TEAL Physics and Mathematics Documentation

MIT Center for Educational Computer Initiatives

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This document provides descriptions of the physics and mathematics for many of the applets and visualizations created for the Technology Enabled Active Learning (TEAL)/Studio Physics Project at MIT and for related NSF and Foundation grants that preceded or followed that project.

Sample code is included, as appropriate. All units are SI unless otherwise indicated. In the translating formulas into code, factors of $\frac{\mu_o}{4\pi}$ and $\frac{1}{4\pi\varepsilon_o}$ are usually dropped, unless otherwise indicated.

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1 Supporting Mathematics

1.1 Special Functions

1.1.1 The Complete Elliptic Integral

The general complete elliptic integral is

$$cel(k_c, p, a, b) \equiv \int_0^{\pi/2} \frac{(a\cos^2\beta + b\sin^2\beta)\,d\beta}{(\cos^2\beta + p\sin^2\beta)\sqrt{\cos^2\beta + k_c{}^2\sin^2\beta}} \tag{1.1.1.1}$$

Some useful relations for *cel* are:

$$cel(k_c,1,1,1) = \int_0^{\pi/2} \frac{d\beta}{\sqrt{\cos^2\beta + k_c{}^2\sin^2\beta}} = \int_0^{\pi/2} \frac{d\beta}{\sqrt{1-k^2\sin^2\beta}} = K(k^2) \tag{1.1.1.2}$$

$$k^2 \equiv 1 - k_c{}^2 \tag{1.1.1.3}$$

$$cel(k_c,1,1,k_c) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\beta}\;d\beta = E(k^2) \tag{1.1.1.4}$$

where K and E are Legendre's standard forms of the complete elliptic integrals of the first and second kind. Some associated elliptic integrals are

$$D = \int_0^{\pi/2} \frac{\sin^2\beta\;d\beta}{\sqrt{1-k^2\sin^2\beta}} = \frac{K-E}{k^2} = cel(k_c,1,0,1) \tag{1.1.1.5}$$

$$B = \int_0^{\pi/2} \frac{\cos^2\beta\;d\phi}{\sqrt{1-k^2\sin^2\beta}} = \frac{E-k_c{}^2K}{k^2} = cel(k_c,1,1,0) \tag{1.1.1.6}$$

1.1.2 Code for Computing The Complete Elliptical Integral

```
package teal.math;

public class SpecialFunctions {
/**
   * Elliptic integrals:
   *    This algorithm for the calculation of the complete elliptic
   *    integral (CEI) is presented in papers by Ronald Bulirsch,
   *    Numerical Calculation of Elliptic Integrals and
```

```
 *    Elliptic Functions, Numerische Mathematik 7,
 *    78-90 (1965) and Ronald Bulirsch: Numerical Calculation
 *    of Elliptic Integrals and Elliptic Functions III,
 *    Numerische Mathematik 13,305-315 (1969).  The definition
 *    of the complete elliptic integral is given in equation (1.1.1.1)
 *    of the document " TEAL Physics and Mathematics Documentation "
 */

public static double ellipticIntegral(double kcc, double pp, double aa, double bb, double
accuracy) {
      double ca, kc, p, a, b, e, m, f, q, g;
      ca = accuracy;
      kc = kcc;
      p = pp;
      a = aa;
      b = bb;
      if ( kc != 0.0 )
      {
         kc = Math.abs(kc);
         e = kc;
         m = 1.0;
         if (p > 0.)
         {
                  p = Math.sqrt(p);
                  b = b/p;
         }
         else
         {
                  f = Math.pow(kc,2.0);
                  q = 1.-f;
                  g = 1.-p;
                  f = f-p;
                  q = q*(b-a*p);
                  p = Math.sqrt(f/g);
                  a = (a-b)/g;
                  b = -q/(p*Math.pow(g,2.0)) + a*p;
         }
         f = a;
         a = b/p + a;
         g = e/p;
         b = 2.0*(f*g + b);
         p = p + g;
         g = m;
         m = m + kc;

         while (Math.abs(g - kc) > g*ca)
```

```
            {
                kc = 2.0*Math.sqrt(e);
                e = kc*m;
                f = a;
                a = b/p + a;
                g = e/p;
                b = 2.0*(f*g + b);
                p = p + g;
                g = m;
                m = m + kc;
            }

            return (Math.PI / 2.)*(a*m + b)/(m*(m + p));

        }

        else
        {
            return 0.0;
        }
    }

}
```

1.2 Geometry

1.2.1 Vector Transformations For An EM Object With An Axis Of Symmetry

We frequently want to find the coordinates of an observation point in a "primed" coordinate system centered on an electromagnetic object that has an axis of symmetry, with the z prime axis along the axis of symmetry. This occurs, for example, when the expression for the field of that object takes on an especially simple form in this "primed" coordinate system (e.g., the field of a point magnetic dipole).

To get the coordinates of an arbitrary observation point in this primed coordinate system, we do the following. Let **M** be the symmetry axis of the electromagnetic object (for example, the magnetic dipole moment vector). Let $\mathbf{X}_{object}$ be the position of the object, and **X** the position of the observation point. Define the vectors

$$\hat{\mathbf{Z}}' = \mathbf{M}/M \qquad \mathbf{R} = \mathbf{X} - \mathbf{X}_{object} \qquad \mathbf{R}_{perp} = \mathbf{R} - (\mathbf{R}\cdot\hat{\mathbf{Z}}')\hat{\mathbf{Z}}' \qquad \hat{\boldsymbol{\rho}}' = \mathbf{R}_{perp}/R_{perp} \qquad (1.2.1.1)$$

Then if we compute $\hat{\boldsymbol{\rho}}'\cdot\mathbf{R}$ and $\hat{\mathbf{Z}}'\cdot\mathbf{R}$, we have the coordinates of our observation point in a frame in which the electromagnetic object is at the origin, the z prime axis is along the symmetry axis of the object, and the ρ prime axis is in the plane defined by the symmetry axis **M** and the direction to the observer **R**, and perpendicular to the symmetry axis. We then use these coordinates to calculate by means of relatively simple formulas the components of the field in this coordinate system, say $B_{\rho'}$ and $B_{z'}$. We then reconstruct the field in our "unprimed" original coordinate system using

$$\mathbf{B} = B_{z'}\,\hat{\mathbf{Z}}' + B_{\rho'}\,\hat{\boldsymbol{\rho}}' \tag{1.2.1.2}$$

1.3 Flux Functions

1.3.1 General Considerations for Axisymmetric Configurations

When studying field lines in the case of an axisymmetric configuration, i.e., when the vectors do not depend on the azimuth angle ϕ of the cylindrical coordinate system (ρ,ϕ, z), it is useful to consider their associated flux functions. To this end, consider two classes of vectors: *poloidal* vectors, say **V**, and *toroidal* vectors, say **W.** By definition, a poloidal vector **V** lies in the ρz plane, and thus has two components: V_ρ and V_z. In contrast, a toroidal vector **W** has only one component, W_ϕ, pointing along the azimuth ϕ, so that its field lines close on themselves. In other words, **W** is divergence free. If we assume in addition that the poloidal vector **V** is also divergence free, then it is easy to show that the curl of a toroidal vector generates a poloidal vector (and *vice versa*), viz.:

$$\mathbf{V} = \nabla\times\mathbf{W} = \nabla W_\phi \times\hat{\phi} + W_\phi \nabla\times\hat{\phi} = V_\rho\hat{\rho} + V_z\hat{z} \tag{1.3.1.1}$$

where

$$V_\rho = -\frac{\partial W_\phi(\rho,z)}{\partial z} \qquad\qquad V_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho W_\phi(\rho,z)\right] \tag{1.3.1.2}$$

We now define the scalar *flux function* $F(\rho,z)$ of **V** to be flux of **V** passing through a circle of radius ρ at height z concentric with the z-axis, e.g.

$$F(\rho,z) = \int_{surface}\mathbf{V}\cdot\mathbf{dA} = \int_{surface}\mathbf{V}\cdot\hat{\mathbf{z}}\;dA = \int_{surface}\left(\nabla\times\mathbf{W}\right)\cdot\hat{\mathbf{z}}\quad dA \tag{1.3.1.3}$$

Using Stokes Theorem, we can transform the surface integral to a line integral, giving

$$F(\rho,z) = \oint_{line} W_\phi\;\rho\;d\phi = 2\pi\,\rho\,W_\phi \tag{1.3.1.4}$$

A surface on which F is constant is an axially symmetrical shell containing the lines of force of **V**. Comparing equations (1.3.1.2) and (1.3.1.4) shows that **V** is related to its flux function F by

$$V_\rho = -\frac{1}{2\pi\rho}\frac{\partial F(\rho,z)}{\partial z} \qquad\qquad V_z = \frac{1}{2\pi\rho}\frac{\partial F(\rho,z)}{\partial\rho} \tag{1.3.1.5}$$

1.3.2 The Time Dependence of Field Lines In Magneto-quasi-statics

We discuss the concept of field line motion in magneto-quasi-statics, and how to define that motion in a physically meaningful, but not unique, way. Consider the following thought experiment. We have a solenoid carrying current provided by the *emf* of a battery. The axis of the solenoid is vertical. We place the entire 9.2ratus on a cart, and move the cart horizontally at a constant velocity **V**. Our intuition is that the magnetic field lines associated with the currents in the solenoid should move with their source, i.e., with the cart.

How do we make this intuition quantitative? First, we realize that in the laboratory frame there will be a "motional" electric field given by $\mathbf{E} = -\mathbf{V}\times\mathbf{B}$. We then imagine placing a low energy test electric charge in the magnetic field of the solenoid, at its center. The charge will gyrate about the field and the center of gyration will move in the laboratory frame because it $\mathbf{E}\times\mathbf{B}$ drifts ($\mathbf{v} = \mathbf{E}\times\mathbf{B}/B^2$) in the $-\mathbf{V}\times\mathbf{B}$ electric field. This $\mathbf{E}\times\mathbf{B}$ drift velocity is just **V**. That is, the test electric charge "hugs" the "moving" field line, moving at the velocity our intuition expects. In the more general case (e.g., two sources of field moving at different velocities), the motion we choose has the same physical basis. That is, the motion of a given field line is what we would observe in watching the motion of low energy test electric charges spread along that magnetic field line.

We also use this definition of the motion of file lines in situations that are not quasistatic, for example dipole radiation in the induction and radiation zones. In this case (but not in the quasistatic cases) the calculated motion of the field lines is non-physical, as their speed exceeds that of light in some regions. However, animations of the field line motion using the definition above are still useful. For example, the direction of the direction of field line motion so defined indicates the direction of energy flow.

