Asset Pricing and the Credit Market

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This article studies the central role of the credit market. We show that the credit market facilitates optimal risk sharing by allowing less risk-averse investors to take on levered positions and consume more risk. The equilibrium amount behaves procyclically when aggregate consumption is low but countercyclically when it is high. The varying size of the credit market modifies the amount of risk sharing, which in turn influences asset prices such as expected stock returns, stock return volatility, and the term structure of interest rates. Our article provides a frictionless benchmark for the role and the behavior of the credit market. (*JEL* G11, G12)

The credit sector plays a central role in financial markets. Recent events such as the mortgage market crisis, the massive deleveraging of the financial sector, and the credit crunch in both the consumer and business sectors of the economy indicate that the availability of credit (or lack thereof) can have first-order effects on asset values and the broader economy. The bubble and bust of the credit market are often attributed to market frictions (e.g., Holmstrom and Tirole 1996; Kiyotaki and Moore 1997) and blamed as a potential cause of financial excess and the following crisis. Yet, there is little theoretical basis for what the efficient levels of credit should be in an economy and how it varies through economic cycles. In this article, we consider a frictionless financial market with a meaningful credit sector and study how it relates to economic cycles and asset prices. Our goal is to provide a simple benchmark for the behavior of the credit market.

We extend the canonical asset-pricing framework (e.g., Lucas 1978; Cox, Ingersoll, and Ross 1985) to include a credit sector. In particular, we consider a model with heterogeneous agents who rely on both the credit and asset markets to achieve optimal risk sharing. In such a model, the credit sector expands and contracts in response to changes in agents' risk-sharing needs over economic cycles. We use the model to examine how the amount of credit, determined endogenously in the market, facilitates risk sharing among investors and influences the behavior of stock and bond prices.

We are grateful for helpful comments and suggestions from John Cochrane, Jun Liu, Hanno Lustig, Monika Piazzesi, and Pedro Santa-Clara, for excellent research assistance from Dongyan Ye, and for comments from participants of the 15th Annual CFE Workshop on Derivative Securities & Risk Management at Columbia and seminars at Barclays Global Investors, the CFA Institute, Claremont-McKenna College, the FDIC, the University of Colorado, and Vanderbilt University. Send correspondence to Jiang Wang, E62-614, 100 Main Street, Cambridge, MA 02139, USA; telephone: 617-253-2632. E-mail: wangj@mit.edu.

doi:10.1093/rfs/hhs086

For parsimony, we consider two classes of investors with different levels of risk aversion, each represented by a representative agent. In such an economy, the equilibrium consumption allocation is such that the more risk-averse agent's consumption is less risky than the aggregate endowment (or consumption) while the less risk-averse agent's consumption is more risky. As a result, the more risk-averse agent ends up with the lion's share of total consumption in bad states of the economy (when aggregate consumption is low) while giving up much of his share in the good states.[1] Such optimal risk sharing is achieved by investors' dynamic trading strategies in the securities market. We provide explicit equilibrium solutions for investors' optimal credit and asset positions and security prices. Analyzing their activities in the credit and asset markets and the resulting prices leads to several interesting results.

First, the credit market is essential in facilitating this optimal risk sharing. In particular, the more risk-averse agent provides credit to the less risk-averse agent, allowing him to take on levered positions in the stock and thus bear more risk. In return, the more risk-averse agent switches part of his portfolio into debt, receiving a stream of safe cash flows in the form of interest payments. As a result, the size of the credit market varies drastically with market "demographics," i.e., the wealth distribution between the two agents. When the wealth is too skewed toward one agent, which is the case when the economy is in extremely good or bad states, the credit market becomes minuscule. This is because the agent with little wealth can no longer accommodate the borrowing or lending needs of the other agent. For intermediate states of the economy, however, the size of the credit market becomes substantial, allowing sufficient leverage for the less risk-averse agent to take on more risk.[2]

Consequently, the relative size of the credit market, measured by the ratio of the amount of credit in the market to the value of all assets or *market leverage ratio*, exhibits interesting dynamics. At low levels of aggregate consumption (low states), the market leverage ratio behaves procyclically. However, at high levels of aggregate consumption (high states), the market leverage ratio becomes countercyclical.

If we associate the financial sector as the credit supplier in our model and the rest of the economy as credit consumers, our analysis shows that, absent of frictions, the efficient level of leverage ratio should be procyclical for low and moderate states of the economy but turn countercyclical for high states of the economy. This implies that cyclical behavior in the size of the credit market

[1] The optimal consumption allocation in a complete market can be dated back to Arrow (1964) and Debru 1959 (see also Wilson 1968). Dumas 1989 and Wang 1996 derive the allocation in settings similar to ours.

[2] It should be pointed out that the credit market also facilitates intertemporal smoothing of consumption, which is different from consumption smoothing across different states of the economy. In our setting, even in the absence of risk, the two agents will still borrow and lend, to smooth their consumption over their life cycle, except that in this case the behavior of the credit market will be simple and deterministic. Thus, the credit market is driven by both the life cycle and the economy cycle, which gives rise to the risk of the overall economy. Since the emphasis of this article is on how the credit market evolves over economic cycles, we focus on the risk-sharing role of the credit market throughout the article, omitting the relatively simple role it plays for intertemporal smoothing.

by itself need not be a symptom of market failure and thus the basis for policy intervention.

Second, we show that the relative size of the credit market is closely related to the behavior of asset prices. Under calibrated parameter values, we find that stock return volatility comoves with the market's leverage ratio, defined as the total amount of credit in the market normalized by the total size of the market. This is in part because as leverage reaches its maximum, agents' wealth and consumption shares become most sensitive to changes of the economy, which leads to more volatile stock prices. In fact, we show that when both agents are present, despite the expanded risk-sharing opportunities provided by the credit and stock market, the equilibrium stock price volatility can be higher than its level when only one of the agents is present. In addition, we find that when the leverage ratio is maximized, the interest rate becomes more stable, which also leads to an overall upward-sloping term structure for interest rates.

Moreover, we show that under i.i.d. shocks to the economy, the resulting market demographics, shaped by agents' risk-sharing strategies, evolve in non-trivial ways. This leads to rich patterns in stock and bond returns. For example, the dividend yield, risk premium, and Sharpe ratio of the stock typically behave countercyclically. In extremely bad states of the economy, however, the stock's risk premium can turn procyclical, becoming negatively correlated with dividend yield. Under certain parameter values, the dividend yield and the expected return of the stock can both be nonmonotonic with respect to the level of the market and behave procyclically. Stock returns also display interesting forms of heteroscedasticity. Return volatility is highly persistent over time, procyclical in low states of the economy but countercyclical in high states.

A key contribution of the article is to establish a fundamental link between asset prices and quantities in the market, especially the amount of credit. The primary empirical implication of the model is that changes in the size of the credit sector are highly informative about shifts in the demographics of the market, which, in turn, drive the behavior of asset prices. More specifically, changes in the size of the credit sector should be linked to time variation in the equity premium. Thus, information about the size of the credit market may prove useful in forecasting excess stock returns.

We test this empirical implication using the standard predictive regression framework familiar from the asset-pricing literature. Specifically, we regress one-year (non-overlapping) excess returns on the CRSP value-weighted index for the 1953-to-2010 period on a number of variables that previous research has suggested may have predictive ability for excess stock returns: lagged stock returns, the dividend yield, Lettau and Ludvigson's (2001) *cay* measure, and the short-term interest rate. We then introduce several measures of the size of the credit sector into the regression and examine how the predictive ability of the regression changes.

The results are striking. By themselves, the lagged stock return, dividend yield, *cay*, and short-term rate variables result in a predictive regression for the one-year horizon with an adjusted R^2 of about 22%. When the credit sector variables are introduced, however, the adjusted R^2 for the regression increases to nearly 33%. These results demonstrate both the theoretical and empirical importance of the credit sector in asset pricing.

In summary, by analyzing the transactions among investors in the traditional asset-pricing framework, we identify a number of key roles that credit plays in the economy. Credit markets are crucial in facilitating risk sharing among diverse agents, and leveraging and deleveraging in financial markets can be understood in the broader context of the dynamic replication strategies agents use to synthesize “macro options” in the financial markets. Furthermore, the credit market has a unique informational role since its endogenously determined size is a reflection of economic fundamentals that affect financial market returns over multi-year horizons. These results clearly have important implications for current policy debates about the use of public sector debt in providing credit to industries such as mortgage, banking, insurance, automobile, student loan, credit card, etc.

Several articles have considered the impact of heterogeneity in investors’ risk aversion on asset pricing. Dumas (1989) and Wang (1996) use a two-agent setting to examine how heterogeneity gives rise to time-varying risk aversion in the aggregate and the resulting behavior of the short-term interest rates. Further allowing the feature of “keeping up with the Joneses” in investor preferences in a similar setting, Chan and Kogan (2002) consider the dynamics of stock prices. The existing articles focus primarily on the behavior of asset prices and thus consider mostly the aggregate behavior of the market, through the behavior of the representative agent of the economy. They do not examine individual agents’ trading activities.[3]

Our analysis substantially exceeds what was achieved in these articles. In particular, we are able to obtain the full equilibrium solution, including all asset prices and individual portfolio policies. This allows us to significantly extend this line of research in several important directions. First, instead of considering only prices, we can analyze both prices and quantities, especially the interaction between the two. To link prices with quantities is of essential importance to models with heterogeneous investors—it is where these models can produce new, distinctive, and testable predictions beyond those from a representative model. After all, in a complete market, there always exists a representative-agent representation for an economy with heterogeneous agents that yields identical pricing relations from the fundamentals. Thus, empirical tests of these models have to turn to the additional predictions, which must

[3] It is worth pointing out that Dumas (1989) considers a linear production economy, in which output and consumption are part of the equilibrium outcome. He numerically solves the equilibrium interest rate and consumption dynamics. The stock price stays at one given the linear production technology. Wang (1996) considers a pure-exchange economy. He solves in closed form the equilibrium interest rate but only characterizes the stock price in aggregate consumption (which equals aggregate endowment) and agents’ preferences. In both articles, no explicit solution was provided for agents’ equilibrium security holdings.

concern disaggregated variables such as quantities. By solving and analyzing in closed form investors' portfolio behavior together with prices, we are able to produce various predictions directly connecting the two.

Second, our article focuses on the role of the credit market, in particular, how the amount of credit is related to the degree of risk sharing and the resulting stock and bond prices. Moreover, we explore some of the empirical predictions of the model. Especially, we show that the amount of credit endogenously generated in the market contains useful information about the risk premium of the stock. In addition, even with limited degrees of freedom, we are able to produce a rich set of results concerning both the dynamics of stock and bond prices that are compatible with the empirical findings, at least for some sets of calibrated parameters.

The objective of our article is to demonstrate how the amount of credit in the market facilitates risk sharing in the market and affects asset prices. For this purpose, we rely on a parsimonious model, which allows us to solve for both equilibrium prices *and* quantities in closed form. This enables us to uncover the underlying mechanism connecting credit and asset prices in a more precise and clear fashion. Our focus is thus mainly on the qualitative nature of the impact of the credit market rather than quantitative properties of the model, such as the levels of risk premium, interest rates, etc. As the literature on the calibration of aggregate pricing models illustrates, richer structure is needed to reconcile its quantitative implications with the data.[4] Such an extension of our analysis here is beyond the scope of this article and left for future work.

This article is organized as follows. Section 1 describes the model. Section 2 presents a single-agent version of the model as a benchmark for comparison. Section 3 solves the equilibrium for the two-agent model. Section 4 discusses the equilibrium consumption allocation. Section 5 analyzes how the credit market helps the two agents achieve the optimal risk sharing. Section 6 examines the behavior of asset prices in the model and the interactions with borrowing and lending in the credit market. Section 7 considers the trading activity in the stock market. Section 8 reports our exploratory empirical work on the link between the size of the credit market and stock market returns. Section 9 concludes. All proofs are provided in the Appendix. For completeness, we have included several results from previous articles. They will be attributed to the original author(s) when formally presented.

1. The Model

The primary goal of this article is to explore the fundamental connection between activities in the credit market and asset prices; we are more interested in the qualitative implications of such a connection rather than quantitative

[4] See, for example, Mehra and Prescott (1985); Hansen and Jagannathan (1991); Heaton and Lucas (1996); and Campbell and Cochrane (1999).

predictions. Thus, to maintain parsimony in the economic setting, we consider for tractability and clarity. We will return to potential enrichments at the end of this section.

We consider a pure-exchange economy similar to Wang (1996). The economy is endowed with a flow of a single perishable consumption good, which also serves as the numeraire. We denote the rate of endowment flow as X_t and assume that it follows a geometric Brownian motion,

$$dX_t = \mu X_t dt + \sigma X_t dZ_t, \tag{1}$$

where $X_0 > 0$, $\mu \geq 0$ and $\sigma > 0$ are constants, and Z_t is a standard Wiener process.[5] The process X_t is positive with probability one and, conditional on X_t, $X_{t+\tau}$ with $\tau \geq 0$ is lognormally distributed.

There exists a market where shares of the aggregate endowment (the "stock") are traded. A share of the stock yields a dividend flow at rate X_t. The total number of shares of the stock in the economy then equals one. In addition, there exists a "money market" where a locally riskless security is traded (i.e., investors can borrow from or lend to each other without default). As is standard, we assume that this riskless security is in zero net supply in the economy. Let P_t denote the price of the stock and r_t the instantaneous riskless interest rate.

Investors in this economy can trade competitively in the securities market and consume the proceeds. Let C_t be an investor's consumption rate, N_t his holdings of the stock, and M_t his holdings of the riskless security. The consumption and trading strategies $\{C_t, (N_t, M_t)\}$ are adapted processes satisfying the standard integrability conditions, that is, $\forall\, T \in [0, \infty)$,

$$\int_0^T C_t\, dt < \infty, \quad \int_0^T |\, M_t\, r_t\, dt + N_t\, (X_t\, dt + dP_t)| < \infty, \quad \int_0^T N_t^2\, d[P_t] < \infty, \tag{2}$$

where $[P_t]$ denotes the quadratic variation process of P_t.[6] The investor's wealth process, defined by $W_t = M_t + N_t\, P_t$, must be positive with probability one, and conform to the stochastic differential equation,

$$dW_t = r_t\, M_t\, dt + (X_t\, dt + dP_t)\, N_t - C_t\, dt. \tag{3}$$

The requirement that wealth be positive is to rule out arbitrage opportunities following Dybvig and Huang 1988). Let Θ denote the set of consumption/trading strategies that satisfy the above conditions.

There are two classes of identical investors in the economy, denoted as 1 and 2. Both classes are initially endowed with only shares of the stock. The initial endowment of shares for the classes of investors are $1-n$ and n,

[5] Throughout the article, equalities or inequalities involving random variables are always in the sense of almost surely with respect to the underlying probability measure.

