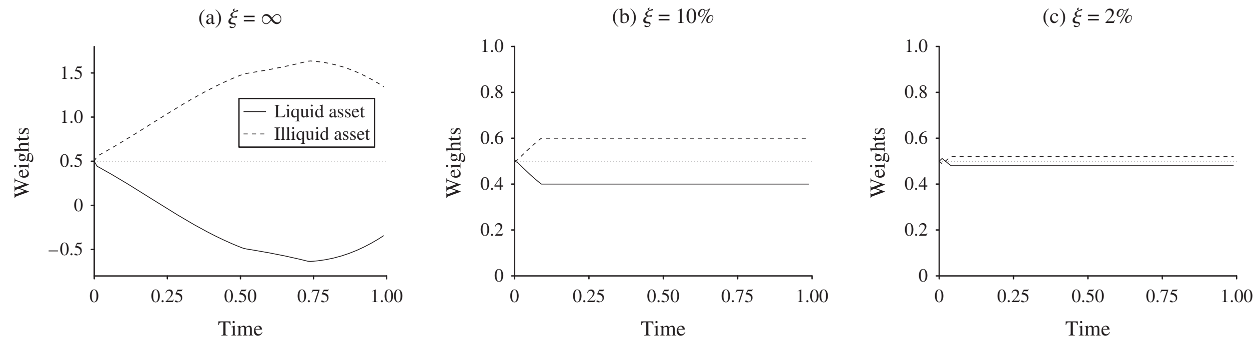


Figure 15. Asset Weights Over Time for the Scenarios Depicted in Figure 14



Assume that the manager's initial optimal portfolio allocation is 50/50, and that there is some underlying benefit (such as portfolio diversification) to preserve this optimal split during the execution window. All three cases in Figure 13 show that the manager could be left overexposed to the illiquid asset during the execution window, as its weight can move above the optimal 0.5 line.

A simple way to mitigate the undesirable exposure is to constrain the admissible order quantities at each trading period. For instance, one can restrict each asset's weight to an interval, $w_{i,n} \in [w_i^* - \xi, w_i^* + \xi]$, where w_i^* is the desired weight targeted in asset i and $\xi \in [0, \infty)$ controls the desired margin of error. The parameter ξ is chosen by the manager and can be thought of as the degree of tolerance to weight variability. Figures 14 and 15 show the impact of different tolerance parameters on the OEPs and weight profiles. As $\xi \to 0$, both asset weights converge to the 0.5 line, and the OEPs converge to a single strategy. Interestingly, the unique optimal strategy is a weighted combination of the two individual unconstrained OEPs of each asset. Further, it is in the strict interior of the two.

Adding these types of constraints to the problem reduces the set of feasible execution strategies, leading to increased execution costs over the unconstrained global optimum. This raises the question of how costly it is to synchronize the portfolio in this fashion. We define the synchronization cost as the expense one would have to incur over the most efficient (lowest cost) outcome, in order to maintain a targeted weight profile during the execution process.

In our examples, the synchronization cost for $\xi = 10\%$ is equal to 18 bps, whereas in the worst-case scenario ($\xi \to 0$), it is equal to 38 bps. The latter represents the maximum amount the manager would expect to pay, in excess of the most efficient outcome, in order to remain fully in line with the optimal targeted weight allocation throughout the entire execution window.

6. Conclusion

Controlling price impact is of central importance in portfolio management, and is particularly crucial in practical situations where managers need to execute large positions in multiple assets. We have studied the multiasset execution problem demonstrating that it is far from being a simple extension to the single-asset case. Assets can interact in complex ways, and these interactions can substantially affect the aggregate portfolio execution cost and risk. Understanding the exact nature of these interactions requires an extensive market microstructure model that can adequately capture coupled supply and demand dynamics at the order book level.

Our results suggest that managing execution at the portfolio level needs to take account of links in both risk and liquidity across assets. In the presence of such links, we find that managers can improve execution efficiency by engaging in a series of nontrivial buy and sell trades in multiple assets simultaneously. The trades are nontrivial in the sense that they may require temporarily trading positively correlated assets in the same direction, or even overshooting one's portfolio target during the execution window.

These results extend to portfolios with heterogeneous liquidity across assets. There, the liquidity differential between assets can lead to complex strategies which utilize the liquid asset to improve execution efficiency at the portfolio level. However, we also find that these advanced strategies can leave managers overexposed to illiquid assets during the execution. This synchronization risk can be mitigated by introducing constraints that can synchronize the portfolio trades, at the cost of reduced execution efficiency. This led to the concept of synchronization cost—a measure which allows managers to trade off these two factors, based on their individual preferences.

Perhaps an even more compelling takeaway is that advanced strategies can be optimal for simple and common execution objectives (such as the liquidation of a single asset in the portfolio). This implies that it may be crucial for managers to systematically take into account cross asset interactions in risk and liquidity in their risk-management and trading decisions. It also implies that market regulators should be aware of the increased liquidity needs this can lead to, if deployed on a large-scale basis.

Acknowledgments
The authors are grateful to David Brown, Victor DeMiguel, Vivek Farias, Arseniy Kukanov, Ruben Lobel, Costis Maglaras, Ciamac Moallemi, and Nikolaos Trichakis, as well as seminar participants from the 2011 Annual INFORMS Meeting in Charlotte, the Management Science seminar at Rutgers University, the Operations Management seminar at the Wharton School, University of Pennsylvania, the 2012 Annual Manufacturing and Service Operations Management meeting at Columbia University, the Information, Risk, and Operations Management seminar at the University of Texas at Austin McCombs School of Business, AQR Capital Management, Morgan Stanley, and Weiss Asset Management for insightful comments.

Appendix
The appendix is structured as follows: Section A.1 contains some additional results; Section A.2 contains proofs for some preliminary results; Section A.3 contains the proof of the concavity result; Section A.4 contains proofs for the dynamic programming solution; Section A.5 contains proofs for the equivalent static QP.

A.1. Additional Results
A.1.1. Effect of the Equilibrium Bid-Ask Spread/Proportional Transaction Costs
Figure A.1 highlights the sensitivity of the inactive asset OEP, to its steady-state bid-ask spread, in the example from Section 4.2. The OEP is plotted with values $s_2 = 0, 50, 100$ and 200 bps. The reference point is the inactive asset's initial mid-price, $v_{2,0} = \$1$, so that 100 bps corresponds to a bid-ask spread of 1¢. At a spread of 200 bps, any trading activity in the inactive asset is halted completely. The associated costs and total volume traded in the inactive asset are provided in Table A.1.

Figure A.1. Effect of the Steady-State Bid-Ask Spread on the Inactive Asset OEP from Figure 6(b)

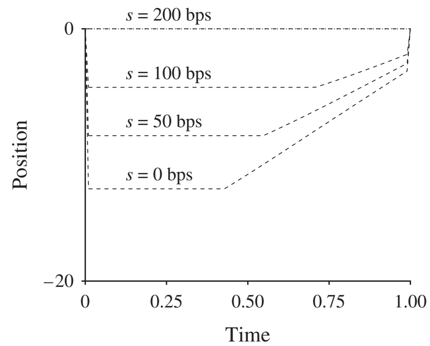


Table A.1. Expected Costs and Volume Traded for Different Values of s

Bid-ask spread s (bps)	0	50	100	200
Expected costs (\$)	2.12	2.41	2.68	3.17
Total volume in inactive asset (shares)	25.4	17.0	9.2	—

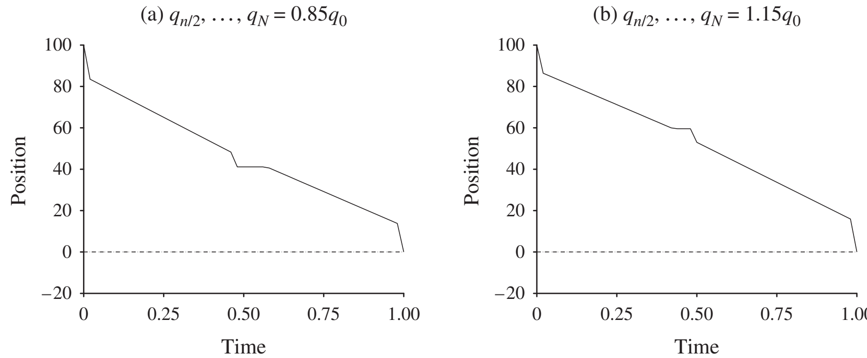
A.1.2. Intraday Liquidity Variations and Time-Dependent Parameters
Figure A.2 shows the sensitivity of a single-asset OEP for a RN manager ($\alpha = 0$) with a time-varying view on the order book densities. The fact that liquidity can predictably change at different time scales has been empirically documented (see, e.g., Chordia et al. 2001). We plot the OEP for respective changes in the value of q, both lower (Figure A.2(a)), and higher (Figure A.2(b)), in the interval $[N/2, N]$.

There is a significant change of trading velocity both immediately preceding and following the change in liquidity. Furthermore, temporary "dead-zones" emerge around the time of the change in liquidity, where it becomes optimal to halt all trading activity. Intuitively, these indicate that the manager should wait until the liquidity changes are fully absorbed by the order books and supply/demand converges to the new regime before finishing off the remaining orders.