To calculate field line motion consistent with our definition above, we need to insure that our velocity field is flux preserving (Stern, 1966; Vasyliunas, 1972; Rossi and Olbert, 1970). For any general vector field **G***(x,y,z,t)*, the rate of change of the flux of that field through an open surface *S* bounded by a contour *C* which moves with velocity **v***(x,y,z,t)* is given by

$$\frac{d}{dt}\int_S \mathbf{G}\cdot\mathbf{dA} = \int_S \frac{\partial\mathbf{G}}{\partial t}\cdot\mathbf{dA} + \int_S (\nabla\cdot\mathbf{G})\mathbf{v}\cdot\mathbf{dA} - \oint_C (\mathbf{v}\times\mathbf{G})\cdot\mathbf{dl} \tag{1.3.2.1}$$

If we apply this equation to **B***(x,y,z,t)* and use $\nabla\cdot\mathbf{B} = 0$ and $\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}$, we have

$$\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{dA} = -\oint_C (\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\mathbf{dl} \tag{1.3.2.2}$$

If we then define the motion of our contours so that the magnetic flux through the surfaces they bound is constant as a function of time, and consider circular contours and fields with azimuthal symmetry, then equation (1.3.2.2) guarantees that their motion satisfies $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$, which is the same as $\mathbf{v} = \mathbf{E} \times \mathbf{B} / B^2$, assuming that **v** and **B** are perpendicular. We can make this assumption since there is no meaning to the motion of a field line parallel to itself). This is just the drift velocity of low energy test electric monopoles that we refer to above. This definition of field line motion is not unique (see Vasyliunas, 1972).

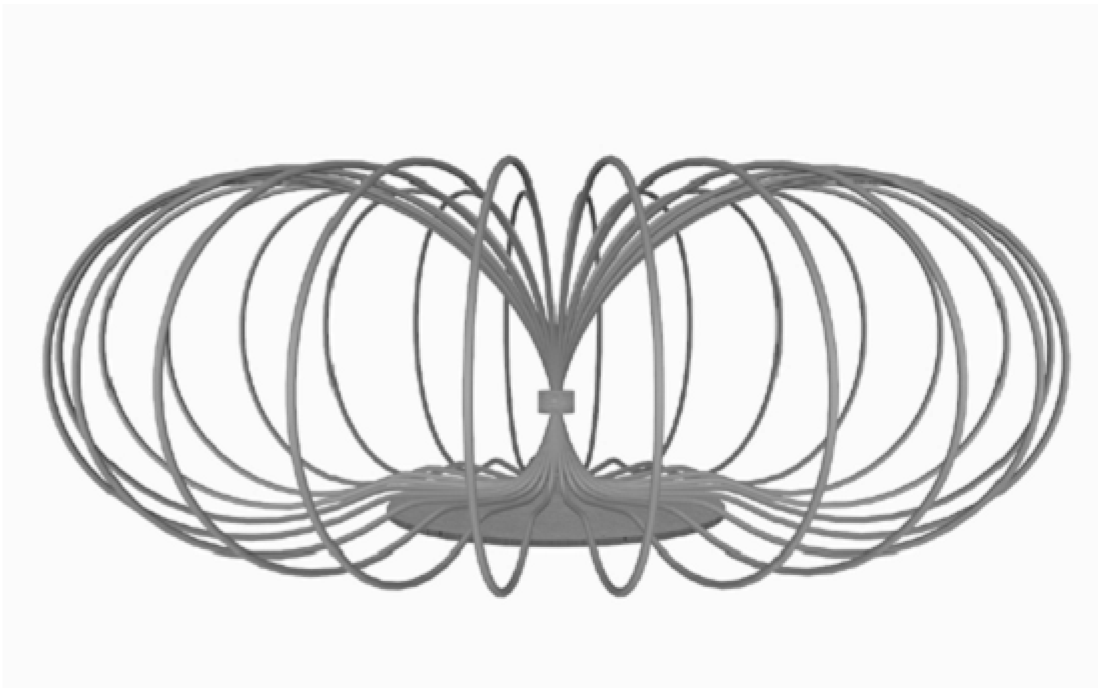

Figure 1.3-1: A magnet levitating above a disk with zero resistance.

Field Lines Originating From A Singularity: In the situation that our field lines originate from a singularity, constructing their time dependence is straightforward. Consider the motion of the field lines of a magnet levitating above a disk with zero resistance (Figure 1.3-1). The magnet is constrained to move only on the axis of the disk, and the dipole moment of the magnet is also constrained to be parallel to that axis. Eddy currents in the disk will repel the magnet, and at some point there will be a balance between the downward force of gravity and the upward force of repulsion. We then consider small displacements about this equilibrium position, which will be periodic. The field lines themselves are given by Davis and Reitz (1971), and have azimuthal symmetry. How do we trace the *motion* of a field line?

We do this by starting our integration very close to the magnet at a constant angle from the vertical axis, following a given field line out from that point. To animate a line, we use the same starting angle at every point in the oscillation. The field line traced out will be different when the magnet is at different distances from the disk. But consider the flux inside any open surface whose bounding contour is defined by the intersection of a horizontal plane and the field line when rotated azimuthally. This open surface will have constant flux inside it, since $\nabla \cdot \mathbf{B} = 0$. Since we always start from close to the singularity at the same angle, this constant flux will be the same for every instant of time. Since the left hand side of (1.3.2.2) is zero by construction, the right hand side must also be zero, and by symmetry the integrand must be zero as well. Therefore our field line motion as we have constructed it reflects the drift motion of low energy test electric charges spread along it.

Field Lines Not Originating From A Singularity: In this case the construction of the time evolution of the field lines is more complicated, and we use the flux functions defined above. Our flux function for the magnetic field

$$F(\rho, z, t) = \int_{surface} \mathbf{B}(\rho, z, t) \cdot \mathbf{dA} \tag{1.3.2.3}$$

(cf. 1.3.1.3) is now time-dependent. If we choose successive field lines as time evolves such that they have the same (constant) value of the flux function, then (1.3.2.2) is again satisfied, and again in symmetric situations the field line motion so defined is such that $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$, or equivalently, $\mathbf{v} = \mathbf{E} \times \mathbf{B} / B^2$. Another way of stating this is that if we look at isocontours of the flux function $F(\rho, z, t)$, then these isocontours trace out the time-dependent motion of the field lines.

Although the derivation that we have sketched above is elegant, it is perhaps also reassuring to do this in a manner that shows the same thing in more detail, although in a much clumsier way. We construct such a proof in Appendix 8.2.

As an example of this process, consider the time dependence of a field line when we have two rings of current (Figure 1.3-2) separated by a vertical distance of 10. The radius of the bottom ring is 10, with a dipole moment vector of 1.0 $\hat{\mathbf{z}}$, and the radius of the top ring is 5, with a dipole moment that varies in time. The innermost field line in Figure 1.3-2 corresponds to a dipole moment vector of the top ring of 0.1 $\hat{\mathbf{z}}$, at a flux function value of 0.5. The middle field line corresponds to the same flux function value of 0.5, but for a dipole moment vector of the top ring of 0.5 $\hat{\mathbf{z}}$. The outermost field line is again at the same value of the flux function, but for a dipole moment of the top ring of 1.0 $\hat{\mathbf{z}}$. The sequence of field lines is what we expect for the time evolution of this field line as the dipole moment vector of the top ring grows in time from 0.1 $\hat{\mathbf{z}}$ to 1.0 $\hat{\mathbf{z}}$. That is, this would be the path traced out by low energy charged particles spread along the field line as the dipole moment of the top ring grows in time.

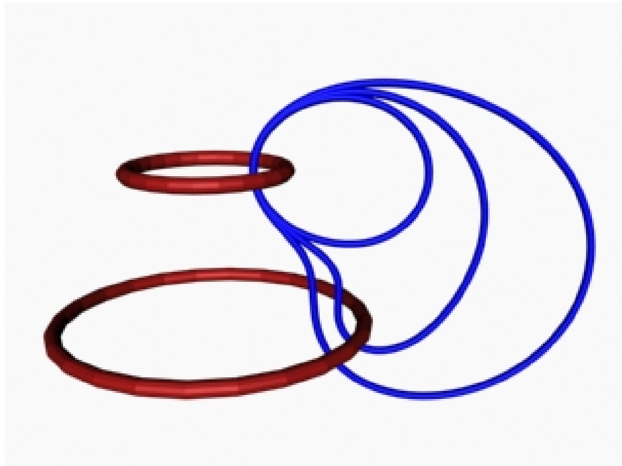

Figure 1.3-2: Isocontour levels of the flux functions of two rings as the dipole moment of the top ring increases.

2 Electrostatics

2.1 Two Dimensional Electrostatics (and Magnetostatics)

2.1.1 General Considerations

It is well known that in two Cartesian dimensions that solving potential problems in electrostatics due to a discrete number of line charges has many correspondences with the theory of analytic functions of a complex variable (Morse and Feshbach 1953). In particular, consider the analytic (except for a discrete number of singularities at the location of the line charges) function *G(Z)* of the complex variable $Z = x + iy$ (*Z* is *not* the spatial *z* coordinate), where *x* and *y* are the Cartesian coordinates of the two-dimensional problem. If we can find an *G(Z)* whose real part is the electrostatic potential Φ for the problem, then the electric field lines are given by the isocontours of the imaginary part of *G*.

For completeness, we sketch why this is true. Let $G(Z)=U(x,y)+\mathrm{i}\,V(x,y)$, where *U* and *V* are real functions of *x* and *y*. For *G(Z)* to be analytic at a point *Z*, its derivative must exist and be the same whether we approach the point *Z* in the complex plane along the x-axis or along the y-axis. That is, we must have

$$\frac{dG(Z)}{d(Z)} = \lim_{\Delta x\to 0}\frac{G(x+\Delta x, y) - G(x,y)}{\Delta x} = \lim_{i\Delta y\to 0}\frac{G(x, y+\Delta y) - G(x,y)}{i\Delta y} \tag{2.1.1.1}$$

Using the definition of *G* in terms of *U* and *V*, and equating real and imaginary parts in equation (2.1.1.1), we have

$$\left[\frac{\partial U(x,y)}{\partial x} = \frac{\partial V(x,y)}{\partial y}\right] \quad and \quad \left[\frac{\partial U(x,y)}{\partial y} = -\frac{\partial V(x,y)}{\partial x}\right] \tag{2.1.1.2}$$

from which it can easily be deduced that both *U* and *V* are solutions to Laplace's equation in two-dimensions, almost everywhere.