[6] See Karatzas and Shreve (1988) for a discussion of the quadratic variation process of a given stochastic process.

respectively. The initial number of shares optimally chosen by each class at time zero, of course, need not equal their initial endowments. Investors in each class choose their consumption and investment strategies to maximize their lifetime expected utility. The preferences of the two classes of investors are

$$E_t\left[\int_0^{\infty} e^{-\rho\tau}\,\frac{C_{1,t+\tau}^{1-\gamma}}{1-\gamma}\,d\tau\right], \tag{4a}$$

$$E_t\left[\int_0^{\infty} e^{-\rho\tau}\,\frac{C_{2,t+\tau}^{1-2\gamma}}{1-2\gamma}\,d\tau\right], \tag{4b}$$

respectively, where γ is a positive constant. $C_{1,t}$ and $C_{2,t}$ denote the total consumption of the first and second classes of investors, respectively. Thus, the first and second classes of investors have constant relative risk aversion (CRRA) of γ and 2γ, respectively.

We further impose several conditions on the model's parameter values. The first condition is the growth condition,

$$\rho > \max\{0,\ (1-\gamma)(\mu-\tfrac{1}{2}\gamma\sigma^2),\ (1-2\gamma)(\mu-\gamma\sigma^2)\}. \tag{5}$$

It ensures that investors' expected utilities are uniformly bounded given the aggregate consumption process in Equation (1). In addition, we need the following set of conditions:

$$\sqrt{(\mu-\tfrac{1}{2}\sigma^2)^2+2\rho\sigma^2} + (\mu-\tfrac{1}{2}\sigma^2) - 2\gamma\sigma^2 > 0, \tag{6a}$$

$$\sqrt{(\mu-\tfrac{1}{2}\sigma^2)^2+2\rho\sigma^2} - (\mu-\tfrac{1}{2}\sigma^2) + (\gamma-1)\sigma^2 > 0. \tag{6b}$$

These conditions guarantee that the stock and bond prices behave properly.[7]

In specifying the securities markets, we have only introduced the stock and the locally riskless security as traded securities. As will be shown later, the stock and the riskless security are sufficient to dynamically complete the securities market in the sense of Harrison and Kreps (1979). Arbitrary consumption plans (satisfying certain integrability conditions) can be financed by continuous trading in the stock and the riskless security. Allowing additional securities will not affect the nature of the equilibrium. Thus, in deriving the market equilibrium, we will consider the securities market as consisting of only the stock and the riskless security. Simple arbitrage arguments can then be used to price other securities if they exist.

We have assumed that there are only two classes of investors in the economy and that they behave competitively in the market. Since investors within each

[7] Given CRRA preferences and the process for the aggregate endowment, both classes of investors' marginal utility and stock payoffs are unbounded from above. Thus, parameter restrictions are needed to ensure that the prices of certain securities such as the stock and bonds are well defined.

class have the same isoelastic preferences, we can represent each class with a single representative investor who has the same preferences as the individual investors and the total endowment of each class (e.g., Rubinstein 1974). In deriving the equilibrium, we can then treat the economy as populated with the two representative investors who behave competitively. In the remainder of the article, we treat the two representative investors generically and simply refer to them as the more risk-averse and less risk-averse agents, whom we also refer to as agent 1 and 2.

Market equilibrium in this economy consists of a pair of price processes $\{P_t, r_t\}$ and the consumption-trading strategies $\{C_{i,t}, (N_{i,t}, M_{i,t}),\ i=1,2\}$ such that the agents' expected lifetime utilities are maximized subject to their respective wealth dynamics in Equation (3), and the securities markets clear:

$$N_{1,t} + N_{2,t} = 1, \tag{7a}$$

$$M_{1,t} + M_{2,t} = 0. \tag{7b}$$

Before we move on, a few words are in order about the model. The basic setting is canonical (e.g., Black and Scholes 1973; Cox, Ingersoll, and Ross 1985; Mehra and Prescott 1985). By explicitly introducing two classes of investors, the model attempts to capture the need to borrow and lend for risk sharing and the link credit market and asset valuation. The model's simplicity allows our analysis to be tractable, clean, and to better demonstrate the underlying economic forces driving the credit market and its influence on asset prices.

Naturally, the simplicity also carries limitations. For example, the presence of only two classes of investors limits the richness in interactions among diverse investors. The time-additivity and constant relative risk aversion prevent a closer fit of the model to the data. They also lead to undesirable asymptotic properties of the economy (Wang 1996; Chan and Kogan 2002). The single-state variable of the economy, the level of aggregate consumption, also imposes too tight of a connection between different aspects of the market (e.g., instantaneous changes in all asset prices are perfectly correlated). In addition, the assumption of a complete and frictionless financial market also simplifies the role of the credit sector. We hope that the interesting results we obtain in the simple model provide a strong motivation to consider the role of the credit market in more general settings.

It is also worth pointing out that in our model, default does not occur in equilibrium. This is mainly due to the fact that agents have infinite marginal utility at zero consumption. Thus, they will do their best to maintain positive wealth. With complete markets, this implies no default. Under continuous-time setting with diffusive information flow, agents can achieve high leverage without worrying about default, as continuous trading allows them to avoid it.

2. The Single-agent Equilibrium

Before presenting the results for the two-agent model, we first review the trading and asset-pricing implications of the familiar single-representative-agent model in our setting. In fact, as shown by Stapleton and Subrahmanyam (1990), this is the well-known situation considered by Black and Scholes (1973). The results from the single-agent model then provide a benchmark for comparison to those from the two-agent model.

The single-agent model is nested within the two-agent model by assuming that only one of the two agents, say the less risk-averse agent, is present in the market. Thus, the less risk-averse agent is initially endowed with all of the shares of stock in the economy, $1-n=1$ or $n=0$.[8] The agent maximizes his expected lifetime utility through consumption and investment choices $\{C_t,(N_t,M_t)\}$. Here, for brevity, we have omitted the subscript i in denoting agent i. In equilibrium, however, the agent's consumption C_t must equal the aggregate amount of dividends, X_t. Similarly, market clearing implies that the agent holds all of the shares of the stock and does not borrow or lend; $N_t=1$ and $M_t=0$. This latter feature makes the trading implications of the single-agent model simple, as there is no trading in equilibrium and the agent never changes the number of shares of the stock or the riskless asset in his portfolio.

The equilibrium price S_t of a security with payoff $\{D_s, s\geq 0\}$ can be obtained directly from the Euler equation,

$$S_t = E_t\left[\int_0^{\infty} e^{-\rho\tau}\left(\frac{C_{t+\tau}}{C_t}\right)^{-\gamma} D_{t+\tau}\, d\tau\right] = E_t\left[\int_0^{\infty} e^{-\rho\tau}\left(\frac{X_{t+\tau}}{X_t}\right)^{-\gamma} D_{t+\tau}\, d\tau\right], \tag{8}$$

where the second equality follows from $C_t=X_t$. We have the following result:

Lemma 1. (Stapleton and Subrahmanyam) In the single-agent economy,the equilibrium stock price is given by

$$P_t = \frac{1}{\rho-\kappa}\, X_t, \tag{9}$$

where

$$\kappa = (1-\gamma)(\mu-\tfrac{1}{2}\sigma^2) + \tfrac{1}{2}(1-\gamma)^2\sigma^2, \tag{10}$$

and the riskless interest rate is given by

$$r_t = \rho + \mu\gamma - \tfrac{1}{2}\gamma(1+\gamma)\sigma^2, \tag{11}$$

which is a constant.

[8] The parallel case where the more risk-averse agent is endowed with all the shares of stock is given by simply replacing γ with 2γ throughout all of the formulas in this section.

In this economy, the price-dividend ratio, defined as

$$Y_t = P_t / X_t, \tag{12}$$

is constant in the single-agent economy, i.e., $Y_t = 1/(\rho - \kappa)$. Its inverse, $y_t = X_t / P_t$, is simply the dividend yield of the stock, which is $\rho - \kappa$.

An application of Itô's Lemma to Equation (9) gives the dynamics for the stock price,

$$dP_t = P_t \, (\mu \, dt + \sigma \, dZ_t). \tag{13}$$

Thus, the stock price inherits the geometric Brownian motion dynamics of the underlying dividend process. In this single-agent economy, the agent's wealth W_t equals the value of his stock holdings, $W_t = M_t + N_t P_t = P_t$.

Clearly, the single-agent market exhibits some simple properties. For example, the interest rate is constant and the stock returns are i.i.d. In particular, the expected return on the stock is $\mu + y_t = \mu + \rho - \kappa$ and its return volatility is σ, both constant over time. Moreover, stock returns are serially uncorrelated. As we see below, these are no longer true when both agents are present in the market.

3. The Two-agent Equilibrium

In this section, we present the closed-form solutions for equilibrium consumption, asset prices, and portfolio choices for the two-agent model. We explore the economic intuition and implications of these results more fully in the subsequent sections, and provide the proofs and derivations in the Appendix.

The equilibrium is derived in three steps. First, relying on the complete securities market in our model, we solve for the equilibrium allocation of consumption between the two agents from its Pareto optimality. Second, using the Euler equation for the agents, we compute the equilibrium stock price and interest rate that support the equilibrium allocation. Finally, by analyzing the agents' portfolio policies financing their consumption, we obtain their equilibrium holdings of the stock and the riskless security.

3.1 Consumption

In the two-agent economy, the sum of the agents' consumption streams must equal the aggregate dividends, i.e., $C_{1,t} + C_{2,t} = X_t$. An allocation $C_{1,t}, C_{2,t}$ is Pareto optimal if, and only if, there exists $\alpha \in [0,1]$ such that $C_{1,t}, C_{2,t}$ solves the problem:

$$\max_{C_{1,t}+C_{2,t} \le X_t} E_0 \left[\int_0^\infty e^{-\rho t} \left[\alpha \frac{C_{1,t}^{1-\gamma}}{1-\gamma} + (1-\alpha) \frac{C_{2,t}^{1-2\gamma}}{1-2\gamma} \right] dt \right]. \tag{14}$$

The solution, first obtained by Wang (1996), is given in Proposition 1.

Proposition 1. (Wang) In the two-agent economy, the equilibrium consumption allocation is given by

$$C_{1,t} = X_t - \frac{2}{b}\left(\sqrt{1+bX_t}-1\right), \qquad C_{2,t} = X_t - C_{1,t} = \frac{2}{b}\left(\sqrt{1+bX_t}-1\right), \tag{15}$$

where $b=4\left(\frac{\alpha}{1-\alpha}\right)^{1/\gamma}$.

To simplify notation, we denote the more risk-averse agent's consumption $C_{2,t}$ simply as C_t. Thus, in equilibrium, the less risk-averse agent's consumption is $C_{1,t}=X_t-C_t$.

The demographics of the market are characterized by the relative consumption levels of the two agents. Let s_t denote the less risk-averse agent's share of total consumption. We have

$$s_t = \frac{X_t - C_t}{X_t} = \frac{\sqrt{1+bX_t}-1}{\sqrt{1+bX_t}+1}. \tag{16}$$

As we will see below, s_t is an important variable in characterizing the behavior of the economy. We analyze its properties in Section 5.

3.2 Asset prices

Given the equilibrium consumption allocation, we now compute the stock price and the interest rate that support the equilibrium. The Euler equation for the more risk-averse agent leads to the following equation for the price of a security with payoff $\{D_s, s \geq 0\}$:

$$S_t = E_t\left[\int_0^\infty e^{-\rho\tau}\left(\frac{C_{t+\tau}}{C_t}\right)^{-2\gamma} D_{t+\tau}\, d\tau\right], \tag{17}$$

where C_t is given in Equation (15). The equilibrium stock price and riskless interest rate are given in closed form in Proposition 2.

Proposition 2. In equilibrium, the price-dividend ratio of the stock is

$$\frac{P_t}{X_t} = Y_t = a_1 - a_2\, s_t\, F(1,-\theta,3+\theta-2\gamma;s_t) - a_3\,(1-s_t)\, F(1,\lambda,2\gamma+2\lambda;1-s_t), \tag{18}$$

where $F(a,b,c;z)$ is the standard hypergeometric function,

$$\theta = \frac{\psi+(\mu-\frac{1}{2}\sigma^2)}{\sigma^2}, \qquad \lambda = \frac{\psi-(\mu-\frac{1}{2}\sigma^2)}{\sigma^2}, \qquad \psi = \sqrt{(\mu-\tfrac{1}{2}\sigma^2)^2+2\rho\sigma^2}, \tag{19}$$

and

$$a_1 = \frac{\theta+\lambda-\gamma}{\psi(\gamma+\lambda-1)(1+\theta-2\gamma)}, \tag{20a}$$

$$a_2 = \frac{2\gamma}{\psi(1+\theta-2\gamma)(2+\theta-2\gamma)}, \tag{20b}$$

$$a_3 = \frac{\gamma}{\psi(\gamma+\lambda-1)(2\gamma+2\lambda-1)}. \tag{20c}$$

The equilibrium riskless rate is

$$r_t = \rho + \frac{2\mu\gamma}{(1+s_t)} - \frac{\gamma(2\gamma+3)\sigma^2}{(1+s_t)^2} + \frac{2\gamma\sigma^2}{(1+s_t)^3}. \tag{21}$$

The result on the riskless rate was obtained in Wang (1996), while the solution of the equilibrium stock price is new.

3.3 Optimal leverage and stock holding

In equilibrium, the more risk-averse agent's wealth W_t is simply the present value of his consumption stream, which is given as follows:

Lemma 2. The wealth of the more risk-averse agent is

$$\frac{W_t}{X_t} = w_t = b_1\,(1-s_t) - b_2\,s_t(1-s_t)\,F(1,1-\theta,3+\theta-2\gamma;s_t)$$

$$+\, b_3\,s_t(1-s_t)\,F(1,1+\lambda,2\gamma+2\lambda;1-s_t), \tag{22}$$

where

$$b_1 = \frac{\theta+\lambda}{\psi(2\gamma+\lambda-1)(1+\theta-2\gamma)}, \tag{23a}$$

$$b_2 = \frac{2\gamma-1}{\psi(1+\theta-2\gamma)(2+\theta-2\gamma)}, \tag{23b}$$

$$b_3 = \frac{2\gamma-1}{\psi(2\gamma+\lambda-1)(2\gamma+2\lambda-1)}. \tag{23c}$$

The wealth of the less risk-averse agent is given by $P_t - W_t$.

From the wealth of the more risk-averse agent, we can derive his optimal holdings of the stock and the riskless security, denoted by N_t and M_t, respectively. Market clearing then implies that the less risk-averse agent will hold $1-N_t$ shares of the stock and $-M_t$ units of the riskless security.

