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MARKET STRUCTURE, SECURITY PRICES, AND INFORMATIONAL EFFICIENCY

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We consider an economy with an incomplete securities market and heterogeneously informed investors. Each investor trades in the market to hedge the risk to his endowment and to speculate on future security payoffs using his private information. We examine the efficiency of the securities market in allocating risk and transmitting information under different market structures, as defined by the set of securities traded in the market. We show that the introduction of derivative securities can decrease the market's efficiency in revealing information on security payoffs, and increase the equity premium and price volatility in the market.

Keywords: Securities Market, Market Structure, Security Prices, Communication Efficiency

1. INTRODUCTION

A securities market performs two important functions: allocating risk and communicating information among investors [see, e.g., Hayek (1945), Debreu (1959), and Arrow (1964)]. How efficiently the market performs these two functions crucially depends on the market structure, as defined by the set of securities traded in the market. Over time, the structure of the securities market changes as new securities are introduced. In the literature, the impact of these changes on the market's informational efficiency has been studied separately from the impact on its allocational efficiency. For example, in analyzing the informational role of derivative trading, the allocational trade in the market often is specified exogenously (as "noise") [see, e.g., Grossman (1977)]. As pointed out by Grossman (1995), the informational role and the allocational role of the securities market are fundamentally related. This paper focuses on the interaction between the allocational and

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informational roles of the securities market in analyzing the impact of changing market structure.

We consider an economy in which each investor is endowed with nontraded income and private information about security payoffs. The securities market consists of a set of primary securities, including a risk-free security (bond) and a risky security (stock), both with nonzero net supply, and possibly a futures-type derivative security on the stock with zero net supply. An investor trades in the market both to hedge the risk from his nontraded income and to speculate on future security payoffs using his private information. The equilibrium is solved under two different market structures, one with only the primary securities being traded, and the other with both the primary and the derivative securities. We examine how adding a derivative security to the market changes the trading and pricing of primary securities in equilibrium, and how it affects the allocational and informational efficiencies of the market.

When investors have symmetric information, the role of the securities market is primarily to allocate risks among investors. When the market is incomplete, investors are often unable to perfectly hedge their individual risks. Security prices depend on both the aggregate and the individual risks in the economy. Introducing derivative securities creates new hedging opportunities and increases allocational efficiency. As a result, it tends to decrease the equity premium and price volatility.

When investors have asymmetric information, in addition to allocating risk, the market also transmits information among investors through the security prices. Not only does introducing derivative securities change the allocational efficiency, but it also changes the informational efficiency of the market. On the one hand, the prices of new securities provide additional signals for investors to learn about other investors' private information, making the market informationally more efficient. On the other hand, the expanded trading opportunities increase the amount of allocational trade and, therefore, generate additional price movements in the existing securities, making the prices less informative about investors' private information on the asset payoffs. In some cases, the second effect dominates and opening derivative trading reduces the informational efficiency of the market. In contrast to the case of symmetric information, introducing derivative securities can increase the equity premium and price volatility under asymmetric information.

In the model, we analyze the market equilibrium under both incomplete-market structure and asymmetric information. Many authors have considered how market incompleteness affects investor behavior and market equilibrium.[1] In general, the individual optimization problem is difficult to solve in an incomplete market, and the results regarding optimal policies are limited. Analyzing the market equilibrium is more difficult and mostly carried out numerically.[2] The existence of asymmetric information makes the analysis even more formidable.[3] Our approach in this paper is to impose specific restrictions on individual preferences and shock distributions. We sacrifice generality for the benefit of being able to obtain closed-form solutions and analyze in more detail individual portfolio policies, equilibrium security prices, and allocational and informational efficiencies under different market structures. The intuition obtained from the model can be helpful in understanding more general models.

Studies on the informational role of derivative markets include the first formal discussion of Grossman (1977), using a single-period model, and the more recent work of Grossman (1988), Back (1993), and Brennan and Cao (1995), using multiperiod models. All of these papers use the noisy rational expectations framework, in which the allocational trade is introduced exogenously. We, however, use a fully rational expectations framework and explicitly model both the allocational and informational trade in the market. In particular, an investor's demand for derivative securities is derived endogenously from his optimal consumption and investment policies under the new market structure. Hence, we are able to analyze the allocational and informational efficiencies of the securities market, and the interaction between these two functions in a unified framework.

The paper proceeds as follows. Section 2 defines the model and Section 3 gives a general discussion of the equilibrium. Section 4 considers the special case in which the securities market is complete. The equilibrium under symmetric and asymmetric information is analyzed in Sections 5 and 6, respectively. Section 7 concludes. Proofs can be found in the Appendix.

2. THE MODEL

We consider an economy with a continuous time-horizon $[0, \infty)$ and a single good (which is also taken as the numeraire). Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, where Ω is the set of states of nature describing the exogenous environment of the economy, $\mathcal{F}$ is the σ algebra of distinguishable events, and P is the probability measure on $(\Omega, \mathcal{F})$. The uncertainty of the economy is generated by an n-dimensional standard Wiener process defined on $(\Omega, \mathcal{F}, P)$, denoted by w. The exogenous information flow is given by the augmented filtration $\{\mathcal{F}_t : t \in [0, \infty)\}$, which is a set of σ algebras of $\mathcal{F}$ generated by w.

The economy consists of two classes of investors, denoted by $i = 1$ or 2, with population weight ω and $1 - \omega$, respectively. Investors are identical within each class, but are different between the two classes with regard to their endowment and information. For convenience, we also refer to any class-i investor as investor i.[4] The economy is further defined as follows.

2.1. Market Structure

There is a competitive securities market with $m + 1$ traded securities, indexed by $k = 0, 1, \ldots, m$. The menu of traded securities includes:

(0) Security 0 is a risk-free security (bond) that has constant interest rate $r > 0$. The bond price is $B_t = e^{rt} B_0$.

(1) Security 1 is a risky security (stock) that pays a cumulative dividend D_t, with

$$D_t = \int_0^t (G_s\,ds + b_D\,dw_s), \tag{1a}$$

$$G_t = G_0 + \int_0^t e^{a_G(t-s)} b_G\,dw_s, \tag{1b}$$

where a_G is a negative constant and b_D, b_G are constant matrices of proper order. Thus, the dividend paid on the stock from t to $t + dt$ is $dD_t = G_t\,dt + b_D\,dw_t$, where G_t gives the expected dividend rate and $b_D\,dw_t$ the random shock. S_t denotes the stock price at t.

(2) Security 2 is a derivative security that pays a dividend at a rate equal to the current stock price. Thus, the dividend paid from t to $t + dt$ is $S_t\,dt$. This security is similar to a collar contract in the fixed-income market; hence it is called the collar from now on.[5] H_t denotes the collar price at t.

There also may be other securities with payoffs contingent on public information. Except securities 0 and 1, all securities are of zero net supply. The bond (security 0) has infinitely elastic supply at constant interest rate r. The stock (security 1) has total supply of one share per capita. We denote the m-dimensional vectors of cumulative cash flow and prices of all risky securities by $C_t = \text{stack}\{C_{1,t}, \ldots, C_{m,t}\}$ and $P_t = \text{stack}\{P_{1,t}, \ldots, P_{m,t}\}$, where $C_{k,t}$, $P_{k,t}$ are the cumulative cash flow and market price of security $k (k \geq 1)$, respectively.[6] Let

$$dQ_t = dC_t + dP_t - rP_t\,dt$$

be the vector of excess share returns of all risky securities. The first component, $dQ_t^S = dD_t + dS_t - rS_t\,dt$, gives the dollar return on one share of stock financed by borrowing at the risk-free rate. Similarly, the second component, $dQ_t^H = S_t\,dt + dH_t - rH_t\,dt$, gives the excess share return on the collar.

For any claim traded in the market, enforceability requires that its payoff be contingent only on public information. When all investors have perfect information about the underlying state of the economy, the payoff of a security can be made contingent on the realization of the state. In general, however, some investors do not observe the underlying state. Security payoffs then should be made contingent only on the information that is publicly available. Thus, in changing the market structure, we restrict our attention to the set of derivative securities whose payoffs depend only on the market prices of other traded securities. In particular, we consider the collar contract. The two market structures to be examined are

(a) Market structure I—only the stock and the bond are traded,
(b) Market structure II—the collar also is traded in addition to the stock and the bond.

2.2. Endowments

Each investor is initially endowed with one share of the stock and a flow of nontraded income. Investor i's cumulative nontraded income $N_{i,t}$ $(i = 1, 2)$ is given by

$$N_{i,t} = \int_0^t Y_{i,s} b_N \, dw_s, \qquad Y_{i,t} = \beta_{i,Y} Y_t + \beta_{i,Z} Z_t, \tag{2a}$$

$$Y_t = Y_0 + \int_0^t e^{a_Y(t-s)} b_Y \, dw_s, \tag{2b}$$

$$Z_t = Z_0 + \int_0^t e^{a_Z(t-s)} b_Z \, dw_s, \tag{2c}$$

where a_Y, a_Z are negative constants, and $\beta_{i,Y}$, $\beta_{i,Z}$, b_N, b_Y, b_Z are constant matrices of proper order. To fix ideas, we assume that $|a_Y| < |a_Z|$. Thus, Y_t and Z_t correspond, respectively, to the relatively more persistent and the more transitory components of investor 1's exposure to nontraded risk. Investor i's nontraded income from t to $t + dt$ is $dN_{i,t} = Y_{i,t} b_N \, dw_t$, where $b_N \, dw_t$ is the shock to the nontraded income process, and $Y_{i,t}$ determines investor i's exposure to this nontraded risk.[7] For simplicity, we have assumed zero drift for the nontraded-income process. Extending the current model to allow a drift term is possible.

2.3. Information Distribution

Both investors observe the public information, which includes the path of dividend payments and market prices of all traded securities $\{C_s, P_s : 0 \le s \le t\}$. The expected dividend rate G_t and individual investors' exposure to nontraded income $Y_{i,t}$ are private information. To simplify notation, define $X_t = \text{stack}\{G_t, Y_t, Z_t\}$, which fully determines the distribution of future stock payoffs and aggregate nontraded income. We assume that investor 1 observes the realization of X_t. Investor 2, on the other hand, only observes a set of signals about X_t. The signal process U_t (which can be multidimensional) is given by

$$U_t = \int_0^t (a_U X_s \, ds + b_U \, dw_s), \tag{3}$$

where a_U and b_U are constant matrices of proper order.

Let $\mathcal{F}_{i,t}$, $\mathcal{F}_t^{\{P,C\}}$ denote the filtrations generated by the information set of investor i and by the path of prices and dividends at $t \in [0, \infty)$, respectively. Then, $\mathcal{F}_{1,t} = \mathcal{F}_0 \otimes \mathcal{F}^{\{P,C,N_1,X\}}$ and $\mathcal{F}_{2,t} = \mathcal{F}_0 \otimes \mathcal{F}^{\{P,C,N_2,U\}}$, where $\mathcal{F}_0$ is the investors' prior information on $X_0 = \text{stack}\{G_0, Y_0, Z_0\}$. Furthermore, $\mathcal{F}_t^{\{P,C\}} \subseteq \mathcal{F}_{2,t} \subseteq \mathcal{F}_{1,t} = \mathcal{F}_t$. The information of investor 1 (weakly) dominates that of investor 2. When the dominance is strict, we call investor 1 the informed and investor 2 the uninformed. If $X_0 \le \mathcal{F}_0$, a_U is full ranked and $b_U = 0$, $U_{[0,t]}$ fully reveals X_t and investor 2 becomes fully informed as well. This gives one example of symmetric information where $\mathcal{F}_{1,t} = \mathcal{F}_{2,t} = \mathcal{F}_t$.

2.4. Policies and Preferences

For investor i $(i = 1, 2)$, let $\{c_{i,t}, \theta_{i,t} : t \in [0, \infty)\}$ be his consumption and trading policies, where $c_{i,t}\, dt$ is his consumption from t to $t + dt$ and $\theta_{i,t}$ is the m-dimensional vector of his shareholdings in all risky securities at t. His policies are adapted to $\mathcal{F}_{i,t}$. Consumption policies are restricted to integrable processes, and trading policies are restricted to predictable, square-integrable processes with respect to the gain processes of the traded securities [see, e.g., Harrison and Pliska (1981) for a discussion on the requirement of square integrability].[8]

We assume that investors maximize expected utilities of the following form

$$E\left[-\int_t^{\infty} e^{-\rho(s-t)-\gamma c_{i,s}}\, ds \,\Big|\, \mathcal{F}_{i,t}\right], \qquad i = 1, 2, \tag{4}$$

where ρ and γ (both positive) are the time discount coefficient and the absolute risk-aversion coefficient, respectively. This particular form of the utility function helps to solve the equilibrium in closed form.

2.5. Equilibrium Notion

Prices of risky securities are determined by the equilibrium of the economy. The notion of equilibrium is the standard one of rational expectations [see, e.g., Radner (1972)]. It is defined as a price process $\{P_t\}$ under which each investor adopts feasible consumption and trading policies that maximize his expected utility

$$\begin{aligned} J_{i,t} &= \sup_{c_i, \theta_i} E\left[-\int_t^{\infty} e^{-\rho(s-t)-\gamma c_{i,s}}\, ds \,\Big|\, \mathcal{F}_{i,t}\right] \\ \text{s.t.} \quad & dW_{i,t} = (rW_{i,t} - c_{i,t})\, dt + \theta'_{i,t}\, dQ_t + dN_{i,t}, \end{aligned} \tag{5}$$

where $i = 1, 2$, and the market clears

$$\omega\theta_{1,t} + (1-\omega)\theta_{2,t} = 1_{11}^{(m,1)}. \tag{6}$$

Here, $1_{lm}^{(p,q)}$ denotes an *index matrix* of order $(p \times q)$ with its (l, m)th element being 1 and all other elements being 0. The transversality condition of the Merton (1971, 1989) type is imposed on each investor's optimization problem: $\lim_{s\to\infty} E[J_{i,s} \,|\, \mathcal{F}_{i,t}] = 0$.[9] We only consider the stationary equilibrium of the economy.