Now suppose we find an analytic (except for a discrete number of singularities) function *G(Z)* such that the real part of *G(Z)* is the solution to a two-dimensional electrostatic potential problem with discrete sources. That is, for this *G(Z)*, our potential satisfies $\mathrm{Re}[F] = U = \phi$. The electric field lines due to this potential are given by $\mathbf{E} = -\nabla\phi$. Consider the isocontours of $\mathrm{Im}[F] = V$. Let *Y(x)* be an isocontour of *V(x,y)*. Then the change in *V(x,y)* when we move along *Y(x)* must be zero. That is

$$\frac{dV(x,Y(x))}{dx} = \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}\frac{dY}{dx} = 0 \tag{2.1.1.3}$$

which means that

$$\frac{dY}{dx} = -\frac{\partial V}{\partial x} / \frac{\partial V}{\partial y} = \frac{\partial U}{\partial y} / \frac{\partial U}{\partial x} = \frac{\partial \phi}{\partial y} / \frac{\partial \phi}{\partial x} = \frac{E_y}{E_x} \tag{2.1.1.4}$$

where we have used equation (2.1.1.2) above to replace the partials of U with partials of V, the fact that $U = \phi$ by assumption, and $\mathbf{E} = -\nabla \phi$. Equation (2.1.1.4) is exactly what we require for a curve defining an electric field line. Thus if the real part of *G(Z)* is equal to the electrostatic potential, then the isocontours of the imaginary part of *G(Z)* are parallel to the electric field lines.

Note that the scalar function $V(x, y) = \operatorname{Im} F(Z)$ is in some cases related to the flux function we discussed for example in 1.3, as may be seen from the fact that

$$\mathbf{E} = \nabla \times \left[\operatorname{Im} F(Z)\, \hat{\mathbf{z}} \right] = \nabla \times \left[V(x, y)\ \hat{\mathbf{z}} \right] \tag{2.1.1.5}$$

For example, when the system is symmetric about the y-axis, the imaginary part of *G(Z)* is one half of the flux of **E** passing through a rectangle of width $2x$ in the x-direction, located at height y on the y-axis, and centered on that axis, per unit length in the z-direction. More importantly, as can be shown by *explicit* construction, the time evolution of the electric field lines in electro-quasi-statics are the same as the time evolution of the isocontours of the imaginary part of *G(Z)*. We now give three useful examples of these functions in two-dimensional electrostatics, and one related example in two-dimensional magnetostatics.

2.1.2 The 2D Electrostatic (or Magnetostatic) Dipole and Line of 2D Dipoles

Consider a two-dimensional electric dipole, that is a dipole formed by taking a line charge $+\lambda$ and a line charge $-\lambda$ a vector distance **d** apart, with **d** pointing from the negative to the positive line charge. We let **d** go to zero and λ go to infinity in such a way that the product

$$\mathbf{p} = \lambda \mathbf{d} \tag{2.1.2.1}$$

goes to a constant. The quantity **p** is the two-dimensional electric dipole vector. The electric potential for a two dimensional electric dipole, assuming the dipole is at the origin, is

$$\Phi = \frac{1}{2\pi\varepsilon_o} \frac{\mathbf{p} \cdot \mathbf{X}}{(x^2 + y^2)} = \frac{1}{2\pi\varepsilon_o} \left[\frac{p_x x + p_y y}{(x^2 + y^2)} \right] \tag{2.1.2.2}$$

where $\mathbf{X} = x\,\mathbf{i} + y\,\mathbf{j}$. We can write the potential in equation (2.1.2.2) as

$$\Phi = \mathrm{Re}\left[\frac{1}{2\pi\varepsilon_o}\frac{(p_x - ip_y)Z}{x^2+y^2}\right] \tag{2.1.2.3}$$

Therefore our electric field lines for this problem are isocontours of

$$V(x,y) = \mathrm{Im}\left[\frac{1}{2\pi\varepsilon_o}\frac{(p_x - ip_y)Z}{x^2+y^2}\right] = \frac{1}{2\pi\varepsilon_o}\frac{p_x\, y - p_y\, x}{x^2+y^2} = \frac{1}{2\pi\varepsilon_o}\frac{[\mathbf{p}\times\mathbf{X}]_z}{x^2+y^2} \tag{2.1.2.4}$$

These expressions also holds for a two dimensional magnetic dipole.

These expressions can be generalized to a line of 2D dipoles with a given orientation. Consider the following configuration. A line of dipoles extends along the x-axis from *–d/2* to *d/2*. Along this line there is a line of two 2D dipoles with dipole moment per unit length $\mathbf{p} = p_x\hat{\mathbf{i}} + p_y\hat{\mathbf{j}}$. The electric potential for this situation can be derived by integration of (2.1.2.2) to yield

$$\Phi(x,y) = -\frac{1}{2}p_x \ln\left[\frac{(x-d/2)^2+y^2}{(x+d/2)^2+y^2}\right] + p_y \arctan\left[\frac{dy}{x^2+y^2-d^2/4}\right] \tag{2.1.2.5}$$

and the electric field components can be computed by taking the negative gradient of this function. The field lines for this configuration are isocontours of the following function, by integration of (2.1.2.4).

$$V(x,y) = p_x \arctan\left[\frac{dy}{x^2+y^2-d^2/4}\right] + \frac{1}{2}p_y \ln\left[\frac{(x-d/2)^2+y^2}{(x+d/2)^2+y^2}\right] \tag{2.1.2.6}$$

2.1.3 Constant Electric Field

As a second example, consider the electric field lines of a constant field $\mathbf{E}^o$. The electrostatic potential in this case is

$$\Phi = -\mathbf{E^o}\cdot\mathbf{X} = \mathrm{Re}\left[-\left(E^{\mathbf{o}}{}_x x + E^{\mathbf{o}}{}_y y\right)\right] = \mathrm{Re}\left[-\left(E^{\mathbf{o}}{}_x - iE^{\mathbf{o}}{}_y\right)Z\right] \tag{2.1.3.1}$$

where again $\mathbf{X} = x\,\mathbf{i} + y\,\mathbf{j}$, so that the field lines are given by the isocontours of

$$V(x,y) = \mathrm{Im}[-\left(E^{\mathbf{o}}{}_x - iE^{\mathbf{o}}{}_y\right)Z] = -E^{\mathbf{o}}{}_x y + E^{\mathbf{o}}{}_y x = [\mathbf{X}\times\mathbf{E}^o]_z \tag{2.1.3.2}$$

If we want to find the field lines of a two-dimensional dipole in a constant field, we simply add the two functions above appropriate to the two potentials to get the appropriate function for this case.

2.1.4 Line of Charge

Finally, for our third example, consider the electric field lines of a line charge. The electrostatic potential for this case is

$$\Phi = -\frac{\lambda}{2\pi\varepsilon_o}\ln(\sqrt{x^2+y^2}) = \text{Re}\left[-\frac{\lambda}{2\pi\varepsilon_o}\ln(Z)\right] \tag{2.1.4.1}$$

so that the field lines are isocontours of

$$V(x,y) = \text{Im}\left[-\frac{\lambda}{2\pi\varepsilon_o}\ln(Z)\right] = -\frac{\lambda}{2\pi\varepsilon_o}\theta \tag{2.1.4.2}$$

When there are two or more line charges present, one has to be careful using the function in equation (2.1.4.2) because of the branch cut in θ as θ runs from 0 to 2π.

2.1.5 Line of Current

Although strictly speaking we are only considering electrostatics in this section, we point out that we can do much the same thing in magneto-quasi-statics in two dimensions. If the current density **J** is zero except at discrete locations (that is, we have a finite number of line currents running in the z-direction), then $\nabla \times \mathbf{B} = 0$ almost everywhere, and we can write for regions away from the discrete sources that

$$\mathbf{B} = -\nabla\Phi_B \tag{2.1.5.1}$$

where ϕ_B is the magneto-quasi-static potential. Since $\nabla \cdot \mathbf{B} = 0$, we have

$$\nabla^2\Phi_B = 0 \tag{2.1.5.2}$$

In two dimensions, finding the magnetic field lines in magnetostatic problems with a discrete number of line currents can be aided by using the theory of complex variables, as above. In particular, consider the analytic (except for a discrete number of singularities at the sources) function *G(Z)* of the complex variable $Z = x + iy$, where x and y are the cartesian coordinates of the two-dimensional problem. Just as above, and for the same reasons, if we can find an *G(Z)* whose real part is the magnetostatic potential ϕ_B for the problem, then the magnetic field lines are given by the isocontours of the imaginary part of *G(Z)*.

For example, consider the magnetic field of a line current at the origin. The magnetostatic potential for this case is

$$\Phi_B = -\frac{\mu_o I}{2\pi}\theta = \mathrm{Re}\left[i\frac{\mu_o I}{2\pi}\ln(Z)\right] \quad (2.1.5.3)$$

since

$$\mathbf{B} = -\nabla\Phi_B = \hat{\boldsymbol{\theta}}\frac{\mu_o I}{2\pi}\frac{1}{\sqrt{x^2+y^2}}\frac{\partial}{\partial\theta}\theta = \hat{\boldsymbol{\theta}}\frac{\mu_o I}{2\pi\rho} \quad (2.1.5.4)$$

where $\hat{\boldsymbol{\theta}}$ is the unit vector in the azimuthal direction, right-handed about the z axis. Thus the field lines are given by the isocontours of

$$V(x,y) = \mathrm{Im}\left[i\frac{\mu_o I}{2\pi}\ln(Z)\right] = \frac{\mu_o I}{2\pi}\ln(r) \quad (2.1.5.5)$$

2.2 Three Dimensional Electrostatics

2.2.1 Point Charge in A Uniform Electric Field

Suppose we have a point charge at the origin of a spherical polar coordinate system. The electric field of this point charge everywhere except at the location of the charge can be written as

$$\mathbf{E}_{point}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_o}\nabla\mathbf{x}\left\{\frac{(1-\cos\theta)}{\sin\theta}\frac{1}{r}\hat{\phi}\right\} \quad (2.2.1.1)$$

The constant electric field in the z-direction can be written as

$$\mathbf{E}(\mathbf{r},t) = -E_o\,\nabla\mathbf{x}\left\{\frac{1}{2}(r\sin\theta)\,\hat{\phi}\right\} \quad (2.2.1.2)$$

The electrostatic flux function (cf equation (1.3.1.4)) can therefore be written as

$$F(r,\theta) = \frac{q}{2\varepsilon_o}\left(1-\cos\theta_{charge}\right) - \pi\,E_o\,\left(r_{charge}\sin\theta_{charge}\right)^2 \quad (2.2.1.3)$$

2.2.2 Point Charge Being Charged By a Line Current

We consider the situation where we have a point charge at rest at the origin of our coordinate system with a charge $Q(t)$ which is varying with time. The increasing charge is being supplied with current by a line current which is arranged along the $-z$ axis, carrying current $I = dQ/dt$. In the quasi-electro-static approximation, if $\hat{\mathbf{r}}$ is the spherical polar unit vector, the electric field is given by

$$\mathbf{E}(\rho,z,t)=\hat{\mathbf{r}}\frac{Q(t)}{4\pi\,\varepsilon_o\left(z^2+\rho^2\right)} \tag{2.2.2.1}$$

and the magnetic field is given by

$$\mathbf{B}(\rho,z,t)=\hat{\phi}\frac{\mu_o I}{4\pi\,\rho}\left[1-\frac{z}{\left(z^2+\rho^2\right)^{1/2}}\right] \tag{2.2.2.2}$$

We can derive the expression for B given above by considering the line integral of **B** around a circle of radius ρ centered on the z-axis a distance z up that axis, and applying Ampere's Law including the displacement current term.