By definition, the more risk-averse agent's wealth equals the value of his portfolio holdings, $W_t = M_t + N_t P_t$. Following Cox and Huang (1989) and

Wang (1996), this implies that $dW_t = N_t dP_t$ after imposing the self-financing constraint $dM_t + P_t dN_t = 0$. Thus, the ratio of the diffusion coefficients in the dynamics of W_t and P_t can be used to solve for N_t. Once N_t is determined, M_t can be obtained directly from the expression for wealth. Consequently, we have the following:

Proposition 3. The optimal portfolio holdings for the more risk-averse agent are

$$N_t = \frac{\Psi_t}{\Phi_t}, \tag{24a}$$

$$M_t = W_t - N_t P_t, \tag{24b}$$

where

$$\begin{aligned}\Psi_t = {} & \frac{(1+s_t)}{s_t(1-s_t)} w_t - b_1 \\ & - b_2\,(1-2s_t)\,F(1,1-\theta,3+\theta-2\gamma;s_t) - b_2\,s_t(1-s_t) \\ & \quad F'(1,1-\theta,3+\theta-2\gamma;s_t) \\ & + b_3\,(1-2s_t)\,F(1,1+\lambda,2\gamma+2\lambda;1-s_t) - b_3\,s_t(1-s_t) \\ & \quad F'(1,1+\lambda,2\gamma+2\lambda;1-s_t),\end{aligned} \tag{25a}$$

$$\begin{aligned}\Phi_t = {} & \frac{(1+s_t)}{s_t(1-s_t)}\frac{1}{y_t} \\ & - a_2\,F(1,-\theta,3+\theta-2\gamma;s_t) - a_2\,s_t\,F'(1,-\theta,3+\theta-2\gamma;s_t) \\ & + a_3\,F(1,\lambda,2\gamma+2\lambda;1-s_t) + a_3\,(1-s_t)\,F'(1,\lambda,2\gamma+2\lambda;1-s_t),\end{aligned} \tag{25b}$$

y_t is the dividend yield ($y_t = 1/Y_t$), and $F'(a,b,c;z) = (ab/c)F(a+1,b+1,c+1,z)$.

Proposition 2 provides a full solution to the agents' equilibrium security holdings. It thus allows to examine in detail how quantity variables in addition to prices, such as the amount of borrowing/lending and agents' leverage ratios, behave in equilibrium.

4. Consumption Allocation

From the market equilibrium given in the previous section, we now examine the allocation of consumption (and risk) between the two agents, how this allocation is achieved through their trading in the stock and credit markets, and how their trading activity determines the behavior of asset prices.

When only a single agent is present, he consumes the aggregate endowment X_t. When two agents are present, they have to share the aggregate endowment.

Given the difference in their preferences, each will not simply share a constant portion of the aggregate endowment. From Equation (16), the equilibrium allocation is such that the more risk-averse agent consumes the lion's share in bad states, i.e., states with low aggregate endowment, and the less risk-averse agent consumes the major share in the good states.

In illustrating the results of the article, we will use a baseline calibration throughout to make the results easier to compare. Specifically, we assume that the expected dividend growth rate μ is 0.03 and that the volatility of dividend growth σ is 0.10. These values are consistent with the historical properties of imputed corporate dividends (e.g., Longstaff and Piazzesi 2004). We also assume that the subjective time discount rate ρ is 0.01 and the less risk-averse agent has logarithmic preferences, i.e., $\gamma = 2$. As we will see in Section 7, these parameter values lead to asset prices that are broadly compatible with what we see in the data. For example, the interest rate ranges between 3.00% and 4.15%, the stock risk premium ranges between 6% and 7%, stock return volatility ranges between 10.0% and 10.6%, and the term premium of interest rates is close to zero. Finally, we assume that the initial dividend level X_0 and the share allocation between the two agents n are such that $\alpha = \frac{1}{2}$.

Figure 1 plots the two agents' shares of aggregate consumption for different levels of aggregate consumption (endowment). Also plotted is the aggregate consumption X_t as a function of the less risk-averse agent's share, s_t. The left panel of Figure 1 shows that at a given time t, the share of the less risk-averse agent's consumption s_t monotonically increases with the aggregate level of consumption X_t. It starts at zero as X_t is close to zero, but increases as X_t increases and approaches one as X_t goes to infinity. This consumption allocation across different states of the economy is intuitive. As the aggregate endowment decreases, the marginal utility of the more risk-averse agent increases faster than

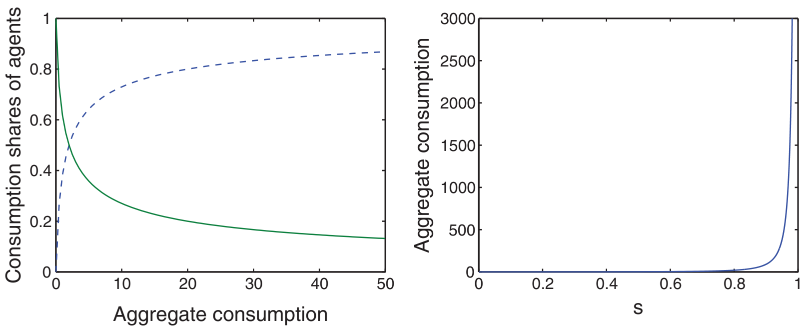

Figure 1
Aggregate consumption and agents' consumption shares
The left panel plots the agents' consumption shares as functions of aggregate consumption, the dashed line for the less risk-averse agent and the solid line for the more risk-averse agent, and the right panel plots aggregate consumption as a function of the less risk-averse agent's consumption share s_t. The parameters are at the benchmark values: $\mu = 0.03$, $\sigma = 0.10$, $\rho = 0.01$, $\gamma = 2$, and $\alpha = 0.5$.

that of the less risk-averse agent. On the other hand, as the aggregate endowment increases, the marginal utility of the more risk-averse agent decreases faster than that of the less risk-averse agent. The optimal consumption is reached when the marginal utilities of the two agents are equal. This is achieved when the more risk-averse agent consumes a relatively larger share in the bad states by claiming a relative smaller share in the good states. The right panel of Figure 1 shows that aggregate consumption X_t is very small in absolute value in the states when the more risk-averse agent dominates the economy (when s_t is small), while the opposite is true when the less risk-averse agent dominates the economy.

Over time, as X_t changes, so does s_t. In particular, s_t evolves as follows

$$ds_t = \mu_{s,t}\, dt + \sigma_{s,t}\, dZ_t, \tag{26}$$

where

$$\mu_{s,t} = \frac{s_t(1-s_t)}{1+s_t}\left[(\mu-\sigma^2) + \frac{1}{(1+s_t)^2}\sigma^2\right], \tag{27a}$$

$$\sigma_{s,t} = \frac{s_t(1-s_t)}{1+s_t}\sigma. \tag{27b}$$

One immediate observation is that in addition to itself, the dynamics of s_t depend only on the parameters governing the aggregate consumption process, i.e., μ and σ; the dynamics do not depend on the initial condition of the economy (i.e., X_0 and n), which only fixes the initial value of s_t. More importantly, the dynamics of s_t do not depend on the parameters concerning the agents' preferences, i.e., ρ and γ. They do, however, depend on the fact that the ratio between the two agents' relative risk aversion is two. This property comes from the fact that given the dynamics of total consumption, the dynamics of s_t are determined by the sharing rule between the two agents. Given that the two agents have constant relative risk aversion and the same time discount rate ρ, the sharing rule depends only on the ratio of their relative risk aversion coefficients. In our model, this ratio is two.

The drift and volatility of s_t imply that it follows a process similar to the class of Wright-Fisher diffusions used in genetics and many other contexts (e.g., Karlin and Taylor 1981). The drift of this process is a ratio of simple polynomials. Depending on parameter values, the drift can be uniformly positive (when $\mu > \frac{3}{4}\sigma^2$), uniformly negative (when $\mu < 0$), or can be positive for values of s_t below some threshold and negative for values greater than that threshold (when $0 < \mu < \frac{3}{4}\sigma^2$). This latter situation implies a certain type of mean-reverting behavior for the process. However, the process does not have a stationary distribution in this situation. The volatility of the process takes its maximum value at $s_t = \sqrt{2} - 1$.

Figure 2 plots both the drift (the left panel) and the volatility of s_t (the right panel) for the baseline parameter values. Clearly, in this case where $\mu - \frac{3}{4}\sigma^2 =$

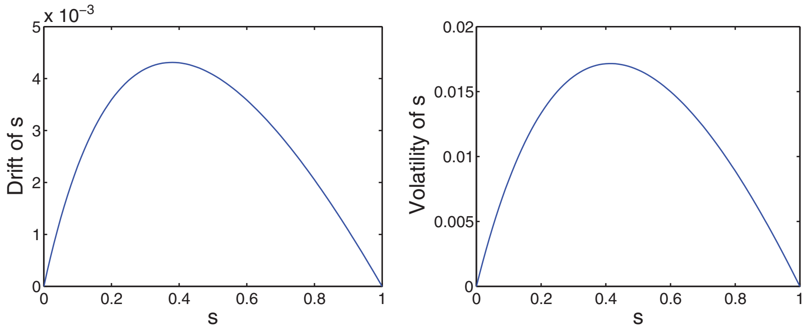

Figure 2
Dynamics of the consumption share of the less risk-averse agent
The left panel plots the drift of the consumption share of the less risk-averse agent $\mu_{s,t}$ as a function of s_t, and the right panel plots the volatility of the consumption share $\sigma_{s,t}$. The parameters are at the benchmark values: $\mu = 0.03$, $\sigma = 0.10$, $\rho = 0.01$, $\gamma = 2$, and $\alpha = 0.5$.

$0.03 - \frac{3}{4}(0.10)^2 = 0.0225 > 0$, the drift of s_t is always positive, indicating that the less risk-averse agent is steadily gaining a share of the economy. The drift increases steeply with s_t for small values of s_t. It peaks as s_t approaches 0.4 and then declines quickly. The volatility of s_t has a simple humped shape. It is zero at the two extreme ends, i.e., when s_t is equal to zero or one, when one of the agents owns the whole economy. It peaks when s_t is around 0.4. As we will see below, the dynamics of s_t are very much related to the risk sharing between the two agents and the resulting market behavior.

5. Risk Sharing and the Credit Market

The consumption share of the two agents, as shown in Figure 1, reveals a striking pattern. The consumption of the less risk-averse agent is a convex function of aggregate consumption, while that of the more risk-averse agent is a concave function. This represents the optimal risk sharing between the two agents given their preferences. In fact, the more risk-averse agent shifts a large part of the aggregate risk, given by the uncertainty in X_t, to the less risk-averse agent. As a result, the risk profile of the less risk-averse agent actually exceeds that of the overall economy.

5.1 Leverage and risk sharing

Risk sharing is achieved through the two agents' trading in the securities market. In particular, it is facilitated by the lending of the more risk-averse agent in the credit market to the less risk-averse agent. As a result, the more risk-averse agent is able to switch his stock holdings into riskless debt, and thus to maintain a less risky wealth profile. This is accommodated by the less risk-averse agent, who issues debt to the more risk-averse agent to finance his own levered purchase of additional stock shares.

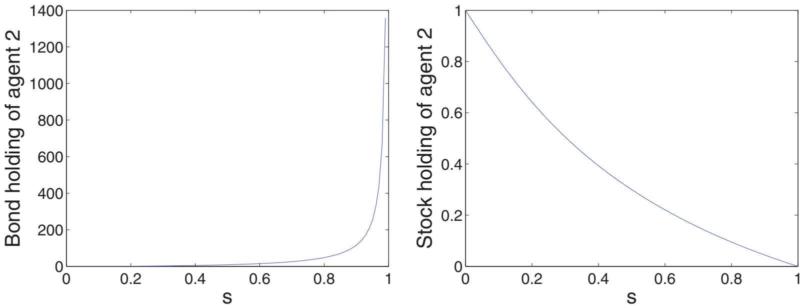

Figure 3
Bond and stock holdings of the more risk-averse agent
The left panel plots the amount of riskless debt the more risk-averse agent (agent 2) holds as a function of s_t, and the right plots the number of stock shares he holds. The parameters are at the benchmark values: $\mu = 0.03$, $\sigma = 0.10$, $\rho = 0.01$, $\gamma = 2$, and $\alpha = 0.5$.

Thus, the credit market plays a critical role in allowing optimal risk sharing among agents with different risk preferences. In the absence of a credit market, each of the agents would have to hold the same portfolio consisting of a 100% weight in the stock. The presence of the credit market allows agents to modify the risk profile of their portfolios by borrowing and lending and thus allocate risk optimally.

Figure 3 plots the debt and stock shares held by the more risk-averse agent as a function of s_t. The left panel shows M_t, the total amount of "short-term" debt, in the form of instantaneous credit, held by the more risk-averse agent. As we can see, for all possible states of the economy (i.e., the whole range of s_t), M_t is positive. That is, the more risk-averse agent is always the lender in the market, lending money to the less risk-averse agent in exchange for safe future payoffs. Of course, the bond position of the less risk-averse agent is simply $-M_t$, which is always negative.

In Figure 3, at low levels of X_t, the consumption share of the less risk-averse agent s_t is close to zero. In these states, the more risk-averse agent owns most of the economy and consumes most of the aggregate endowment. As the left panel of Figure 3 shows, the level of debt is small in these states, as the less risk-averse agent has little wealth to support his borrowing. As X_t increases, the overall wealth of the economy increases. Moreover, the less risk-averse agent also has more wealth. Consequently, he can take on more debt by issuing more bonds to the more risk-averse agent. Indeed, we see that M_t rises quickly with X_t, or equivalently, s_t.

While the increase in the lending of the more risk-averse agent represents a shift in his wealth from the stock to bond, the increase in the borrowing of the less risk-averse agent is used to increase his stock positions. The right panel of Figure 3 plots the stock shares held by the more risk-averse agent N_t. Since the total number of stock shares is normalized to one, the number of shares held by the less risk-averse agent is simply $1-N_t$. Clearly, at low levels of X_t, the more risk-averse agent holds most of the stock. In fact, as mentioned above, he owns most of the economy and consumes the lion's share of the aggregate consumption. As X_t increases, however, his stock holding monotonically decreases. When X_t approaches infinity, s_t approaches one (the less risk-averse agent consumes most of the economy), and N_t approaches zero. In those states, the more risk-averse agent holds most of his wealth in bonds.

As shown in Cox and Huang (1989), an agent's portfolio rebalancing can be interpreted as the dynamic trading strategy that generates "derivative" contracts that deliver the optimal consumption for each date and state. From this perspective, the portfolio strategy of the more risk-averse agent, who sells stock shares for bonds as the stock price rises and buys when the stock price falls, is exactly to achieve the negative convexity in his desired consumption profile. Thus, through the dynamic rebalancing of his stock and bond positions, he is synthetically "selling" call options to the less risk-averse agent. In the vocabulary of the options market, the risk-averse agent is short "gamma," while the less risk-averse agent is long "gamma."

Finding that the risk-averse agent sells options to the less risk-averse agent may seem counterintuitive at first. After all, selling options is generally viewed as a highly risky enterprise. In this equilibrium, however, the more risk-averse agent is not simply selling options outright with a potentially unbounded downside. Rather, the risk-averse agent follows a much more conservative "covered call" strategy by selling options against an underlying stock position. Obviously, the credit market is crucial to allow him to achieve this through his portfolio strategy.

5.2 Optimal portfolio weights

In addition to describing the agents' stock and bond holdings in absolute terms, we also examine them in relative terms. In particular, we consider the relative weight of stock in both agents' portfolios $w_{i,t}$, where

$$w_{1,t} = \frac{(1-N_t)P_t}{(1-N_t)P_t - M_t}, \qquad w_{2,t} = \frac{N_t P_t}{N_t P_t + M_t}. \tag{28}$$

The relative weight of bond in agent i's portfolio is simply $1-w_{i,t}$. Figure 4 plots $w_{1,t}$ and $w_{2,t}$ as a function of s_t.