A.2. Proof of Lemmas 1, 2, and 3
Proof of Lemma 1 (Temporary Price Impact). A buy order of size x being executed against i's ask-side inventory q_i^a, displaces the best ask price from $a_{i,n} \to a^*_{i,n}$, according to $\int_{a_{i,n}}^{a^*_{i,n}(x)} q_i^a(p)\,dp = x$. Combining this expression with Assumption 1, we have

$$\int_{a_{i,n}}^{a^*_{i,n}(x)} q_i^a \mathbf{1}_{p \ge a_{i,n}}\, dp = q_i^a(a^*_{i,n}(x) - a_{i,n}) = x$$
$$\Rightarrow\ a^*_{i,n}(x) = a_{i,n} + \frac{x}{q_i^a}.$$

Figure A.2. RN OEP with an Order Book Density Decrease in (a) and a Density Increase in (b)



Therefore for $x = x^+_{i,n}$, we have $a^*_{i,n} = a_{i,n} + x^+_{i,n}/q^a_i$ and the temporary price impact displacement is defined as $a^*_{i,n} - a_{i,n} = x^+_{i,n}/q^a_i$. The derivation for a sell order follows similar steps. □

Proof of Lemma 2 (Best Prevailing Bid/Ask Prices). We present below an outline of the derivation for the best ask price dynamics the bid price dynamics are derived in a similar way. The best available ask price for asset i at time n is given by

$$a_{i,n} = u_{i,n} + \tfrac{1}{2}s_i + \mathrm{PPI} + \mathrm{TPI},$$

where the first term accounts for the random walk driving the mid-price, the second term accounts for the steady-state bid-ask spread, the third term accounts for the aggregate PPI of all orders up to (but excluding) n, and the fourth term is the order book state vector which accounts for the TPI of all orders up to (and including) n. Following Equation (12) and the definition of the vector $\mathbf{z}_n$ in Section 2.1, the aggregate PPI for asset i can be written as

$$\sum_{k=1}^{n}\sum_{j\in I}\lambda_{ij}(x^+_{j,k-1} - x^-_{j,k-1}) = [\boldsymbol{\Lambda}(\mathbf{z}_0 - \mathbf{z}_n)]_i,$$

where $[\cdot]_i$ returns the ith line of a matrix. Following Assumption 4, the aggregate TPI can recursively be written as $d^a_{i,n} = (d^a_{i,n-1} + \kappa^a_i(x^\pm_{i,n-1}))e^{-\rho^a_i\tau}$, where $\kappa^a_i(x^\pm_{i,n})$ is the net displacement in the ask-side order book resulting from a buy trade in asset i at time n. The net displacement is given by the difference between the TPI and PPI at time n:

$$\kappa^a_i(x^\pm_{i,n}) = \frac{x^+_{i,n}}{q^a_i} - \sum_{j\in I}\lambda_{ij}(x^+_{j,n} - x^-_{j,n}).$$

Using these expressions, and removing the recursion in $d^a_{i,n-1}$, the aggregate TPI can be written as

$$d^a_{i,n} = \sum_{k=1}^{n}\left[\frac{x^+_{i,k-1}}{q^a_i} - \sum_{j\in I}\lambda_{ij}(x^+_{j,k-1} - x^-_{j,k-1})\right]e^{-\rho^a_i(n-k+1)\tau}.$$

Note, the recursive vector form of the aggregate TPI across all assets given in Equation (16) follows immediately from the previous expressions, in particular, $d^a_{i,n} = [(\mathbf{d}^a_{n-1} + \boldsymbol{\kappa}^a\mathbf{x}_{n-1})\mathbf{e}^{-\rho^a\tau}]_i$ and thus $\mathbf{d}^a_n = (\mathbf{d}^a_{n-1} + \boldsymbol{\kappa}^a\mathbf{x}_{n-1})\mathbf{e}^{-\rho^a\tau}$.

Combining the aggregate PPI and TPI terms, and repeating similar steps for the bid side, we obtain the following expressions for the best available ask and bid prices of asset i at each time n:

$$\begin{aligned} a_{i,n} &= u_{i,n} + s_i/2 + \sum_{k=1}^{n}\sum_{j\in I}\lambda_{ij}\delta x_{j,k-1} \\ &\quad + \sum_{k=1}^{n}\left(\frac{x^+_{i,k-1}}{q^a_i} - \sum_{j\in I}\lambda_{ij}\delta x_{j,k-1}\right)e^{-\rho^a_i(n-k+1)\tau}, \\ b_{i,n} &= u_{i,n} - s_i/2 + \sum_{k=1}^{n}\sum_{j\in I}\lambda_{ij}\delta x_{j,k-1} \\ &\quad + \sum_{k=1}^{n}\left(-\frac{x^-_{i,k-1}}{q^b_i} - \sum_{j\in I}\lambda_{ij}\delta x_{j,k-1}\right)e^{-\rho^b_i(n-k+1)\tau}. \end{aligned}$$

where $\delta x_{j,k-1} = (x^+_{j,k-1} - x^-_{j,k-1})$. Extending the above steps to all assets, and using vector notation, we can obtain the final vector forms for the best available ask and bid prices in Equations (15a) and (15b). □

Proof of Lemma 3 (Execution Costs/Revenues). Following an executed order, the associated costs/revenues can simply be calculated by integrating the best available bid/ask prices over the total amount of units executed x. It follows that

$$c_{i,n}(x) = \int_0^x a^*_{i,n}(u)\,du \quad \text{and} \quad r_{i,n}(x) = \int_0^x b^*_{i,n}(u)\,du,$$

where $a^*_{i,n}(x)$ and $b^*_{i,n}(x)$ are given in (7). Specifically, for an incoming order $x = x^+_{i,n}$ or $x = x^-_{i,n}$, we find

$$c_{i,n}(x^+_{i,n}) = \left(a_{i,n} + \frac{x^+_{i,n}}{2q^a_i}\right)x^+_{i,n} \quad \text{and} \quad r_{i,n}(x^-_{i,n}) = \left(b_{i,n} - \frac{x^-_{i,n}}{2q^b_i}\right)x^-_{i,n}.$$

Equivalently, in vector notation: $c_n = \mathbf{x}^{+\prime}_n(\mathbf{a}_n + \mathbf{Q}^a\mathbf{x}^+_n)$ and $r_n = \mathbf{x}^{-\prime}_n(\mathbf{b}_n - \mathbf{Q}^b\mathbf{x}^-_n)$. □

A.3. Proof of Lemma 4—Concavity

This section proves Lemma 4 by showing that the iterative structure of the dynamic problem preserves concavity at all times. The procedure follows closely from the results presented in section 3.2 (operations the preserve convexity) of Boyd and Vandenberghe (2004; hereinafter BV). The next lemma summarizes these results.

Lemma A1 (BV). *Denote by* $\mathbf{x}_{[0,n]} \stackrel{\text{def}}{=} [\mathbf{x}_0; \ldots, \mathbf{x}_n]$ *and* $\mathbf{u}_{[0,n]} \stackrel{\text{def}}{=} [\mathbf{u}_0; \ldots; \mathbf{u}_n]$ *the vector of the first n trade and shock vectors* $\mathbf{x}$ *and* $\mathbf{u}$, *respectively. Further, let* $f\colon \mathbf{R}^{2mn\times mn} \to \mathbf{R}$ *and* $g\colon \mathbf{R}^{2m(n-1)} \to \mathbf{R}$. *Suppose* $f(\mathbf{x}_{[0,n]};\cdot)$ *is concave in* $\mathbf{x}_{[0,n]}$. *Then,*

(a) *for* $\alpha > 0$, *the composite function* $-e^{-\alpha f(\mathbf{x}_{[0,n]};\cdot)}$ *is concave in* $\mathbf{x}_{[0,n]}$;

(b) *for $k \leq n$, the conditional expectation*

$$\mathbb{E}\left[f\left(\mathbf{x}_{[0,n]}; \mathbf{u}_{[0,n]}\right) \mid \mathbf{u}_{[0,k]}\right]$$

is concave in $\mathbf{x}_{[0,n]}$;

(c) *the maximization over element* $\mathbf{x}_n$,

$$g(\mathbf{x}_{[0,n-1]}) \stackrel{\text{def}}{=} \max_{\mathbf{x}_n \geq 0} f(\mathbf{x}_{[0,n]}, \cdot),$$

is concave in $\mathbf{x}_{[0,n-1]}$.

Proof. Part (a) follows from section 3.2.4 (simple composition results with exponentials) of BV. Part (b) follows from section 3.2.1 (infinite nonnegative weighted sums and integrals of convex functions) of BV. Part (c) follows from section 3.2.5, problem 3.16 (minimization over convex sets) of BV. □

Now, consider the dynamic program $J_n = \max_{x_n \geq 0} \mathbb{E}[J_{n+1}]$. At time 0, this can be written in nested expectations:

$$J_0 = \max_{\mathbf{x}_0 \geq 0} \mathbb{E}_0\left[\max_{\mathbf{x}_1 \geq 0} \mathbb{E}_1\left[\ldots \max_{\mathbf{x}_{N-1} \geq 0} \mathbb{E}_{N-1}\left[\max_{\mathbf{x}_N \geq 0} \mathbb{E}_N[-e^{-\alpha W_N}]\right] \ldots\right]\right],$$

where the stochastic wealth function is given by (25):

$$W_N = W_{0^-} - \mathbf{u}'\bar{\boldsymbol{\delta}}'\mathbf{x} - \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_0 - (\mathbf{y}_0'\bar{\mathbf{A}}'\bar{\mathbf{N}})\mathbf{x} - \mathbf{x}'\mathbf{D}\mathbf{x},$$
$$\mathbf{D} \stackrel{\text{def}}{=} \tfrac{1}{2}(\bar{\mathbf{B}}'\bar{\mathbf{N}} + \bar{\mathbf{N}}'\bar{\mathbf{B}}) + \bar{\mathbf{Q}}.$$

Suppose W_N is concave in $\mathbf{x}_{[0,N]}$, for any realizations of $\mathbf{u}_{[0,N]}$. Then, (a) by Lemma A1(a), W_N concave $\Rightarrow -e^{-\alpha W_N}$ concave for $\alpha > 0$; (b) by Lemma A1(b), $-e^{-\alpha W_N}$ concave $\Rightarrow \mathbb{E}_N[-e^{-\alpha W_N}]$ concave; (c) by Lemma A1(c), $\mathbb{E}_N[-e^{-\alpha W_N}]$ concave $\Rightarrow \max_{\mathbf{x}_N \geq 0} \mathbb{E}_N[-e^{-\alpha W_N}] \stackrel{\text{def}}{=} J_N$ concave in $\mathbf{x}_{[0,N-1]}$.

The above shows that all relevant operators preserve concavity. Now consider the next iteration. By Lemma A1(b), J_N concave in $\mathbf{x}_{[0,N-1]}$ implies $\mathbb{E}_{N-1}[J_N]$ concave in $\mathbf{x}_{[0,N-1]}$. By Lemma A1(c), $\mathbb{E}_{N-1}[J_N]$ concave in $\mathbf{x}_{[0,N-1]}$ implies $\max_{\mathbf{x}_{N-1}} \mathbb{E}_{N-1}[J_N] \stackrel{\text{def}}{=} J_{N-1}$ concave in $\mathbf{x}_{[0,N-2]}$. Rolling the argument backwards, we obtain that for all $n \in \{1, \ldots, N\}$, the stage objective J_n is concave in $\mathbf{x}_{[0,n-1]}$.

Therefore, to ensure that the DP is strictly concave, it suffices to impose strict joint concavity in all $\mathbf{x}_n$ on the terminal wealth W_N. Given its quadratic form in (25), this is ensured if $\mathbf{D}$ is positive definite. □

A.4. Proof of Propositions 1 and 2—DP Solution

This section proves Propositions 1 and 2 by solving the DP through induction. Before proceeding, we introduce intermediate results.