The **ExB** magnetic monopole drift velocity in these crossed **E** and **B** fields is given by (θ is the spherical polar angle)

$$\mathbf{V}_{d,B}(\rho,z,t)=c^2\frac{\mathbf{E}\times\mathbf{B}}{E^2}=-\hat{\boldsymbol{\theta}}\frac{r}{Q}\frac{dQ}{dt}\left[\frac{1-\cos\theta}{\sin\theta}\right] \tag{2.2.2.3}$$

The field lines are radial, and since the velocity of the field lines are given by the equation above, we can write for the θ dependence of a given field line that

$$r\frac{d\theta}{dt}=-\frac{r}{Q}\frac{dQ}{dt}\left[\frac{1-\cos\theta}{\sin\theta}\right] \tag{2.2.2.4}$$

This equation can be integrated to show that the time dependence of the angle θ denoting a particular radial field line is given by

$$Q(t)\left(1-\cos\theta(t)\right)=const \tag{2.2.2.5}$$

This is not surprising given the flux function in (2.2.1.3) for the point charge.

3 Magnetostatics

3.1 Magnetic Field Lines Of A Point 3D Dipole

3.1.1 The Vector Potential

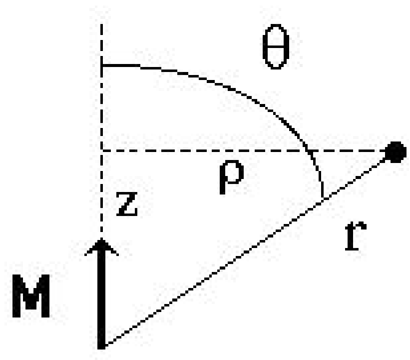

Figure 3.1-1: A Point 3D Magnetic Dipole

For a point 3D dipole with dipole moment **M,** the vector potential **A** is given by

$$\mathbf{A}(\rho,z)=\frac{\mu_o}{4\pi}\frac{\mathbf{X}\times\mathbf{M}}{X^3}=\hat{\phi}\,\frac{\mu_o}{4\pi}\frac{M\sin\theta}{z^2+\rho^2}=\hat{\phi}\,\frac{\mu_o M}{4\pi}\frac{\rho}{\left(z^2+\rho^2\right)^{3/2}} \tag{3.1.1.1}$$

3.1.2 Components of the Magnetic Field

$$\mathbf{B}(\rho,z)=\nabla\times\left[A(\rho,z)\hat{\phi}\right] \tag{3.1.2.1}$$

So

$$B_\rho(\rho,z)=-\frac{\partial}{\partial z}\left[\mathrm{A}(\rho,\mathrm{z})\right]=\frac{\mu_o}{4\pi}M\frac{3\rho z}{(z^2+\rho^2)^{5/2}} \tag{3.1.2.2}$$

and

$$B_z(\rho,z)=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\mathrm{A}(\rho,\mathrm{z})\right]=\frac{\mu_o}{4\pi}M\frac{(2z^2-\rho^2)}{(z^2+\rho^2)^{5/2}} \tag{3.1.2.3}$$

3.1.3 The Flux Function

The flux function for a magnetic dipole (cf. 1.3.1.4 and 3.1.1.1) is

$$F(\rho,z)=\frac{\mu_o}{4\pi}\frac{2\pi M\ \rho^2}{\left(z^2+\rho^2\right)^{3/2}} \tag{3.1.3.1}$$

3.2 Magnetic Field Lines Of A Circular Loop Of Current

3.2.1 The Vector Potential

Consider a loop with radius a carrying current I, with magnetic dipole moment $M = \pi a^2 I$. The vector magnetic potential **A** can be written as (Jackson, Pgs. 178 - 179)

$$\mathbf{A}(\rho, z) = \hat{\phi}\, A(\rho, z) = \hat{\phi}\, \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{\cos\phi \, d\phi}{\sqrt{a^2 + z^2 + \rho^2 - 2a\rho\cos\phi}} \tag{3.2.1.1}$$

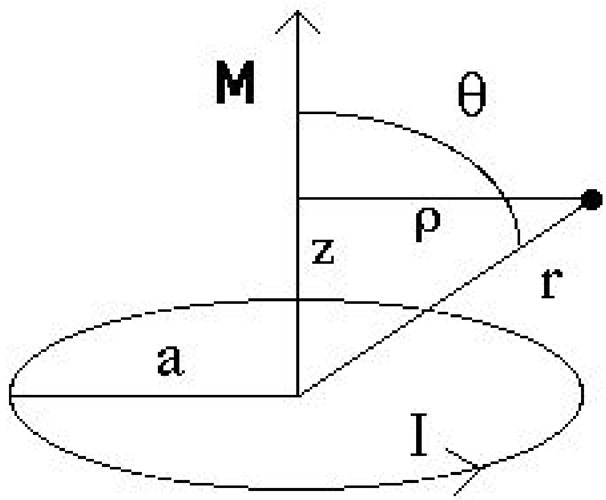

Figure 3.2-1: Loop of current

where ϕ is the azimuthal angle about the axis of the loop, and ρ and z are cylindrical coordinates. If we change variables using $\phi = 2\beta + \pi$, and define the normalized distances $z' \equiv z/a$ and $\rho' \equiv \rho/a$, then this equation becomes

$$A(\rho, z) = -\frac{I\mu_0}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{\cos 2\beta \, d\beta}{\sqrt{1^2 + z'^2 + \rho'^2 + 2\rho'\cos 2\beta}} \tag{3.2.1.2}$$

which (using $\cos 2\beta = \cos^2\beta - \sin^2\beta$) can be written as

$$A(\rho, z) = \frac{I\mu_0}{\pi\sqrt{z'^2 + (1+\rho')^2}} \int_0^{\pi/2} \frac{(\sin^2\beta - \cos^2\beta)\, d\beta}{\sqrt{\cos^2\beta + k_c{}^2 \sin^2\beta}} \tag{3.2.1.3}$$

where

$$k_c{}^2 \equiv \frac{z'^2 + (1-\rho')^2}{z'^2 + (1+\rho')^2} \tag{3.2.1.4}$$

For future use, we also define

$$k^2 \equiv 1 - k_c{}^2 = \frac{4\rho'}{z'^2 + (1+\rho')^2} \tag{3.2.1.5}$$

With this definition, and using the definition of *cel* given in (1.1.1.1), we have

$$A(\rho, z) = \frac{\mu_0}{4\pi} \frac{2M}{\pi\, a^2} \frac{k}{\rho'^{1/2}} cel(k_c, 1, -1, 1) \tag{3.2.1.6}$$

To show the correspondence between this expression and the expression above in (3.1.1.1) for a point dipole, we take the limit that we are far from the ring, which corresponds to the limit that $k^2 << 1$. In this limit,

$$cel(k_c, 1, -1, 1) = \frac{\pi}{16} k^2 \tag{3.2.1.7}$$

and using this expression it can be shown that (3.2.1.6) reduces to (3.1.1.1).

3.2.2 Components of the Magnetic Field and the Electric Field

The components of the magnetic field are given by

$$\mathbf{B}(\rho, z) = \nabla \times \left\lfloor A(\rho, z)\,\hat{\phi} \right\rfloor \tag{3.2.2.1}$$

The radial component is thus (using (3.2.1.1))

$$B_\rho(\rho, z) = -\frac{\partial}{\partial z}\left[A(\rho, z)\right] = -\frac{\partial}{\partial z}\left[\frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{\cos\phi \; d\phi}{\sqrt{a^2 + z^2 + \rho^2 - 2a\rho\cos\phi}}\right] \tag{3.2.2.2}$$

which after some manipulation can be written as

$$B_\rho(\rho, z) = \frac{I\mu_0 z'}{\pi a\left[z'^2 + (1+\rho')^2\right]^{3/2}} \int_0^{\pi/2} \frac{(\sin^2\beta - \cos^2\beta)\; d\beta}{\left(\cos^2\beta + k_c{}^2 \sin^2\beta\right)^{3/2}} \tag{3.2.2.3}$$

Using our definition of *cel* in (1.1.1.1) and (3.2.1.4), and $M = \pi a^2 I$, we can write this as

$$B_\rho(\rho, z) = \frac{\mu_0}{4\pi} \frac{M}{\pi a^3} \frac{z' k^3}{\rho'^{3/2}} \frac{1}{2} cel(k_c, k_c{}^2, -1, 1) \tag{3.2.2.4}$$

Now, the z component of **B** is given by

$$B_z(\rho,z)=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho A(\rho,z)\right]=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\frac{\mu_0\rho Ia}{4\pi}\int_0^{2\pi}\frac{\cos\phi\,d\phi}{\sqrt{a^2+z^2+\rho^2-2a\rho\cos\phi}}\right] \quad (3.2.2.5)$$

which after some manipulation can be written as

$$B_z(\rho,z)=\frac{\mu_0}{4\pi}\frac{M}{\pi a^3}\frac{k}{\rho'^{3/2}}\left\{cel(k_c,1,-1,1)+\frac{1}{2}\left[1+k_c^{\,2}-\left(1-k_c^{\,2}\right)\rho'\right]cel(k_c,k_c^{\,2},-1,1)\right\} \quad (3.2.2.6)$$

To show the correspondence between these expressions and the expressions above in (3.1.1), we need to take the limit that we are far from the ring, which corresponds to the limit that $k^2 << 1$. In this limit, (3.2.1.7) is true, and we also have in this limit that

$$cel(k_c,k_c^{\,2},-1,1)=\frac{3\pi}{16}k^2 \quad (3.2.2.7)$$

Using these limiting expressions it can be shown that our components in (II.B.2.4) and (II.B.2.6) reduce to those of a 3D point dipole given in (II.A.2) far from the loop.