Facilitated by the credit market and the possibility of leverage, the difference between the two agents' portfolios is striking. For the less risk-averse agent, the weight of stock in his portfolio is always above one, reflecting the fact that he is always levered. For small values of s_t, which corresponds to low levels of X_t or bad states of the economy, the stock weight in his portfolio is close to two. In other words, he leverages all his wealth to borrow. In these states, it is the more risk-averse agent who is wealthier, and he can fully accommodate the leverage needs of the less risk-averse agent. From Figure 3, we see that the absolute size of debt is small for small s_t. But it is a large percentage of the less risk-averse agent's portfolio. As s_t increases, the economy moves into good states,

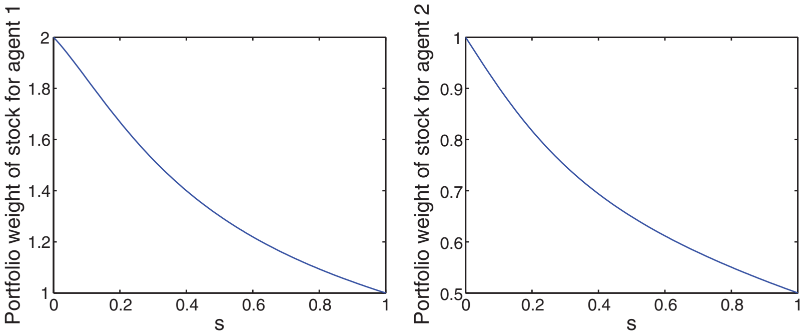

Figure 4
Weight of stock in agents' portfolios
The left panel plots the weight of stock in the portfolio of the less risk-averse agent (agent one) against s_t, and the right panel plots that of the more risk-averse agent (agent two). The parameters are at the benchmark values: $\mu=0.03$, $\sigma=0.10$, $\rho=0.01$, $\gamma=2$, and $\alpha=0.5$.

in which the less risk-averse agent gains a larger share of the total wealth (and consumption). Despite his preference for leverage, less debt would be available as the wealth share of the more risk-averse agent dwindles. Consequently, he is forced to reduce leverage and the weight of the stock in his portfolio decreases. When s_t approaches one, the less risk-averse agent owns most of the economy, which is the stock. The weight of the stock in his portfolio approaches one.

For the more risk-averse agent, the weight of stock in his portfolio is always between zero and one since he holds part of his portfolio in the riskless bond. For small values of s_t (i.e., X_t), he owns most of the economy and thus most of the stock. The debt he holds is only a trivial part of his total portfolio. In these bad states of the economy, the opportunity for risk sharing is very limited and $w_{2,t}$ is close to one. As s_t increases, his share of the total economy decreases. He shifts to safer asset allocations, investing a smaller fraction of his wealth in the stock while investing more in the bond. When s_t approaches one, the less risk-averse agent dominates the economy. In these states, $w_{2,t}$ approaches zero and the more risk-averse agent ends up with all his portfolio in the bond. In other words, he completely avoids the risk of the economy.

The above analysis reveals the central role the credit market plays. The risk sharing between the two agents is achieved by allowing the less risk-averse agent to bear a larger share of the aggregate risk, which is fully reflected in the risk of the stock market. Such a shift is facilitated by the credit market, which allows the less risk-averse agent to borrow capital and take on levered positions in the stock. The amount he can borrow, however, depends on the amount of wealth he has. In the bad states (i.e., s_t is close to zero), he has less wealth and risk sharing is limited. In the good states (i.e., s_t is close to one), the less risk-averse agent controls most of the wealth and thus has abundant collateral. In these states, the risk sharing is more complete as he bears all the risk of the economy and the more risk-averse agent's wealth is all in bonds.

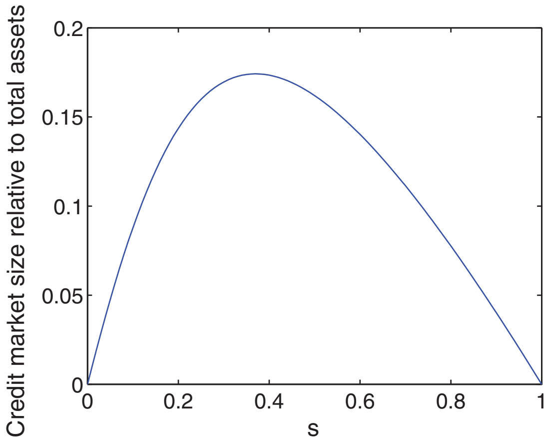

Figure 5
Market-leverage ratio
The figure plots the ratio of the amount of debt outstanding to the total value of the assets in the economy for different values of s_t. The parameters are at the benchmark values: $\mu=0.03$, $\sigma=0.10$, $\rho=0.01$, $\gamma=2$, and $\alpha=0.5$.

5.3 The market-leverage ratio

Given the importance of the credit market, we now consider the ratio of aggregate credit in the market to the total value of assets held by the agents. This ratio, which we denote the market-leverage ratio, is simply M_t/P_t. Intuition suggests that there are some common-sense bounds on the values that this ratio can take in equilibrium. In particular, the market-leverage ratio should be bounded below by zero given the agents' risk-sharing incentives. On the other hand, if the interest-rate payments on the debt exceed the dividend payments received by the less risk-averse agent, he would only be able to avoid default by borrowing further. Even this expedient would appear to have a limit since the total debt payments made by the less risk-averse agent could not exceed the aggregate dividend payments generated by the stock, the positive-net-supply asset in the economy.

Figure 5 plots the market-leverage ratio as a function of s_t. As expected, the market-leverage ratio approaches zero as s_t approaches either zero or one. This follows simply because the aggregate amount of debt in the market depends on the relative size of each agent in the market. If there is effectively only one agent in the market, little or no debt can occur. For intermediate values of s_t, however, the amount of debt in the economy can be substantial. The maximum market-leverage ratio of close to 18 percent occurs around $s_t=0.4$.

The maximum market-leverage ratio implied by the model is very consistent with the U.S. historical experience. Using the Federal Reserve's Z.1 statistical data for the flow of funds accounts of the United States from 1953 to 2006, we find that the market-leverage ratio ranges from a low of about 8% in 1953 to a high of slightly more than 19% in 2005. This historical high of 19% agrees closely with the maximum value implied by our model.

Comparing the market-leverage ratio plotted in Figure 5 for different states of the economy, which are fully characterized by s_t, with the volatility of the less risk-averse agent's consumption share plotted in Figure 2 (the right panel), we find a striking similarity between the two. In particular, the market-leverage ratio peaks at around 0.4 and so does the volatility of s_t (the exact locations of their maxima are slightly different). This is not surprising. Given that optimal risk sharing induces the less risk-averse agent to load up on risk by leveraging, the amount of risk he bears reaches a maximum with the amount of leverage. At this point, his wealth as well as consumption are also most volatile.

5.4 Welfare gains from risk sharing

Both the credit and the stock markets facilitate risk sharing between the agents, which improves their welfare. In this subsection, we explore the significance of this welfare gain. Let n_i be the initial shares of the stock agent i is endowed, X_0 the initial level of dividend (also aggregate consumption), and $C_{i,t}$ his consumption rate at t. In general, $C_{i,t}$ depends on X_0 and n_i, as they jointly determine the agent's budget set and his future consumption levels. Thus, we can express agent i's expected utility at time 0 as follows:

$$V_{i,0}(X_0,n_i)=E_0\left[\int_0^{\infty} e^{-\rho t}\frac{C_{i,t}^{1-\gamma_i}}{1-\gamma_i}dt\right]. \tag{29}$$

In the absence of the bond and stock markets, agents only consume their endowments. For agent i who is endowed with n_i shares of the stock, his consumption will simply be $C_{i,t}=n_i X_t$. The resulting expected utility is

$$V_{i,0}^{NT}(X_0,n_i)=\frac{1}{\rho-\kappa_i}\frac{n_i^{1-\gamma_i}X_0^{1-\gamma_i}}{1-\gamma_i},\quad \kappa_i=(1-\gamma_i)(\mu-\sigma^2/2)+\frac{1}{2}(1-\gamma_i)^2\sigma^2. \tag{30}$$

When the two markets are open, the agents trade and we can again compute the expected utility they achieve in equilibrium by substituting in their consumption given in Equation (15). Some algebra leads to the following:

$$\begin{aligned}V_{1,0}(X_0,n_1)=&\frac{2^{2\gamma-2}b^{1-\gamma}(X_0-C_0)^{3-2\gamma}}{\psi(1-\gamma)X_0}\left\{\left(\frac{X_0}{X_0-C_0}\right)\left(\frac{1}{2\gamma-\lambda-2}+\frac{1}{2+\theta-2\gamma}\right)\right.\\&+(X_0-C_0)\left[\frac{b}{2(2\gamma-2\lambda-2)}-\frac{b}{4(2\gamma-\lambda-2)}\right]\\&\times F(1,1-\lambda,2\gamma-2\lambda-1;s)\end{aligned}$$

$$+(X_0-C_0)\left[\frac{b(2\gamma-2\theta-2)}{4(2+\theta-2\gamma)(3+\theta-2\gamma)}+\frac{b}{2(3+\theta-2\gamma)}\right]$$

$$\left.\times F(1,1-\theta,4+\theta-2\gamma;1-s)\right\}. \tag{31a}$$

$$V_{2,0}(X_0,n_2)=\frac{(X_0-C_0)^{2-2\gamma}}{2\psi(1-2\gamma)X_0}\left\{\left(\frac{X_0}{X_0-C_0}\right)\left(\frac{2}{2\gamma-\lambda-1}+\frac{2}{\theta+1-2\gamma}\right)\right.$$

$$+(X_0-C_0)\left[\frac{b}{2\gamma-2\lambda-1}-\frac{b}{2(2\gamma-\lambda-1)}\right]F(1,1-\lambda,2\gamma-2\lambda;s)$$

$$+(X_0-C_0)\left[\frac{b(2\gamma-2\theta-1)}{2(\theta+1-2\gamma)(\theta+2-2\gamma)}+\frac{b}{\theta+2-2\gamma}\right]$$

$$\left.\times F(1,1-\theta,\theta+3-2\gamma;1-s)\right\}. \tag{31b}$$

Comparing the levels of indirect utilities in these two cases, we can compute the gains from trading.

To be concrete, we consider the following case. We first consider the equilibrium with trading, assuming that each agent is initially endowed with a half share of the stock, i.e., $n=1/2$, and obtain the expected utility each agent achieves in equilibrium by $V_{i,0}(X_0,1/2)$. We then consider his expected utility in the autarky with $(1+\delta_i)(1/2)$ shares of the stock such that it is the same as $V_{i,0}(X_0,1/2)$:

$$V_{i,0}^{NT}(X_0,(1+\delta_i)(1/2))=V_{i,0}(X_0,1/2),\quad i=1,2. \tag{32}$$

Then, δ_i provides a measure of welfare gain from trading for agent i, expressed in terms of percentage of endowment. Table 1 gives the δ for the two agents for different values of σ, the amount of risk in the economy.

From Table 1, we observe several results, which are quite intuitive. First, the gains from trading are always positive for both agents. This result is obvious. With both the bond and stock markets open, the market becomes complete, which allows Pareto optimal allocations in equilibrium. Next, the welfare gain is higher for the more risk-averse agent, i.e., agent two, as expected. Trading allows both agents to smooth consumption over time and across different states of the economy. The agent with higher risk aversion exhibits more concavity in his utility. Consumption smoothing allows higher gains in expected utility. Finally, as the economy becomes riskier, i.e., when the volatility of aggregate endowment increases, the gains from trading increase for both agents, which is expected.

Table 1
Welfare gains from trading

	Welfare Gain from Trading (as Percentage of Initial Endowment)	
Volatility (σ)	Agent 1	Agent 2
0.05	2.806	4.565
0.06	2.826	4.612
0.07	2.876	4.720
0.08	2.981	4.949
0.09	3.197	5.425
0.10	3.650	6.447
0.11	4.689	8.863

This table reports the welfare gain for each of the two agents from being allowed to trade stock and to borrow or lend. The gain is expressed as the additional percentage increase in initial endowment required to give an agent when he is unable to trade the same expected utility when he was able to trade and had an initial endowment of half of the shares of stock. Volatility denotes the risk of the dividend process σ.

6. Asset Prices

We now examine how risk sharing between the two agents influences the behavior of asset prices. Especially, we focus on the role of the credit market and its impact on asset prices.

To put the asset-pricing implications of leverage and risk sharing into perspective, recall that the interest rate is constant and stock returns are i.i.d. through time in the single-agent economy. In contrast, the moments of returns are generally time varying in the two-agent economy. From Equations (18) and (21), we see that both r_t and y_t (the dividend yield) vary with s_t. We will first examine the behavior of the stock price and then the properties of interest rates.

6.1 The stock price and its dynamics

Figure 6 plots the price-dividend ratio Y_t of the stock as a function of s_t for the baseline parameter values. Clearly, the price-to-dividend ratio increases as the economy expands, in a slightly nonlinear fashion. In other words, the price-to-dividend ratio varies procyclically. This is intuitive. As we have seen from their consumption share, in the bad states (s_t small), it is the more risk-averse agent (agent 2) who owns most of the economy and the stock. Hence, the stock valuation will be low for the same dividend level. In the good states of the economy, the less risk-averse agent owns the economy and the lion share of the stock and the stock valuation tends to be high.

Since $P_t = X_t Y_t$ where Y_t is the price-dividend ratio given in Equation (18), Itô's Lemma implies that stock-price dynamics dP_t/P_t can be expressed in terms of dX_t/X_t and dY_t/Y_t. From Equation (1), however, the moments of the dividend process dX_t/X_t are constant, since the dividend follows an i.i.d. geometric Brownian motion. As a result, any variation in the return moments

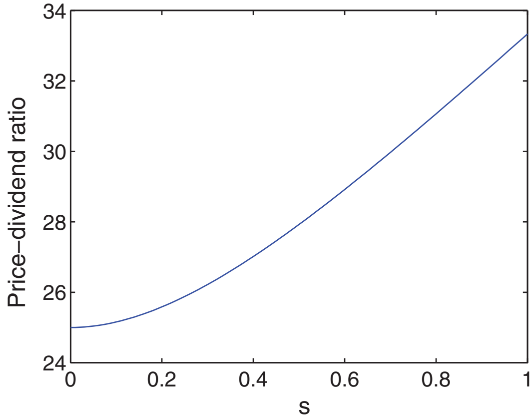

Figure 6
Stock price-dividend ratio
The figure plots the price-dividend ratio as a function of s_t. The parameters are at the benchmark values: $\mu = 0.03$, $\sigma = 0.10$, $\rho = 0.01$, $\gamma = 2$, and $\alpha = 0.5$.

is due entirely to variation in the valuation ratio Y_t, which is fully determined by s_t.