A.4.1. Piecewise Quadratic (PWQ) Optimization

We begin by introducing a preliminary result which will be useful in stating our main results later. Let $\mathbf{x} \in \mathbb{R}^{m_1}$ and $\mathbf{y} \in \mathbb{R}^{m_2}$. Define the following PWQ optimization problem:

$$J(\mathbf{y}) \stackrel{\text{def}}{=} \underset{\mathbf{x} \in D, \mathbf{x} \geq 0}{\text{maximize}} \, f(\mathbf{x}, \mathbf{y}), \tag{A.1}$$

with $f: \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ being a strictly concave, continuous, PWQ function with L pieces indexed by j:

$$f(\mathbf{x}, \mathbf{y}) = f^j(\mathbf{x}, \mathbf{y}) \quad \text{if } (\mathbf{x}, \mathbf{y}) \in D^j,$$
$$\text{where } f^j(\mathbf{x}, \mathbf{y}) = -\tfrac{1}{2}\mathbf{x}'\mathbf{M}^j\mathbf{x} - \mathbf{y}'\mathbf{N}^{j'}\mathbf{x},$$

with associated regions defined by general polyhedral sets D^j (e.g., $D^j = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{A}^j\mathbf{x} \leq \mathbf{B}^j\mathbf{y}\}$), where $D = \{D^j: j \in L\}$ is a polyhedral decomposition of dom f. We have the following result.

Result A1. (a) *The solution to* (A.1), *denoted* $\mathbf{x}^*(\mathbf{y})$, *exists, which is single valued, continuous, and has a piecewise-affine structure, that is, it can be written as*

$$\mathbf{x}^*(\mathbf{y}) = \mathbf{K}(\mathbf{y})\mathbf{y}, \quad \textit{where } \mathbf{K}(\mathbf{y}) \textit{ is piecewise constant.} \tag{A.2}$$

(b) $J(\mathbf{y})$ *is a continuous PWQ function in* $\mathbf{y}$.

(c) $\mathbf{x}^*(\mathbf{y})$ *and* $J(\mathbf{y})$ *are subdifferentiable.*

(d) *The generalized KKT conditions associated with problem* (A.1) *are sufficient to characterize the solution* (A.2), *given by*

(K1) *Optimality*: $\mathbf{0} \in \partial_{\mathbf{x}}[f(\mathbf{x}, \mathbf{y}) + \boldsymbol{\nu}'\mathbf{x}]$, *where* $\boldsymbol{\nu}$ *is the Lagrange nonnegativity multiplier and* $\partial(\cdot)$ *denotes the subdifferential.*

(K2) *Primal feasibility*: $\mathbf{x} \geq 0$.

(K3) *Dual feasibility*: $\boldsymbol{\nu} \geq \mathbf{0}$.

(K4) *Complementary slackness*: $\boldsymbol{\nu} \otimes \mathbf{x} = \mathbf{0}$.

The KKT regularity conditions are automatically satisfied since all constraints are polyhedral.

Proof. Problem (A.1) can be considered as a subcase of the more general PWQ program studied in Patrinos and Sarimveis (2011; hereinafter PS), problem (4), whose properties are given in PS proposition 5. We only need to establish the connection to their setting, namely, the parameter $\mathbf{y} \leftrightarrow \mathbf{p}$, the solution $\mathbf{x}^* \leftrightarrow \mathscr{X}$ and the objective $J(\mathbf{y}) \leftrightarrow V(\mathbf{p})$.[19] Thus, (A.1) is a subcase of the PWQ problem (4) studies in PS.

(a) Result A1(a) follows from PS proposition 5(b), which states that under strict convexity (concavity in our case), the solution to PS problem (4) is single-valued, piecewise affine and (Lipschitz) continuous in $\mathbf{p}$. Thus, (A.2) follows directly from the piecewise affine structure of the solution.[20]

(b) The PWQ structure in Result A1(b) follows directly from PS proposition 5(a). Continuity of $J(\mathbf{y})$ follows directly from the continuity property of the optimal solution.

(c) Subdifferentiability in Result A1(c) follows directly from Results A1(a) and A1(b). In particular, the subdifferentiability of a piecewise linear and a convex PWQ function follows from the fact their subgradients are polyhedral (Rockafellar and Wets 1998, proposition 10.21).

(d) The generalized KKT conditions for subdifferentiable functions are established in the literature, for instance, see Ruszczynski (2006, chap. 3.6, pp. 133–135). □

Note, PWQ problems have been extensively studied in the literature and the results in Patrinos and Sarimveis (2011) and by extension, in Result A1(a)–(d) have been established in more general settings. A classic reference is Rockafellar and Wets (1998, chaps. 10.E and 11.D). The authors provide a formal definition of a PWQ in definition 10.20, establish continuity and subdifferentiability in propositions 10.21 and 11.32(b)(c) (and by extension, in exercise 7.45), and establish the general polyhedral structure of the optimal solution in exercise 10.22 and corollary 11.16.

A.4.2. Terminal Time N

We now turn to our problem. We first examine the problem at the final time N, which reveals the structure of the solution. Following (20), the wealth at the terminal time is

$$W_N = W_{N-1} - (\mathbf{u}_N'\boldsymbol{\delta}' + \mathbf{z}_0'\boldsymbol{\Lambda}\boldsymbol{\delta}' + \mathbf{y}_N'\mathbf{N})\mathbf{x}_N - \mathbf{x}_N'\mathbf{Q}\mathbf{x}_N, \tag{A.3}$$

which is a continuous strictly concave quadratic function of $\mathbf{x}_N$. The assumption in Lemma 4 that W_N is strictly concave in $\mathbf{x}_0, \ldots, \mathbf{x}_N$ trivially implies that W_N is strictly concave in the optimization variable $\mathbf{x}_N$. Further, $\mathbf{u}_N$ is realized at N, thus the terminal optimization problem is deterministic, given by

$$J_N = \underset{\mathbf{x}_N \geq 0}{\text{maximize}}(-e^{-\alpha W_N}) \quad \text{s.t. } \boldsymbol{\delta}'\mathbf{x}_N = \mathbf{z}_N.$$

By monotonicity of the exponential, $\arg\max_{\mathbf{x}_N} -e^{-\alpha W_N} = \arg\max_{\mathbf{x}_N} W_N$, for $\alpha > 0$, which turns our problem into a standard concave linear quadratic problem, with polyhedral constraints. From the equality constraint, the terms $\boldsymbol{\delta}'\mathbf{x}_N = \mathbf{z}_N$, leading to

$$\underset{\mathbf{x}_N \geq 0}{\text{maximize}}\left\{W_{N-1} - \mathbf{u}_N'\mathbf{z}_N - \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_N - \mathbf{y}_N'\mathbf{N}\mathbf{x}_N - \mathbf{x}_N'\mathbf{Q}\mathbf{x}_N\right\},$$
$$\text{s.t. } \boldsymbol{\delta}'\mathbf{x}_N = \mathbf{z}_N.$$

Ignoring the terms that do not depend on the optimization variable (i.e., $W_{N-1} - \mathbf{u}_N'\mathbf{z}_N - \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_N$), the problem at N becomes a special case of the general PWQ problem (A.1), with a single piece $j = L = 1$, and polyhedral constraints $\boldsymbol{\delta}'\mathbf{x}_N = \mathbf{z}_N$ and $\mathbf{x}_N \geq 0$. As the problem is deterministic at time N, the solution $\mathbf{x}_N^*$ is deterministic as well. Also, from Result A1, $\mathbf{x}_N^*$ is continuous in $\mathbf{y}_N$, and can be written like in (A.2).

Though it is not required for the proof, we can take a few more steps to explicitly characterize $\mathbf{x}_N^*$ by using the standard KKT conditions at N. For any asset i, there are three cases to consider: if $z_{i,N} > 0$, then the optimal trade is a buy order: $x_{i,N}^+ = z_{i,N}$, and $x_{i,N}^- = 0$; if $z_{i,N} < 0$, then the optimal trade is a sell order: $x_{i,N}^+ = 0$, and $x_{i,N}^- = -z_{i,N}$; if $z_{i,N} = 0$, then $x_{i,N}^+ = x_{i,N}^- = 0$ and this can be viewed as a buy or a sell order of size 0. Using indicator functions we can compactly write the solution as follows:

$$\mathbf{x}_N^* = \mathbf{k}_N(\mathbf{z}_N)\mathbf{z}_N, \quad \mathbf{k}_N(\mathbf{z}_N) = \begin{bmatrix} \operatorname{diag}[1_{z_{1,N}>0}, \ldots, 1_{z_{m,N}>0}] \\ -\operatorname{diag}[1_{z_{1,N}<0}, \ldots, 1_{z_{m,N}<0}] \end{bmatrix}. \tag{A.4}$$

Note, from the above, one can also see that $\mathbf{x}_N^*$ is continuous in $\mathbf{z}_N$. The regions that define its pieces, $D_N^j(\mathbf{z}_N)$, are given through the inequalities in the indicator functions, and their closure constitutes a polyhedral decomposition of dom $\mathbf{x}_N^*$. Lastly, using the simple transformation $\mathbf{z}_N = \mathbf{i}'\mathbf{y}_N$, with $\mathbf{i} = [\mathbf{0}; \mathbf{I}; \mathbf{0}; \mathbf{0}]$, we can rewrite the solution as:

$$\mathbf{x}_N^* = \mathbf{K}_N(\mathbf{y}_N)\mathbf{y}_N, \quad \mathbf{K}_N(\mathbf{y}_N) = \mathbf{k}_N(\mathbf{i}'\mathbf{y}_N)\mathbf{i}', \tag{A.5}$$

where $\mathbf{K}(\mathbf{y}_N)$ is a piecewise constant matrix, containing 1's and 0's.