A useful limiting form for $A(\rho,z)$ when k^2 is small is

$$A(\rho,z)=\frac{\mu_0}{4\pi}\frac{Ia}{\sqrt{a^2+z^2+\rho^2+2a\rho}}\frac{k^2}{4} \quad (3.2.2.8)$$

In a situation where we have a ring of current moving with velocity **v** and/or with a time changing current in the ring, the non-relativistic expressions for the magnetic field are just those given above, and the electric field is given by the motional electric field and the induced electric field, e.g.

$$\mathbf{E}=-\mathbf{v}\times\mathbf{B}-\frac{\partial\mathbf{A}}{\partial t}=-\mathbf{v}\times\mathbf{B}-\frac{\mu_0}{4\pi}\frac{dI}{dt}\frac{a}{\sqrt{a^2+z^2+\rho^2+2a\rho}}\frac{k^2}{4}\hat{\phi} \quad (3.2.2.9)$$

3.2.3 The Flux Function

The flux function for a circular loop of wire is just (see 1.3.1.4 and 3.2.1.6)

$$F(\rho,z)=\frac{\mu_0}{4\pi}\frac{4M\,k}{a}\rho'^{1/2}\,cel(k_c,1,-1,1) \quad (3.2.3.1)$$

and if we use (3.2.1.5) and (3.2.1.7) we see that the above equation reduces to (3.1.3.1) for a point dipole when we are far from the loop. It is instructive to plot this function in the plane of the loop, from the center of the loop to close to its radius.

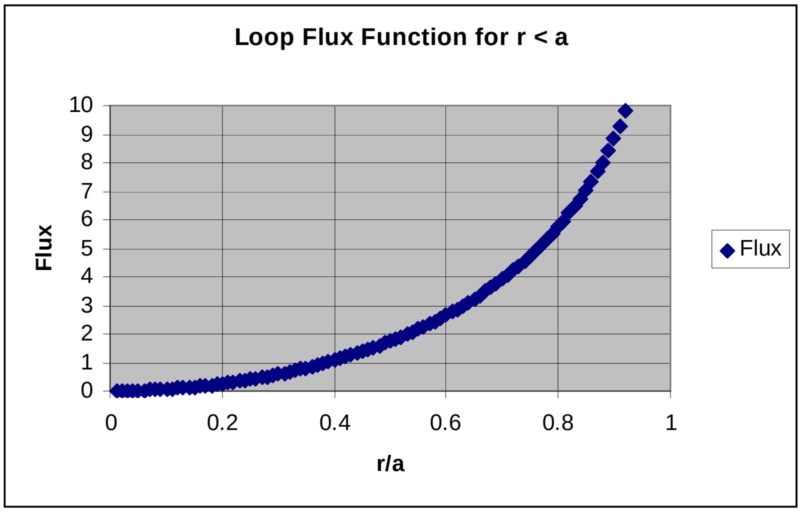

Figure 3.2-2: Loop flux function for *r* < *a*

Figure 3.2-2 gives such a plot, for a dipole moment vector of unity and a radius of unity. The values of the flux function scale inversely as loop radius for a fixed value of **M**. That is, as the radius increases, the value of the flux function decreases in proportion, at a given value of r/a.

We also show in Figure 3.2-3 the dependence of the flux function on radius in the plane of the loop for $r > a$. Figure 3.2-4 shows the field line with a flux function value of 2.0, for a loop with unit dipole moment and unit radius. This field line crosses the plane of the loop at $r = 0.531$ and also at $r = 3.26$, as we would expect given the flux curves in Figures 3.2-2 and 3.2-3.

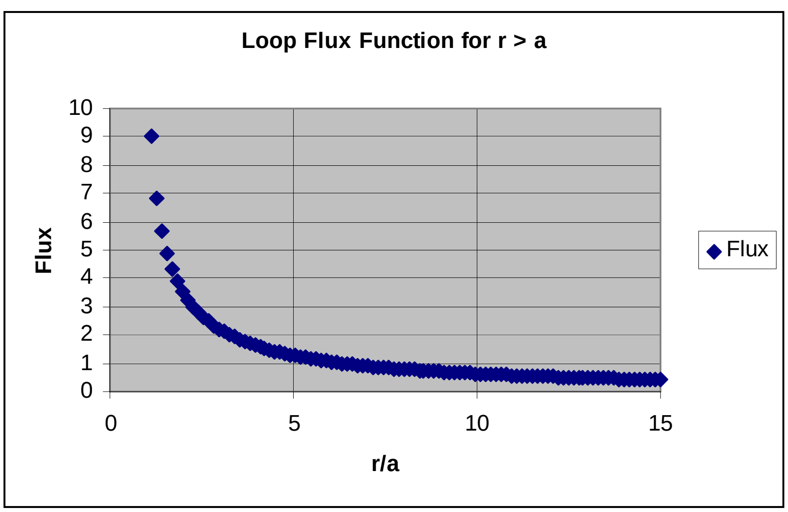

Figure 3.2-3: Loop flux function for *r* > *a*.

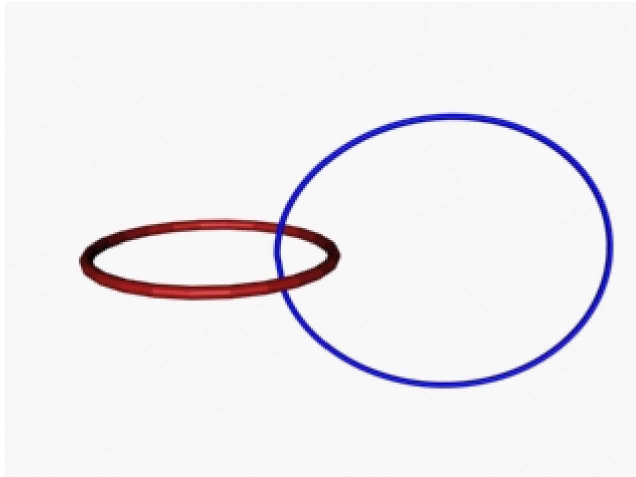

Figure 3.2-4: Field line for a loop with a flux function value of 2.0.

3.3 Magnetic Field of a Line of 3D Magnetic Dipoles

3.3.1 The Potential of a 3D Dipole

The potential and field of a single 3D dipole at the origin with dipole moment **m** is given by

$$\Phi_M = \frac{\mathbf{m}\cdot\mathbf{n}}{r^2} = \frac{\mathbf{m}\cdot\mathbf{r}}{r^3} \qquad \mathbf{B} = -\nabla\Phi_M = \frac{3\mathbf{n}\left(\mathbf{m}\cdot\mathbf{n}\right)-\mathbf{m}}{r^3} \tag{3.3.1.1}$$

3.3.2 The Potential of a Line of 3D Dipoles

We assume the line starts at the origin and lies along the x-axis, ending at $x = L$. We have

$$\Phi_M = \int_o^L \frac{\left(m_x(x-x') + m_y y + m_z z\right)}{\left((x-x')^2 + y^2 + z^2\right)^{3/2}}\, dx' \tag{3.3.2.1}$$

$$\Phi_M = m_x \int_o^L \frac{(x-x')\,dx'}{\left((x-x')^2 + y^2 + z^2\right)^{3/2}} + \left(m_y y + m_z z\right)\int_o^L \frac{dx'}{\left((x-x')^2 + y^2 + z^2\right)^{3/2}} \tag{3.3.2.2}$$

$$\int_o^L \frac{(x-x')\,dx'}{\left((x-x')^2 + y^2 + z^2\right)^{3/2}} = \left.\frac{1}{\left((x-x')^2 + y^2 + z^2\right)^{1/2}}\right|_0^L = \frac{1}{\left((x-L)^2 + y^2 + z^2\right)^{1/2}} - \frac{1}{\left(x^2 + y^2 + z^2\right)^{1/2}} \tag{3.3.2.3}$$

$$\int_o^L \frac{dx'}{\left((x-x')^2+y^2+z^2\right)^{3/2}} = \frac{1}{\left(y^2+z^2\right)}\int_o^L \frac{dx'/\left(y^2+z^2\right)^{1/2}}{\left(\frac{(x-x')^2}{y^2+z^2}+1\right)^{3/2}} \tag{3.3.2.4}$$

$$\tan\eta = (x-x')/\left(y^2+z^2\right)^{1/2} \qquad \frac{d\eta}{\cos^2\eta} = -\frac{dx'}{\left(y^2+z^2\right)^{1/2}} \tag{3.3.2.5}$$

$$\int_o^L \frac{dx'}{\left((x-x')^2+y^2+z^2\right)^{3/2}} = -\frac{1}{\left(y^2+z^2\right)}\int_{\tan^{-1}x/\left(y^2+z^2\right)^{1/2}}^{\tan^{-1}(x-L)/\left(y^2+z^2\right)^{1/2}} \frac{d\eta}{\cos^2\eta\left(\tan^2\eta+1\right)^{3/2}} \tag{3.3.2.6}$$

$$\int_o^L \frac{dx'}{\left((x-x')^2+y^2+z^2\right)^{3/2}} = -\frac{1}{\left(y^2+z^2\right)}\int_{\tan^{-1}x/\left(y^2+z^2\right)^{1/2}}^{\tan^{-1}(x-L)/\left(y^2+z^2\right)^{1/2}} \cos\eta d\eta = -\frac{\sin\eta}{\left(y^2+z^2\right)}\Bigg|_{\tan^{-1}x/\left(y^2+z^2\right)^{1/2}}^{\tan^{-1}(x-L)/\left(y^2+z^2\right)^{1/2}} \tag{3.3.2.7}$$

$$\sin\eta = \frac{1}{\sqrt{1+1/\eta^2}} = \frac{\eta}{\sqrt{1+\eta^2}} \tag{3.3.2.8}$$

$$\int_o^L \frac{dx'}{\left((x-x')^2+y^2+z^2\right)^{3/2}} = -\frac{1}{\left(y^2+z^2\right)}\left[\frac{(x-L)}{\sqrt{(x-L)^2+y^2+z^2}} - \frac{x}{\sqrt{x^2+y^2+z^2}}\right] \tag{3.3.2.9}$$