Given Equation (18) and the dynamics of s_t in Equations (26–27), the stock-price dynamics are given in the following proposition:

Proposition 4. The equilibrium stock-price dynamics are given by

$$\frac{dP_t}{P_t} = \left(\phi_t\, \mu_{s,t} + \tfrac{1}{2}\, \xi_t\, \sigma_{s,t}^2\right) dt + \phi_t\, \sigma_{s,t}\, dZ_t, \tag{33}$$

where

$$\begin{aligned} \phi_t &= (1+s_t)/[s_t(1-s_t)] \\ &\quad - a_2\, y_t\, F(1,-\theta,3+\theta-2\gamma;s_t) - a_2\, y_t\, s_t\, F'(1,-\theta,3+\theta-2\gamma;s_t) \\ &\quad + a_3\, y_t\, F(1,\lambda,2\gamma+2\lambda;1-s_t) + a_3\, y_t\, (1-s_t)\, F'(1,\lambda,2\gamma+2\lambda;1-s_t), \end{aligned} \tag{34a}$$

$$\begin{aligned} \xi_t &= 2\,\phi_t\,(1+s_t)/[s_t(1-s_t)] - 2/[s_t^2(1-s_t)^2] \\ &\quad - 2a_2\, y_t\, F'(1,-\theta,3+\theta-2\gamma;s_t) - a_2\, y_t\, s_t\, F''(1,-\theta,3+\theta-2\gamma;s_t) \\ &\quad - 2a_3\, y_t\, F'(1,\lambda,2\gamma+2\lambda;1-s_t) - a_3\, y_t\, (1-s_t)\, F''(1,\lambda,2\gamma+2\lambda;1-s_t), \end{aligned} \tag{34b}$$

and the derivatives F' and F'' are given by the simple differentiation formula for hypergeometric functions, $F'(a,b,c;z) = (ab/c)F(a+1,b+1,c+1;z)$.

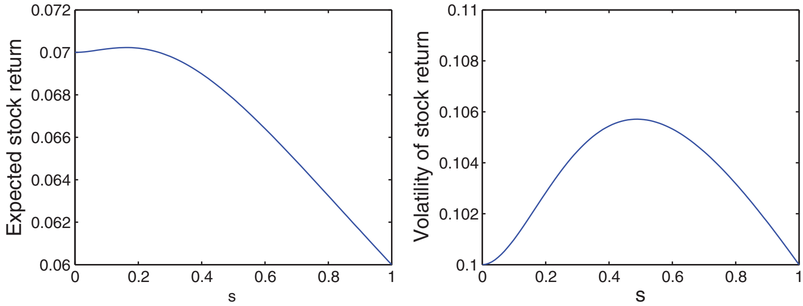

Figure 7
Expected return and volatility of the stock
The left panel plots the expected return of the stock as a function of s_t, and the right panel plots the stock return volatility. The parameters are at the benchmark values: $\mu=0.03$, $\sigma=0.10$, $\rho=0.01$, $\gamma=2$, and $\alpha=0.5$.

Although there are numerous terms in the drift and volatility terms, it is obvious that the stock-price dynamics are explicitly a function of s_t, the market demographics.

6.2 The expected return and return volatility of the stock

Given the stock-price dynamics, we now analyze the behavior of stock returns, especially their conditional distributions. Since the stock price follows a diffusion process and the dividend yield follows a smooth process, the instantaneous stock returns are conditionally normal. Thus, we need only to focus on their first two moments: the expected return and return volatility.

The expected return for the stock, which we denote by q_t, is just the sum of the dividend yield y_t and the expected price appreciation $E[dP_t/P_t]$. From Equation (33), this can be expressed as

$$q_t = y_t + \phi_t \, \mu_{s,t} + \tfrac{1}{2}\, \xi_t \, \sigma_{s,t}^2. \tag{35}$$

Given the smooth nature of the dividend, stock return volatility comes solely from price volatility. From Equations (33–34), it is given by $\sigma_t = |\phi_t \; \sigma_{s,t}|$. For the baseline parameters, Figure 7 plots both the expected return of the stock (the left panel) and its return volatility (the right panel) as a function of s_t.

For most values of s_t in Figure 7, in particular for s_t greater than 0.16, the stock's expected return decreases with s_t. That is, the expected return behaves in a countercyclical manner for most of the states of the economy. The same intuition behind the behavior of the price-to-dividend ratio applies here. In low consumption states, the more risk-averse agent owns most of the stock, the stock valuation is low, and its expected return is high. In high consumption states, the less risk-averse agent owns most of the stock, the valuation is high, and the expected return is low. Hence, overall, the stock's expected return decreases as the economy expands. This behavior also implies a negative correlation between the expected return and the return of the stock itself.

However, this pattern is not uniform. When the economy falls into very low consumption states (when s_t falls below 0.16), the expected stock return can behave procyclically—it comoves with the overall level of the market. This is mainly due to the fact that the risk of the stock, as measured by its price volatility, increases quickly with the consumption level. This has the effect of increasing the stock's expected return. The possibility of this rich relation between the stock's expected return and price levels should be taken into account when analyzing the empirical behavior of these two variables.

Next, we examine stock return volatility and how it varies with the state of the economy. Recall that in the single-agent economy, the volatility of stock returns is just the volatility of the dividend process σ. This is true independent of the level of risk aversion of the representative single agent. In contrast, the stock return volatility can differ significantly from the volatility of dividends when both agents are present in the market.

The right panel of Figure 7 plots the volatility of stock returns as a function of s_t. As expected, stock return volatility approaches the volatility of the dividend process, which is 0.10, as s_t approaches either of the limiting values of zero or one. For all other values of s_t, however, the volatility of stock returns diverges from the volatility of dividends. In fact, in this case, stock return volatility is always higher than the dividend volatility of 0.10. This implies that when both agents are present in the market and have more risk sharing, the stock price actually becomes more volatile than if only one of them is present. The volatility of stock returns reaches its maximum of over 0.106 when s_t is between 0.40 and 0.50.

The nonmonotonic behavior of stock return volatility with s_t (and thus X_t) implies a rich pattern of heteroscedasticity for stock returns. In particular, in the region of s_t exceeding 0.5, stock return volatility is negatively correlated with changes in the stock price. That is, the volatility increases as the market drops. This is compatible with the empirical relation between stock market returns and return volatility (e.g., Black 1976; Nelson 1991). Figure 5 also shows, however, that for small values of s_t (less than 0.4), i.e., when the economy is in low consumption states, the correlation between stock volatility and return can be positive.

6.3 The risk premium and Sharpe ratio of the stock

We now examine the expected excess return or (instantaneous) risk premium on the stock, which is defined by

$$\pi_t = q_t - r_t, \tag{36}$$

where r_t is the instantaneous interest rate, and its Sharpe ratio is π_t/σ_t. Figure 8 plots these two quantities as functions of s_t. As we see from the left panel, for a wide range of s_t, in general the expected excess return decreases with the

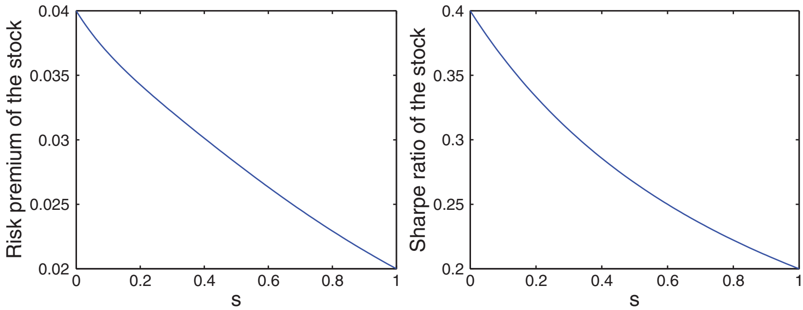

Figure 8
Expected excess return and Sharpe ratio of the stock
The left panel plots the expected excess return of the stock as a function of s_t, and the right panel plots the Sharpe ratio. The parameters are at the baseline values: $\mu=0.03$, $\sigma=0.10$, $\rho=0.01$, $\gamma=2$, and $\alpha=0.5$.

level of the market. This suggests that the time variation of the risk premium is also countercyclical. Comparing the behavior of the expected excess return of the stock and that of the dividend yield, which is the inverse of the price-to-dividend ratio, we see a positive relation between the two. This is consistent with the empirical evidence on the positive correlation between dividend yield and future stock returns (Fama and French 1988; Campbell and Shiller 1988a, 1988b; among others).

The right panel of Figure 8 further shows that in contrast to the more complex behavior of the expected excess return, the Sharpe ratio of the stock exhibits a simple countercyclical pattern. Among the parameter values we have explored, the countercyclical behavior of the Sharpe ratio seems to be quite robust. This is consistent with the empirical evidence presented by Ferson and Harvey (1991), among others.

6.4 Interest rates and bond prices

We now turn our attention to the behavior of interest rates and bond prices. The instantaneous interest rate is given in Equation (21). The behavior of interest rates in a market with heterogeneous agents is analyzed in detail by Wang (1996) in a model similar to ours. Although the interest rate stays constant when only one of the agents is present in the market, it becomes stochastic when both are present. The solution we obtain for the agents' equilibrium portfolio policies allows us to further link quantities in the market, such as the total amount of credit with the behavior of interest rates and bond prices.

In addition to the instantaneous interest rate, we can also compute the prices of long-term bonds and their yields, as in Wang (1996). Especially, we want to consider the price of a consol bond that pays a continuous interest flow at a rate of one. Its price is given in Proposition 5.

Proposition 5. The price of a consol bond is

$$B_t = E_t\left[\int_0^\infty e^{-\rho\tau}\left(\frac{C_{t+\tau}}{C_t}\right)^{-2\gamma} d\tau\right]$$
$$= a'_1 - a'_2\, s_t\ F(1,1-\theta,2+\theta-2\gamma;s_t) - a'_3\,(1-s_t)$$
$$F(1,\lambda+1,2\gamma+2\lambda+2;1-s_t), \tag{37}$$

where

$$a'_1 = \frac{\theta+\lambda-\gamma}{\psi(\gamma+\lambda)(\theta-2\gamma)},\ a'_2 = \frac{2\gamma}{\psi(1+\theta-2\gamma)(\theta-2\gamma)},\ a'_3 = \frac{\gamma}{\psi(\gamma+\lambda)(2\gamma+2\lambda+1)}. \tag{38}$$

The yield to maturity on the consol bond, which represents an average yield on long-term bonds, can then be defined by

$$l_t = \frac{1}{B_t}. \tag{39}$$

The difference between the long-term bond yield and the instantaneous interest rate r_t, $l_t - r_t$, gives a measure of the term spread for bond yields.

Figure 9 illustrates the behavior of short- and long-term interest rates under the baseline parameter values. The top left panel plots the instantaneous interest rate as a function of s_t. Note that at $s_t = 0$, r_t reaches the limiting interest rate $r^{(2)}$ when only the more risk-averse agent is present in the market, which is 0.03. Similarly, at $s_t = 1$, r_t approaches the limiting interest rate $r^{(1)}$ when only the less risk-averse agent is present, which is 0.04. Overall, in this case r_t increases with the level of aggregate consumption (or stock price).[9]

The top right panel of Figure 9 shows the term spread of interest rates for different states of the economy. First recall that $s_t = 0$ and 1 correspond to the single-agent economies where the term spread is zero because in these cases the interest rate is constant and thus the term structure of interest rates is flat. When both agents are present, this is no longer the case. For the parameters considered in this case, the term spread is always negative, although relatively small in magnitude. It reaches its minimum value of -7.2 basis points when s_t is around 0.18. This implies that overall the term structure is downward sloping. Moreover, the slope of the term structure is countercyclical for $s_t < 0.18$ and becomes pro-cyclical for $s_t > 0.18$.

Obviously, the term spread we observe in this case is very small in magnitude. This is in part because for the parameter values we use, the level and the

9 This overall relation between r_t and s_t is sensitive to the parameter values, which determine $r^{(1)}$ and $r^{(2)}$. If $r^{(2)} < r^{(1)}$, as is the case here, r_t decreases with X_t overall (but not necessarily monotonically). If $r^{(2)} > r^{(1)}$, the opposite is true.

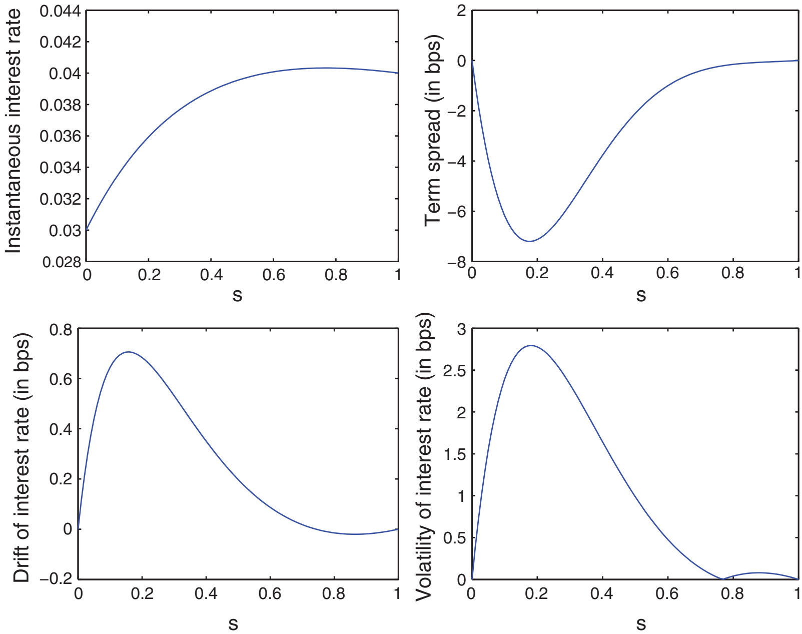

Figure 9
Bond yields and interest rate dynamics
The top two panels plot the instantaneous interest rate (the left panel) and the term spread of interest rates (the right panel) against s_t, respectively. The bottom two panels plot the drift (the left panel) and volatility (the right panel) of the instantaneous interest rate, respectively. The parameters are at the baseline values: $\mu=0.03$, $\sigma=0.10$, $\rho=0.01$, $\gamma=2$, and $\alpha=0.5$.

variability of the interest rate are both low, roughly consistent with the data. The low volatility of the interest rate will limit the slope of the term structure in general. Given the limited degrees of freedom we have in the model, we focus less on the magnitudes of various effects and more on their qualitative features.

Our results indicate that the relation between the term spread and the aggregate state of the economy can be quite rich. Two factors drive the shape of the term structure: the expectation about future interest rates and the risk premium associated with their uncertainty. In order to see how expectations about future interest rates behave, we plot in the bottom two panels of Figure 9 the drift of the instantaneous interest rate (the left panel) and its volatility (the right panel), respectively. It is not surprising that the drift of the interest rate is overall positive. It is highly nonlinear, however. In particular, it increases with s_t for small values of s_t, peaks around $s_t=0.15$, and then starts decreasing, turning slightly negative for $s_t>0.77$.

As in the discussion of stock returns, the behavior of the instantaneous interest rate and the term spread in our model also depends on the parameter values. For other parameter values, the interest rate can be decreasing with s_t

(when $r^{(2)} > r^{(1)}$) and the term spread can be positive for some states of the economy. For brevity, we omit the discussions of these cases.

6.5 Leverage and asset prices

In the discussions above, we focus on the behavior of asset prices alone. In this subsection, we examine the joint behavior of asset prices and the amount of leverage in the market. From the results in Section 6.3, we see that the leverage ratio in the market varies nonmonotonically with the level of aggregate consumption, increasing first and then decreasing. It peaks in the intermediate states of the economy when both agents are significant players in the market. Comparing the behavior of the leverage with the behavior of stock and bond prices shown above, we see interesting links between them.