2.6. Further Simplifying Assumptions

To be more specific, we assume that the n-dimensional Wiener process w has the following decomposition:

$$w_t = \text{stack}\{w_{D,t}, w_{G,t}, w_{Y,t}, w_{Z,t}, w_{N,t}, w_{U,t}\},$$

where all components are standard Wiener processes (need not be one-dimensional) and mutually independent. Furthermore,

$$b_D = \sigma_D(\iota, 0, 0, 0, 0, 0), \qquad b_G = \sigma_G(0, \iota, 0, 0, 0, 0),$$
$$b_N = \sigma_N\left(\kappa_{DN}\iota, 0, 0, 0, \sqrt{1-\kappa_{DN}^2\iota}, 0\right), \qquad b_Y = \sigma_Y(0, 0, \iota, 0, 0, 0),$$
$$b_Z = \sigma_Y(0, 0, 0, \iota, 0, 0), \qquad b_U = \sigma_U(0, 0, 0, 0, 0, \iota),$$

where ι denotes identity matrices of proper order. The above specification about the underlying shocks to the economy has simple interpretations. For example, $w_{D,t}$ and $w_{G,t}$ characterize the shocks to stock dividends, $w_{Y,t}$ and $w_{Z,t}$ characterize the shocks to individual investor's exposure to the nontraded risk. The above assumptions about the b's impose specific structure on the correlation among the state variables. In particular, the stock dividends are correlated with the nontraded income when $\kappa_{DN} \neq 0$. To fix ideas, we maintain the assumption that $\kappa_{DN} > 0$ throughout this paper. The specific correlation structure assumed here simplifies our analysis without great loss of generality on the points we want to make.

To guarantee the existence of an equilibrium, we impose the following parameter restrictions:

$$\sigma_N\sigma_Y < \frac{r-2a_Y}{2\sqrt{2}r\gamma}, \qquad \sigma_N\sigma_Z < \frac{r-2a_Z}{2\sqrt{2}r\gamma}. \tag{7}$$

Equation (7) requires that the variability in the nontraded income is not too large.

The economy as defined above exhibits the following features. First, the securities market is, in general, incomplete [see, e.g., Harrison and Kreps (1979) for a formal definition of market completeness]. Second, the existence of nontraded income and its correlation with returns on traded securities generate allocational trade in the market. Third, the existence of private information on future security payoffs gives rise to the informational trade between the two classes of investors. In particular, class-1 investors speculate in the market on the basis of their private information and expect to earn excess returns.

For future convenience, we introduce some notation. For any state variable that investor 2 does not directly observe, $\hat{\cdot} = E[\cdot \mid \mathcal{F}_{2,t}]$ denotes his conditional expectation. In particular, $\hat{X}_t = E[X_t \mid \mathcal{F}_{2,t}]$ denotes investor 2's conditional expectation of X_t and $o^{(2)} = E[(\hat{X}_t - X_t)^2 \mid \mathcal{F}_{2,t}]$ is the conditional variance.

Let $\beta_i = \text{stack}\{0, \beta_{i,Y}, \beta_{i,Z}\}$, $i = 1, 2$. (Here, we use 0 to denote matrices of zeros without specifying their order, which can be inferred from the context.) Investor i's nontraded income then can be expressed as

$$N_{i,t} = \int_0^t \beta_i' X_s b_N \, dw_s, \tag{8}$$

where $i = 1, 2$.

For any two random variables e_p and e_q, where $de_k = a_k\, dt + b_k\, dw_t$, $k = p, q$, let $\sigma_{kl} = b_k b_l'$ denote the instantaneous cross-variation between e_k and e_l, $\sigma_k^2 =$

σ_{kk}, $\kappa_{kl} = \sigma_{kk}^{-1/2}\sigma_{kl}\sigma_{ll}^{-1/2}$ be the instantaneous cross correlation (assuming that σ_{kk} is positive definite) [see, e.g., Karatzas and Shreve (1988) for a discussion on cross-variation processes].

3. GENERAL DISCUSSION ON EQUILIBRIUM

We now provide a general discussion on the equilibrium of the economy as defined in Section 2. As mentioned earlier, $X_t = \text{stack}\{G_t, Y_t, Z_t\}$ fully determines the distributions of future stock payoffs and nontraded income. Not directly observing X_t, investor 2 relies on his expectation $\hat{X}_t$ (and possibly other moments) in forming his trading policy. Consequently, the equilibrium of the economy depends not only on the true value of X_t, but also on investor 2's conditional expectation $\hat{X}_t$. Define $\Delta_t = \hat{X}_t - X_t$ to be the estimation error of investor 2. Let $X_{1,t} = \text{stack}\{1, X_t, \Delta_t\}$ and $X_{2,t} = \text{stack}\{1, \hat{X}_t\}$.

We restrict our attention to the linear, stationary equilibria of the economy in which security prices are linear, time-independent functions of X_t and $\hat{X}_t$ only. In particular, we can express the prices and cumulative payoffs of the traded securities as follows:

$$P_t = \lambda^P X_{1,t}, \tag{9a}$$

$$C_t = \int_0^t \left(\lambda^C X_{1,s}\, ds + b_C\, dw_s\right), \tag{9b}$$

where $\lambda^P = (\lambda_0^P, \lambda_X^P, \lambda_\Delta^P)$ and $\lambda^C = (\lambda_0^C, \lambda_X^C, \lambda_\Delta^C)$. For investor 2, observing $P_{[0,t]}$ is equivalent to observing $(\lambda_X^P - \lambda_\Delta^P)X_{[0,t]}$. Thus, $\mathcal{F}_{2,t} = \mathcal{F}_t^{\{\Phi\}}$, where $\Phi_t = \text{stack}\{C_t,$ $(\lambda_X^P - \lambda_\Delta^P)X_t, Y_{2,t}, U_t\}$. We now can compute $\hat{X}_t$, given $\mathcal{F}_{2,t}$. First,

$$dX_t = a_X X_t\, dt + b_X\, dw_t,$$

where $a_X = \text{diag}\{a_G, a_Y, a_Z\}$ and $b_X = \text{stack}\{b_G, b_Y, b_Z\}$. Next,

$$d\Phi_t = \left(a_0^\Phi + a_{\hat{X}}^\Phi \hat{X}_t + a_X^\Phi X_t\right) dt + b_\Phi\, dw_t,$$

where $a_0^\Phi = \text{stack}\{\lambda_0^C, 0, 0, 0\}$, $a_{\hat{X}}^\Phi = \text{stack}\{\lambda_\Delta^C, 0, 0, 0\}$, $a_X^\Phi = \text{stack}\{(\lambda_X^C - \lambda_\Delta^C),$ $(\lambda_X^P - \lambda_\Delta^P)a_X, \beta_2' a_X, a_U\}$, and $b_\Phi = \text{stack}\{b_C, (\lambda_X^P - \lambda_\Delta^P)b_X, \beta_2' b_X, b_U\}$. We then have the following result [see, e.g., Lipster and Shriyayev (1977)]:

LEMMA 1. *Given the security prices and payoffs* (9), *investor* 2*'s expectation of* X_t *in a stationary state is given by*

$$d\hat{X}_t = a_X \hat{X}_t\, dt + k(d\Phi_t - E[d\Phi_t \mid \mathcal{F}_{2,t}]), \tag{10a}$$

$$0 = \left[a_X o^{(2)} + o^{(2)} a_X'\right] + b_X b_X' - k\,(b_\Phi b_\Phi')\,k', \tag{10b}$$

where $k = (o^{(2)} a_X^{\Phi\prime} + b_X b_\Phi')(b_\Phi b_\Phi')^{-1}$.

Furthermore,

$$dX_{i,t} = a_{i,X} X_{i,t}\, dt + b_{i,X}\, dw_{i,t}, \qquad i = 1, 2, \tag{11}$$

where $a_{1,X} = \operatorname{diag}\{0, a_X, a_X - ka_X^{\Phi}\}$, $b_{1,X} = \operatorname{stack}\{0, b_X, kb_\Phi - b_X\}$, $a_{2,X} = \operatorname{diag}\{0, a_X\}$, $b_{2,X} = \operatorname{stack}\{0, kb_\Phi\}$, $dw_{1,t} = dw_t$ and $dw_{2,t} = b'_\Phi(b_\Phi b'_\Phi)^{-1}(d\Phi_t - E[d\Phi_t \mid \mathcal{F}_{2,t}])$.

From (9) and (10), the excess share returns on the risky securities can be expressed as

$$dQ_t = dC_t + dP_t - rP_t\, dt = a_{i,Q} X_{1,t}\, dt + b_{i,Q}\, dw_{i,t}, \qquad i = 1, 2, \tag{12}$$

where $a_{1,Q} = \lambda^C + \lambda^P(a_{1,X} - r\iota)$, $b_{1,Q} = b_Q = b_C + \lambda^P b_{1,X}$, $a_{2,Q} = \tilde{\lambda}^C + \tilde{\lambda}^P(a_{2,x} - r\iota)$, and $b_{2,Q} = b_C + \tilde{\lambda}^P b_{2,X}$ with $\tilde{\lambda}^C = (\lambda_0^C, \lambda_X^C)$ and $\tilde{\lambda}^P = (\lambda_0^P, \lambda_X^P)$. Because $X_{i,t}$ follows a Gaussian Markov process under $\mathcal{F}_{i,t}$, it fully characterizes investor i's current and future investment opportunities and endowments.

Given (9), (10), and (12), we can solve for individual consumption and trading policies. The results are given in the following lemma.

LEMMA 2. *Given* (9), (10), *and* (12), *investor i's optimal policies and value function are*

$$\theta_{i,t} = h_i X_{i,t}, \qquad c_{i,t} = rW_{i,t} - \frac{1}{2\gamma} X'_{i,t} v_i X_{i,t} - \frac{1}{\gamma}\ln r, \tag{13a}$$

$$J_{i,t} = -e^{-\rho t - r\gamma W_{i,t} + \frac{1}{2} X'_{i,t} v_i X_{i,t}}, \tag{13b}$$

where $i = 1, 2$ and

$$h_i = (r\gamma b_Q b'_Q)^{-1}\,(a_{i,Q} + b_Q b'_{i,X} v_i - r\gamma b_Q b'_N \beta'_i) \tag{14}$$

if v_i solves

$$(r\gamma)^2 h'_i\,(b_Q b'_Q)\, h_i - (r\gamma\beta_i b_N + v_i b_{i,X})(r\gamma\beta_i b_N + v_i b_{i,X})'$$
$$+ r v_i - \left(a'_{i,X} v_i + v_i a_{i,X}\right) - \overline{v}_i 1_{11}^{(d_i,d_i)} = 0. \tag{15}$$

Here, $\overline{v}_i = 2(r - \rho - r\ln r) + \operatorname{tr}(b'_{i,X} v_i b_{i,X})$, $d_1 = 7$, *and* $d_2 = 4$.

Given investor i's trading policy in (13), the market clearing condition in (6) can be written as

$$\omega h_1 + (1 - \omega) h_2 \tau = 1_{11}^{(m,1)}, \tag{16}$$

where $\tau = (\iota, \operatorname{stack}\{0, \iota\})$.

The price function in (9), the solution to investor 2's expectations in Lemma 1, the solution to both investors' optimal policies in Lemma 2, and the market-clearing condition (16) fully characterize a linear, stationary equilibrium of the economy if it exists. The following theorem states the conditions under which the equilibrium exists.

THEOREM 1. *For the economy defined in Section* 2, *there exists* (*generically*) *a linear stationary equilibrium under both market structure I and II for* ω *close to* 1. *In the equilibrium, security prices and payoffs have the form of* (9); *the class*-2 *investors' expectations satisfy* (10); *investors' optimal policies are given in* (13); *and the coefficients* λ, o, h, v_i, $i = 1, 2$, *solve the system* (10b), (14), (15), *and* (16).

Several comments on the existence result follow. First, the condition that ω is close to one is needed for technical reasons in proving the existence of an equilibrium. Our proof is based on a continuity argument. When $\omega = 1$, investors are identical and all-informed, the market is effectively complete [see, e.g., Lucas (1978)]. A unique, linear, stationary equilibrium exists. By showing that the system to be solved for an equilibrium is nondegenerate at $\omega = 1$, we can prove that its solution also exists for ω close to one. The proof itself, however, does not say how close to one ω needs to be. Second, the existence is only in the generic sense. This means that an equilibrium exists for all parameter values in the parameter space, *except* possibly a measure zero set. Because of the large number of parameters and the particular approach used in the proof, we are unable to establish if this set is actually empty, which would give us absolute existence. Third, except at $\omega = 1$, we have little knowledge concerning the uniqueness of the solution.