So inserting (3.3.2.9) and (3.3.2.3) into (3.3.2.2)

$$\begin{aligned}\Phi_M &= m_x\left[\frac{1}{\left((x-L)^2+y^2+z^2\right)^{1/2}} - \frac{1}{\left(x^2+y^2+z^2\right)^{1/2}}\right] \\ &\quad -\frac{\left(m_y y+m_z z\right)}{\left(y^2+z^2\right)}\left[\frac{(x-L)}{\sqrt{(x-L)^2+y^2+z^2}} - \frac{x}{\sqrt{x^2+y^2+z^2}}\right]\end{aligned} \tag{3.3.2.10}$$

To check limits, let L go to zero. Then

$$\begin{aligned}\Phi_M &= -m_x L\frac{\partial}{\partial x}\left[\frac{1}{\left(x^2+y^2+z^2\right)^{1/2}}\right] + \frac{\left(m_y y+m_z z\right)L}{\left(y^2+z^2\right)}\frac{\partial}{\partial x}\left[\frac{x}{\sqrt{x^2+y^2+z^2}}\right] \\ &= \left[\frac{+m_x L\,x}{\left(x^2+y^2+z^2\right)^{3/2}}\right] + \frac{\left(m_y y+m_z z\right)L}{\left(y^2+z^2\right)}\left[\frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{x^2}{\left(x^2+y^2+z^2\right)^{3/2}}\right] \\ &= \left[\frac{+m_x L\,x}{\left(x^2+y^2+z^2\right)^{3/2}}\right] + \frac{\left(m_y y+m_z z\right)L}{\left(y^2+z^2\right)\left(x^2+y^2+z^2\right)^{3/2}}\left[x^2+y^2+z^2-x^2\right]\end{aligned} \tag{3.3.2.11}$$

$$\Phi_M = \frac{\left(m_x\, x + m_y y + m_z z\right) L}{\left(x^2 + y^2 + z^2\right)^{3/2}} \tag{3.3.2.12}$$

In the limit that L goes to infinity, we have

$$\Phi_M = -\left[\frac{m_x}{\left(x^2 + y^2 + z^2\right)^{1/2}}\right] + \frac{\left(m_y y + m_z z\right)}{\left(y^2 + z^2\right)}\left[\frac{x}{\sqrt{x^2 + y^2 + z^2}} + 1\right] \tag{3.3.2.13}$$

We can generalize this to arbitrary orientations of the line. If **t** is the unit vector along the direction of the line, and the line begins at the origin, then

$$\Phi_M = -\left[\frac{\mathbf{m}\cdot\hat{\mathbf{t}}}{r}\right] + \frac{\left(\mathbf{m}\cdot\mathbf{r} - \hat{\mathbf{t}}\left(\hat{\mathbf{t}}\cdot\mathbf{r}\right)\left(\hat{\mathbf{t}}\cdot\mathbf{m}\right)\right)}{\left|\left(\mathbf{r} - \hat{\mathbf{t}}\left(\hat{\mathbf{t}}\cdot\mathbf{r}\right)\right)\right|^2}\left[\frac{\hat{\mathbf{t}}\cdot\mathbf{r}}{r} + 1\right] \tag{3.3.2.14}$$

3.3.3 The Field of a Line of 3D Dipoles along the x-axis

We go back to the expression given in (3.3.2.13) to get the components of the field.

$$B_x = -\frac{\mathbf{m}\cdot\mathbf{r}}{r^3} \tag{3.3.3.1}$$

$$\begin{aligned} B_y &= -\frac{\partial}{\partial y}\left\{-\left[\frac{m_x}{\left(x^2 + y^2 + z^2\right)^{1/2}}\right] + \frac{\left(m_y y + m_z z\right)}{\left(y^2 + z^2\right)}\left[\frac{x}{\sqrt{x^2 + y^2 + z^2}} + 1\right]\right\} \\ &= -\frac{m_x y}{\left(x^2 + y^2 + z^2\right)^{3/2}} - \left[\frac{x}{\sqrt{x^2 + y^2 + z^2}} + 1\right]\left[\frac{m_y}{\left(y^2 + z^2\right)} - \frac{2\left(m_y y + m_z z\right) y}{\left(y^2 + z^2\right)^2}\right] \\ &\quad + \frac{\left(m_y y + m_z z\right)}{\left(y^2 + z^2\right)}\frac{x\, y}{\left(x^2 + y^2 + z^2\right)^{3/2}} \end{aligned} \tag{3.3.3.2}$$

$$B_y = -\frac{m_x y}{r^3} - \left[\frac{x}{r} + 1\right]\frac{1}{\left(y^2 + z^2\right)^2}\left[m_y\left(z^2 - y^2\right) - 2 m_z z y\right] + \frac{\left(m_y y + m_z z\right)}{\left(y^2 + z^2\right)}\frac{x\, y}{r^3} \tag{3.3.3.3}$$

$$B_z = -\frac{\partial}{\partial z}\left\{-\left[\frac{m_x}{\left(x^2+y^2+z^2\right)^{1/2}}\right]+\frac{\left(m_y y+m_z z\right)}{\left(y^2+z^2\right)}\left[\frac{x}{\sqrt{x^2+y^2+z^2}}-1\right]\right\}$$

$$= -\frac{m_x z}{\left(x^2+y^2+z^2\right)^{3/2}}-\left[\frac{x}{\sqrt{x^2+y^2+z^2}}+1\right]\left[\frac{m_z}{\left(y^2+z^2\right)}-\frac{2\left(m_y y+m_z z\right)z}{\left(y^2+z^2\right)^2}\right] \qquad (3.3.3.4)$$

$$+\frac{\left(m_y y+m_z z\right)}{\left(y^2+z^2\right)}\frac{x\,z}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$B_z = -\frac{m_x z}{r^3}-\left[\frac{x}{r}+1\right]\frac{1}{\left(y^2+z^2\right)^2}\left[m_z\left(y^2-z^2\right)-2m_y yz\right]+\frac{\left(m_y y+m_z z\right)}{\left(y^2+z^2\right)}\frac{x\,z}{r^3} \qquad (3.3.3.5)$$

Or all together

$$B_x = -\frac{\mathbf{m}\cdot\mathbf{r}}{r^3}$$
$$B_y = -\frac{m_x y}{r^3}-\left[\frac{x}{r}+1\right]\frac{1}{\left(y^2+z^2\right)^2}\left[m_y\left(z^2-y^2\right)-2m_z zy\right]+\frac{\left(m_y y+m_z z\right)}{\left(y^2+z^2\right)}\frac{x\,y}{r^3} \qquad (3.3.3.6)$$
$$B_z = -\frac{m_x z}{r^3}-\left[\frac{x}{r}+1\right]\frac{1}{\left(y^2+z^2\right)^2}\left[m_z\left(y^2-z^2\right)-2m_y yz\right]+\frac{\left(m_y y+m_z z\right)}{\left(y^2+z^2\right)}\frac{x\,z}{r^3}$$

To put this in coordinate independent form, define

$$\mathbf{r}_{\perp t} = \mathbf{r}-\hat{\mathbf{t}}\left(\hat{\mathbf{t}}\cdot\mathbf{r}\right) \qquad \mathbf{m}_{\perp t} = \mathbf{m}-\hat{\mathbf{t}}\left(\hat{\mathbf{t}}\cdot\mathbf{m}\right) \qquad (3.3.3.7)$$

$$\mathbf{w} = \hat{\mathbf{t}}\times\left[\mathbf{m}\times\mathbf{r}_{\perp t}\right] = -\mathbf{r}_{\perp t}\left(\hat{\mathbf{t}}\cdot\mathbf{m}\right) \qquad (3.3.3.8)$$

$$B_x = -\frac{\mathbf{m}\cdot\mathbf{r}}{r^3}\hat{\mathbf{t}}_x$$
$$B_y = \frac{w_y}{r^3}-\left[\frac{\hat{\mathbf{t}}\cdot\mathbf{r}}{r}+1\right]\frac{\left[\mathbf{m}_{\perp t}r_{\perp t}^2-2\mathbf{r}_{\perp t}\left(\mathbf{r}_{\perp t}\cdot\mathbf{m}_{\perp t}\right)\right]_y}{\left|\left(\mathbf{r}-\hat{\mathbf{t}}\left(\hat{\mathbf{t}}\cdot\mathbf{r}\right)\right)\right|^4}+\frac{\mathbf{r}_{\perp t}\cdot\mathbf{m}}{\left|\left(\mathbf{r}-\hat{\mathbf{t}}\left(\hat{\mathbf{t}}\cdot\mathbf{r}\right)\right)\right|^2}\frac{\left(\hat{\mathbf{t}}\cdot\mathbf{r}\right)\left(\mathbf{r}_{\perp t}\right)_y}{r^3} \qquad (3.3.3.9)$$
$$B_z = \frac{w_z}{r^3}-\left[\frac{\hat{\mathbf{t}}\cdot\mathbf{r}}{r}+1\right]\frac{\left[\mathbf{m}_{\perp t}r_{\perp t}^2-2\mathbf{r}_{\perp t}\left(\mathbf{r}_{\perp t}\cdot\mathbf{m}_{\perp t}\right)\right]_z}{\left|\left(\mathbf{r}-\hat{\mathbf{t}}\left(\hat{\mathbf{t}}\cdot\mathbf{r}\right)\right)\right|^4}+\frac{\mathbf{r}_{\perp t}\cdot\mathbf{m}}{\left|\left(\mathbf{r}-\hat{\mathbf{t}}\left(\hat{\mathbf{t}}\cdot\mathbf{r}\right)\right)\right|^2}\frac{\left(\hat{\mathbf{t}}\cdot\mathbf{r}\right)\left(\mathbf{r}_{\perp t}\right)_z}{r^3}$$