The top left panel of Figure 10 illustrates the link between the expected return on the stock and the market leverage ratio. Clearly, the relationship is not one-to-one. The lower half represents the relationship for relatively low levels of aggregate consumption ($s_t < 0.4$), while the upper half represents the relationship for high levels of aggregate consumption ($s_t > 0.4$). On average, we see that there is a positive relationship between leverage ratio and expected stock return.

The reason behind such a positive relationship is indicated in the top right panel of Figure 10, which shows how the stock price volatility varies with market leverage ratio. Here, we see a clear and strong positive relationship: when leverage ratio is high, the stock price becomes highly volatile. The strong positive correlation between leverage and stock volatility suggests that even though leverage helps achieve optimal risk sharing overall, it can substantially increase the local volatility of the stock market. Consequently, it also increases the expected return on the stock.

The bottom left panel of Figure 10 plots the instantaneous interest rate against the market leverage ratio. Again, we see a positive, although relatively weak, relationship between the two variables. Also, the relationship is much stronger (the lower part of the curve) for bad or intermediate states of the economy ($s_t < 0.6$) than good states of the economy ($s_t > 0.7$). The bottom right panel of Figure 10 shows the relationship between interest rate volatility and market leverage ratio. It exhibits a strong positive correlation, again much stronger (the upper part of the curve) for bad and intermediate states of the economy.

Given the frictionless setting we have, the equilibrium leverage level is efficient, i.e., the amount of leverage needed to achieve optimal risk sharing. Our results above lead to several observations. First, the efficient leverage level behaves procyclically over a wide range of economic states (the low and medium levels of aggregate consumption). This improves risk sharing. Second, deleveraging occurs either when the economy shrinks or when it expands over a certain threshold. The latter situation happens because in very good states of the economy, the lending agent becomes insignificant in his share of the total wealth and his credit capacity shrinks relative to the size of the economy.

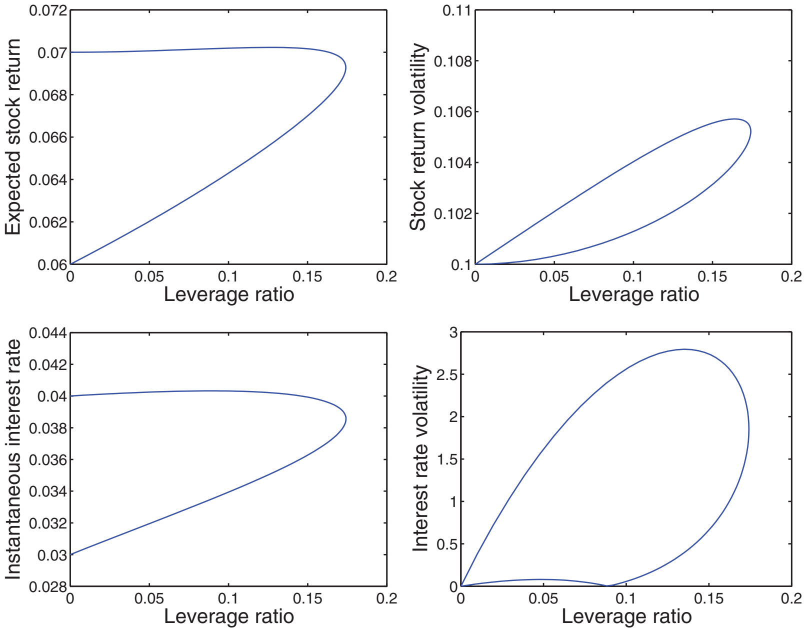

Figure 10
Stock and bond prices and market leverage ratio
The top two panels plot the expected return on the stock (the left panel) and its return volatility (the right panel) against market leverage ratio M_t/P_t, respectively. The bottom two panels plot the instantaneous interest rate (the left panel) and its volatility (the right panel) against market leverage ratio, respectively. The parameters are at the baseline values: $\mu=0.03$, $\sigma=0.10$, $\rho=0.01$, $\gamma=2$, and $\alpha=0.5$.

Third, although leverage facilitates risk sharing, it in general increases the volatility of asset prices, at least the stock price and the interest rate. Thus, the positive relationship between leverage and price variability is by no means a symptom of market inefficiency. Instead, it can well be the natural outcome of efficient risk sharing facilitated by high levels of leverage.

Note that in our current model, a single-state variable, either X_t or s_t, is driving the economy. As a result, the derived variables, e.g., moments of the interest rate, the stock returns, the market leverage ratio, are all functions of the state variable. Hence, they are perfectly correlated locally. The relationships derived from the model are all part of the equilibrium outcome, and it is hard to separate them.

6.6 Further discussions on the behavior of asset prices

Under the baseline parameter values, our model produces relatively simple patterns for stock price and interest rate behavior that are largely compatible with the empirical observations. However, these patterns are by no means unique. In fact, under different parameter values, our model can lead to a variety of behaviors for stock price and interest rate dynamics.

Instead of presenting an extensive analysis of various possible return patterns in our model, we consider another set of parameter values, which are also reasonable in matching the data, and show that they can lead to quite different properties of the equilibrium. Our purpose here is merely to illustrate the richness in the model's predictions. In particular, we let the growth rate of the aggregate dividend μ be 0.05 and its volatility σ be 0.15. Moreover, we let the time discount rate ρ be 0.08 and the relative risk aversion of the less risk-averse agent remain at $\gamma = 2$. For these parameter values, our model leads to an interest rate between 5.0% and 11.2%, a risk premium for the stock between 9.0% and 4.5%, and a stock return volatility around 15%.

Figure 11 illustrates the various properties of the stock price under this set of parameter values. The top left panel plots the stock's price-dividend ratio Y_t for different states of the economy. In contrast to the baseline case, the price-dividend ratio is no longer monotonic in s_t in this case. In fact, it is countercyclical for all s_t less than 0.6. That is, the price-dividend ratio can actually decrease with the level of the market. But for extremely good states of the economy, i.e., when s_t exceeds 0.6, the correlation between the two turns positive. The price-dividend ratio has a minimum value of roughly 9.1 at around $s_t = 0.6$.

The high price-dividend ratio for small values of s_t might seem puzzling, given that in these states it is agent 2, the more risk-averse agent, who dominates the market (having large wealth and consumption share). The reason this can occur is due to the fact that under the parameter values, the interest rate is low for small values of s_t (i.e., $r^{(2)} = 0.05 < r^{(1)} = 0.11$).

The top right panel of Figure 11 shows the behavior of the expected return on the stock q_t. It shows an opposite dependence on s_t as the price-dividend ratio. In particular, for s_t between 0.0 and 0.7, the expected return of the stock is procyclical—it increases with the level of the market. For s_t greater than 0.7, however, the expected return becomes countercyclical.

This nonmonotonic behavior makes it clear that the expected return in the two-agent economy is not just a weighted average of the expected returns of the two extreme cases when only the more risk-averse agent or the less risk-averse agent populates the market. In these cases, the expected stock return would be 0.1445 and 0.1575, respectively. Figure 11 shows that the expected return of the stock in the two-agent model can lie outside the bounds given by the limiting one-agent economies implied by allowing s_t to approach zero or one. This result parallels those described in Wang (1996) for the riskless interest rate.

The bottom left panel of Figure 11 plots the expected excess return of the stock against s_t. It is interesting that it decreases monotonically with s_t. The difference in the behavior of the expected excess return and that of the expected return is caused by the riskless interest rate, which is in general increasing with s_t under the current parameter values, as mentioned above.

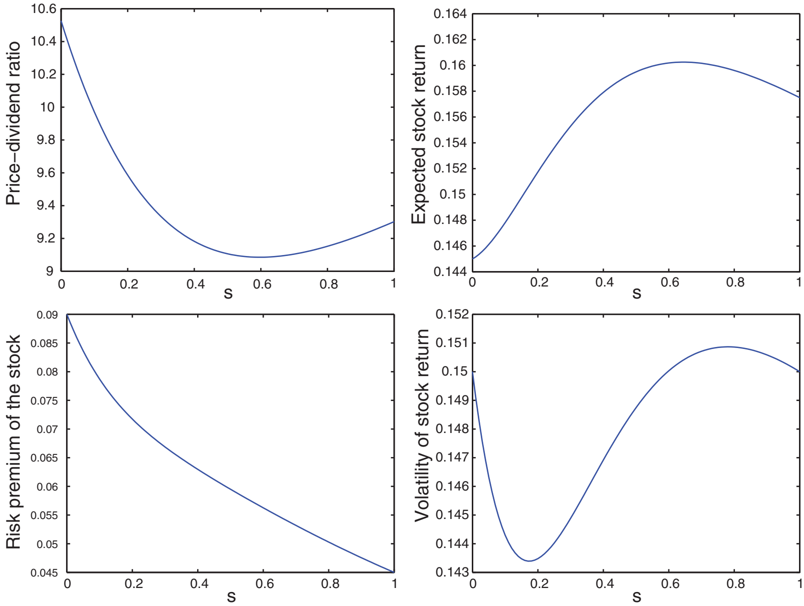

Figure 11
Stock price behavior under alternative parameter values
The top two panels plot the price-dividend ratio (the left panel) and the expected return (the right panel) of the stock, respectively, as functions of s_t. The bottom two panels plot the expected excess return (the left panel) and return volatility (the right panel) of the stock, respectively. The parameters are at the alternative values: $\mu=0.05$, $\sigma=0.15$, $\rho=0.08$, $\gamma=2$, and $\alpha=0.5$.

What is the most striking is the behavior of stock return volatility, which is shown in the bottom right panel of Figure 11. Its dependence on the state of the economy is highly nonlinear. When s_t is small, the volatility decreases with the level of the stock market. It reaches a minimum when s_t is around 0.18. For s_t between 0.18 and 0.8, the volatility of the stock is positively related to the level of its price. After reaching its maximum at s_t around 0.8, the volatility becomes negatively related to s_t again. The fact that stock return volatility can be lower than the fundamental volatility σ, its value in the single-agent economy, shows that risk sharing between the two agents can help reduce price volatility under certain circumstances.

7. Trading Activity

As we discussed in Sections 5 and 6, risk sharing between the two agents with different risk preferences is achieved through their trading in the securities market. In particular, in our model it is accomplished by allowing the less risk-averse agent to borrow in the credit market and then take on a levered position in the stock market. Moreover, such a levered position is not static, but rather dynamic. As the economy evolves, the desire for borrowing and lending also changes for both agents. Consequently, both agents follow dynamic trading strategies to replicate their desired consumption profiles. Their equilibrium trading strategies drive not only asset prices, as we elaborated in the previous section, but also the trading activities both in the credit and the stock market. In Section 6, we examined the amount of credit generated endogenously in the market and its behavior. In this section, we turn our attention to the stock market and analyze our model's implications for stock trading activity and how it behaves.

In a continuous-time setting like ours, with the diffusive nature of the information flow, trading volume in the conventional sense is not properly defined. In fact, it would be infinite. This is because the local variation of the underlying shocks is unbounded, and so is the agents' security holdings. Trading costs have to be part of the analysis in order to study volume in a rigorous manner (e.g., Lo, Mamaysky, and Wang 2004). Such a treatment is beyond the scope of this article. Instead, we use an alternative measure for the amount of trading activity in the market. In particular, given the stock holding of an agent N_t (e.g., the more risk-averse agent), we use its absolute volatility $\sigma_{N,t}$ to gauge his trading activity. Given that the less risk-averse agent's stock holding is $1-N_t$, our measure of trading activity does not depend on which agent we follow.

From the stock holdings of the more risk-averse agent given in Equations (24–25), some algebra yields the following expression for $\sigma_{N,t}$:

$$V_t = \sigma_{N,t} = \left| \left[\frac{1}{\Phi_t} \frac{d\Psi_t}{ds_t} - \frac{\Psi_t}{\Phi_t^2} \frac{d\Phi_t}{ds_t} \right] \sigma_{s,t} \right|, \tag{40}$$

where $\sigma_{s,t}$ is the volatility of s_t given in Equation (27). Figure 12 plots our measure of stock trading activity V_t for different values of s_t.

Not surprisingly, trading activity exhibits the same unimodal pattern as the market-leverage ratio. In the two extremes, i.e., when $s_t = 0$ or 1, the market is dominated by one of the agents and there is no trading. Somewhere in the middle range of s_t, trading is most intense, as both agents have large needs to share risk and are also compatible in size to accommodate each other.

The behavior of stock trading activity shown in Figure 12 has several interesting implications. First, the level of trading activity evolves smoothly in the state space, but can differ substantially in different parts. This implies that it can be highly persistent over time. When the economy moves into those states with high trading activity, say, when s_t falls between 0.05 and 0.30, it will stay there for a while and so will trading activity. Second, in some states of the economy, in particular when s_t is relatively small, trading is procyclical, while in other states, i.e., when s_t is relatively large, it can be countercyclical. This rich relation between trading activity and changes in the price level of the stock market may help explain the complex empirical patterns between them (e.g., Karpoff 1987; Gallant, Rossi, and Tauchen 1992). Third, comparing the

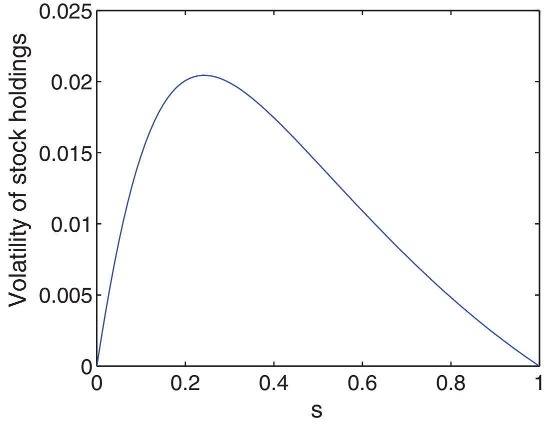

Figure 12
Stock trading activity
The figure plots the volatility of agents' stock holdings as a measure of trading activity for different values of s_t. The parameters are at the baseline values: $\mu=0.03$, $\sigma=0.10$, $\rho=0.01$, $\gamma=2$, and $\alpha=0.5$.

behavior of trading activity and stock return volatility, we see a strong positive relation between the two. Trading is particularly active when return volatility is high. This is one of the most robust patterns about trading activity observed in the data (see Karpoff 1987). Fourth, comparing the behavior of stock trading activity and leverage in the market (Figure 5), we also see a close relation between these two variables. In particular, trading in the stock market peaks as the market-leverage ratio approaches its maximum. This is intuitive, given that in our model leverage is used by the less risk-averse agent to finance his stock purchases.

8. Empirical Results

The key difference between the standard single-agent framework and the two-agent model developed in this article is that the distribution of wealth among agents becomes an important state variable that drives the equilibrium. While the notion that heterogeneity affects asset pricing is certainly not new, taking heterogeneous-agent models to the data has traditionally proven difficult precisely because agent heterogeneity is not directly observable, at least at the aggregate level.