The actual solution to the equilibrium is obtained by numerically solving the system (10b), (15), and (16). Recognizing the nature of our existence result, we always start from the point $\omega = 1$ in our numerical algorithm, and decrease it gradually to reach desired values of $\omega \in [0, 1]$. This helps us to find a solution and stay on the same solution branch if multiple solutions exist. We have explored extensively in the parameter space following the above approach, and finding a numerical solution was quite easy.

Most of our analysis is based on numerical illustrations. Given the large number of parameters in the model, only the results for a small range of parameter values are presented for brevity. The parameter values are chosen to be compatible with Campbell and Kyle's (1993) estimated price process, which has a linear form similar to ours. The remaining degrees of freedom are used to fix a particular set of parameter values that generate simultaneously all results in this paper. As a cost, some of the effects may seem small for this set of parameter values even though they can be larger for other parameter values. When a particular result under consideration changes qualitatively with certain parameters, we try to show the changes by varying the relevant parameters in the numerical illustrations or to discuss them verbally. In particular, we focus on two parameters: σ_Y and ω, where σ_Y is the instantaneous variability in investor 1's exposure to nontraded income and ω is the population weight of investor 1. These two parameters capture the heterogeneity between the two investors and its relative importance to the equilibrium. Obviously, our exploration of the parameter space and the results presented in the paper are by no means exhaustive.

In our analysis, we make an additional assumption about the distribution of nontraded income among investors. It is clear that the nature of the equilibrium depends on the distribution of risky endowments among investors. For some distributions of the nontraded income, the equilibrium becomes fully revealing under market structure II. In this case, the introduction of derivative trading clearly improves the informational efficiency of the market. In the remainder of the paper, however, we focus on those distributions of the nontraded income under which the equilibrium is nonfully revealing under both market structures I and II. In this case, the impact of derivative trading on the informational efficiency of the market is less obvious. In particular, we set $\beta_1 = \text{stack}\{0, 1, 1\}$ and $\beta_2 = \text{stack}\{0, 0, 0\}$ in (8). In other words, only investor 1 is endowed with nontraded income.

4. CASE OF COMPLETE MARKET

We first consider the case in which the market is effectively complete. In particular, we consider the case when $\omega = 1$ and the economy is populated only by class-1 investors. This case provides some basic understanding about the model, which is useful in analyzing more general cases.

When the securities market is complete (or effectively complete), the equilibrium allocation is Pareto optimal and does not depend on the actual market structure as long as it satisfies the spanning property [see, e.g., Duffie and Huang (1985)]. Solving individual optimization problem and market clearing condition, we have the following theorem.

THEOREM 2. *When $\omega = 1$, the economy has a unique linear, stationary equilibrium of the form in Theorem 1. In particular, the stock price is*

$$S_t = \lambda^S X_t = \frac{1}{r - a_G} G_t + \lambda_0^S + \lambda^S Y_t + \lambda^S Z_t, \tag{17}$$

the investors' value function is $J_t = -e^{-\rho t - r\gamma W_t + (1/2)\tilde{X}_t' v \tilde{X}_t}$, and their optimal consumption policy is $c_t = rW_t - (1/2\gamma)\tilde{X}_t' v \tilde{X}_t - (1/\gamma)\ln r$, where $\tilde{\lambda}^S$ and v are constant matrices of proper order given in Appendix A.3. For any derivative security ($k > 1$) with cumulative payoff $C_t = \int_0^t (f(X_s, s)\,ds + b_C\,dw_s)$, if its price is twice differentiable with respect to X_t and once differentiable with respect to t, then $P(X_t, t)$ satisfies the following equation:

$$\begin{aligned} rP &= \partial_t P + f + X' a_X' \partial_X P + \tfrac{1}{2}\text{tr}\left(\sigma_{XX}\partial_X^2 P\right) \\ &\quad - [r\gamma\lambda^S b_X + (r\gamma\beta_1 b_N - v b_X) X_t]\,(b_X' \partial_X P + b_C'), \end{aligned} \tag{18}$$

where $\partial_X P$ denotes the vector of first-order derivatives of P with respect to elements of X, $\partial_X^2 P$ the matrix of second-order derivatives, and $\partial_t P$ its derivative with respect to t.[10]

To better understand the nature of the equilibrium, we consider a special case when $Z_t = 0\,\forall\, t$, and Y_t fully characterizes the exposure to nontraded risk. In this case, we have $v = \mathrm{diag}\{v_{00}, v_{YY}, 0\}$, where

$$v_{YY} = \left(2\sigma_Y^2\right)^{-1}\left[(r - 2a_Y) - \sqrt{(r - 2a_Y)^2 - 4(r\gamma)^2\sigma_N^2\sigma_Y^2}\right] > 0,$$
$$v_{00} = (1/r)\sigma_Y^2 v_{YY} + r\gamma\lambda_0^S + (2/r)(r - \rho - r\ln r),$$

and

$$\lambda_Y^S = -r\gamma\sigma_{DN}\big/\left(r - a_Y - \sigma_Y^2 v_{YY}\right),$$
$$\lambda_0^S = -\gamma\left[\frac{1}{(r - a_G)^2}\sigma_G^2 + \sigma_D^2 + \left(\lambda_Y^S\sigma_Y\right)^2\right].$$

Furthermore, v_{YY}, $|\lambda_0^S|$, and $|\lambda_Y^S|$ increase with σ_Y. It follows that an investor's optimal consumption c_t decreases with his exposure to nontraded risk Y_t, reflecting the investor's precautionary saving. When the investor faces higher risk in his future nontraded income, his marginal utility for future consumption increases. Under constant interest rate, he decreases current consumption to save more for future consumption.

The stock price is a simple linear function of the underlying state variables, $S_t = [1/(r - a_G)]G_t + \lambda_0^S + \lambda_Y^S Y_t$, where $[1/(r - a_G)]G_t$ gives the expected value of the stock's future cash flow discounted at the risk-free rate, and $\lambda_0^S + \lambda_Y^S Y_t$ is the risk premium on the stock. The constant component of the risk premium λ_0^S is proportional to the investors' risk aversion and the instantaneous variance of the stock price. This is because an investor's consumption covaries linearly with the stock price (because his wealth does). The covariance between consumption changes and stock returns depends linearly on the variance of stock returns. So does the risk premium.

The time-varying component of the risk premium is linear in Y_t with proportionality coefficient λ_Y^S. It is easy to show that $r - a_Y - \sigma_Y^2 v_{YY} > 0$. Hence, λ_Y^S has the opposite sign of σ_{DN}. Note that σ_{DN} being positive implies positive correlation between shocks to an investor's nontraded income and shocks to the stock payoff. When $Y_t > 0$, the investor has a positive exposure to the nontraded risk. Investing in the stock then becomes less desirable given a positive σ_{DN}. In equilibrium, the stock price has to decrease with Y_t. Thus, λ_Y^S is negative. Furthermore, as the volatility σ_Y of the aggregate exposure to nontraded risk increases, an investor's expected utility becomes more sensitive to the changes in his exposure to nontraded risk (i.e., v_{YY} increases). The stock price becomes more sensitive to Y_t and $|\lambda_Y^S|$ increases.

The price of a derivative security must satisfy the pricing equation (18) with appropriate boundary conditions. As an example, we solve for the collar price. It pays a dividend at a rate equal to current stock price. H_t should be a function of X_t only, independent of the calendar time t, i.e., $H_t = H(X_t)$. Given that X_t follows a Gaussian Markov process, it can be shown that $H(\cdot)$ is linear. Thus,

$$H_t = H(X_t) = \lambda^H X_t,$$

where $\lambda^H = (\lambda_0^H, \lambda_G^H, \lambda_Y^H, \lambda_Z^H)$ is a constant matrix. Substituting this into equation (18), we obtain

$$\lambda^H = \lambda^S \left[r\iota - a_X + (r\gamma)\sigma_{XX}\lambda^{S'} 1_{11}^{(1,4)} + (r\gamma\sigma_{XN} - \sigma_{XY}v_{YY})\, 1_{13}^{(1,4)} \right]^{-1},$$

which fully specifies the equilibrium collar price. (The matrix in the square bracket is full ranked.)

5. CASE OF SYMMETRIC INFORMATION

We now consider the case when $\mathcal{F}_{1,t} = \mathcal{F}_{2,t} = \mathcal{F}_t$ and all investors are fully informed about the underlying state of the economy. This is a special case of the general model when investor 2's private signal U_t is fully informative of the unobserved state variables (e.g., when a_U is full ranked and $b_U = 0$). This case allows us to focus on the change in allocational trade among investors and its impact on the equilibrium risk allocation and security prices when the market structure changes. The result in this section serves as a benchmark when we introduce informational trade in Section 6. Note that under both market structures I and II, the market is incomplete.

5.1. Equilibrium Under Market Structure I

Under market structure I, only the stock and the bond are traded. Let $\tilde{X}_t = \text{stack}\{1, Y_t, Z_t\}$, $a_{\tilde{X}} = \text{diag}\{0, a_Y, a_Z\}$ and $b_{\tilde{X}} = \text{stack}\{0, b_Y, b_Z\}$. Then, $d\tilde{X}_t = a_{\tilde{X}}\tilde{X}_t\, dt + b_{\tilde{X}}\, dw_t$. Applying Theorem 1 to this case yields the following corollary:

COROLLARY 1. *Under market structure I, when $\mathcal{F}_{i,t} = \mathcal{F}_t\ \forall\, t$, $i = 1, 2$, $\beta_1 = (0, 1, 1)'$, $\beta_2 = (0, 0, 0)'$, and ω is close to one, the economy has a linear, stationary equilibrium in which the stock price and investors' policies are*

$$S_t = \frac{1}{(r - a_G)} G_t + \tilde{\lambda}^S \tilde{X}_t,$$

$$\theta_{i,t}^S = h_i \tilde{X}_t, \qquad c_{i,t} = rW_{i,t} - (1/2\gamma)\tilde{X}_t' v_i \tilde{X}_t - (1/\gamma)\ln r,$$

where $i = 1, 2$, $\tilde{\lambda}^S$, h_i, v_i are constant matrices determined by equations (14)–(16) *with $a_Q = \tilde{\lambda}^S(a_{\tilde{X}} - r\iota)$ and $b_Q = \tilde{\lambda}^s b_{\tilde{X}} + [1/(r - a_G)]b_G + b_D$.*

We first examine the equilibrium trading strategy for investor 1. From market clearing, we can easily infer the strategy for investor 2. From Corollary 1, investor 1's stockholding is a linear function of Y_t and Z_t, that is, $\theta_{1,t}^S = h_{1,0}^S + h_{1,Y}^S Y_t + h_{1,Z}^S Z_t$. First note that it does not depend on G_t, the expected future stock payoff. Under symmetric information, any information on G_t is fully reflected in the current stock price. Because investors have constant absolute risk aversion and their demand for risky securities is independent of wealth, they have no incentive to trade as G_t changes. The stockholding, however, does depend on Y_t and Z_t,

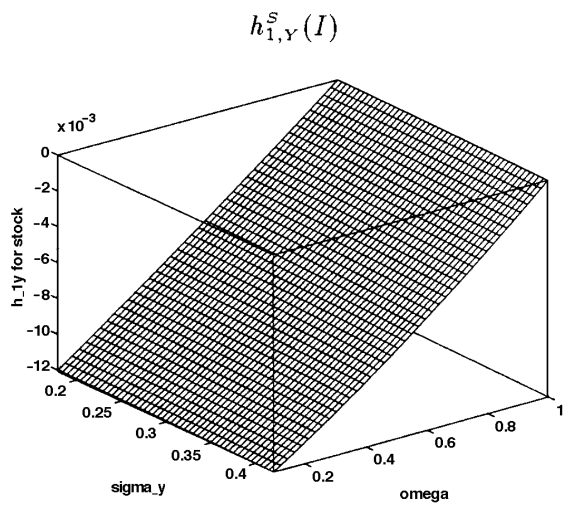


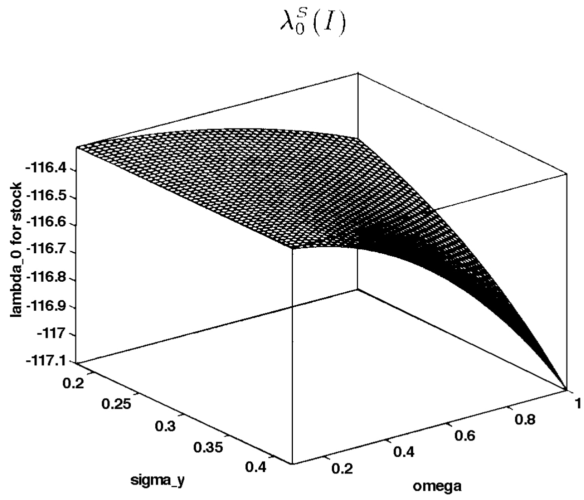
FIGURE 1. Hedging intensity for investor 1 under symmetric information and market structure I [parameters have the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.06$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.6$, $\sigma_Z = 0.5$, $\sigma_N = 0.22$, $\kappa_{DN} = 0.5$, $\beta_1 = (0, 1, 1)'$, and $\beta_2 = (0, 0, 0)'$].

because investor 1 uses the stock to hedge his nontraded risk. Given the linear form of investor 1's stockholding, it is sufficient to look at the coefficients $h^S_{1,Y}$ and $h^S_{1,Z}$, which characterize the intensity of his hedging trade in response to changes in his exposure to nontraded risk.