For convenience, we introduce the following quantity:

$$Q_N(\mathbf{y}_N) = \mathbf{y}_N'\mathbf{N}\mathbf{x}_N^* + \mathbf{x}_N^{*\prime}\mathbf{Q}\mathbf{x}_N^* = \mathbf{y}_N'\hat{\mathbf{M}}_N\mathbf{y}_N,$$

where $\mathbf{x}_N^* = \mathbf{K}_N\mathbf{y}_N$ and

$$\hat{\mathbf{M}}_N = \tfrac{1}{2}((\mathbf{N}\mathbf{K}_N + \mathbf{K}_N'\mathbf{Q}\mathbf{K}_N) + (\mathbf{N}\mathbf{K}_N + \mathbf{K}_N'\mathbf{Q}\mathbf{K}_N)'). \tag{A.6}$$

Here, $Q_N(\mathbf{y}_N)$ represents the cash (net of fundamental value) generated by the optimal trade $\mathbf{x}_N^*$ at $t = N$—the first equation expresses it in terms of the optimal policy at $t = N$ and the second equation expresses it in terms of the underlying state variables. The wealth W_N in (A.3) now can be expressed as

$$W_N(\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}, \mathbf{x}_N^*) = W_{N-1} - \mathbf{u}_N'\mathbf{z}_N - \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_N - Q_N(\mathbf{y}_N) \stackrel{\text{def}}{=} \hat{W}_N. \tag{A.7}$$

We refer to $W_N(\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}, \mathbf{x}_N^*) = \hat{W}_N$ as the "wealth-to-go," i.e, the wealth under the optimal strategy at terminal time N. By Lemma A1(c), strict concavity of $W_N(\cdot)$ in $\mathbf{x}_0, \ldots, \mathbf{x}_N$ guarantees strict concavity of $\hat{W}_N$ in $\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}$. The value function can then be expressed as

$$J_N(W_{N-1}, \mathbf{y}_N, \mathbf{u}_N) = -e^{-\alpha\hat{W}_N}. \tag{A.8}$$

By Lemma A1(a), strict concavity of $\hat{W}_N$ implies strict concavity of J_N in $\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}$. From Result A1, $\mathbf{x}_N^*$ is continuous and subdifferentiable in $\mathbf{y}_N$. It then follows that $Q_N(\mathbf{y}_N)$ is also a continuous and subdifferentiable PWQ in $\mathbf{y}_N$.

Two important observations are in order: First, we will refer to the value function in (A.8) as *separable*, in the sense that the noise term $\mathbf{u}_N'\mathbf{z}_N$ appears as an additive linear term inside the exponential. Separability will be key to ensure that the optimal policy remains deterministic at all n.

A.4.3. Induction

Following the above results for N, we propose that they continue to hold at time $n+1$:

(P0) At time $n+1$, we have

$$\mathbf{x}_{n+1}^* \stackrel{\text{def}}{=} \mathbf{K}_{n+1}\mathbf{y}_{n+1}, \tag{A.9a}$$
$$J_{n+1} \stackrel{\text{def}}{=} -e^{-\alpha\hat{W}_{n+1}}, \tag{A.9b}$$
$$\begin{aligned} \hat{W}_{n+1} &\stackrel{\text{def}}{=} W_n - \mathbf{u}_{n+1}'\mathbf{z}_{n+1} - \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_{n+1} - Q_{n+1}(\mathbf{y}_{n+1}), \\ Q_{n+1}(\mathbf{y}_{n+1}) &\stackrel{\text{def}}{=} \mathbf{y}_{n+1}'\hat{\mathbf{M}}_{n+1}\mathbf{y}_{n+1}, \end{aligned} \tag{A.9c}$$

and J_{n+1} and $\hat{W}_{n+1}$ are strictly concave in $\mathbf{x}_{[0,n]} = [\mathbf{x}_0; \ldots; \mathbf{x}_n]$, where $Q_{n+1}(\mathbf{y}_{n+1})$ is PWQ, continuous, subdifferentiable, and its pieces, defined by polyhedral sets $D_{n+1}^j(\mathbf{y}_{n+1})$, constitute a polyhedral decomposition of dom Q_{n+1}.

Clearly, (P0) holds for $n+1 = N$. We want to show that (P0) being true at $n+1$ implies that it is also true at n:

(P1) $\mathbf{x}_n^*$ retains the piecewise linear form of (A.9a), and is continuous and subdifferentiable; and

(P2) J_n retains the separable form of (A.9b), and is continuous, subdifferentiable.

A.4.4. Proof of (P1)—The Optimal Trade at n

The value function at n can be obtained through the value function at $n+1$ from

$$\begin{aligned} J_n &= \underset{\mathbf{x}_n \geq 0}{\text{maximize}}\, \mathrm{E}_n[J_{n+1}] \\ &= \underset{\mathbf{x}_n \geq 0}{\text{maximize}}\, \mathbb{E}_n[-e^{-\alpha\hat{W}_{n+1}}] \stackrel{\text{def}}{=} -e^{-\alpha\hat{W}_n}. \end{aligned} \tag{A.10}$$

We proceed to express $\hat{W}_{n+1}(W_n, \mathbf{y}_{n+1}, \mathbf{u}_{n+1})$ as a function of the optimization variable $\mathbf{x}_n$ and the current state dynamics. First, we deal with W_n. Following (20), we have

$$W_n = W_{n-1} - (\mathbf{u}_n'\boldsymbol{\delta}' + \mathbf{z}_0'\boldsymbol{\Lambda}\boldsymbol{\delta}' + \mathbf{y}_n'\mathbf{N})\mathbf{x}_n - \mathbf{x}_n'\mathbf{Q}\mathbf{x}_n.$$

Plugging this expression for W_n into $\hat{W}_{n+1}$ in (A.9b) gives

$$\begin{aligned} \hat{W}_{n+1} = W_{n-1} &- (\mathbf{u}_n'\boldsymbol{\delta}' + \mathbf{z}_0'\boldsymbol{\Lambda}\boldsymbol{\delta}' + \mathbf{y}_n'\mathbf{N})\mathbf{x}_n - \mathbf{x}_n'\mathbf{Q}\mathbf{x}_n \\ &- \mathbf{u}_{n+1}'\mathbf{z}_{n+1} - \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_{n+1} - \mathbf{y}_{n+1}'\hat{\mathbf{M}}_{n+1}\mathbf{y}_{n+1}. \end{aligned}$$

First, we deal with the term $\mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_{n+1} = \mathbf{z}_0'\boldsymbol{\Lambda}(\mathbf{z}_n - \boldsymbol{\delta}'\mathbf{x}_n) = \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_n - \mathbf{z}_0'\boldsymbol{\Lambda}\boldsymbol{\delta}'\mathbf{x}_n$. This leads to

$$\begin{aligned} \hat{W}_{n+1} = W_{n-1} &- (\mathbf{u}_n'\boldsymbol{\delta}' + \mathbf{y}_n'\mathbf{N})\mathbf{x}_n - \mathbf{x}_n'\mathbf{Q}\mathbf{x}_n - \mathbf{u}_{n+1}'\mathbf{z}_n \\ &- \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_n - \mathbf{y}_{n+1}'\hat{\mathbf{M}}_{n+1}\mathbf{y}_{n+1}. \end{aligned}$$

Next, we deal with the random walk term $-\mathbf{u}'_{n+1}\mathbf{z}_{n+1}$ using the dynamics for $\mathbf{u}_{n+1}$ and $\mathbf{z}_{n+1}$:

$$\begin{aligned}-\mathbf{u}'_{n+1}\mathbf{z}_{n+1} &= -(\mathbf{u}'_n + \boldsymbol{\epsilon}'_{n+1})(\mathbf{z}_n - \boldsymbol{\delta}'\mathbf{x}_n)\\ &= -\mathbf{u}'_n\mathbf{z}_n + \mathbf{u}'_n\boldsymbol{\delta}'\mathbf{x}_n - \boldsymbol{\epsilon}'_{n+1}(\mathbf{z}_n - \boldsymbol{\delta}'\mathbf{x}_n).\end{aligned}$$

Substituting this expression back into $\hat{W}_{n+1}$, we have

$$\begin{aligned}\hat{W}_{n+1} &= W_{n-1} - (\cancel{\mathbf{u}'_n\boldsymbol{\delta}'} + \mathbf{y}'_n\mathbf{N})\mathbf{x}_n - \mathbf{x}'_n\mathbf{Q}\mathbf{x}_n - \mathbf{u}'_n\mathbf{z}_n + \cancel{\mathbf{u}'_n\boldsymbol{\delta}'\mathbf{x}_n}\\ &\quad - \boldsymbol{\epsilon}'_{n+1}(\mathbf{z}_n - \boldsymbol{\delta}'\mathbf{x}_n) - \mathbf{z}'_0\boldsymbol{\Lambda}\mathbf{z}_n - \mathbf{y}'_{n+1}\hat{\mathbf{M}}_{n+1}\mathbf{y}_{n+1}\\ &= W_{n-1} - \mathbf{y}'_n\mathbf{N}\mathbf{x}_n - \mathbf{x}'_n\mathbf{Q}\mathbf{x}_n - \mathbf{u}'_n\mathbf{z}_n - \boldsymbol{\epsilon}'_{n+1}(\mathbf{z}_n - \boldsymbol{\delta}'\mathbf{x}_n)\\ &\quad - \mathbf{z}'_0\boldsymbol{\Lambda}\mathbf{z}_n - \mathbf{y}'_{n+1}\hat{\mathbf{M}}_{n+1}\mathbf{y}_{n+1}.\end{aligned}$$

Next, we need to express $\mathbf{y}_{n+1}$ as a function of the state at n using the state dynamics (19), that is, $\mathbf{y}_{n+1} = \mathbf{A}\mathbf{y}_n + \mathbf{B}\mathbf{x}_n$, which gives

$$\begin{aligned}\hat{W}_{n+1} &= W_{n-1} - \mathbf{z}'_0\boldsymbol{\Lambda}\mathbf{z}_n - \mathbf{y}'_n\mathbf{N}\mathbf{x}_n - \mathbf{x}'_n\mathbf{Q}\mathbf{x}_n - (\mathbf{A}\mathbf{y}_n + \mathbf{B}\mathbf{x}_n)'\\ &\quad \cdot \hat{\mathbf{M}}_{n+1}(\mathbf{A}\mathbf{y}_n + \mathbf{B}\mathbf{x}_n) - \mathbf{u}'_n\mathbf{z}_n - \boldsymbol{\epsilon}'_{n+1}(\mathbf{z}_n - \boldsymbol{\delta}'\mathbf{x}_n).\end{aligned}$$