In this article, we have shown that the credit market allows for risk sharing among the agents in the model. In general, the more equal the distribution of wealth in the economy, the greater the amount of leverage. An immediate corollary of our results is that changes in the size of the credit sector (which are observable) provide direct information about changes in the relative wealth of the two classes of agents (which are not directly observable). Thus, the model delivers the testable empirical implication that changes in the size of the credit sector should be associated with changes in key asset-pricing measures such as expected returns.

To explore this empirical implication of the model, we focus on the relation between the equity premium and the size of the credit sector. Instead of conducting a direct test of the model, we adopt a more empirical approach in our analysis, which is guided by the theoretical predictions. Such a strategy is motivated by several considerations. First, our model is very parsimonious for its theoretical clarity at the cost of empirical flexibility. It is meant to demonstrate the important link between credit and asset prices qualitatively rather than to produce quantitative predictions. Overly emphasizing the latter really goes beyond the scope of the model. Second, tying the empirical analysis too closely to the model forces us to see the data through the lens of the model, which is restrictive and can even be distortive. A more flexible empirical approach following the direction of the model allows the data to speak for itself. It can be more informative in pointing to further extensions of the theory and empirical work. Third, at a more practical level, the data on market quantities, such as the size of credit market, are very coarse. Much more detailed work is needed in gathering and sorting out the data. Our attempt here is to take a rough but first cut at the data. Of course, the results we obtain, if any, are merely suggestive, but should also lend additional justification for the analysis pursued in this article.

Since the equity premium is itself not directly observable, we will use the standard approach of estimating predictive vector autoregressions (VARs) in which ex post excess stock market returns are regressed on ex ante credit sector measures. Intuitively, if time variation in the equity premium is correlated with changes in the size of credit sector, then these measures should have predictive power for subsequent excess stock returns.

As the measure of excess stock market returns, we use the excess return on the CRSP value-weighted index. The data consist of the annual excess (non-overlapping) returns for the 59-year period from 1952 to 2010 (data provided by courtesy of Ken French).

There is an extensive and rapidly growing literature on stock return predictability that is far too lengthy for us to review in depth. We note, however, that there are a number of economic measures identified in the literature that appear to have some predictive power for excess stock returns. Our approach will be to include four of the variables that appear prominently in the forecasting literature, and then evaluate whether credit sector information has incremental forecasting power for excess returns in the VARs.

These four variables are the lagged excess return for the CRSP value-weighted index, the Lettau and Ludvigson's (2001) *cay* measure, the annual dividend yield for the CRSP value-weighted index, and the short-term interest rate. The inclusion of the lagged excess return is motivated by the extensive empirical literature on the returns from momentum strategies. Examples of this literature include DeBondt and Thaler (1985), Lo and MacKinlay (1988), Poterba and Summers (1988), Jegadeesh and Titman (1993), and many others.[10] The inclusion of the *cay* measure is motivated by the evidence in Lettau and Ludvigson that the consumption-wealth ratio is a strong predictor of stock returns. The annual *cay* measure is the average of quarterly data from Martin Lettau's website. The inclusion of the dividend yield in the VARs is motivated by the results of Fama and French (1988), Goyal and Welch (2003), Cochrane (2008), Lettau and Van Nieuwerburgh (2008), and many others (data obtained from the Bloomberg system). The use of the short-term interest rate (also obtained by the courtesy of Ken French) follows from Ang and Bekaert (2007), among others.[11]

To capture changes in the size of the credit sector, we use two variables in the VARs that reflect the size of the household and corporate credit markets in the economy. The first is the ratio of aggregate consumer credit to aggregate consumer durables. The two variables used to compute these ratios appear as lines 34 and 7 of Table B.100, "Balance Sheet of Households and Nonprofit Organizations," reported in statistical release Z.1, "Flow of Funds Accounts of the United States," by the Federal Reserve Board. The second measure is the ratio of total corporate credit to total corporate net worth, given as line 36 in Table B.102, "Balance Sheet of Nonfarm Nonfinancial Corporate Business," reported in statistical release Z.1, "Flow of Funds Accounts of the United States," by the Federal Reserve Board.[12]

In Figure 13, we plot the two data series. Several broad features are apparent. First, over the sample period, there has been an overall increase in the relative size of the credit sector for both consumers and corporations. Second, for the two largest recessions in this period (1981–1982 and 2008–2009), we see substantial reduction in the credit market. Third, consumer credit seems to exhibit more short-term variability, while corporate credit mainly exhibits lower-frequency fluctuations. We also notice that although the two series co-vary with the broad economic cycles, they are far from being perfectly synchronized.

Table 2 reports the results from the VARs for the excess return predictive regressions. The first regression specification includes only the lagged excess return, *cay* measure, the dividend yield, and the short-term rate. Consistent with the results in the literature, we find that the combination of these four ex ante variables has significant in-sample predictive power for the CRSP value-weighted excess returns at the one-year horizon. The R^2 for the regression is 0.2748; the adjusted R^2 is 0.2190.

The predictive power for the VAR model increases significantly when the ex ante consumer credit ratio (measured at time $t-1$) and the change in the consumer credit ratio (from $t-2$ to $t-1$) are used to forecast the excess stock

[10] Also see Grinblatt, Titman, and Wermers (1995) and Chan, Jegadeesh, and Lakonishok (1996).

[11] We are grateful to the referee for suggesting the use of this variable in the VARs.

[12] We also investigated whether the ratio of household mortgage debt to total household real estate value had predictive power for excess stock returns. This variable was subsumed by the consumer credit ratio.

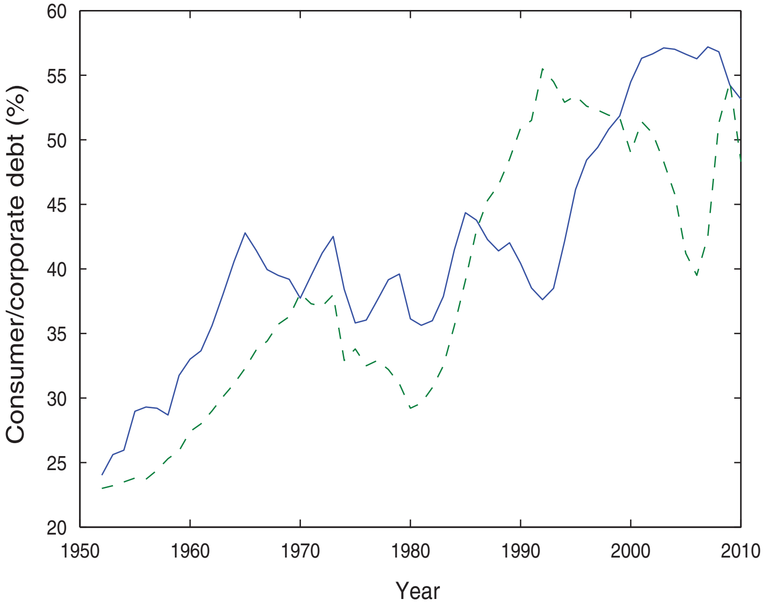

Figure 13
Aggregate credit of consumers and corporations
The figure plots the relative size of consumer credit (solid line) and corporate credit (dashed line) for the years 1952 to 2010. The relative size of consumer credit is given by the aggregate consumer credit normalized by aggregate consumer durables; the relative size of corporate credit is given by the aggregate corporate credit normalized by total corporate net worth.

return for year t. In particular, the adjusted R^2 increases to 0.3030. Similarly, the predictive power of the VAR model increases when the corporate credit ratio and the change in this ratio are included, with the adjusted R^2 increasing to 0.2690. When both the consumer credit ratio and the corporate credit ratios are included along with their first differences, the R^2 of the VAR model increases to 0.4255 and the adjusted R^2 becomes 0.3278. This R^2 is on the same order of magnitude as those reported by Cochrane and Piazzesi (2005) in forecasting one-year excess returns on Treasury bonds using a vector of forward rates. Finally, Table 2 also reports Newey-West t-statistics for the explanatory variables. These t-statistics need to be interpreted carefully, however, given the well-known biases associated with lagged persistent variables in predictive regressions (see Stambaugh 1999). The results show that both the consumer credit ratio and the first difference of the corporate bond ratio are statistically significant even after controlling for the other predictive variables.

9. Conclusion

The credit market does not play a significant role in the standard single-representative-agent model in asset pricing. In this article, we allow for two

Table 2
Results from the predictive regressions for excess stock returns

Predictive Variable	Regression Coefficients			
Intercept	0.05067	−58.48035	−21.17657	−54.40700
	[0.01]	[−2.51]	[−1.01]	[−2.15]
ExRet	−0.18610	−0.05351	−0.16860	−0.09407
	[−2.19]	[−0.58]	[−1.76]	[−0.95]
Cay	5.15571	6.44780	4.80937	8.79286
	[3.38]	[4.54]	[2.98]	[4.53]
DivYld	5.33282	13.01834	9.47150	12.65582
	[2.69]	[3.36]	[2.42]	[2.76]
r	−1.86801	−3.38905	−2.76572	−3.39682
	[−2.86]	[−3.72]	[−2.60]	[−3.53]
Consumer		1.00531		1.40095
		[2.78]		[3.14]
Δ Consumer		−1.45725		−1.72875
		[−1.13]		[−1.32]
Corporate			0.35186	−0.46802
			[0.93]	[−1.02]
Δ Corporate			−1.74964	−1.68189
			[−1.90]	[−2.15]
R^2	0.2748	0.3791	0.3487	0.4255
Adjusted R^2	0.2190	0.3030	0.2690	0.3278
F-Statistic	–	4.2228	2.8850	3.1397
P-value	–	0.0203	0.0654	0.0228

This table reports the results from the regression of annual excess returns for the CRSP value-weighted index on the ex ante predictive variables. ExRet denotes the excess return on the CRSP value-weighted index, Cay is the Lettau-Ludvigson Cay measure, DivYld is the dividend yield for the CRSP value-weighted index, r is the one-month Treasury bill rate, Consumer is the ratio of total consumer credit to consumer durables, and Corporate is the ratio of total nonfarm nonfinancial corporate debt to total nonfarm nonfinancial corporate new worth. The F-statistic and its P-value test whether the consumer and/or corporate credit variables have significant incremental predictive power after controlling for the other variables. The Newey-West t-statistics in parentheses are based on three lags. The data consist of annual observations for the 1952–2010 period ($N=59$).

classes of investors with different levels of risk aversion and solve in closed form for equilibrium consumption levels, portfolio choices, and asset prices. In this setting, agents borrow and lend to each other to achieve optimal risk sharing and a meaningful credit sector arises.

In this model, the "demographics" of the market, as measured by the relative wealth of the agents, emerges as a key state variable driving the market. We show that borrowing and lending between the less and more risk-averse investors facilitate risk sharing between them and shape the evolution of their relative wealth, which induces significant time variation in expected asset returns and return volatility. An immediate implication of this market interaction is that changes in the amount of credit in the market reveal information about changes in the relative wealth of the agents, and therefore, about changes in return moments. We take this empirical implication to the data and show that variables measuring changes in the size of the credit sector have significant predictive power for excess stock returns, even after controlling for previously documented predictive variables. Our results provide strong support for the empirical implications of the model.

Since the main goal of the article is to demonstrate, both theoretically and empirically, the importance of the credit market in understanding asset prices, rather than a comprehensive treatment of asset pricing, we focus on the key elements in this analysis, namely the interaction of different agents in the financial market, and maintain parsimony along other dimensions. This approach leaves the model with several limitations. For example, only two classes of agents are considered, preferences are limited to the time-additive and constant relative risk-aversion class, and aggregate consumption growth is i.i.d. These restrictive assumptions lead to several undesirable features of the model, such as that most variables are highly correlated (they are driven by one source of risk) and their distributions are not stationary (in general, the less risk-averse agent tends to dominate the economy in the long run). They also constrain the model in its ability to fit details of the data. Nonetheless, it should be clear from our analysis that the fundamental link as illustrated by the simple model is always present in the market and its implications are important. Further extensions can be considered to enrich the model in relaxing some of its restrictions.[13] These extensions, however, are left for future work.

Appendix

A.1 Solution to the Single-agent Model

In equilibrium, $C_t = X_t$ for the representative agent. Substituting this into the Euler equation gives

$$\frac{P_t}{X_t} = \int_0^\infty e^{-\rho\tau} \, \frac{E_t[X_{t+\tau}^{1-\gamma}]}{X_t^{1-\gamma}} \, d\tau. \tag{A1}$$

From Equation (1),

$$X_{t+\tau} = X_t \, \exp\left[(\mu - \tfrac{1}{2}\sigma^2)\tau + \sigma\,(Z_{t+\tau} - Z_t)\right], \tag{A2}$$

which implies

$$E_t[X_{t+\tau}^{1-\gamma}] = X_t^{1-\gamma} \, \exp\left[(1-\gamma)(\mu - \tfrac{1}{2}\sigma^2)\tau + \tfrac{1}{2}(1-\gamma)^2\sigma^2\tau\right]. \tag{A3}$$

Substituting into Equation (A1) gives

$$\frac{P_t}{X_t} = \int_0^\infty e^{-(\rho-\kappa)\tau} \, d\tau, \tag{A4}$$

where $\rho > \kappa$, which then implies Equation (9).

Denote the value of a riskless zero-coupon bond with maturity $\Delta\tau$ as $e^{-r\Delta\tau}$. The agent's first-order conditions imply

$$e^{-r\Delta\tau} = e^{-\rho\Delta\tau} \, \frac{E[X_{t+\Delta\tau}^{-\gamma}]}{X_t^{-\gamma}}. \tag{A5}$$

Using Equation (A2) to represent $X_{t+\Delta\tau}^{-\gamma}$ and taking expectations gives an expression similar to Equation (A3), which is then substituted into Equation (A5),

$$e^{-r\Delta\tau} = e^{-\rho\Delta\tau} \, \exp\left[-\gamma(\mu - \tfrac{1}{2}\sigma^2)\,\Delta\tau + \tfrac{1}{2}\gamma^2\sigma^2\Delta\tau\right]. \tag{A6}$$

Taking the logarithm and letting $\Delta\tau \to 0$ gives Equation (13).

[13] For example, by introducing habits, Chan and Kogan (2002) show that a similar model can become stationary and match many aspects of the price data. They do not consider how prices are linked to quantities in the market.

A.2 Solution to the Two-agent Model

A. Equilibrium consumption allocation

To be a solution for the problem in Equation (14), an allocation $C_{1,t}, C_{2,t}$ must satisfy the optimality condition,

$$E_0\left[\int_0^\infty e^{-\rho t}\left[\alpha\, C_{1,t}^{-\gamma} - (1-\alpha)\, C_{2,t}^{-2\gamma}\right]dt\right] = 0, \tag{A7}$$

for all $t > 0$. Substituting the solutions for $C_{1,t}$ and $C_{2,t}$ given in Equation (15) shows that they satisfy this optimality condition. The relative weight of the two agents, α, is determined by the initial conditions of the economy, in particular X_0 and agents' endowment of shares, given by n. The condition to determine α is given later.