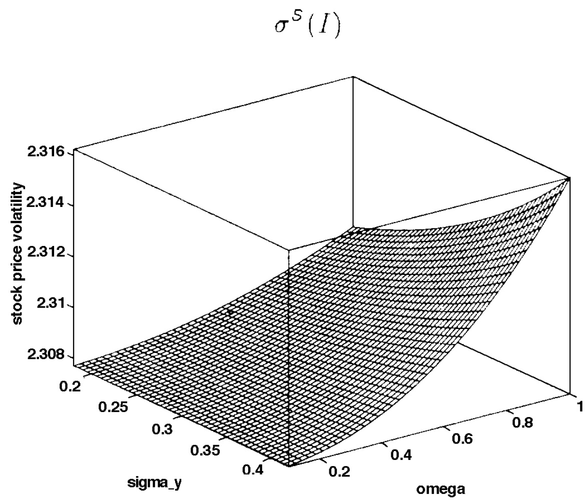
Figure 1 plots $h^S_{1,Y}$ for different values of σ_Y and ω. Because Y_t and Z_t play similar roles in the model, we focus only on Y_t from now on. Consider the situation when $Y_t > 0$ and investor 1 has a positive exposure to the nontraded risk. Because the stock dividends are positively correlated with the nontraded income ($\sigma_{DN} > 0$), investor 1 reduces his stockholding to hedge his nontraded risk. By doing so, he reduces the overall variability of his wealth. For example, when the nontraded income is low, it is more likely that the dividend on the stock is also low. The hedging position in the stock (by selling the stock) then yields a high payoff, which compensates for the low level of nontraded income. This implies that $h^S_{1,Y}$ must be negative as Figure 1 confirms. Furthermore, as σ_Y increases, investor 1's marginal utility becomes more sensitive to changes in his exposure to nontraded risk. Given a level of his exposure (i.e., a value of Y_t), he tends to hedge more aggressively using the stock. Thus, $|h^S_{1,Y}|$ increases with σ_Y.

When $\omega = 1$, the economy is populated only by class-1 investors. Nobody takes the opposite side for his hedging trade. Hence, $h^S_{1,Y}$ approaches zero. As ω decreases, more class-2 investors are present to make the market, which allows class-1 investors to hedge more aggressively, and $|h^S_{1,Y}|$ increases.

We now consider stock risk premium and price volatility. Figure 2 plots λ^S_0 and σ^S for different values of σ_Y and ω. As σ_Y increases, investor 1 trades more aggressively in the stock to hedge his nontraded risk. Consequently, the stock price



A)



B)

FIGURE 2. Stock risk premium and price volatility under symmetric information and market structure I [other parameters have the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.05$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.2$, $\sigma_Z = 0.4$, $\sigma_N = 0.22$, $\kappa_{DN} = 0.6$, $\beta_1 = (0, 1, 1)'$, and $\beta_2 = (0, 0, 0)'$].

becomes more sensitive to Y_t and the absolute value of λ_Y^S increases. Given that stock price is linear in $\lambda_Y^S Y_t$, as σ_Y increases, the price becomes more volatile due both to its increased sensitivity to Y_t and to the increased volatility of Y_t. As the stock price becomes more volatile, investors require a higher risk premium, leading to a higher value of $|\lambda_0^S|$.

The dependence of λ_0^S and σ^S on ω is also quite intuitive. For larger values of ω, the population of investor 1 is larger and so is the aggregate exposure to nontraded risk (for given values of Y_t and Z_t). The stock price becomes more sensitive to changes in Y_t and Z_t. Thus, the stock demands a higher premium and exhibits larger price volatility.

5.2. Equilibrium Under Market Structure II

In addition to the stock and the bond, a derivative security (collar) also is traded under market structure II. Investors can achieve a larger set of possible payoffs. In particular, they can construct trading strategies to better hedge their nontraded risk. The new trading opportunities created by the introduction of the collar contract certainly affect the stock price and the equilibrium allocation. From Theorem 1, we have

COROLLARY 2. *Under market structure II, when $\mathcal{F}_{i,t} = \mathcal{F}_t \,\forall\, t$ and ω is close to* 1, *the economy has a linear*, *stationary equilibrium in which the security prices and investors' consumption and security holdings are*

$$S_t = \frac{1}{(r - a_G)} G_t + \tilde{\lambda}^S \tilde{X}_t, \qquad H_t = \frac{1}{(r - a_G)^2} G_t + \tilde{\lambda}^H \tilde{X}_t,$$

$$\begin{pmatrix} \theta_{i,t}^S \\ \theta_{i,t}^H \end{pmatrix} = h_i \tilde{X}_t, \qquad c_{i,t} = r W_{i,t} - \frac{1}{2\gamma} \tilde{X}_t' v_i \tilde{X}_t - \frac{1}{\gamma} \ln r,$$

where $i = 1, 2$, and $\tilde{\lambda}^S$, $\tilde{\lambda}^H$, h_i, and v_i are determined by equations (14)–(16) *with*

$$a_Q = \tilde{\lambda}(a_{\tilde{X}} - r\iota) + \text{stack}\{0, \tilde{\lambda}^S\}$$

and

$$b_Q = \text{stack}\{\tilde{\lambda}^S, \tilde{\lambda}^H\} b_{\tilde{X}} + \text{stack}\left\{ \frac{1}{r - a_G}, \frac{1}{(r - a_G)^2} \right\} b_G + \text{stack}\{b_D, 0\}.$$

Figure 3 plots $h_{1,Y}^S$ and $h_{1,Y}^H$, the trading intensities in both markets for investor 1. To understand investor 1's trading behavior, we still consider the situation when $Y_t > 0$, that is, the investor has a positive exposure to the nontraded risk. Given the positive correlation between stock dividends and his nontraded income, investor 1 wants to reduce his stockholding to hedge his nontraded risk. However, returns on the stock depend not only on the realization of current dividends, but also on changes in its price. Hence, the stock does not provide a perfect hedge. By going short in the stock, investor 1 exposes himself to the risk of future price changes. This risk, which is unrelated to the risk to be hedged, is called the basis risk of the hedging instrument. The existence of basis risk makes the stock a less attractive hedging vehicle and limits investor 1's hedging trade.

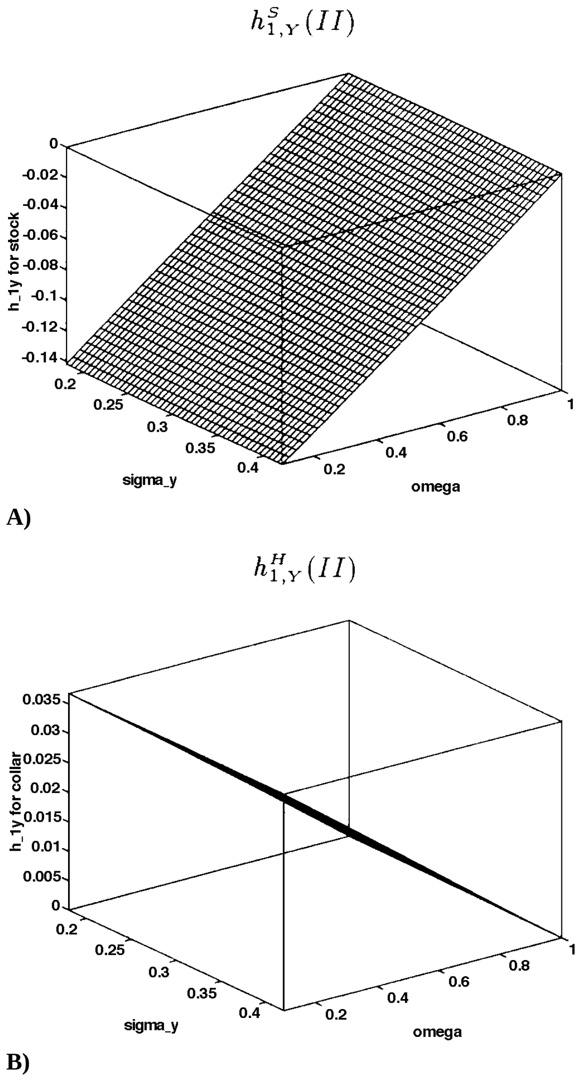


Figure 3. Hedging intensity for investor 1 under symmetric information and market structure II [parameters have the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.06$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.6$, $\sigma_Z = 0.5$, $\sigma_N = 0.22$, $\kappa_{DN} = 0.5$, $\beta_1 = (0, 1, 1)'$, and $\beta_2 = (0, 0, 0)'$].

With the introduction of the collar, whose price is positively correlated to the stock price, investor 1 can use the collar to offset the basis risk in his hedging position. In the above example, when $Y_t > 0$, investor 1 can take a long position in the collar to hedge the basis risk in his short position in the stock. Indeed, Figure 3 shows that $h^H_{1,Y}$ has the opposite sign of $h^S_{1,Y}$. By combining the stock and the collar, investor 1 is now able to establish a hedging position (against his nontraded risk) with much less basis risk. (The remaining basis risk is due to the imperfect correlation between the stock dividend and the nontraded income, and between the stock price and the collar price.) The reduction in basis risk increases investor 1's hedging intensity. Comparing Figures 1 and 3A shows that the absolute value of $h^S_{1,Y}$ is much larger under market structure II than under market structure I.

Figure 4 shows how the stock risk premium and price volatility change with σ_Y and ω. Qualitatively, their behavior is similar to that under market structure I. For brevity, we have omitted the premium and price volatility of the collar, which varies with σ_Y and ω in a fashion similar to that of the stock.

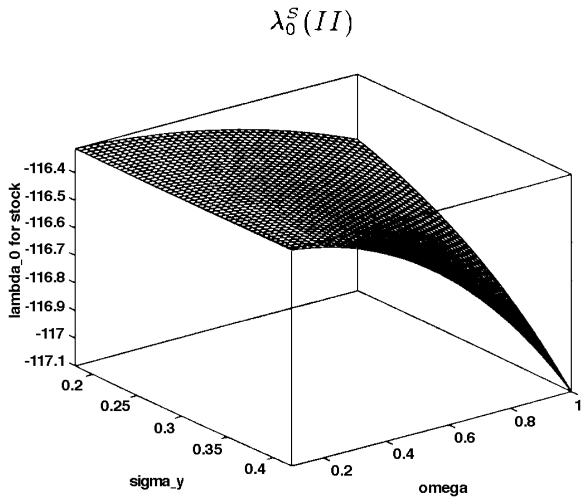
5.3. Comparing Market Structures I and II

To see more clearly the impact of collar trading on the equilibrium, we plot in Figure 5 the differences in stock risk premium and price volatility between market structures I and II. From the above discussion, opening collar trading allows an investor to better hedge his nontraded risk, and reduces the individual risk he bears. As a result, the stock price becomes less sensitive to the changes in individual exposure to nontraded risk. Figure 5A shows that the risk premium on the stock, as measured by $|\lambda^S_0|$, decreases as the derivatives market opens, and Figure 5B shows that the stock price volatility also decreases.

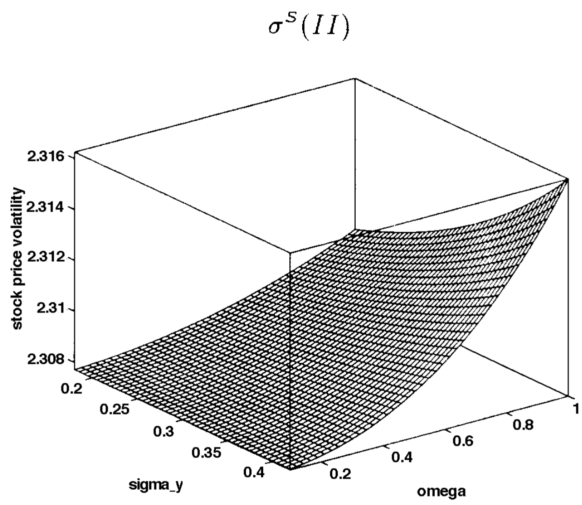
The impact of collar trading on stock premium and price volatility depends on the heterogeneity among the two classes of investors. It is negligible when ω approaches 1 or 0, but becomes significant when ω is in the middle range of [0, 1]. For $\omega = 1$ or 0, the economy is populated only by class-1 or class-2 investors, respectively. There is no heterogeneity among investors, the market is effectively complete, and the introduction of collar has no impact on the equilibrium. When ω is in the middle range of [0, 1], the heterogeneity among investors becomes significant, and so is the impact of opening collar trading. For a given value of ω, the heterogeneity increases with σ_Y. Thus, the impact of trading on stock premium and price volatility is increasing with σ_Y. For the parameter values shown in Figure 5, opening collar trading always reduces the stock premium and price volatility. However, for large values of σ_Y, we have found cases in which collar trading can increase the stock risk premium.