Given $\mathcal{F}_n$ (including $\mathbf{u}_n$ and $\mathbf{y}_n$), the only remaining source of risk is $\boldsymbol{\epsilon}'_{n+1}$, which is normally distributed. Thus $\hat{W}_{n+1}$ is normally distributed and we have

$$\begin{aligned}J_n &= \underset{\mathbf{x}_n \geq 0}{\text{maximize}}\, \mathrm{E}_n[-e^{-\alpha\hat{W}_{n+1}}]\\ &= \underset{\mathbf{x}_n \geq 0}{\text{maximize}} -e^{-\alpha(\mathbb{E}_n[\hat{W}_{n+1}] - (1/2)\alpha\,\mathrm{Var}_n[\hat{W}_{n+1}])}.\end{aligned}$$

By the induction definition (A.9b), $J_n \stackrel{\text{def}}{=} -e^{-\alpha\hat{W}_n}$. Then, by monotonicity of the exponential, and identification, the optimization problem above is equivalent to

$$\hat{W}_n \stackrel{\text{def}}{=} \underset{\mathbf{x}_n \geq 0}{\text{maximize}}\, \mathbb{E}_n[\hat{W}_{n+1}] - \tfrac{1}{2}\alpha\,\mathrm{Var}_n[\hat{W}_{n+1}]. \tag{A.11}$$

Computing the mean and variance given $\mathcal{F}_n$ is straightforward:

$$\begin{aligned}\mathbb{E}_n[\hat{W}_{n+1}] &= W_{n-1} - \mathbf{z}'_0\boldsymbol{\Lambda}\mathbf{z}_n - \mathbf{y}'_n\mathbf{N}\mathbf{x}_n - \mathbf{x}'_n\mathbf{Q}\mathbf{x}_n - \mathbf{u}'_n\mathbf{z}_n\\ &\quad - (\mathbf{A}\mathbf{y}_n + \mathbf{B}\mathbf{x}_n)'\hat{\mathbf{M}}_{n+1}(\mathbf{A}\mathbf{y}_n + \mathbf{B}\mathbf{x}_n),\\ \mathrm{Var}_n[\hat{W}_{n+1}] &= (\mathbf{z}_n - \boldsymbol{\delta}'\mathbf{x}_n)'(\tau\boldsymbol{\Sigma})(\mathbf{z}_n - \boldsymbol{\delta}'\mathbf{x}_n)\\ &= (\mathbf{i}'\mathbf{y}_n - \boldsymbol{\delta}'\mathbf{x}_n)'(\tau\boldsymbol{\Sigma})(\mathbf{i}'\mathbf{y}_n - \boldsymbol{\delta}'\mathbf{x}_n).\end{aligned} \tag{A.12}$$

Before computing the solution, we can further simplify the stage objective

$$\tilde{f}_n \stackrel{\text{def}}{=} \mathbb{E}_n[\hat{W}_{n+1}] - \tfrac{1}{2}\alpha\,\mathrm{Var}_n[\hat{W}_{n+1}]$$

by dropping the terms that do not affect the optimization over $\mathbf{x}_n$ (i.e., the additive terms $W_{n-1} - \mathbf{z}'_0\boldsymbol{\Lambda}\mathbf{z}_n - \mathbf{u}'_n\mathbf{z}_n - \mathbf{y}'_n(\mathbf{A}'\hat{\mathbf{M}}_{n+1}\mathbf{A} + \frac{1}{2}\alpha\tau\mathbf{i}\boldsymbol{\Sigma}\mathbf{i}')\mathbf{y}_n$). We refer to this simplified objective as f_n. This allows us to conveniently write the manager's optimization problem at n in compact form

$$\underset{\mathbf{x}_n \geq 0}{\text{maximize}}\, f_n(\mathbf{x}_n, \mathbf{y}_n) \stackrel{\text{def}}{=} -\tfrac{1}{2}\mathbf{x}'_n\mathbf{M}_n\mathbf{x}_n - \mathbf{y}'_n\mathbf{N}_n\mathbf{x}_n, \tag{A.13}$$

where

$$\begin{aligned}\mathbf{M}_n &= \left(\mathbf{B}'\hat{\mathbf{M}}_{n+1}\mathbf{B} + \mathbf{Q} + \tfrac{1}{2}\alpha\tau\boldsymbol{\delta}\boldsymbol{\Sigma}\boldsymbol{\delta}'\right)\\ &\quad + \left(\mathbf{B}'\hat{\mathbf{M}}_{n+1}\mathbf{B} + \mathbf{Q} + \tfrac{1}{2}\alpha\tau\boldsymbol{\delta}\boldsymbol{\Sigma}\boldsymbol{\delta}'\right)', \qquad \text{(A.14a)}\\ \mathbf{N}_n &= \mathbf{N} + \mathbf{A}'(\hat{\mathbf{M}}_{n+1} + \hat{\mathbf{M}}'_{n+1})\mathbf{B} - \alpha\tau\mathbf{i}\boldsymbol{\Sigma}\boldsymbol{\delta}'. \qquad \text{(A.14b)}\end{aligned}$$

From (A.13), it is clear that we are dealing with a deterministic maximization problem over a piecewise quadratic and continuous objective function, with nonnegativity constraints on the control variables. All noise terms have been canceled out or are irrelevant with respect to the optimization. Thus, the optimal maximizer, if it exists, is deterministic.

A.4.5. Solution at Time n

To characterize the solution, we need to show that the problem fits the framework of PWQ program described in Section A.4. First, we show that the stage problem preserves concavity. We have the following result.

Result A2. *If $\hat{W}_{n+1}$ is strictly concave in $\mathbf{x}_{[0,n]}$, then* (a) *$\tilde{f}_n$ is strictly concave in $\mathbf{x}_{[0,n]}$, and therefore f_n is strictly concave in the optimization variable $\mathbf{x}_n$, and* (b) *$\hat{W}_n$ is strictly concave in $\mathbf{x}_{[0,n-1]}$.*

Proof. (a) The objective $\tilde{f}_n$ is a sum of expectation and variance terms. By Lemma A1(b), $\hat{W}_{n+1}$ strictly concave in $\mathbf{x}_{[0,n]}$ implies $\mathbb{E}_n[\hat{W}_{n+1}]$ strictly concave in $\mathbf{x}_{[0,n]}$. Next, notice that $\mathrm{Var}_n[\hat{W}_{n+1}]$ is a quadratic form in $\mathbf{x}_{[0,n]}$. In particular, from (1), $\mathbf{z}_n - \boldsymbol{\delta}'\mathbf{x}_n = \mathbf{z}_0 - \sum_{i=0}^{n}\boldsymbol{\delta}'\mathbf{x}_i \stackrel{\text{def}}{=} \mathbf{z}_0 - \boldsymbol{\delta}'_{[0,n]}\mathbf{x}_{[0,n]}$, where $\boldsymbol{\delta}'_{[0,n]} \stackrel{\text{def}}{=} \mathbf{diag}(\boldsymbol{\delta}', \ldots, \boldsymbol{\delta}')$ and $\mathbf{z}_0$ is constant. Substituting this into (A.12) gives

$$\mathrm{Var}_n[\hat{W}_{n+1}] = (\mathbf{z}_0 - \boldsymbol{\delta}'_{[0,n]}\mathbf{x}_{[0,n]})'(\tau\boldsymbol{\Sigma})(\mathbf{z}_0 - \boldsymbol{\delta}'_{[0,n]}\mathbf{x}_{[0,n]}).$$

Given $\boldsymbol{\Sigma}$ is a covariance matrix, it is positive semidefinite, and since $\tau > 0$ is a positive scalar, $\mathrm{Var}_n[\hat{W}_{n+1}]$ is convex in $\mathbf{x}_{[0,n]}$. Hence $-\frac{1}{2}\alpha\,\mathrm{Var}_n[\hat{W}_{n+1}]$ is concave in $\mathbf{x}_{[0,n]}$, for $\alpha > 0$. As $\tilde{f}_n$ is the sum of a strictly concave function with a concave function, it is strictly concave in $\mathbf{x}_{[0,n]}$. It follows the simplified objective f_n is strictly concave in $\mathbf{x}_n$.

(b) From (A.11) and Result A2(a), $\hat{W}_n \stackrel{\text{def}}{=} \max_{\mathbf{x}_n} \tilde{f}_n$, with $\tilde{f}_n$ strictly concave in $\mathbf{x}_{[0,n]}$. Therefore, by Lemma A1(c), $\hat{W}_n$ is strictly concave in $\mathbf{x}_{[0,n-1]}$. □

Second, we need to show that the pieces of f_n constitute a polyhedral decomposition.

Result A3. *The regions of $f_n(\mathbf{x}_n, \mathbf{y}_n)$ constitute a polyhedral decomposition* (*in $\mathbf{y}_n$*) *of the domain of f_n.*

Proof. Observe that from the induction assumption (P0), the pieces of Q_{n+1} (and hence of $\hat{\mathbf{M}}_{n+1}, \mathbf{M}_n, \mathbf{N}_n$) constitute a polyhedral decomposition in $\mathbf{y}_{n+1}$. This implies that the regions can generally be written as

$$D^j_{n+1} = \left\{(\mathbf{x}_{n+1}, \mathbf{y}_{n+1}) \mid \mathbf{A}^j_{n+1}\mathbf{x}_{n+1} \leq \mathbf{B}^j_{n+1}\mathbf{y}_{n+1}\right\}.$$

From the induction assumption (A.9a), $\mathbf{x}^*_{n+1} = \mathbf{K}_{n+1}\mathbf{y}_{n+1}$, and from the system dynamics, $\mathbf{y}_{n+1} = \mathbf{A}\mathbf{y}_n + \mathbf{B}\mathbf{x}_n$, thus

$$D^j_{n+1} = \left\{(\mathbf{x}_n, \mathbf{y}_n) \mid \mathbf{A}^j_{n+1}\mathbf{K}_{n+1}(\mathbf{A}\mathbf{y}_n + \mathbf{B}\mathbf{x}_n) \leq \mathbf{B}^j_{n+1}(\mathbf{A}\mathbf{y}_n + \mathbf{B}\mathbf{x}_n)\right\},$$

which is clearly polyhedral in $\mathbf{y}_n$. Furthermore, the additional positivity requirement $\mathbf{x}_n \geq 0$ is trivially polyhedral in $\mathbf{y}_n$ (without loss, it can be written as $\mathbf{x}_n \geq 0\mathbf{y}_n$). Thus $f_n(\mathbf{x}_n, \mathbf{y}_n)$ is PWQ and its regions constitute a polyhedral decomposition. □

Following Result A3, Result A1 holds, implying that the optimal trade can be written as

$$\mathbf{x}^*_n = \mathbf{K}_n(\mathbf{y}_n)\mathbf{y}_n,$$

for some piecewise constant $\mathbf{K}_n(\mathbf{y}_n)$. Lastly, from Result A1, $\mathbf{x}^*_n$ is continuous and subdifferentiable with respect to $\mathbf{y}_n$. This completes the induction property (P1). □

Before moving on to show (P2), we can provide some additional guidance as to how the structure of the solution emerges through the generalized KKT conditions in

Result A1(d). We provide an outline below and refer interested readers to the literature (e.g., Patrinos and Sarimveis 2011) that has developed efficient algorithms to deal with the required computations.