$$\mathbf{B} = -\frac{\left[\left(\mathbf{m}\cdot\mathbf{r}\right)\hat{\mathbf{t}} + \mathbf{r}_{\perp t}\left(\hat{\mathbf{t}}\cdot\mathbf{m}\right)\right]}{r^3} - \left[1+\frac{\hat{\mathbf{t}}\cdot\mathbf{r}}{r}\right]\frac{\left[\mathbf{m}_{\perp t}r_{\perp t}^2 - 2\mathbf{r}_{\perp t}\left(\mathbf{r}_{\perp t}\cdot\mathbf{m}_{\perp t}\right)\right]}{r_{\perp t}^4} + \frac{\left(\mathbf{r}_{\perp t}\cdot\mathbf{m}\right)}{r_{\perp t}^2}\frac{\left(\hat{\mathbf{t}}\cdot\mathbf{r}\right)\mathbf{r}_{\perp t}}{r^3} \tag{3.3.3.10}$$

$$\mathbf{B} = -\frac{\left[\left(\mathbf{m}\cdot\mathbf{r}\right)\hat{\mathbf{t}} + r_{\perp t}\hat{\mathbf{r}}_{\perp t}\left(\hat{\mathbf{t}}\cdot\hat{\mathbf{m}}\right)\right]}{r^3} - \left[1+\frac{\hat{\mathbf{t}}\cdot\mathbf{r}}{r}\right]\frac{\left[\mathbf{m}_{\perp t} - 2\hat{\mathbf{r}}_{\perp t}\left(\hat{\mathbf{r}}_{\perp t}\cdot\mathbf{m}_{\perp t}\right)\right]}{r_{\perp t}^2} + \left(\hat{\mathbf{r}}_{\perp t}\cdot\mathbf{m}\right)\frac{\left(\hat{\mathbf{t}}\cdot\mathbf{r}\right)\hat{\mathbf{r}}_{\perp t}}{r^3} \tag{3.3.3.11}$$

$$\mathbf{B} = -\frac{\left[\left(\mathbf{m}\cdot\mathbf{r}\right)\hat{\mathbf{t}} + \hat{\mathbf{r}}_{\perp t}\left\{r_{\perp t}\left(\hat{\mathbf{t}}\cdot\mathbf{m}\right) - \left(\hat{\mathbf{t}}\cdot\mathbf{r}\right)\left(\hat{\mathbf{r}}_{\perp t}\cdot\mathbf{m}\right)\right\}\right]}{r^3} - \left[1+\frac{\hat{\mathbf{t}}\cdot\mathbf{r}}{r}\right]\frac{\left[\mathbf{m}_{\perp t} - 2\hat{\mathbf{r}}_{\perp t}\left(\hat{\mathbf{r}}_{\perp t}\cdot\mathbf{m}_{\perp t}\right)\right]}{r_{\perp t}^2} \tag{3.3.3.12}$$

4 Faraday's Law

4.1 The Falling Ring

4.1.1 The Equation of Motion

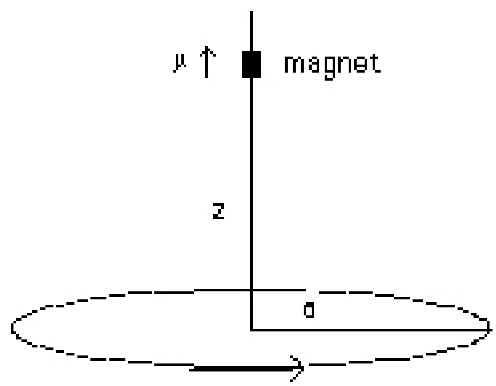

Figure 4.1-1: Geometry of the falling ring

We have a 3D dipole with dipole moment $\vec{\mu} = \mu\hat{\mathbf{z}}$. It moves on the axis of a circular loop of radius *a*, resistance *R*, inductance *L*, with inductive time constant *L/R*. It moves downward under the influence of gravity. The equation of motion is

$$m\frac{d^2z}{dt^2} = -mg + \mu\frac{dB_z}{dz} \tag{4.1.1.1}$$

where B_Z is the field due the current *I* in the ring (positive in the direction show in Figure 4.1-1)[1]. The expression for B_Z is

$$B_z = \frac{\mu_o I a^2}{2(a^2+z^2)^{3/2}} \tag{4.1.1.2}$$

so that equation (4.1.1.1) is

$$m\frac{d^2z}{dt^2} = -mg + \frac{\mu\mu_o I a^2}{2}\frac{d}{dz}\frac{1}{(a^2+z^2)^{3/2}} \tag{4.1.1.3}$$

or

[1]This is the appropriate equation for both the situation of the ring at rest and the magnet moving, or the magnet at rest and the ring moving--the mass *m* switches from the mass of the magnet to the mass of the ring, depending on the situation.

$$m\frac{d^2z}{dt^2} = -mg - \frac{3\mu\mu_o I a^2}{2}\frac{z}{(a^2+z^2)^{5/2}} \tag{4.1.1.4}$$

4.1.2 Determining An Equation for I from Faraday's Law

Faraday's Law is

$$\oint \mathbf{E}\cdot\mathbf{dl} = -\frac{d}{dt}\int \mathbf{B}\cdot\mathbf{dA} = -\frac{d}{dt}\int \mathbf{B}_{dipole}\cdot\mathbf{dA} - L\frac{dI}{dt} \tag{4.1.2.1}$$

and if $\mathbf{E} = \rho\mathbf{J}$, then

$$\oint \mathbf{E}\cdot\mathbf{dl} = \oint \rho\mathbf{J}\cdot\mathbf{dl} = I\oint \rho dl/A = IR\text{, with } R = \oint \rho dl/A \tag{4.1.2.2}$$

so that

$$IR = -L\frac{dI}{dt} - \frac{d}{dt}\int \mathbf{B}_{dipole}\cdot\mathbf{dA} \tag{4.1.2.3}$$

We need to determine the magnetic flux through the ring due to the dipole field. To do this we calculate the flux through a spherical cap of radius $\sqrt{a^2+z^2}$ with an opening angle given θ given by $\sin\theta = a/\sqrt{a^2+z^2}$ (this is the same as the flux through the ring because $\nabla\cdot\mathbf{B} = 0$).

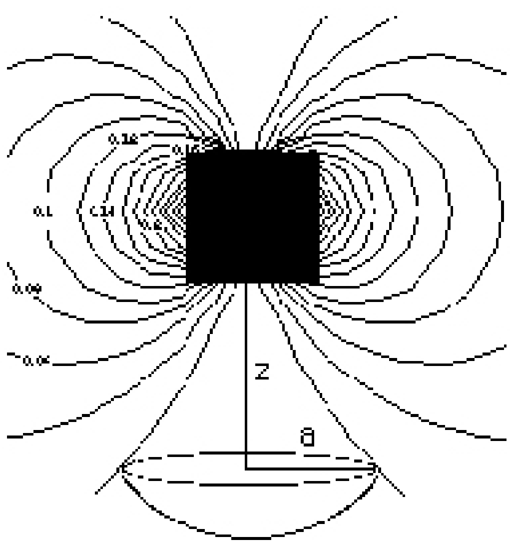

Figure 4.1-2: Dipole Flux Through a Polar Cap

The flux through a spherical cap only involves the radial component of the dipole field, given by

$$B_z = \frac{\mu_o}{2\pi}\frac{\mu\cos\theta}{r^3} \tag{4.1.2.4}$$

Our expression for the flux is thus

$$\int \mathbf{B}_{dipole}\cdot\mathbf{dA} = \int \frac{\mu_o}{2\pi}\frac{\mu\cos\theta}{r^3}2\pi r^2\sin\theta d\theta$$

which can be integrated to give

$$\int \mathbf{B}_{dipole}\cdot\mathbf{dA} = -\frac{\mu_o\mu}{r}\int\cos\theta d\cos\theta = -\frac{\mu_o\mu}{2r}\left(\cos^2\theta - 1\right) = \frac{\mu_o\mu}{2r}\sin^2\theta = \frac{\mu_o\mu}{2}\frac{a^2}{\left(a^2+z^2\right)^{3/2}} \tag{4.1.2.5}$$

Inserting (4.1.2.5) into (4.1.2.2) yields

$$IR = -L\frac{dI}{dt} - \frac{d}{dt}\frac{\mu_o\mu}{2}\frac{a^2}{\left(a^2+z^2\right)^{3/2}} \tag{4.1.2.6}$$

We assume that μ is constant, with $\mu = M_o$, so that (4.1.2.6) becomes

$$IR = -L\frac{dI}{dt} + \frac{3\mu_o a^2 M_o}{2}\frac{z}{\left(a^2+z^2\right)^{5/2}}\frac{dz}{dt} \tag{4.1.2.7}$$

Equations (4.1.1.4) and (4.1.2.7) are our coupled ordinary differential equations which determine the dynamics of the situation.

4.1.3 Dimensionless Form of the Equations

We now put these equations into dimensionless form. We measure all distances in terms of the distance *a*, and all times in terms of the time $\sqrt{a/g}$. Let

$$z' = \frac{z}{a} t' = \frac{t}{\sqrt{a/g}} I' = \frac{I}{I_o}, where I_o = \frac{mga^2}{\mu_o M_o} \tag{4.1.3.1}$$

The time $\sqrt{a/g}$ is roughly the time it would take the magnet to fall under the influence of gravity through a distance a starting from rest. The current I_O is roughly the current in the ring that is required to produce a force sufficient to offset gravity when the magnet is a distance a above the ring. In terms of these variables, our equations (4.1.1.4) and (4.1.2.6) are

$$\frac{d^2z'}{dt'^2} = -1 - \frac{3}{2}\frac{z'I'}{(1+z'^2)^{5/2}} \tag{4.1.3.2}$$

$$I' = -\frac{L}{R}\sqrt{\frac{g}{a}}\frac{dI'}{dt'} + \frac{3(\mu_o M_o)^2}{2mga^3R}\sqrt{\frac{g}{a}}\frac{z'}{\left(1+z'^2\right)^{5/2}}\frac{dz'}{dt'} \tag{4.1.3.3}$$

We introduce the four dimensionless parameters

$$\alpha = \frac{R}{L}\sqrt{\frac{a}{g}} \qquad \beta = \frac{\mu_o M_o}{LI_o a} = \frac{(\mu_o M_o)^2}{Lmga^3} \qquad \lambda = \frac{L}{\mu_o a} \qquad D = \sqrt{\frac{g}{a}}\frac{\mu_o a}{R} = \frac{1}{\lambda\alpha} \tag{4.1.3.4}$$

and we define the reduced flux rate function *F(z')* as

$$F(z') = \frac{3}{2}\frac{z'}{(1+z'^2)^{5/2}} \tag{4.1.3.5}$$

Note that we can write the reference current I_O as

$$\frac{I_o a^2}{M_o} = \frac{mga^4}{\mu_o M_o^{\,2}} = \frac{1}{\lambda\beta} \tag{4.1.3.6}$$