6. CASE OF ASYMMETRIC INFORMATION

In Section 5, we discussed the impact of changing market structure on equilibrium risk allocation and security prices when investors trade only for allocational reasons. In the presence of asymmetric information, changing market structure not only changes risk allocations, but also changes the information revealed through the security prices. Comparing the equilibrium under asymmetric information with that under symmetric information, we can see the interaction between the informational and allocational functions of the market. We maintain the assumption



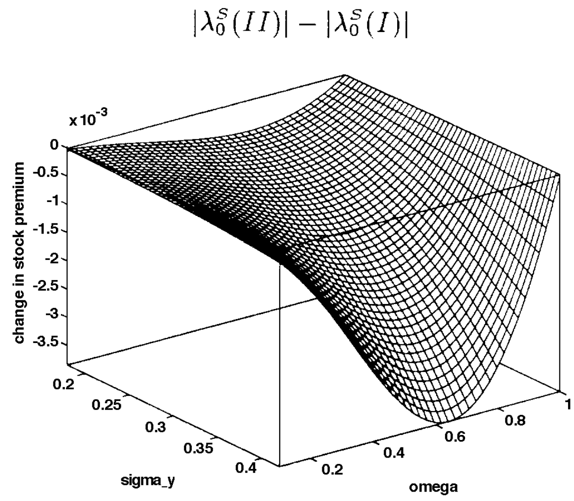
A)



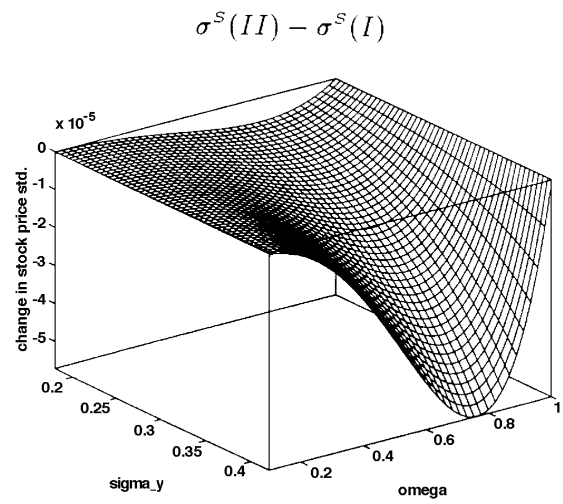
B)

FIGURE 4. Stock risk premium and price volatility under symmetric information and market structure II [parameters have the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.06$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.6$, $\sigma_Z = 0.5$, $\sigma_N = 0.22$, $\kappa_{DN} = 0.5$, $\beta_1 = (0, 1, 1)'$, and $\beta_2 = (0, 0, 0)'$].

that only investor 1 is informed and exposed to nontraded income. Without loss of generality, assume investor 2's private signal U_t contains no useful information. Therefore, he learns his information only from the cash flows and market prices of the traded securities. Formally, $\mathcal{F}_{2,t} = \mathcal{F}_t^{\{P,C\}} \subseteq \mathcal{F}_{1,t} = \mathcal{F}_t$. The actual information content of $\mathcal{F}_{2,t}$ now crucially depends on the market structure.



A)



B)

FIGURE 5. Change of stock risk premium and price volatility from market structure I to II under symmetric information [parameters have the following values: $\rho = 0.1, \gamma = 20, r = 0.06$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.6$, $\sigma_Z = 0.5$, $\sigma_N = 0.22$, $\kappa_{DN} = 0.5$, $\beta_1 = (0, 1, 1)'$, and $\beta_2 = (0, 0, 0)'$].

6.1. Equilibrium Under Market Structure I

The equilibrium of the economy under market structure I again can be stated as a special case of the general results in Theorem 1. As noted in Section 3, the state vector in general includes investor 2's estimation $\hat{X}_t$ of the unobserved state variables $X_t = \text{stack}\{G_t, Y_t, Z_t\}$. The stock price reveals a linear combination of these variables: $(\lambda_X^S - \lambda_\Delta^S)X_t$. Thus, $(\lambda_X^S - \lambda_\Delta^S)\hat{X}_t = (\lambda_X^S - \lambda_\Delta^S)X_t$, or $(\lambda_X^S - \lambda_\Delta^S)\Delta_t = 0$. Only two degrees of uncertainty remain in investor 2's estimation error. In particular, $\Delta_Z = \hat{Z}_t - Z_t$ can be expressed as a linear combination of $\Delta_G = \hat{G}_t - G_t$ and $\Delta_Y = \hat{Y}_t - Y_t$. Incorporating this observation into our description of the equilibrium, we have the following corollary.

COROLLARY 3. *Under market structure I, when* $\mathcal{F}_{1,t} = \mathcal{F}_t$, $\mathcal{F}_{2,t} = \mathcal{F}_t^{\{P,C\}}$, $\beta_1 = (0, 1, 1)'$, $\beta_2 = (0, 0, 0)'$, *and* ω *is close to* 1, *the economy has a linear stationary equilibrium in which investor i's policies and the stock price are*

$$S_t = \frac{1}{(r - a_G)} G_t + \tilde{\lambda}^S \tilde{X}_{1,t},$$

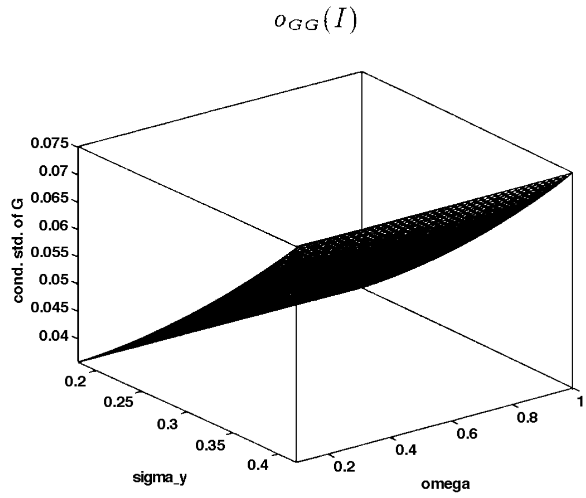
$$\theta_{i,t}^S = h_i \tilde{X}_{i,t}, \qquad c_{i,t} = rW_{i,t} - (1/2\gamma)\tilde{X}_{i,t}{}' v_i \tilde{X}_{i,t} - (1/\gamma)\ln r,$$

where $X_{1,t} = \text{stack}\{1, Y_t, Z_t, \Delta_G, \Delta_Y\}$, $\tilde{X}_{2,t} = \text{stack}\{1, \hat{Y}_t, \hat{Z}_t\}$; $\lambda^S = (\lambda_0^S, \lambda_0^S, \lambda_Z^S, \lambda_{\Delta G}^S, \lambda_{\Delta Y}^S)$, $h_1 = (h_{1,0}, h_{1,Y}, h_{1,Z}, h_{1,\Delta G}, h_{1,\Delta Y})$, $h_2 = (h_{2,0}, h_{2,Y}, h_{2,Z})$, v_1, *and* v_2 *are determined by* (10b), (14), (15), *and* (16).

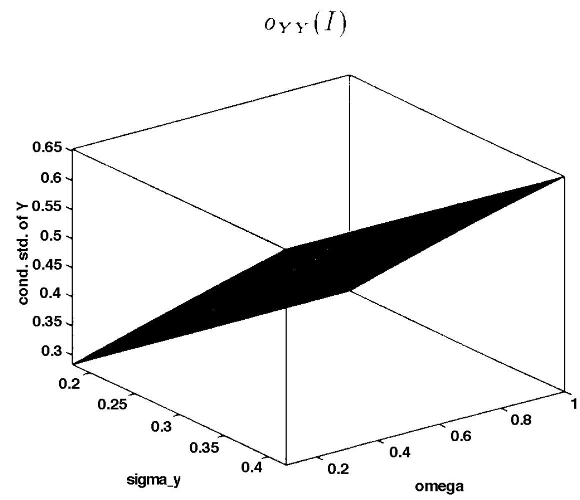
An important characteristic of the equilibrium is the information asymmetry between the two investors, measured by the conditional standard deviation of investor 2's estimation of the unobserved state variables. Let $o_{GG} = \{E[(\hat{G}_t - G_t)^2 \mid \mathcal{F}_{2,t}]\}^{1/2}$ denote the information asymmetry concerning future stock payoffs, and $o_{YY} = \{E[(\hat{Y}_t - Y_t)^2 \mid \mathcal{F}_{2,t}]\}^{1/2}$ and $o_{ZZ} = \{E[(\hat{Z}_t - Z_t)^2 \mid \mathcal{F}_{2,t}]\}^{1/2}$ denote the information asymmetry concerning investor 1's hedging need.

Figure 6 plots o_{GG} and o_{YY} against σ_Y and ω. The plot for o_{ZZ} is omitted here because its behavior is similar to that of o_{YY}. Increasing ω, the population weight of class-1 (informed) investors, has two offsetting effects on the information asymmetry between the two classes of investors. On one hand, as more informed traders take speculative positions, more information is incorporated into the prices. On the other hand, there is also more hedging trade in the market because only class-1 investors are endowed with nontraded income. The increase in hedging trade introduces additional movements in the stock price that are unrelated to its payoffs, making prices less informative about class-1 investors' private information on future stock payoffs. The net change in o_{GG} as ω increases depends on which of these two effects dominates. In the case shown in Figure 6A, o_{GG} increases with ω. The change of o_{YY} can be analyzed similarly.

Increasing σ_Y, the volatility of investor 1's exposure to nontraded income always increases o_{GG}. Higher σ_Y gives rise to more volatile hedging trade from investor 1, and thus reduces the amount of information revealed through stock trading on future stock payoffs. The impact of increasing σ_Y on o_{YY} is, however, ambiguous. On the one hand, the higher σ_Y increases the unconditional uncertainty about Y_t. On the other hand, the price of the stock, now more driven by investor 1's hedging trade, also becomes more informative about Y_t. The trade-off between these two effects determines the net change in o_{YY} when σ_Y increases. For the current set of parameters, the first effect dominates and o_{YY} increases with σ_Y.



A)



B)

FIGURE 6. Conditional standard deviation of G_t and Y_t for investor 2 under market structure I and asymmetric information (parameters have the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.06$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.6$, $\sigma_Z = 0.5$, $\sigma_N = 0.22$, $\kappa_{DN} = 0.5$, $\beta_1 = (0, 1, 1)'$, and $\beta_2 = (0, 0, 0)'$].

Figure 7 illustrates how the stock risk premium and price volatility change with σ_Y and ω. Comparing it with Figure 2, we note that the behavior of the equilibrium price under asymmetric information resembles that under symmetric information for large ω, but differs significantly when ω is small. In particular, for small values of ω, the risk premium $|\lambda_0^S|$ and price volatility σ_S both increase with σ_Y under

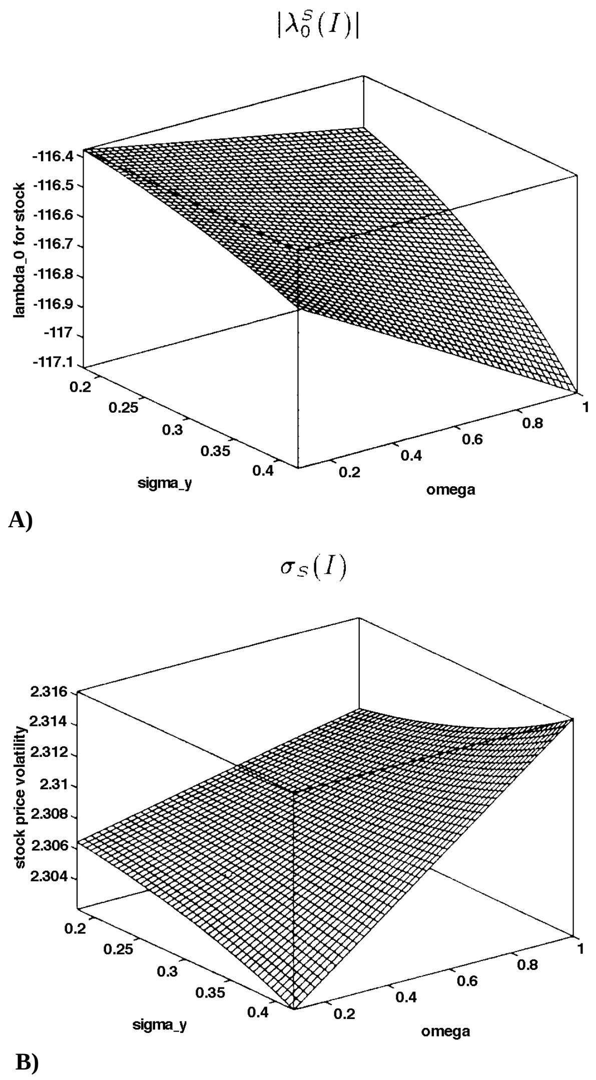


FIGURE 7. Stock risk premium and price volatility under market structure I and asymmetric information [parameters have the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.06$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.6$, $\sigma_Z = 0.5$, $\sigma_N = 0.22$, $\kappa_{DN} = 0.5$, $\beta_1 = (0, 1, 1)'$, and $\beta_2 = (0, 0, 0)'$].

symmetric information (see Figure 2). But under asymmetric information, the risk premium increases with σ_Y and price volatility decreases (see Figure 7).

Increasing σ_Y has two effects on the stock price under asymmetric information. First, similar to the case of symmetric information, it tends to increase the stock risk premium and price volatility because of the increase in allocational trade. The effect is stronger the larger that ω is, because more class-1 investors are trading the stock to hedge their nontraded risk. Second, increasing σ_Y reduces the amount of information that investor 2 can extract from the security prices about future stock payoffs, as reflected by the increase in o_{GG} shown in Figure 6. Hence, investor 2 demands a higher premium on the stock to compensate for the increase in his perceived uncertainty. Also, less information tends to reduce the variability of investor 2's expectation of G_t and thus reduces the price volatility. Thus, increasing σ_Y tends to increase the risk premium and decrease the price volatility. This information effect is stronger when ω is small and the information asymmetry between the two classes of investors is large. The net impact of increasing σ_Y depends on the combination of the allocational and informational effects. When ω is large, the allocational effect dominates and the equilibrium stock price behaves similar to that under symmetric information. For small values of ω, the information effect dominates, the stock risk premium increases with σ_Y, and its price volatility decreases with σ_Y.