Let $\boldsymbol{\nu}_n = [\boldsymbol{\nu}_n^+; \boldsymbol{\nu}_n^-]$ be the associated Lagrange nonnegativity multipliers at n. We have the following generalized KKT conditions:

(K1) Optimality: $\mathbf{0} \in \partial_{\mathbf{x}_n}[f_n(\mathbf{x}_n, \mathbf{y}_n) + \boldsymbol{\nu}_n'\mathbf{x}_n]$.

(K2–K4) $\mathbf{x}_n \geq 0$, $\boldsymbol{\nu}_n \geq \mathbf{0}$ and $\boldsymbol{\nu}_n \otimes \mathbf{x}_n = \mathbf{0}$.

To observe how the piecewise affine structure emerges from the above KKT conditions, it is helpful to split the domain of f between differentiable points, and nondifferentiable points which can occur at the boundaries between regions D_{n+1}^j.

In the differentiable regions, the subdifferential condition (K1) reduces to a standard first-order condition (FOC) on the gradients:

$$\mathbf{0} = \mathbf{M}_n\mathbf{x}_n + \mathbf{N}_n'\mathbf{y}_n - \boldsymbol{\nu}_n, \tag{A.15}$$

which leads to a standard piecewise linear solutions. To see this, note that we can write without loss of generality:

$$\mathbf{x}_n = \begin{bmatrix} \mathbf{x}_n^{(+)} \\ \mathbf{0} \end{bmatrix}, \quad \boldsymbol{\nu}_n = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\nu}_n^{(+)} \end{bmatrix},$$

where $[\mathbf{x}_n]^{(+)}$ denotes the subvector of nonzero trades, $[\boldsymbol{\nu}_n]^{(+)}$ denotes the subvector of nonzero multipliers. Clearly, $\boldsymbol{\nu}_n \otimes \mathbf{x}_n = 0$. We can then write (A.15) as[21]

$$\mathbf{0} = \begin{bmatrix} \mathbf{M}_{n,11} & \mathbf{M}_{n,12} \\ \mathbf{M}_{n,21} & \mathbf{M}_{n,22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_n^{(+)} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{N}_{n,1}' \\ \mathbf{N}_{n,2}' \end{bmatrix} \mathbf{y}_n - \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\nu}_n^{(+)} \end{bmatrix}. \tag{A.16}$$

This leads to the following linear structure for $\mathbf{x}_n$ and $\boldsymbol{\nu}_n$ in this region:

$$\mathbf{x}_n = \begin{bmatrix} -\mathbf{M}_{n,11}^{-1}\mathbf{N}_{n,1}' \\ \mathbf{0} \end{bmatrix} \mathbf{y}_n, \quad \boldsymbol{\nu}_n = \begin{bmatrix} \mathbf{0} \\ -\mathbf{M}_{n,12}\mathbf{M}_{n,11}^{-1}\mathbf{N}_{n,1}' + \mathbf{N}_{n,2}' \end{bmatrix} \mathbf{y}_n.$$

Therefore, the solution in this region clearly has the piecewise linear structure of (A.2). In fact, the piecewise linear structure also applies to the multiplier.

Next, we deal with any nondifferentiable point, which can occur at the boundaries of regions D_{n+1}. At any such a boundary point $\mathbf{x}_n^C$, if $\mathbf{0} \in \partial_{\mathbf{x}_n^C}[f_n(\mathbf{x}_n^C, \mathbf{y}_n) + \boldsymbol{\nu}_n'\mathbf{x}_n^C]$, then it is the global maximum. From Result A3, regions D_{n+1} constitute a polyhedral decomposition in $\mathbf{y}_n$ and hence any point $\mathbf{x}_n^C$ at the boundary of these regions is necessarily polyhedral in $\mathbf{y}_n$. Therefore in this region, any optimal solution $\mathbf{x}_n^b$ has the same general structure of (A.2).

A.4.6. Proof of (P2)—The Value Function at n.

The next step is to compute the value function J_n. Substituting the optimal trade policy $\mathbf{x}_n^* = \mathbf{K}_n\mathbf{y}_n$ into the objective we just optimized in (A.13) yields

$$f_n^* = -\tfrac{1}{2}\mathbf{x}_n^{*\prime}\mathbf{M}_n\mathbf{x}_n^* - \mathbf{y}_n'\mathbf{N}_n\mathbf{x}_n^* = -\mathbf{y}_n'\left(\tfrac{1}{2}\mathbf{K}_n'\mathbf{M}_n\mathbf{K}_n + \mathbf{N}_n\mathbf{K}_n\right)\mathbf{y}_n.$$

Plugging this into J_n and adding back the irrelevant terms omitted preoptimization, we obtain:

$$J_n = -e^{-\alpha\hat{W}_n}$$

with

$$\hat{W}_n \stackrel{\text{def}}{=} \max_{\mathbf{x}_n \geq 0}\left\{\tilde{f}_n = (W_{n-1} - \mathbf{u}_n'\mathbf{z}_n - \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_n - \mathbf{y}_n'\hat{\mathbf{M}}_n\mathbf{y}_n)\right\}, \quad \text{where}$$

$$\begin{aligned}\hat{\mathbf{M}}_n &= \tfrac{1}{2}\mathbf{K}_n'\mathbf{M}_n\mathbf{K}_n + \tfrac{1}{2}(\mathbf{N}_n\mathbf{K}_n + \mathbf{K}_n'\mathbf{N}_n') + \mathbf{A}'\hat{\mathbf{M}}_{n+1}\mathbf{A} \\ &\quad + \tfrac{1}{2}\alpha\tau\mathbf{i}\boldsymbol{\Sigma}\mathbf{i}'.\end{aligned} \tag{A.17}$$

As the last step, let

$$Q_n(\mathbf{y}_n) = \mathbf{y}_n'\hat{\mathbf{M}}_n\mathbf{y}_n.$$

Then we can write the value function as

$$\begin{aligned}&J_n(W_{n-1}, \mathbf{u}_n, \mathbf{y}_n) \\ &\quad = -e^{-\alpha\hat{W}_n} = -\exp\left[-\alpha(W_{n-1} - \mathbf{u}_n'\mathbf{z}_n - \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_n - Q_n(\mathbf{y}_n))\right].\end{aligned}$$

Thus, the value function retains the same form in (A.9b). Lastly, we need to verify concavity and continuity. For concavity, from Result A2(b), $\hat{W}_n$ is strictly concave in $\mathbf{x}_{[0,n-1]}$, and thus by Lemma A1(a), J_n is strictly concave in $\mathbf{x}_{[0,n-1]}$ for $\alpha > 0$. The recursion thus preserves concavity at all iterations (concavity at the terminal time N was shown in Section A.4). For continuity, following Result A1, the value function is continuous everywhere. This concludes the induction property (P2).

It follows by induction that (P1) and (P2) hold for all time periods, and the proof is complete for Propositions 1 and 2. □

A.5. Proof of Proposition 3—Static QP Equivalence

Proposition 1 allows us to reformulate the problem (21) as a large, but static, quadratic program. We prove this in two steps: First, Section A.5 shows that under Proposition 1, static and dynamic formulations are equivalent at all times. Second, Section A.5 reformulates the static problem into a QP.

A.5.1. DP/Static Equivalence

Let V_n represent the static problem, truncated to arbitrary time $n \in \{0, \ldots, N\}$, that is,

$$V_n = \max_{\mathbf{x}_n, \ldots, \mathbf{x}_N \geq 0} \mathbb{E}_n\left[-e^{-\alpha W_N}\right], \tag{A.18}$$

where $\mathbb{E}_n[\cdot] = \mathbb{E}[\cdot \mid \mathcal{F}_n]$. We have the following result.

Corollary 1. *Under Proposition 1, the static and dynamic formulations are fully equivalent, that is,* $J_n = V_n$, $\forall\, n \in \{0, \ldots, N\}$.

Proof. We proceed by induction, starting at the terminal time N. The equality $V_N = J_N$ is trivial to show: we have $V_N = \max_{\mathbf{x}_N \geq 0} \mathbb{E}_N[-e^{-\alpha W_N}] = \max_{\mathbf{x}_N \geq 0} -e^{-\alpha W_N} = J_N$, where the second equality follow from the fact that $\mathbf{u}_N$ is realized at time N, and the third equality follows from the definition of J_N.

At an arbitrary time n, assume

$$V_{n+1} = J_{n+1}. \tag{A.19}$$

We want to show this implies $V_n = J_n$. We have

$$\begin{aligned}J_n &= \max_{\mathbf{x}_n \geq 0} \mathbb{E}_n[J_{n+1}^*] = \max_{\mathbf{x}_n \geq 0} \mathbb{E}_n[V_{n+1}] \\ &= \max_{\mathbf{x}_n \geq 0} \mathbb{E}_n\left[\max_{\mathbf{x}_{n+1}, \ldots, \mathbf{x}_N \geq 0} \mathbb{E}_{n+1}[-e^{-\alpha W_N}]\right],\end{aligned}$$

where the first equality follows from the induction assumption (A.19) and the second follows from the definition of V_{n+1} in (A.18) with $n \to n+1$. From Proposition 1, $\mathbf{x}_{n+1}, \ldots, \mathbf{x}_N$ are path independent, hence the max and expectation operators at are commutable at n. This leads to the following:

$$\begin{aligned} J_n &= \max_{\mathbf{x}_n \geq 0} \max_{\mathbf{x}_{n+1},\ldots,\mathbf{x}_N \geq 0} \mathbb{E}_n[\mathbb{E}_{n+1}[-e^{-\alpha W_N}]] \\ &= \max_{\mathbf{x}_n,\ldots,\mathbf{x}_N \geq 0} \mathbb{E}_n[\mathbb{E}_{n+1}[-e^{-\alpha W_N}]] = \max_{\mathbf{x}_n,\ldots,\mathbf{x}_N \geq 0} \mathbb{E}_n[-e^{-\alpha W_N}] = V_n, \end{aligned}$$

where the third equality follows from the law of iterated expectations (i.e., tower property) and the fourth follows from the definition of V_n in (A.18). This allows us to conclude that $J_0 = V_0$ by induction. Hence, any optimal solution to the static problem is also an optimal solution to the dynamic problem. □