B. The stock price

To solve for the stock price P_t, we substitute the more risk-averse agent's optimal consumption in Equation (17),

$$\begin{aligned} P_t &= \left(\sqrt{1+bX_t}-1\right)^{2\gamma}\; E_t\left[\int_0^\infty e^{-\rho\tau}\,\frac{X_{t+\tau}}{\left(\sqrt{1+bX_{t+\tau}}-1\right)^{2\gamma}}\,d\tau\right], \\ &= b^{-2\gamma}(\sqrt{1+bX_t}-1)^{2\gamma}\; E_t\left[\int_0^\infty e^{-\rho\tau}\, X_{t+\tau}^{1-2\gamma}\,(\sqrt{1+bX_{t+\tau}}+1)^{2\gamma}\, d\tau\right]. \end{aligned} \tag{A8}$$

We rewrite $X_{t+\tau}$ as $X_t\, e^u$, where u is normally distributed with mean $\hat{\mu}\,\tau = (\mu - \frac{1}{2}\sigma^2)\tau$ and variance $\sigma^2\tau$. Substituting in the normal density gives

$$\begin{aligned} P_t &= b^{-2\gamma}\left(\sqrt{1+bX_t}-1\right)^{2\gamma}\; X_t^{1-2\gamma}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}e^{(1-2\gamma)u}\left(\sqrt{1+bX_te^u}+1\right)^{2\gamma} \\ &\quad\times\int_0^\infty e^{-\rho\tau}\,\tau^{-1/2}\exp\left[\frac{-u^2+2\hat{\mu}u\tau-\hat{\mu}^2\tau^2}{2\sigma^2\tau}\right]d\tau\; du, \\ &= b^{-2\gamma}\left(\sqrt{1+bX_t}-1\right)^{2\gamma}\; X_t^{1-2\gamma}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}\; e^{(1-2\gamma)u}\;\left(\sqrt{1+bX_te^u}+1\right)^{2\gamma} \\ &\quad\times e^{\hat{\mu}u/\sigma^2}\int_0^\infty \tau^{-1/2}\;\exp\left[\frac{-u^2}{2\sigma^2}\,\frac{1}{\tau}-\frac{(\hat{\mu}^2+2\sigma^2\rho)\tau}{2\sigma^2}\right]d\tau\; du, \\ &= b^{-2\gamma}\left(\sqrt{1+bX_t}-1\right)^{2\gamma} X_t^{1-2\gamma}\int_{-\infty}^{\infty}\frac{2}{\sqrt{2\pi\sigma^2}}e^{(1-2\gamma+\hat{\mu}/\sigma^2)u}\left(\sqrt{1+bX_te^u}+1\right)^{2\gamma} \\ &\quad\times\sqrt{\frac{|u|}{\psi}}\;K_{1/2}\left(\frac{|u|\psi}{\sigma^2}\right)\,du, \end{aligned} \tag{A9}$$

where $\psi = \sqrt{\hat{\mu}^2+2\rho\sigma^2}$, $K_{1/2}(\cdot)$ is the modified Bessel function (see Abramowitz and Stegun 1970, Chapter 10), and the last expression follows from Gradshteyn and Ryzhik (2000), 3.471.9. In turn, Gradshteyn and Ryzhik 8.469.3 implies

$$K_{1/2}\left(\frac{|u|\psi}{\sigma^2}\right) = \sqrt{\frac{\pi\sigma^2}{2|u|\psi}}\exp\left(\frac{-|u|\psi}{\sigma^2}\right). \tag{A10}$$

Substituting this in Equation (A9) gives

$$P_t = \frac{b^{-2\gamma}}{\psi}\;\left(\sqrt{1+bX_t}-1\right)^{2\gamma}\; X_t^{1-2\gamma}\;(I_1 + I_2), \tag{A11}$$

where

$$I_1=\int_0^{\infty}\exp[(1-2\gamma-\lambda)u]\,\left(\sqrt{1+bX_te^u}+1\right)^{2\gamma}\,du,$$

$$I_2=\int_{-\infty}^{0}\exp[(1-2\gamma+\theta)u]\,\left(\sqrt{1+bX_te^u}+1\right)^{2\gamma}\,du, \tag{A12}$$

and

$$\lambda=\frac{\psi-\hat{\mu}}{\sigma^2}\geq 0, \qquad \theta=\frac{\psi+\hat{\mu}}{\sigma^2}\geq 0. \tag{A13}$$

Define a new variable $w=\sqrt{1+bX_te^u}-1$, and let $\eta=\sqrt{1+bX_t}$. Changing variables gives

$$I_1=2^{1-\lambda}\,(bX_t)^{2\gamma+\lambda-1}\int_{\eta-1}^{\infty}(w+1)\,(1+\tfrac{1}{2}w)^{-\lambda}\,w^{-\lambda-2\gamma}\,dw, \tag{A14a}$$

$$I_2=2^{\theta+1}\,(bX_t)^{2\gamma-\theta-1}\int_{0}^{\eta-1}(w+1)\,(1+\tfrac{1}{2}w)^{\theta}\,w^{\theta-2\gamma}\,dw. \tag{A14b}$$

In turn,

$$\begin{aligned}I_1+I_2=&\,2^{1-\lambda}\,(bX_t)^{2\gamma+\lambda-1}\int_{\eta-1}^{\infty}w^{-\lambda-2\gamma}\,(1+\tfrac{1}{2}w)^{-\lambda}\,dw\\&+2^{1-\lambda}\,(bX_t)^{2\gamma+\lambda-1}\int_{\eta-1}^{\infty}w^{1-\lambda-2\gamma}\,(1+\tfrac{1}{2}w)^{-\lambda}\,dw\\&+2^{\theta+1}\,(bX_t)^{2\gamma-\theta-1}\int_{0}^{\eta-1}w^{\theta-2\gamma}\,(1+\tfrac{1}{2}w)^{\theta}\,dw\\&+2^{\theta+1}\,(bX_t)^{2\gamma-\theta-1}\int_{0}^{\eta-1}w^{1+\theta-2\gamma}\,(1+\tfrac{1}{2}w)^{\theta}\,dw.\end{aligned} \tag{A15}$$

Applying Gradshyteyn and Ryzhik 3.194.1 and 3.194.2, which requires $1+\theta-2\gamma>0$ and $2\gamma+2\lambda-2>0$, and then using Abramowitz and Stegun (1970) 15.3.4, we have

$$\begin{aligned}I_1+I_2=&\,\frac{2(\eta+1)^{2\gamma-1}}{2\gamma+2\lambda-1}\,F(1,\lambda,2\gamma+2\lambda;2/(\eta+1))\\&+\frac{2(\eta+1)^{2\gamma-1}(\eta-1)}{2\gamma+2\lambda-2}\,F(1,\lambda,2\gamma+2\lambda-1;2/(\eta+1))\\&+\frac{2(\eta+1)^{2\gamma-1}}{1+\theta-2\gamma}\,F(1,-\theta,2+\theta-2\gamma;(\eta-1)/(\eta+1))\\&+\frac{2(\eta+1)^{2\gamma-1}(\eta-1)}{2+\theta-2\gamma}\,F(1,-\theta,3+\theta-2\gamma;(\eta-1)/(\eta+1)).\end{aligned} \tag{A16}$$

To simplify the expression, we apply Abramowitz and Stegun 15.2.20,

$$\begin{aligned}F(1,\lambda,2\gamma+2\lambda-1;2/(\eta+1))=&\,\frac{\eta+1}{\eta-1}-\frac{2(2\gamma+\lambda-1)}{(\eta-1)(2\gamma+2\lambda-1)}\\&\times F(1,\lambda,2\gamma+2\lambda;2/(\eta+1)),\end{aligned} \tag{A17a}$$

$$\begin{aligned}F(1,-\theta,2+\theta-2\gamma;(\eta-1)/(\eta+1))=&\,\frac{\eta+1}{2}-\frac{(1+\theta-1\gamma)(\eta-1)}{2+\theta-2\gamma}\\&\times F(1,-\theta,3+\theta-2\gamma;(\eta-1)/(\eta+1)).\end{aligned} \tag{A17b}$$

Substituting these expressions in the solution for I_1+I_2, substituting I_1+I_2 into (A11), and then collecting terms gives

$$\begin{aligned}P_t=&\frac{X_t}{\psi(\gamma+\lambda-1)}+\frac{X_t}{\psi(1+\theta-2\gamma)}\\&-\frac{2\gamma(X_t-C_t)}{\psi(1+\theta-2\gamma)(2+\theta-2\gamma)}\;F(1,-\theta,3+\theta-2\gamma;1-C_t/X_t)\\&-\frac{\gamma C_t}{\psi(\gamma+\lambda-1)(2\gamma+2\lambda-1)}\;F(1,\lambda,2\gamma+2\lambda;C_t/X_t),\end{aligned}\tag{A18}$$

after substituting out η. Dividing this expression through by X_t and using the definition of s_t gives the price-dividend ratio in Equation (18) and defines the constants a_1, a_2, and a_3.

C. The instantaneous interest rate and consol price

The riskless rate r is again given by the more risk-averse agent's first-order condition for a short-term riskless bond,

$$e^{-r\Delta\tau}=e^{-\rho\Delta\tau}\;E_t\left[\left(\frac{C_{t+\Delta\tau}}{C_t}\right)^{-2\gamma}\right],\tag{A19}$$

which becomes

$$e^{-r\Delta\tau}=e^{-\rho\Delta\tau}\;\frac{E_t\left[\left(\sqrt{1+bX_{t+\Delta\tau}}-1\right)^{-2\gamma}\right]}{\left(\sqrt{1+bX_t}-1\right)^{2\gamma}}.\tag{A20}$$

Applying Itô's Lemma to $\left(\sqrt{1+bX_{t+\Delta\tau}}-1\right)^{-2\gamma}$, taking expectations in the numerator above, and then allowing $\Delta\tau\to 0$ gives

$$r_t=\rho+\frac{4\mu\gamma(1+bX_t)-\gamma b\sigma^2X_t}{2(1+bX_t)(2-C_t/X_t)}-\frac{\gamma(2\gamma+1)\sigma^2}{(2-C_t/X_t)^2}.\tag{A21}$$

Expressing X_t in terms of s_t and substituting gives Equation (21).

To solve for the price of a consol bond, we substitute the solution for C_t into Equation (33), which gives

$$B_t=\left(\sqrt{1+bX_t}-1\right)^{2\gamma}E_t\left[\int_0^\infty \mathrm{e}^{-\rho\tau}\left(\sqrt{1+bX_te^u}-1\right)^{-2\gamma}d\tau\right].\tag{A22}$$

This expression is very similar to Equation (A8) and can be evaluated by following the same steps used in deriving P_t above. In doing this, the additional parameter restrictions $\theta>2\gamma$ and $\gamma+\lambda>0$ are required to ensure the existence of a finite solution for the consol price. Combining these parameter restrictions with those following Equation (A15), we have the sufficient parameter conditions in Equation (6) to guarantee finite stock and consol prices.

D. Optimal portfolios

The more risk-averse agent's wealth W_t is the present value of his consumption stream:

$$W_t=E_t\left[\int_0^\infty e^{-\rho s}\left(\frac{C_{t+\tau}}{C_t}\right)^{-2\gamma}C_{t+\tau}\;d\tau\right].\tag{A23}$$

After substituting for C_t, this becomes

$$W_t=\frac{2}{b}\left(\sqrt{1+bX_t}-1\right)^{2\gamma}E_t\left[\int_0^\infty e^{-\rho\tau}\left(\sqrt{1+bX_te^u}-1\right)^{1-2\gamma}\;d\tau\right].\tag{A24}$$

This expression is similar to Equation (A8), and the closed-form solution in Equation (22) can be obtained by following the same steps used in deriving P_t above.

To solve for N_t, we note that the ratio of the diffusion coefficients for W_t and P_t is simply W_X/P_X. These derivatives are easily obtained from Equations (18) and (22) using the differentiation formula for the hypergeometric function, $F'(a,b,c'z)=(ab/c)F(a+1,b+1,c+1;z)$. The value of M_t follows from the identity $M_t = W_t - N_t P_t$.

E. Stock-price dynamics

To obtain stock-price dynamics, we note that $P_t = X_t Y_t$. Furthermore,

$$X_t = \frac{4}{b}\frac{s_t}{(1-s_t)^2}. \tag{A25}$$

Thus, P_t can be expressed exclusively as a function of s_t. A straightforward application of Itô's Lemma gives the expressions in Equations (29–30).

F. The determination of α

The initial wealth of the more risk-averse agent is $n P_0$. From Equation (A23),

$$n P_0 = W_0 = \frac{2}{b}\left(\sqrt{1+bX_0}-1\right)^{2\gamma} E_0\left[\int_0^\infty e^{-\rho t}\left(\sqrt{1+bX_t}-1\right)^{1-2\gamma} dt\right]. \tag{A26}$$

Substituting in Equation (A8) for P_0, we have

$$n\left[\int_0^\infty e^{-\rho t}\left(\sqrt{1+bX_t}-1\right)^{-2\gamma} X_t\, dt\right] = \frac{2}{b} E_0\left[\int_0^\infty e^{-\rho t}\left(\sqrt{1+bX_t}-1\right)^{1-2\gamma} dt\right]. \tag{A27}$$

Since the conditional expectations on the two sides of this equation depend only on X_0 and b, this equation determines b in terms of n and X_0. Since

$$b = 4\left(\frac{\alpha}{1-\alpha}\right)^{1/\gamma}, \tag{A28}$$

we obtain α.

G. The measure of trading activity $\sigma_{N,t}$

From the stock holding of the more risk-averse agent given in Equations (24–25) and the dynamics of s_t given in Equations (26–27), by Itô's Lemma we have $\sigma_{N,t} = N_s\, \sigma_{s,t}$.

H. The agents' utility functions

Focusing first on the utility function for the second agent, the utility function can be determined by substituting the equilibrium consumption for the second agent given in Equation (15) into Equation (4b):

$$\begin{aligned} V_{2,0}(X_0,n_2) &= E_0\left[\int_0^\infty \frac{e^{-\rho\tau}}{1-2\gamma}\left(\frac{2}{b}\right)^{1-2\gamma}\left(\sqrt{1+bX_\tau}+1\right)^{1-2\gamma} d\tau\right] \\ &= \frac{1}{1-2\gamma}\left(\frac{2}{b}\right)^{1-2\gamma} E_0\left[\int_0^\infty e^{-\rho\tau}(bX_0e^u)^{1-2\gamma}\left(\sqrt{1+bX_0e^u}+1\right)^{2\gamma-1} d\tau\right], \end{aligned} \tag{A29}$$

where u is the same normal random variable introduced after Equation (A8). Substituting the density function for u and then integrating with respect to τ gives expressions that are analogous to Equation (A8), which leads to

$$V_{2,0}(X_0,n_2)=\frac{1}{\psi(1-2\gamma)}(2X_0)^{1-2\gamma}(I_1+I_2), \tag{A30}$$

where

$$I_1=\int_0^\infty \exp\,[(1-2\gamma+\lambda)u]\left(\sqrt{1+bX_0e^u}+1\right)^{2\gamma-1}du, \tag{A31a}$$

$$I_2=\int_0^\infty \exp\,[(1-2\gamma+\theta)u]\left(\sqrt{1+bX_0e^u}+1\right)^{2\gamma-1}du. \tag{A31b}$$

The derivation now follows almost exactly as in Equations (A13–16), resulting in Equation (31b). The derivation of the utility function for the first agent is the same as for the second agent.

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