6.2. Equilibrium Under Market Structure II

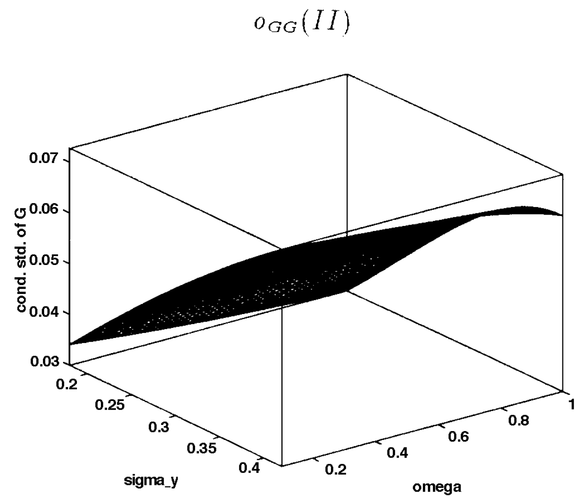
As the collar is introduced, the prices of the stock and the collar provide two endogenous signals about $X_t = (G_t, Y_t, Z_t)'$, in particular, $(\lambda_X^S - \lambda_\Delta^S)\Delta_t = 0$ and $(\lambda_X^H - \lambda_\Delta^H)\Delta_t = 0$. Thus, only one degree of uncertainty remains in investor 2's estimation, and Δ_Y, Δ_Z can be expressed as linear functions of Δ_G. We have the following corollary.

COROLLARY 4. *Under market structure II, when $\mathcal{F}_{1,t} = \mathcal{F}_t$, $\mathcal{F}_{2,t} = \mathcal{F}_t^{\{C,P\}}$, $\beta_1 = (0, 1, 1)'$, $\beta_2 = (0, 0, 0)'$, and ω is close to 1, the economy has a linear, stationary equilibrium in which*

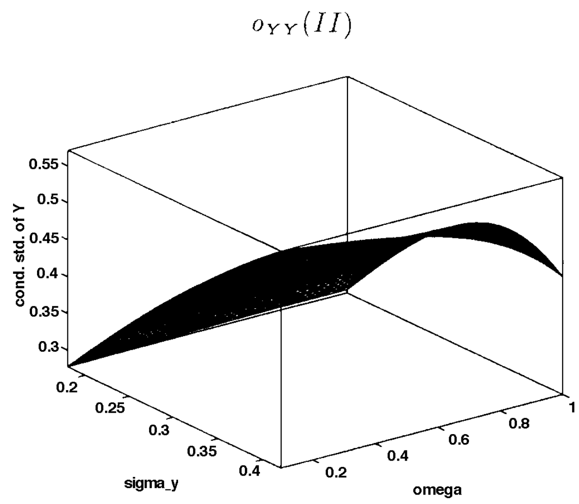
$$S_t = \frac{1}{(r - a_G)} G_t + \tilde{\lambda}^S \tilde{X}_{1,t}, \qquad H_t = \frac{1}{(r - a_G)^2} G_t + \tilde{\lambda}^H \tilde{X}_{1,t},$$
$$\theta_{i,t} = h_1 \tilde{X}_{1,t}, \qquad c_{i,t} = r W_{i,t} - (1/2\gamma) \tilde{X}_{i,t}{}' v_i \tilde{X}_{i,t} - (1/\gamma) \ln r,$$

where $i = 1, 2$, $\tilde{X}_{1,t} = \text{stack}\{1, Y_t, Z_t, \Delta_G\}$, $\tilde{X}_{2,t} = \text{stack}\{1, \hat{Y}_t, \hat{Z}_t\}$; $\tilde{\lambda}^S = (1, \lambda_Y^S, \lambda_Z^S, \lambda_{\Delta G}^S)$, $\tilde{\lambda}^H = (1, \lambda_Y^H, \lambda_Z^H, \lambda_{\Delta G}^H)$, $h_1 = (h_{1,0}, h_{1,Y}, h_{1,Z}, h_{1,\Delta G})$, $h_2 = (h_{2,0}, h_{2,Y}, h_{2,Z})$, v_1, and v_2 are determined by (10b), (14), (15), *and* (16).

Again, we first examine the information asymmetry between the two investors. Figure 8 plots o_{GG} and o_{YY} for different values of σ_Y and ω. The intuition obtained under market structure I applies here as well. Now that investor 2 receives signals both from the stock price and from the collar price, in addition to the two effects in the Section 6.1, increasing σ_Y has a third effect on information asymmetry. That is, as σ_Y changes, the difference between the two signals can also change, making the combination of the two signals more or less informative. For example, each price itself can become less informative about G_t as σ_Y increases. However, the



A)

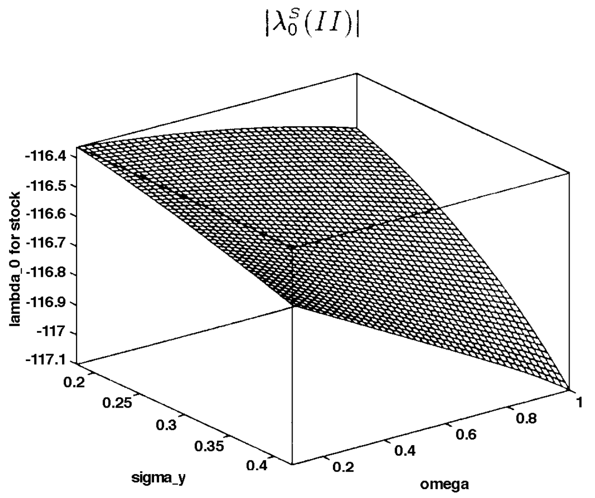


B)

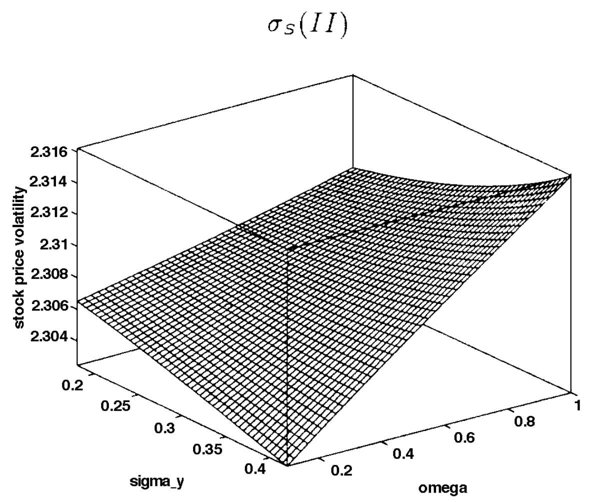
FIGURE 8. Conditional standard deviation of G_t and Y_t for investor 2 under market structure II and asymmetric information [parameters have the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.06$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.6$, $\sigma_Z = 0.5$, $\sigma_N = 0.22$, $\kappa_{DN} = 0.5$, $\beta_1 = (0, 1, 1)'$, and $\beta_2 = (0, 0, 0)'$].

two signals can become less correlated, and jointly become more informative. As a result, o_{GG} can either increase or decrease with σ_Y.

Figure 9 plots the risk premium and price volatility of the stock for different values of σ_Y and ω. They behave in a similar way to that under market structure I.



A)



B)

Figure 9. Stock risk premium and price volatility under market structure II and asymmetric information [parameters have the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.06$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.6$, $\sigma_Z = 0.5$, $\sigma_N = 0.22$, $\kappa_{DN} = 0.5$, $\beta_1 = (0, 1, 1)'$, and $\beta_2 = (0, 0, 0)'$].

6.3. Comparison Between Market Structures I and II

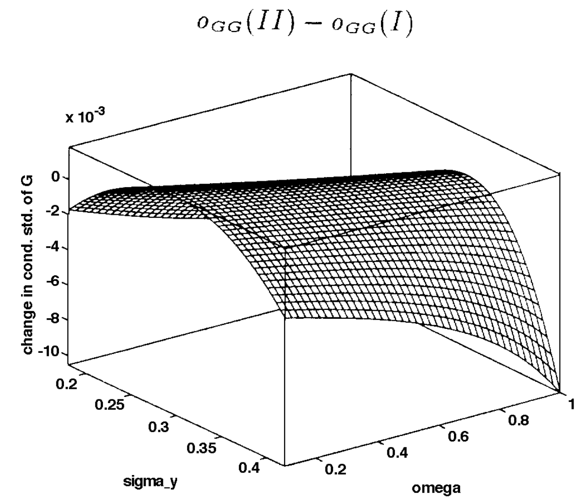
We now compare the equilibrium under market structures I and II and examine the impact of derivative trading on the informational efficiency of the securities market. Under the assumption that the allocational trade (or noise trade) is exogenously specified and unchanged as new securities are introduced, Grossman (1977) argues that adding securities to the market should improve its informational efficiency because more private information is revealed through a larger set of prices. In general, however, the allocational trade cannot be exogenously specified, especially when we consider changing the market structure. As we have shown under symmetric information, investors change their allocational trade significantly when the collar is introduced. An increase in allocational trade can introduce additional stock price movements that are unrelated to future payoffs, making the price less informative. In the case that the loss of information from the stock price exceeds the gain of information from the collar price, the information asymmetry in the market can increase when the collar is introduced. Thus, opening derivative trading can reduce the informational efficiency of the market.

To verify this intuition, we plot in Figure 10 the changes of o_{GG} and o_{YY} when market structure changes from I to II. At $\omega = 1$, the economy is populated only by class-1 investors. We can view this economy as if the allocational trade were exogenously specified, because, being the only type of investors in the economy, investor 1 is forced to hold his endowment under both market structures. The intuition in Grossman (1977) applies in this case and the information asymmetry, as measured by o_{GG}, decreases after opening the collar market.

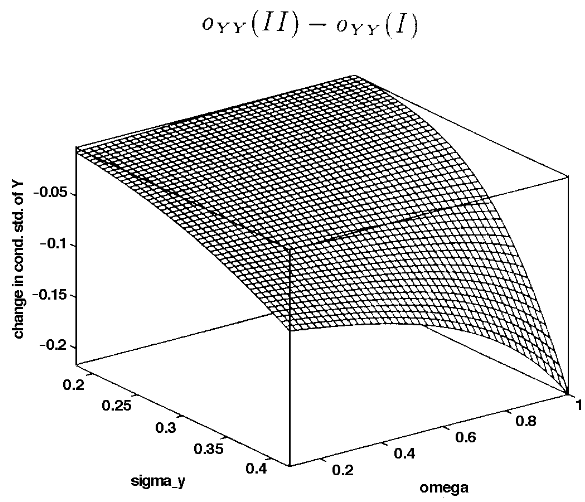
At $\omega < 1$, with investor 2 making the market, investor 1 endogenously determines his allocational trade based on the market structure. The smaller the ω, the more significantly investor 1 changes his allocational trade after collar trading opens. For most values of σ_Y and ω under consideration, o_{GG} decreases from market structure I to II, indicating that the information asymmetry between the investors decreases after introducing the collar because the collar price provides new information to the uninformed investors. However, for certain values of ω and σ_Y, especially when ω is small, the introduction of collar trading can increase the information asymmetry between the two classes of investors on future stock payoffs, as reflected by the increase in o_{GG}. This is the case when the information loss from the stock price exceeds the information gain from the collar price.

In general, o_{YY} decreases from market structure I to II. This is not surprising because the addition of collar allows investor 1 to more actively hedge his nontraded risk. The increase in his hedging activity reveals more information about his hedging need through the stock and collar prices.

We now examine the difference in equilibrium stock price between market structures I and II. Figure 11 illustrates the changes of stock risk premium and price volatility when the market structure changes from I to II. At $\omega = 1$, changing market structure does not change the equilibrium allocation and security prices because the market is effectively complete. For $\omega < 1$, opening the collar market has two effects on the equilibrium security prices. On the one hand, similar to the case of symmetric information, it allows an investor to better hedge his nontraded risk and thus reduces the stock risk premium and price volatility. This allocational effect is negligible as ω approaches 1 or 0, because introducing derivative security has little impact on the equilibrium prices when the economy is dominated by one



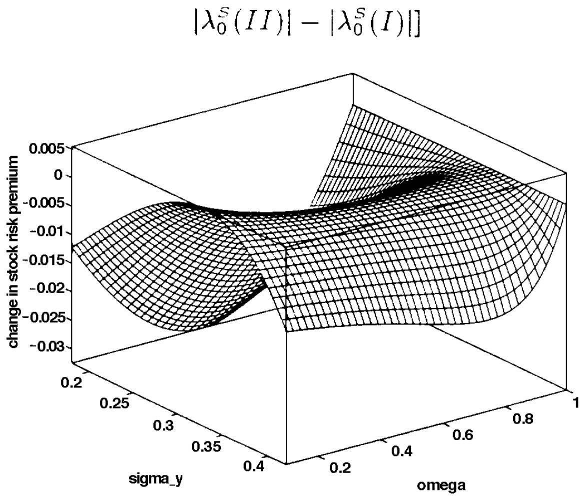
A)



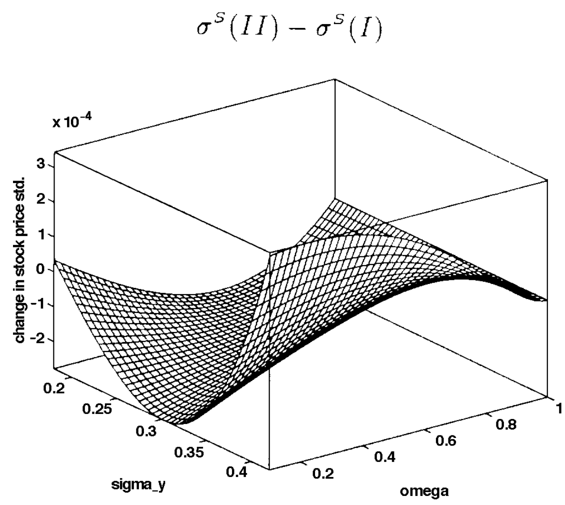
B)

FIGURE 10. Change in information asymmetry, as measured by the conditional standard deviation of G_t and Y_t for investor 2 from market structure I to II [parameters have the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.06$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.6$, $\sigma_Z = 0.5$, $\sigma_N = 0.22$, $\kappa_{DN} = 0.5$, $\beta_1 = (0, 1, 1)'$, and $\beta_2 = (0, 0, 0)'$].

class of investors (see Figure 5). On the other hand, opening the collar market changes the amount of information about future stock payoffs that uninformed investor 2 extracts from prices. As discussed under market structure I, if o_{GG} increases, the information effect tends to increase the stock risk premium and decrease the price volatility. If o_{GG} decreases, the opposite applies.