A.5.2. QP Reformulation

Next, we show how V_0 can be transformed into a QP. From (25), the stochastic terminal wealth function can be compactly written as

$$W_N = W_{0^-} - \mathbf{u}'\bar{\boldsymbol{\delta}}'\mathbf{x} - \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_0 - (\mathbf{y}_0'\bar{\mathbf{A}}'\bar{\mathbf{N}})\mathbf{x} - \mathbf{x}'\mathbf{D}\mathbf{x}.$$

Using Proposition 1, we can treat the optimal controls as deterministic variables. It follows that the only source of uncertainty in the problem is the random walk, implying that the manager's total postexecution wealth is normally distributed. We have that $W_N \sim \mathcal{N}(\mu_{W_N}, \sigma^2_{W_N})$, where

$$\begin{aligned} \mu_{W_N} = \mathbb{E}_0[W_N] = {} & W_{0^-} - \mathbf{u}_0'\bar{\mathbf{I}}\bar{\boldsymbol{\delta}}'\mathbf{x} \\ & - \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_0 - (\mathbf{y}_0'\bar{\mathbf{A}}'\bar{\mathbf{N}})\mathbf{x} - \mathbf{x}'\mathbf{D}\mathbf{x} \end{aligned} \quad \text{(A.20)}$$

and

$$\sigma^2_{W_N} = \mathrm{Var}_0[W_N] = \mathbf{x}'\bar{\boldsymbol{\delta}}\boldsymbol{\Sigma}_{\mathbf{u}}\bar{\boldsymbol{\delta}}'\mathbf{x}, \quad \text{(A.21)}$$

where $\mathbf{u}_0'$ contains the initial asset prices (assumed > 0) and $\boldsymbol{\Sigma}_{\mathbf{u}}$ is the covariance matrix of the noise process vector $\mathbf{u} = [\mathbf{u}_0; \ldots; \mathbf{u}_N]$.

A consequence of the normal distribution is that we can establish equivalence between the manager's exponential utility and the mean-variance objective often used in the execution literature. This follows directly from the identity $\mathbb{E}[e^{\alpha W}] = e^{\mathbb{E}[\alpha W] + \frac{1}{2}\alpha^2 \mathrm{Var}[W]}$, for any normally distributed W, and from the monotonicity of the exponential. The manager's original optimization problem over his exponential utility can thus be equivalently written as

$$\max_{\mathbf{x} \in S^x} \mu_{W_N} - \tfrac{1}{2}\alpha\sigma^2_{W_N}, \quad \text{(A.22)}$$

where the feasible set $S^x = \{\mathbf{x} \mid \mathbf{x} \geq 0, \bar{\mathbf{I}}\bar{\boldsymbol{\delta}}'\mathbf{x} = \mathbf{z}_0\}$. The first condition is the positivity constraint on the inputs, and the second ensures that all inputs sum to the manager's total order size $\mathbf{z}_0$. Using this equivalent form and the Equations (A.20) and (A.21), the manager's optimization problem becomes

$$\max_{\mathbf{x} \in S^x} W_{0^-} - \mathbf{u}_0'\mathbf{z}_0 - \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_0 - (\mathbf{y}_0'\bar{\mathbf{A}}'\bar{\mathbf{N}})\mathbf{x} - \mathbf{x}'\big(\mathbf{D} + \tfrac{1}{2}\alpha\bar{\boldsymbol{\delta}}\boldsymbol{\Sigma}_{\mathbf{u}}\bar{\boldsymbol{\delta}}'\big)\mathbf{x}.$$

The above problem can be equivalently written as a minimization problem over the risk-adjusted execution shortfall (i.e., risk-adjusted net execution cost) by subtracting the constant $(W_{0^-} - \mathbf{u}_0'\mathbf{z}_0)$, that is, the initial wealth and preexecution value of the portfolio, and multiplying the objective by (-1). The problem then becomes

$$\min_{\mathbf{x} \in S^x} \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_0 + \mathbf{y}_0'\bar{\mathbf{A}}'\bar{\mathbf{N}}\mathbf{x} + \mathbf{x}'\big(\mathbf{D} + \tfrac{1}{2}\alpha\bar{\boldsymbol{\delta}}\boldsymbol{\Sigma}_{\mathbf{u}}\bar{\boldsymbol{\delta}}'\big)\mathbf{x}.$$

Let $\mathbf{D}^* = \mathbf{D} + \frac{1}{2}\alpha\bar{\boldsymbol{\delta}}\boldsymbol{\Sigma}_{\mathbf{u}}\bar{\boldsymbol{\delta}}'$. Since we have $\mathbf{x}'\mathbf{D}^*\mathbf{x} = \mathbf{x}'((\mathbf{D}^* + \mathbf{D}^{*\prime})/2)\mathbf{x}$, we set the symmetric form $\frac{1}{2}\bar{\mathbf{D}} = ((\mathbf{D}^* + \mathbf{D}^{*\prime})/2)$. So finally, the optimization problem is equivalent to

$$\underset{\mathbf{x} \in S^x}{\text{minimize}} \big\{ \tfrac{1}{2}\mathbf{x}'\bar{\mathbf{D}}\mathbf{x} + (\mathbf{y}_0'\bar{\mathbf{A}}'\bar{\mathbf{N}})\mathbf{x} + \mathbf{z}_0'\boldsymbol{\Lambda}\mathbf{z}_0 \big\}. \quad \square$$

Endnotes

[1] For example, Perold (1988) shows that execution can reduce returns, leading to a significant "implementation shortfall."

[2] See also Almgren (2009), Lorenz and Almgren (2012), He and Mamaysky (2005), and Schied and Schoeneborn (2009). For empirical foundations, see Bouchaud et al. (2009) for a survey as well as the references in Alfonsi et al. (2008, 2010) and Obizhaeva and Wang (2013). For studies on how trade size affects prices, see Chan and Fong (2000), Chan and Lakonishok (1995), Chordia et al. (2002), and Dufour and Engle (2000).

[3] See also Alfonsi et al. (2008, 2010), Bayraktar and Ludkovski (2011), Chen et al. (2013), Cont et al. (2010), Obizhaeva and Wang (2013), Maglaras et al. (2015), and Predoiu et al. (2011). In other related work, Rosu (2009) develops a full equilibrium game theoretic framework and characterizes several important empirically verifiable results based on a model of a limit order market for one asset. Moallemi et al. (2012) develop an insightful equilibrium model of a trader facing an uninformed arbitrageur and show that optimal execution strategies can differ significantly when strategic agents are present in the market.

[4] The existence of cross asset price-impact effects has been empirically documented and theoretically justified. It can simply result from dealers' attempts to manage their inventory fluctuations; see for example, Chordia and Subrahmanyam (2004) and Andrade et al. (2008). Kyle and Xiong (2001) show that correlated liquidity shocks due to financial constraints can lead to cross liquidity effects. King and Wadhwani (1990) argue that in the presence of information asymmetry among investors, correlated information shocks can lead to cross asset liquidity effects among fundamentally related assets. Fleming et al. (1998) show that portfolio rebalancing trades from privately informed investors can lead to cross impact in the presence of risk aversion, even between assets fundamentally uncorrelated. Pasquariello and Vega (2015) develop a stylized model and provide empirical evidence suggesting that cross impact may stem from the strategic trading activity of sophisticated speculators who are trying to mask their informational advantage. Hasbrouck and Seppi (2001) find that both returns and order flows can be characterized by common factors. Lastly, evidence of comovement stemming from sentiment-based views has been studied in Barberis et al. (2005).

[5] A more concrete example of how price-impact models can be integrated in a broader portfolio selection problem can be found in Iancu and Trichakis (2014), which focuses on the multiaccount portfolio optimization problem. A discussion regarding the applicability of advanced cross asset strategies and how they relate to agency trading and *best execution* constraints also can be found in the same paper.

[6] Disentangling cross impact from correlation for individual securities is a challenging statistical problem which is beyond the scope of this paper. Empirical estimation of cross impact is an active area of research for high-frequency trading firms and also could be an interesting direction for future academic research. See Schneider and Lillo (2017) for a recent study.

[7] See Moallemi and Sağlam (2013) for a study regarding the optimal placement of limits orders. See Maglaras et al. (2017) for a study on order placement in fragmented markets.

[8] The best available ask price, $a_{i,n}$, is the lowest price at which a market buy order could (partially or fully) be executed at time n. Similarly, the best available limit bid price $b_{i,n}$ is the highest price at which a market sell order could be executed.

[9] We refer to Alfonsi et al. (2010) for a discussion about general types of density functions and to Predoiu et al. (2011) for an equivalence between discrete and continuous models. A queuing-based approach can be found in Cont et al. (2010).

[10] Although the random walk assumption implies a nonzero probability of negative prices, it is not a concern in our framework given the short-term horizon of optimal execution problems in practice. As such, this assumption is commonly used in this literature.

[11] We focus on a single buy order, implying $x^-_{i,n} = 0$, but the results are directly applicable to sell orders as well.

[12] We do not illustrate the impact of the random walk here to keep the figures clear. In order words, we are holding $u_{i,n}$ constant.

[13] The linearity assumption on the permanent price-impact function is consistent with theorem 1 of Huberman and Stanzl (2004), which provides conditions under which the price-impact model does not admit arbitrage and price manipulation strategies.

[14] Assumption 4 could be relaxed with alternative functional form specifications. The exponential form has the advantage of only requiring a single parameter to describe the replenishment process, keeping the problem tractable. Further, this form has been adopted in previous literature and is in line with several empirical findings on the order book replenishment process. See, for example, Biais et al. (1995) for a detailed empirical study.

[15] To see this, note that $\mathbf{D}$ is positive (semi)definite in Lemma 4. As $\boldsymbol{\Sigma}_{\mathbf{u}}$ is a covariance matrix, it is positive semidefinite by definition, hence $\bar{\mathbf{D}}$ is a sum of positive (semi)definite matrices and given $\alpha > 0$, is thus positive (semi)definite.

[16] Note, this does not imply that the actual bid-ask spread is zero during the execution process. Unsurprisingly, increasing the steady-state bid-ask spread leads to higher overall execution costs, reducing the applicability of advanced trading strategies. A detailed analysis is provided in Appendix A.1.