A)



B)

FIGURE 11. Change in stock risk premium and price volatility from market structure I to II under asymmetric information [parameters have the following values: $\rho = 0.1$, $\gamma = 20$, $r = 0.06$, $a_G = -0.2$, $a_Y = -0.2$, $a_Z = -0.5$, $\sigma_D = 0.7$, $\sigma_G = 0.6$, $\sigma_Z = 0.5$, $\sigma_N = 0.22$, $\kappa_{DN} = 0.5$, $\beta_1 = (0, 1, 1)'$, and $\beta_2 = (0, 0, 0)'$].

The net impact of opening collar trading depends on the trade-off between the allocational and informational effects. For small ω, the informational effect dominates. If the information asymmetry concerning the future stock payoffs increases (i.e., o_{GG} increases), stock risk premium can increase after introducing the collar.

For ω around 0.1 and σ_Y around 0.3, o_{GG} increases after opening the collar market (see Figure 10A) and the risk premium $|\lambda_0^S|$ increases (see Figure 11A). Note that for the same set of parameter values, the stock risk premium decreases under symmetric information (see Figure 5A). On the other hand, if opening collar trading decreases the informational asymmetry (or o_{GG} decreases), stock price volatility tends to increase, because investor 2's expectation about G_t becomes more volatile. Figures 10A and 11B show that for ω around 0.2 and $\sigma_Y > 0.4$, o_{GG} decreases after introducing the collar (see Figure 10A) and the stock price volatility σ^S increases. This is in contrast to the case of symmetric information in which σ_S decreases after introducing the collar (see Figure 5B).

7. CONCLUDING REMARKS

We analyze the impact of derivative trading on the allocational and informational efficiencies of the securities market, using a specific model within the fully rational expectations framework. We show that the introduction of derivative securities not only provides additional sources of information to the less informed investors through the derivative prices, but also changes the information content of the existing security prices by changing the allocational trade. The net impact of introducing derivatives on the informational efficiency of the securities market depends on the interaction between these two effects. In particular, the market may become less efficient informationally when derivative securities are introduced. We also show that introducing derivatives can increase the stock risk premium and price volatility under asymmetric information.

Although the intuitions obtained from our model are general, the model itself contains several restrictive assumptions. In what follows, we provide some further comments on these assumptions.

The model assumes that the risk-free security yields constant returns, independent of market demands. This assumption is needed to solve the model. We can justify this assumption by viewing the economy under consideration as a small economy that has access to an outside bond market. We can also modify the model to avoid this assumption. For example, we can define the model on a finite time horizon and assume that the security payoffs, endowments, and consumptions occur only on the terminal date. We then can use the risk-free security as the numeraire, whose return is zero by definition.[11] Solving such a model is similar to solving the current model. The drawback of a finite-horizon model is that it is no longer stationary; solving the equilibrium is possible but tedious. We do not expect the results to be very different from those obtained in the current setting.

The model also assumes that investors have constant absolute risk aversion, which has the unattractive feature that it allows negative consumption and exhibits no income effect on the individual demand for risky securities. However, under this preference, an investor's holding of risky securities is independent of his wealth, as are the equilibrium prices. The solution for an equilibrium then is simplified greatly.

Another feature of the model is that both stock dividend and nontraded income have Brownian-motion components. This has two implications. First, the instantaneous dividend and nontraded income have significant probability of being negative, regardless of the parameter values. Second, the cumulative income process (including both dividends and nontraded income) has infinite variation, whereas the cumulative consumption process is (required to be) absolutely continuous. This difference in the nature of income process and consumption process makes it infeasible for an investor to buy and hold (in which case, consumption would equal income). Although unattractive, these implications are merely byproducts of the continuous-time specification. The fact that they do not arise in a discrete-time counterpart of the model implies that economically they are not important.[12]

The choice of collar contract as the derivative security is fairly arbitrary and mainly for convenience. More generally, we have given no justification for why the securities market is incomplete in our model. Although it is beyond the scope of this paper to provide an explicit justification, the reasons are quite obvious. Because an investor's nontraded income is private information, certain contracts, such as those with payoffs contingent on the realization of the nontraded income, are informationally infeasible.[13] In the absence of these contracts, the market is incomplete. Modeling the actual process of introducing derivative securities is also beyond the scope of this paper. We provide no rationale as to why a certain contract (e.g., the collar) is introduced and why no additional contracts are introduced. There is, however, a growing literature addressing these issues [see, e.g., Allen and Gale (1994) and Duffie and Rahi (1995)].

NOTES

1. For example, He and Pearson (1991) and Karatzas et al. (1991) examine the existence and characterization of optimal consumption and investment policies (with finite horizon) under an incomplete market. Merton (1971), Duffie et al. (1993), He and Pagès (1993), Svensson and Werner (1993), Koo (1994a, b), Cuoco (1995), among others, consider the problem when investors also have nontraded income.

2. For equilibrium pricing models with nontraded income, see, e.g., Scheinkman and Weiss (1986), Marcet and Singleton (1990), Telmer (1993), Lucas (1994), Detemple (1995), and Heaton and Lucas (1996).

3. Wang (1993) and Detemple (1994) solve multiperiod pricing models under asymmetric information with specific assumptions about preferences and shock distributions. Judd and Bernardo (1994) consider numerical solutions to equilibrium models under asymmetric information.

4. Given that investors are identical within each class, they can be aggregated into a single representative investor.

5. The collar contract defined here represents a series of bets on future stock prices. For positive stock prices ($S_t > 0$), the long side of the contract receives payments at rate S_t, whereas for negative stock prices, the short side receives payments at rate $-S_t$. Note that payments here are in the form of continuous flows instead of discrete lumps.

6. Here, the following notations: diag$\{e_1, e_2, \ldots, e_k\}$, $(e_1, e_2, \ldots, e_k)$, and stack$\{e_1, e_2, \ldots, e_k\}$ denote, respectively, the diagonal matrix, row matrix, and column matrix for a set of elements (of proper order) $e_1, e_2, \ldots, e_k$; $(\cdot)'$ denotes the transpose of a matrix and trace$(\cdot)$ denotes its trace.

7. We have assumed that both investors are exposed to the same set of nontraded risks in the economy. Solving the model with additional independent shocks is straightforward, and the qualitative nature of our results does not change.

8. Here, the integrability on $[0, \infty)$ is defined to be integrable over $[0, T]\,\forall\, T \geq 0$.

9. For the infinite horizon control problem to have well-posed solutions, appropriate boundary conditions are needed. Imposing the above transversality condition is equivalent to the following procedure: First, solve the control problem with finite horizon and a bequest function of the terminal wealth in the same form of the utility function, and then let the terminal data go to infinity.

10. For $X_t = (1, G_t, Y_t, Z_t)'$, its variable elements are G_t, Y_t, Z_t. Let $X_i, i = 1, \ldots, n$, be the variable elements of X, then

$$\partial_X P = \text{stack}\left\{\frac{\partial P}{\partial X_1}, \ldots, \frac{\partial P}{\partial X_n}\right\} \quad \text{and} \quad \partial_X^2 P = \left\{\frac{\partial^2 P}{\partial X_i \partial X_j}\right\}.$$

11. When there is only terminal consumption, the interest rate in the usual sense is not well defined here. Making this assumption allows us to avoid dealing with the interest rate instead of endogenizing it. See Grossman and Zhou (1996) for an example.

12. A discrete-time counterpart of our model has been used by Hong (1996). The nature of the equilibrium in the discrete-time setting is very similar to that in the continuous-time setting.

13. In our setting, investors within the same class are assumed to have the same nontraded income. Thus, information on nontraded income is shared among investors within the same class, but not across classes. We can easily introduce an idiosyncratic component in each investor's nontraded income to make it unobservable to other investors.

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APPENDIX

A.1. PROOF OF LEMMA 2

Given the return processes specified in (11) and (12), it is easy to verify that $J_{i,t}$ in (13b) solves the Bellman equation and satisfies the specified transversality condition when v_i solves (15). In this case, $J_{i,t}$ gives investor i's value function. It then follows that $c_{i,t}$ and $\theta_{i,t}$ in (13a) are investor i's optimal policies. The investors' optimization problem is now reduced to solving the algebraic matrix equation in (15). For any two square matrices m and n, we denote $m \geq n$ (m larger than n) if $m - n$ is positive semidefinite. The strict inequality applies when $m - n$ is positive definite. Define three matrices m_2, m_1, and m_0 by

$$\begin{aligned} m_2 &= b_{i,X}\left[\iota - b'_Q(b_Q b'_Q)^{-1} b_Q\right] b'_{i,X}, \\ m_1 &= b_{i,X} b'_Q \sigma_{QQ}^{-1}(a_Q - r\gamma b_Q b'_N \beta'_i) + [(r/2)\iota - a_{i,X}], \\ m_0 &= (a_{i,Q} - r\gamma b_Q b'_N \beta'_i)' \sigma_{QQ}^{-1}(a_{i,Q} - r\gamma b_Q b'_N \beta'_i) - (r\gamma\sigma_N)^2 \beta_i \beta'_i. \end{aligned} \tag{A.1}$$

Note that both m_2 and m_0 are symmetric. Equation (15) then can be expressed as

$$m_0 + m'_1 v + v m_1 - v m_2 v = 0. \tag{A.2}$$

Equation (A.2) is called the algebraic Riccati equation (ARE). The ARE in general has multiple solutions. We need the smallest solution for v to maximize the value function. The following lemma on ARE is useful.

LEMMA A.1. [Willems and Callier (1991)] *If m_2 and m_0 are positive definite*, (A.2) *has a unique largest* (*smallest*) *solution that is symmetric and positive* (*negative*) *definite.*

Let $v = \text{stack}\{(v_{00}, \tilde{v}_0), (\tilde{v}'_0, \tilde{v})\}$, where $\tilde{v}$ is the submatrix of v that corresponds to the variable part of X_t (v_{00} corresponds to the constant part and $\tilde{v}_0$ the cross part). Also, let $\tilde{m}_0$, $\tilde{m}_1$, $\tilde{m}_2$ denote the submatrices of m_0, m_1, and m_2 that correspond to $\tilde{v}$. Equation (A.2) then gives the following equation for $\tilde{v}$:

$$\tilde{m}_0 + \tilde{m}'_1 \tilde{v} + \tilde{v}\tilde{m}_1 - \tilde{v}\tilde{m}_2\tilde{v} = 0, \tag{A.3}$$

which is also an ARE. We only need to solve for $\tilde{v}$, because given $\tilde{v}$, solving $\tilde{v}_0$ and v_{00} is straightforward. Thus, we need $\tilde{m}_2$ and $\tilde{m}_0$ to be positive definite for the existence of a solution to the optimization problem. Given that $\iota \geq b'_Q(b_Q b'_Q)^{-1} b_Q$ for any b_Q, $\tilde{m}_2$ is positive definite. The existence of a solution $\tilde{v}$ now only requires $\tilde{m}_0$ to be positive. The following lemma is immediate.

LEMMA A.2. *In the absence of nontraded income*, *the investor's control problem given by* (15) *has a unique solution.*

Proof. Given the linear price function, $m_0 = a'_Q \sigma_{QQ}^{-1} a_Q$ in the absence of nontraded income. Here $a_Q = \tilde{\lambda}^P(a_{\tilde{X}} - r\iota) + \tilde{\lambda}^C$. Because both σ_{QQ}^{-1} and $-(a_{\tilde{X}} - r\iota)$ are symmetric positive definite, $a'_Q \sigma_{QQ}^{-1} a_Q$ is also symmetric and positive definite. Thus, m_0 is positive definite, and so is $\tilde{m}_0$. By Lemma A.1, there exists a solution to the optimization problem of the conjectured form. ■

In the presence of nontraded income, to ensure the existence of a solution, we need additional conditions to guarantee $\tilde{m}_0$ positive definite.