[17] The rest of the parameters used in this section are the volatilities $\sigma_1 = \sigma_2 = 0.05$, the order book densities $q_1 = q_2 = 1{,}500$, the replenishment rates $\rho_1 = \rho_2 = 5$, and the permanent impact parameters $\lambda_{11} = \lambda_{22} = 1/(3q_1)$. These parameters are used to generate all the figures, unless otherwise specified.

[18] The portfolio is initially equally weighted, consisting of 100 shares of each asset. Unless otherwise specified, the rest of the parameters used in this section are $v_{1,0} = v_{2,0} = 1$, $\sigma_1 = \sigma_2 = 0.05$, $\lambda_{11} = \lambda_{22} = 1/(3q_1)$, and $\lambda_{12} = \lambda_{21} = 0$.

[19] For completeness, match the objective functions by changing max to min, and observe $j \leftrightarrow k$, $\mathbf{Q}_k \leftrightarrow \mathbf{M}^j$, $\mathbf{R}'_k \leftrightarrow \mathbf{N}^{j'}$, $\mathbf{S}_k = \mathbf{0}$, $\mathbf{q}_k = 0$, $\mathbf{r}_k = 0$, $s_k = 0$. Matching the constraints: $C^j \leftrightarrow D^j$, $\mathbf{A}^j \leftrightarrow \mathbf{A}_k$, $\mathbf{B}^j \leftrightarrow \mathbf{B}_k$, and $\mathbf{b}_k = 0$. The authors also require the objective function to be proper, a property that is trivially satisfied in the case of strict concavity, as long as the function's effective domain is nonempty, dom $f \neq \emptyset$ (Kumar 1991, proposition 2.1.5). Also note, the additional positivity constraint in our setting $\mathbf{x} \geq 0$ is without loss. This could be readily incorporated in the definition of the polyhedral regions D^j, however, we explicitly separate it out for later convenience.

[20] Any single-valued piecewise affine function can be written like in (A.2) for a properly chosen piecewise constant $\mathbf{K}(\mathbf{y})$ and vector $\mathbf{y}$. To see this, consider that if $\mathscr{X}$ (in the PS setting) is single-valued and piecewise affine, then by definition, each piece can be written as $\mathscr{X}^k(\mathbf{p}) = \mathbf{Z}^k\mathbf{p} + \mathbf{c}^k$ if $\mathbf{p} \in D^k(\mathbf{p})$, for some constants $\mathbf{Z}^k, \mathbf{c}^k$, and where the regions $D^k(\mathbf{p})$ combine to form a polyhedral decomposition. Through a change of variable, letting $\mathbf{y} = [1\ \mathbf{p}]$ and $\mathbf{K}^k = [\mathbf{c}^k\ \mathbf{Z}^k]$, observe that $\mathscr{X}^k(\mathbf{p}) = \mathbf{Z}^k\mathbf{p} + \mathbf{c}^k = [\mathbf{c}^k\ \mathbf{Z}^k][1\ \mathbf{p}']' = \mathbf{K}^k\mathbf{y}$. Then, define the piecewise constant $\mathbf{K}(\mathbf{y}) = \mathbf{K}^k$ if $\mathbf{y} \in D^k(\mathbf{y})$, which leads to the desired result in (A.2).

[21] Note that since (A.16) is reexpressed in terms of the re-sorted vectors $\mathbf{x}_n$ and $\boldsymbol{v}_n$, this implies that the corresponding $\mathbf{M}_n$ and $\mathbf{N}_n$ have also undergone the necessary transformations such that (A.15) continues to hold.

References

Alfonsi A, Schied A, Fruth A (2010) Optimal execution strategies in limit order books with general shape functions. *Quant. Finance* 10(2):143–157.

Alfonsi A, Schied A, Schulz A (2008) Constrained portfolio liquidation in a limit order book model. *Banach Center Publ.* 83:9–25.

Almgren R (2009) Optimal trading in a dynamic market. Working paper, New York University, New York.

Almgren R, Chriss N (2000) Optimal execution of portfolio transactions. *J. Risk* 3(2):5–29.

Andrade S, Chang C, Seasholes M (2008) Trading imbalances, predictable reversals, and cross-stock price pressure. *J. Financial Econom.* 88(2):406–423.

Barberis N, Shleifer A, Wurgler J (2005) Comovement. *J. Financial Econom.* 75(2):283–317.

Basak S, Chabakauri G (2010) Dynamic mean-variance asset allocation. *Rev. Financial Stud.* 23(8):2970–3016.

Bayraktar E, Ludkovski M (2011) Optimal trade execution in illiquid markets. *Math. Finance* 21(4):681–701.

Bertsimas D, Lo A (1998) Optimal control of execution costs. *J. Financial Markets* 1(1):1–50.

Bertsimas D, Hummel P, Lo A (1999) Optimal control of execution costs for portfolios. *Comput. Sci. Engrg.* 1(6):40–53.

Biais B, Hillion P, Spatt C (1995) An empirical analysis of the limit order book and the order flow in the paris bourse. *J. Finance* 50(5):1655–1689.

Borrelli F, Bemporad A, Morari M (2017) *Predictive Control for Linear and Hybrid Systems* (Cambridge University Press, Cambridge, UK).

Bouchaud J, Farmer J, Lillo F (2009) How markets slowly digest changes in supply and demand. Hens T, Schenk-Hoppé KR, eds. *Handbook of Financial Markets* (North-Holland, Amsterdam), 57–160.

Boyd S, Vandenberghe L (2004) *Convex Optimization* (Cambridge University Press, Cambridge, UK).

Brown DB, Smith JE (2011) Dynamic portfolio optimization with transaction costs: Heuristics and dual bounds. *Management Sci.* 57(10):1752–1770.

Brown DB, Carlin B, Lobo S (2010) Optimal portfolio liquidation with distress risk. *Management Sci.* 56(11):1997–2014.

Chan K, Fong WM (2000) Trade size, order imbalance, and the volatility-volume relation. *J. Financial Econom.* 57(2):247–273.

Chan L, Lakonishok J (1995) The behavior of stock prices around institutional trades. *J. Finance* 50(4):1147–1174.

Chen N, Kou S, Wang C (2013) Limit order books with stochastic market depth. Working paper.

Chordia T, Subrahmanyam A (2004) Order imbalance and individual stock returns: Theory and evidence. *J. Financial Econom.* 72(3):485–518.

Chordia T, Roll R, Subrahmanyam A (2001) Market liquidity and trading activity. *J. Finance* 56(2):501–530.

Chordia T, Roll R, Subrahmanyam A (2002) Order imbalance, liquidity, and market returns. *J. Financial Econom.* 65(1):111–130.

Cont R, Stoikov S, Talreja R (2010) A stochastic model for order book dynamics. *Oper. Res.* 58(3):549–563.

Dufour A, Engle RF (2000) Time and the price impact of a trade. *J. Finance* 55(6):2467–2498.

Engle R, Ferstenberg R (2007) Execution risk. *J. Portfolio Management* 33(2):34–45.

Fleming J, Kirby C, Ostdiek B (1998) Information and volatility linkages in the stock, bond and money markets. *J. Financial Econom.* 49(1):111–137.

Hasbrouck J, Seppi D (2001) Common factors in prices, order flows, and liquidity. *J. Financial Econom.* 59(3):383–411.

He H, Mamaysky H (2005) Dynamic trading policies with price impact. *J. Econom. Dynam. Control* 29(5):891–930.

Huberman G, Stanzl W (2004) Price manipulation and quasi-arbitrage. *Econometrica* 72(4):1247–1275.

Huberman G, Stanzl W (2005) Optimal liquidity trading. *Rev. Finance* 9(2):165–200.

Iancu D, Trichakis N (2014) Fairness and efficiency in multiportfolio optimization. *Oper. Res.* 62(6):1283–1301.

King M, Wadhwani S (1990) Transmission of volatility between stock markets. *Rev. Financial Stud.* 3(1):5–33.

Kumar S (1991) *Recent Developments in Mathematical Programming* (CRC Press, Boca Raton, Florida).

Kyle A, Xiong W (2001) Contagion as a wealth effect. *J. Finance* 56(4):1401–1440.

Lorenz J, Almgren R (2012) Mean-variance optimal adaptive execution. *Appl. Math. Finance* 18(5):395–422.

Maglaras C, Moallemi CC, Zheng H (2015) Optimal execution in a limit order book and an associated microstructure market impact model. Working paper, Columbia Business School, New York.

Maglaras C, Moallemi C, Zheng H (2017) Queueing dynamics and state space collapse in fragmented limit order book markets. Working paper, Columbia Business School, New York.

Moallemi C, Sağlam M (2013) The cost of latency in high-frequency trading. *Oper. Res.* 61(5):1070–1086.

Moallemi C, Sağlam M (2017) Dynamic portfolio choice with linear rebalancing rules. *J. Financial Quant. Analysis* 52(3):1247–1278.

Moallemi C, Park B, Van Roy B (2012) Strategic execution in the presence of an uninformed arbitrageur. *J. Financial Markets* 15(4): 361–391.

Obizhaeva A, Wang J (2013) Optimal trading strategy with supply/ demand dynamics. *J. Financial Markets* 16(1):1–32.

Parlour C, Seppi D (2008) Limit order markets: A survey. Boot A, Thakor AV, eds. *Handbook of Financial Intermediation and Banking* (Elsevier Science, Amsterdam), 63–96.

Pasquariello P, Vega C (2015) Strategic cross-trading in the U.S. stock market. *Rev. Finance* 19(1):229–282.

Patrinos P, Sarimveis H (2011) Convex parametric piecewise quadratic optimization: Theory and algorithms. *Automatica* 47(8): 1770–1777.

Perold A (1988) The implementation shortfall: Paper versus reality. *J. Portfolio Management* 14(3):4–9.

Predoiu S, Shaikhet G, Shreve S (2011) Optimal execution in a general one-sided limit-order book. *SIAM J. Financial Math.* 2(1):183–212.

Rockafellar T, Wets R (1998) *Variational Analysis* (Springer, Berlin).

Rosu I (2009) A dynamic model of the limit order book. *Rev. Financial Stud.* 22(11):4601–4641.

Ruszczynski A (2006) *Nonlinear Optimization* (Princeton University Press, Princeton, NJ).

Schied A, Schoeneborn T (2009) Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets. *Finance and Stochastics* 13(2):181–204.

Schneider M, Lillo F (2017) Cross impact and no-dynamic-arbitrage. Working paper, Scuola Normale Superiore, Pisa, Italy.