A.2. PROOF OF THEOREM 1

We now prove Theorem 1 by proving the existence of a solution to the system (10b), (15), and (16). The proof follows three steps. First, for a fixed parameter ($\omega = 1$), a unique solution to the system is shown to exist. Then we show that at the solution, the determinant of the Jacobian matrix is *generically* nondegenerate (to be defined). Finally, using the Implicit Function Theorem, we conclude that the system has a solution in a neighborhood of the initial parameter (i.e., $\omega = 1$). We start by providing some auxiliary definitions and results.

DEFINITION A.1. *Let $\mathcal{D}$ be an open set in $\Re^n$. A function $f:\mathcal{D} \to \Re^m$ is called generically nondegenerate if the n-dimensional Lebesgue measure of its zero set $\{x\colon f(x) = 0\}$ is zero.*

LEMMA A.3. (Implicit Function Theorem). *Let $\mathcal{D}$ be an open set in $\Re^{m+n}$ containing the point (x_0, y_0) where $x_0 \in \Re^m$ and $y_0 \in \Re^n$. Suppose that $F:\mathcal{D} \to \Re^m$ is continuous with continuous first partial derivatives in $\mathcal{D}$, and*

$$F(x_0, y_0) = 0 \qquad and \qquad det[\nabla_x F(x_0, y_0)] \neq 0. \tag{A.4}$$

Then positive numbers ϵ_x and ϵ_y can be chosen so that

1. *The direct product of the closed balls $\overline{B_m(x_0, \epsilon_x)}$ and $\overline{B_n(y_0, \epsilon_y)}$ with centers at x_0, y_0 and radii ϵ_x and ϵ_y, respectively, is in $\mathcal{D}$;*
2. *ϵ_x and ϵ_y are such that for each $y \in B_n(y_0, \epsilon_y)$ there is a unique $x \in B_m(x_0, \epsilon_x)$ satisfying $F(x, y) = 0$. If f is the function from $B_n(y_0, \epsilon_y)$ to $B_m(x_0, \epsilon_x)$ defined by these ordered pairs (x, y), then $F[f(y), y] = 0$; furthermore, f and all its partial derivatives are continuous on $B_n(y_0, \epsilon_y)$.*

A proof of Lemma A3 can be found in Protter and Morrey (1991).

LEMMA A.4. *Let $f:\mathcal{D} \to \Re$ be a real analytical function, where $\mathcal{D} = \mathcal{D}_1 \otimes \cdots \otimes \mathcal{D}_n$ is an open subset of $\Re^n$. Let $\mathcal{N} = \{x \in \mathcal{D}: f(x) = 0\}$ be its zero set. Then either $\mathcal{N} = \mathcal{D}$ or $\mu_n(\mathcal{N}) = 0$ where μ_n is the n-dimensional Lebesgue measure.*

Proof. We will prove by induction. First, $\mathcal{N}$ is closed and therefore measurable. For $n = 1, \mathcal{N}$ is either finite, or has an accumulation point. In the latter case, the function f is identically zero on $\mathcal{D}$ [see Ahlfors (1979)]. Noting that any finite set has zero Lebesgue measure concludes this part of the proof. Let us assume that the conclusion of the lemma holds for certain $k \geq 1$ and prove it for $n = k+1$. Denoting f as a function of two variables, $f(t, x)$, on $\mathcal{D}_1 \otimes \mathcal{D}_{-1}$, where $\mathcal{D}_{-1} \equiv \mathcal{D}_2 \otimes \cdots \otimes \mathcal{D}_{k+1}$. We see that f is a real analytical function in both t and x separately as well. Consider the set $\mathcal{S} = \{t \in \mathcal{D}_1 : \forall x \in \mathcal{D}_2, f(t, x) = 0\}$. For $t \notin \mathcal{S}$, we have $\int_{\mathcal{D}_{-1}} f(t, x)\, dx = 0$ by the inductive assumption. If set $\mathcal{S}$ is finite, it is of zero Lebesgue measure in $\mathcal{D}_1$. Thus, $\mu_n(N) = \int_{\mathcal{D}_1} \int_{\mathcal{D}_{-1}} 1_{f(t,x)=0}\, dx\, dt = 0$ by Fubini's theorem [see, e.g., Doob (1991)]. If, on the other hand, $\mathcal{S}$ is not finite, then it has an accumulation point. From the result of $n = 1$, we see that for any fixed $x \in \mathcal{D}_{-1}$, $f(t, x)$ is identically zero in $\mathcal{D}_1$, and therefore identically zero on $\mathcal{D} = \mathcal{D}_1 \otimes \mathcal{D}_{-1}$. This concludes the proof. ■

We are now ready to show the existence of equilibrium at $\omega = 1$.

LEMMA A.5. *At* $\omega = 1$, *if the parameters satisfy assumption* (7), *then there exists a solution to the system* (10b), (15), *and* (16) *under both market structures I and II.*

Proof. At $\omega = 1$, class-2 investors have population-weight zero, hence have no impact on the equilibrium prices. We derive the equilibrium in this case in two steps: first, to find a price process at which class-1 investors' demand for traded securities equals the supply, and then to show that class-2 investors' expectations and optimal policies have the proposed solution under the given price process. The first step is completed in Appendix A.3 and we only focus on the second step here. In particular, we show the existence of a solution to (10b) and (15), taking the price process (9) as given. Let $o = \alpha\tilde{o}\alpha'$, where $\tilde{o}$ is a full-ranked submatrix of o. Simple algebra shows that equation (10b) reduces to an ARE as defined in (A.2) with

$$m_2 = \alpha' a_X^{\Phi\prime}(b_\Phi b_\Phi')^{-1} a_X^\Phi \alpha,$$

$$m_1 = \left\{\alpha'\left[a_X^{\Phi\prime}(b_\Phi b_\Phi')^{-1}(b_X b_\Phi')' + a_X'\right]\right\}\Big|_{(3-m)\times(3-m)},$$

$$m_0 = \left\{b_X\left[b_\Phi'((b_\Phi b_\Phi')^{-1} b_\Phi\right]b_X' + b_X b_X'\right\}\Big|_{(3-m)\times(3-m)},$$

where $|_{(3-m)\times(3-m)}$ means taking the first $(3-m)\times(3-m)$ submatrix. Obviously, $m_2 = (\alpha'\lambda)m(\alpha'\lambda)'$ where $m = a_X'\sigma_{\Phi\Phi}^{-1}a_X$ is symmetric and positive definite and m_0 is also positive definite; by Lemma A.1, (10b) has a unique positive definite solution. Furthermore, because investor 2 has no nontraded income, Lemma A.2 implies that (15) has a solution for v_2. This completes our proof. ■

Now we show the existence of a solution to the system (10b), (15), and (16) for ω close to 1. To conform to the notation in Lemma A.3, we reformulate the system as $F(x;\, p, q) = 0$, where $F = (F_1, F_2, F_3, F_4)$ with F_1, F_2 corresponding to (15), F_3 corresponding to (16), and F_4 to (10b). Let $[\cdot]$ denote the column vector of all nonidentical entries of a matrix. Define $x = [[v_1], [v_2], \lambda^P, [o]]$, where $[v_1], [v_2]$ are the coefficients in investors' value functions, λ^P is the coefficient of the price processes, and o is the uninformed investor's conditional variance of the unobserved state variables. Let $p = \omega$, the population weight of class-1 investors, and $q = [r, \gamma, \rho, a_G, a_Y, a_Z, \sigma_D, \sigma_G, \sigma_Y, \sigma_Z, \sigma_N, [\kappa]]$ ($[\kappa]$ is the vector of covariance coefficients among all shocks) be the vector of all fixed parameters in the system. From Lemma A.5, the system has a solution at $w = 1$. In other words, for any fixed set of parameters q, $\exists\, x_0 = x_0(q)$, such that $F(x_0;\, p_0, q) = 0$ for $p_0 = 1$. As the second step, we show that the Jacobian matrix is nondegenerate at (x_0, p_0) for any set of parameters q. Given the high dimensionality of q, the calculation would be messy and tedious. Instead, we show that the set of q at which the Jacobian is degenerate has measure zero. Let $f(q) = \det[\nabla_x F(x_0;\, p_0, q)]$, then $f(q)$ is a real analytical function. It is easy to find a q_0 such that $f(q_o) \neq 0$; therefore, the Jacobian is generically nondegenerate at (x_0, p_0) by Lemma A.4. Applying the Implicit Function Theorem, we conclude that there exists a solution to system (10b), (15), and (16) for ω close to 1. This completes the proof of Theorem 1.

A.3. PROOF OF THEOREM 2

When $\omega = 1$, the economy is populated only with class-1 investors and is thus equivalent to the economy with one representative agent who has total exposure $Y_t + Z_t$ to nontraded risk. We conjecture a linear equilibrium with price process $P_t = \lambda^P X_t$. The representative agent solves his control problem as defined in (15) with market clearing condition $h = 1_{1,1}^{m,4}$. Define

$\tilde{v} = \text{stack}\{(v_{YY}, v_{YZ}), (v_{YZ}, v_{ZZ})\}$. We can rewrite (15) in the form of (A.2) with coefficients $m_2 = \text{diag}\{\sigma_Y, \sigma_Z\}, m_1 = \text{diag}\{(r/2) - a_Y, (r/2) - a_Z\}, m_0 = r\gamma\sigma_N(1, 1)$. Let $d_0 = -r^2\gamma^2\sigma_N^2$, $d_1 = (r - 2a_Y)^2/4 + d_0\sigma_Y^2$, $d_2 = (r - 2a_Z)^2/4 + d_0\sigma_Z^2$, and $d_4 = (d_1 d_2 - \sigma_Y^2\sigma_Z^2 d_0^2)^{1/2}$. The condition (7) ensures that $d_1 d_2 - \sigma_Y^2\sigma_Z^2 d_0^2 \geq 0$. We can solve $\tilde{v}$ in closed form:

$$v_{YZ} = \pm\sqrt{\frac{d_1 + d_2 \pm 2d_4}{[(d_1 - d_2)/d_0]^2 + 4\sigma_Y^2\sigma_Z^2}},$$

$$v_{YY} = \frac{1}{2\sigma_Y^2}\left[(r - 2a_Y) + d_0 v_{YZ}^{-1} + (d_1 - d_2)\Big/ d_3^{-1} v_{YZ}\right],$$

$$v_{ZZ} = \frac{1}{2\sigma_Z^2}\left[(r - 2a_Z) + d_0 v_{YZ}^{-1} - (d_1 - d_2) d_3^{-1} v_{YZ}\right],$$

$$v_{0Y} = v_{0Z} = 0 \quad \text{and} \quad v_{00} = r\gamma\lambda_0^s + (1/r)\left(\sigma_Y^2 v_{YY} + \sigma_Z^2 v_{ZZ}\right) + (2/r)(r - \rho - r\ln r).$$

Note that we have four different roots for $\tilde{v}$ corresponding to the four choices of v_{YZ}. We denote $\tilde{v}^-$ as the one corresponding to

$$v_{YZ} = \sqrt{\frac{d_1 + d_2 - 2d_4}{[(d_1 - d_2)/d_0]^2 + 4\sigma_Y^2\sigma_Z^2}}.$$

It is easy to check that the difference matrix between $\tilde{v}^-$ and any other root is negative definite. Thus, $\tilde{v}^-$ solves the value function $J_t = -e^{-\rho t - r\gamma W_t + (1/2)X_t' v X_t}$. Having solved $\tilde{v}$, we can easily solve λ^P from the market clearing condition and $v = \text{diag}\{v_{00}, \tilde{v}\}$. The equilibrium stock price is

$$S_t = [1/(r - a_G)]G_t + \tilde{\lambda}^S \tilde{X}_t,$$

where $\tilde{\lambda}^S = (\tilde{\lambda}_0^S, \lambda_Y^S, \lambda_Z^S)$ and $\tilde{X}_t = \text{stack}\{1, Y_t, Z_t\}$

$$\lambda_Y^S = \frac{-r\gamma\sigma_{DN}\left[v_{YZ}\sigma_Z^2 + (r - a_Z - \sigma_Z^2 v_{ZZ})\right]}{\left(r - a_Z - \sigma_Z^2 v_{ZZ}\right)\left(r - a_Y - \sigma_Y^2 v_{YY}\right) - v_{YZ}^2\sigma_Y^2\sigma_Z^2},$$

$$\lambda_Z^S = \frac{-r\gamma\sigma_{DN}\left[v_{YZ}\sigma_Y^2 + \left(r - a_Y - \sigma_Y^2 v_{YY}\right)\right]}{\left(r - a_Z - \sigma_Z^2 v_{ZZ}\right)\left(r - a_Y - \sigma_Y^2 v_{YY}\right) - v_{YZ}^2\sigma_Y^2\sigma_Z^2},$$

$$\lambda_0^S = -\gamma\left[\sigma_D^2 + \frac{1}{(r - a_G)^2}\sigma_G^2 + \lambda_Y^2\sigma_Y^2 + \lambda_Z^2\sigma_Z^2\right].$$

The optimal consumption policy is

$$c_t = rW_t - (1/2\gamma)\tilde{X}_t' v \tilde{X}_t - (1/\gamma)\ln r.$$

Using optimal consumption policy, we can solve the price for any traded security by

$$rP_t\,dt = E_t\left[\frac{u'(c_{t+dt}, t + dt)}{u'(c_t, t)}(dC_t + dP_t)\right],$$

where u' denotes investor's marginal utility, and $C_t = \int_0^t f(X_t, t)\,dt + b_C\,dw_s$ is the cumulative payoff for the security. Using Ito's Lemma, we can get the differential equation (18) for any price process $\{P_t\}$.