The parameters have the following physical meanings. The quantity α is the ratio of the free fall time to the inductive time constant--if α is very large, inductive effects are negligible. The quantity β is roughly the ratio of the current due to induction alone, assuming the resistance is zero (Φ_{dipole}/L with $\Phi_{dipole} \approx \frac{\mu_o M_o}{a}$), to the reference current I_O. With these definitions, our equations become

$$\frac{d^2z'}{dt'^2} = -1 - F(z')I' \tag{4.1.3.7}$$

$$\frac{dI'}{dt'} = -\alpha I' + \beta F(z')\frac{dz'}{dt'} \tag{4.1.3.8}$$

If we define the speed $v' = dz'/dt'$, then we can write three coupled first-order ordinary differential equations for the triplet (z', I', v'), as

$$dz'/dt' = v' \qquad (4.1.3.9)$$

$$\frac{dI'}{dt'} = -\alpha I' + \beta F(z')\frac{dz'}{dt'} \qquad (4.1.3.10)$$

$$\frac{dv'}{dt'} = -1 - F(z')I' \qquad (4.1.3.11)$$

4.1.4 Conservation of Energy

We assume that μ is constant. If we multiply (4.1.1.4) by $v = \frac{dz}{dt}$ and (4.1.2.7) by *I*, after some algebra, we find that

$$\frac{d}{dt}\left[\tfrac{1}{2}mv^2 + mgz + \tfrac{1}{2}LI^2\right] = -I^2 R \qquad (4.1.4.1)$$

which expresses conservation of energy for the falling magnet plus the magnetic field of the ring. The dimensionless form of this equation is

$$\frac{d}{dt'}\left[\tfrac{1}{2}v'^2 + z' + \frac{1}{4\beta}I'^2\right] = -\frac{\alpha}{2\beta}I'^2 \qquad (4.1.4.2)$$

Suppose the resistance of the ring (the superconducting case) is zero (i.e., $\alpha = 0$). In this case, (4.1.4.2) becomes, with one integration

$$I' = C - \frac{\beta}{\left(1 + z'^2\right)^{3/2}} \qquad (4.1.4.3)$$

If we impose boundary conditions that $I' = 0$ when $t' = 0$, with $z' = z_o'$ and $v' = v_o'$ at $t' = 0$, then this is

$$I' = \beta\left[\frac{1}{\left(1 + z_o'^2\right)^{3/2}} - \frac{1}{\left(1 + z'^2\right)^{3/2}}\right] \qquad (4.1.4.4)$$

Using this equation, (4.1.4.2) for the conservation of energy becomes

$$\tfrac{1}{2}v'^2 + \left\{ (z' - z'_o) + \frac{\beta}{4}\left[\frac{1}{\left(1+z'^2_o\right)^{3/2}} - \frac{1}{\left(1+z'^2\right)^{3/2}} \right]^2 \right\} = 0 \qquad (4.1.4.5)$$

4.1.5 Numerical Solutions

A magnet falls through a copper ring. At the ring, the speed of the magnet decreases. When the magnet is through the ring, the magnet resumes free fall. We show a numerical solution to equations (4.1.3.9) through (4.1.3.11) above, appropriate to this case. The initial conditions (z', I', v') for the solution plotted below are (2,0,0). The values of (α, β) are (10,100). Below we plot position as a function of time and current as a function of time (using our dimensionless parameters).

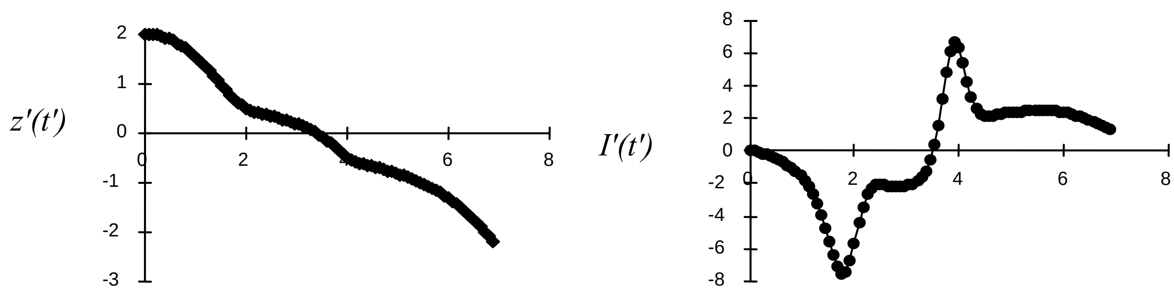

Figure 4.1-3: Numerical Solutions for the Falling Ring Equations

The behavior of these solutions is what we expect. When the magnet reaches a distance of about *a* from the ring, it slows down, because of the increasing current in the ring, which repels the magnet. As the magnet passes through the ring, the current reverses direction, now attracting the magnet from above, which also slows the magnet. Finally the magnet falls far enough that the current in the ring goes to zero, and the magnet is again in free fall.

These are approximate solutions only, using an Excel spreadsheet with an Euler integration scheme. In the final animations, we a fourth order Runge-Kutta scheme to integrate the equations with high accuracy.

4.1.6 The Topology of The Field

How do we plot the field configuration given solutions for z' and I'? The absolute current is given by (cf. equations (4.1.3.3) and (4.1.3.4))

$$I = I_o I' = \frac{1}{\lambda\beta}\frac{M_o}{a^2} I' \qquad (4.1.6.1)$$

How much freedom do we have in choosing the absolute value of the current once we have solved our dimensionless equations? And in particular how does that freedom affect the topology of the magnetic field? One measure of the shape of the total field is the ratio of the field at the center of the ring due to the ring to the field at the center of the ring due to the magnet when the magnet is a distance *a* above the ring, i.e.,

$$\frac{B_{ring}\Big|^{\text{center of ring}}}{B_{dipole}\Big|^{\text{at center of ring when dipole at a}}} = \frac{\dfrac{\mu_o I}{2a}}{\dfrac{\mu_o}{2\pi}\dfrac{M_o}{a^3}} \approx \frac{Ia^2}{M_o} \tag{4.1.6.2}$$

where we are dropping numerical factors. Clearly when this ratio varies the overall shape of the total field must vary. If we use (4.1.6.1) in (4.1.6.2) we have

$$\frac{B_{ring}\Big|^{\text{center of ring}}}{B_{dipole}\Big|^{\text{at center of ring when dipole at a}}} \approx \frac{a^2}{M_o} I = \frac{a^2}{M_o}\frac{1}{\lambda\beta}\frac{M_o}{a^2} I' = \frac{I'}{\lambda\beta} \tag{4.1.6.3}$$

The meaning of equation (4.1.6.3) is that the overall shape of the magnetic field topology is totally determined once we make the one remaining choice of the dimensionless constant λ, defined in equation (4.1.3.4) which up to this point we have not chosen (we have only picked values of α and β to solve our dimensionless equation). Once that choice is made, we have no additional freedom to affect the field topology.

5 Circuits

Under construction.

6 The Displacement Current

Under construction.

7 Radiation

7.1 Electric Dipole Radiation

7.1.1 The Electric and Magnetic Fields

The electric field of a time-varying electric dipole **p**(t) is given by

$$\mathbf{E}(\mathbf{r},t) = \underset{\text{quasi - static dipole}}{\frac{\left[3\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}}) - \mathbf{p}\right]}{4\pi\varepsilon_o r^3}} + \underset{\text{induction}}{\frac{\left[3\hat{\mathbf{n}}(\dot{\mathbf{p}}\cdot\hat{\mathbf{n}}) - \dot{\mathbf{p}}\right]}{4\pi\varepsilon_o c r^2}} + \underset{\text{radiation}}{\frac{(\ddot{\mathbf{p}}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}}{4\pi\varepsilon_o r c^2}} \tag{7.1.1.1}$$

where the “dot” above a variable indicates differentiation with respect to time, and the electric dipole moment vector and its time derivative are evaluated at the retarded time $t_{ret} = t - r/c$. With some algebraic effort, it can be shown that this expression can be written as

$$\mathbf{E}(\mathbf{r},t) = \nabla\times\left\{\frac{1}{4\pi\varepsilon_o}\left[\frac{\dot{\mathbf{p}}}{cr} + \frac{\mathbf{p}}{r^2}\right]\times\hat{\mathbf{n}}\right\} \tag{7.1.1.2}$$

Now when we are not at the origin we are in vacuum, so that we have

$$\nabla\times\,\mathbf{B}(\mathbf{r},t) = \frac{1}{c^2}\frac{\partial\mathbf{E}(\mathbf{r},t)}{\partial t} \tag{7.1.1.3}$$

But if we take the time derivative of (7.1.1.2) and multiply by $1/c^2$, we easily have

$$\frac{1}{c^2}\frac{\partial\mathbf{E}(\mathbf{r},t)}{\partial t} = \nabla\times\left\{\frac{1}{c^2}\frac{1}{4\pi\,\varepsilon_o}\left[\frac{\ddot{\mathbf{p}}}{cr} + \frac{\dot{\mathbf{p}}}{r^2}\right]\times\,\hat{\mathbf{n}}\right\} \tag{7.1.1.4}$$

Comparing the two equations above, we see that we must have

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_o}{4\pi}\left[\frac{\ddot{\mathbf{p}}}{cr} + \frac{\dot{\mathbf{p}}}{r^2}\right]\times\,\hat{\mathbf{n}} \tag{7.1.1.4}$$

7.1.2 The Flux Function for a Dipole Oriented Along The Z-Axis

If the dipole moment **p** is always in the z-direction, we can write the electric field as

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_o}\nabla\times\left\{\left[\frac{\dot{p}(t-r/c)}{cr} + \frac{p(t-r/c)}{r^2}\right]\sin\theta\,\hat{\phi}\right\} \tag{7.1.2.1}$$

Equation (7.1.2.1) and the development in 1.3 above imply that the electric field lines of a simple radiating electric dipole system in this case are given by the isocontours of

$$F(r,\theta,t) = 2\pi\, r\sin\theta A(r,\theta,t) = 2\pi\, r\sin\theta \frac{1}{4\pi\varepsilon_o}\left\{\left[\frac{\dot{p}(t-r/c)}{cr} + \frac{p(t-r/c)}{r^2}\right]\sin\theta\right\} \quad (7.1.2.2)$$

or

$$F(r,\theta,t) = \frac{1}{4\pi\varepsilon_o}\left[\frac{\dot{p}(t-r/c)}{c} + \frac{p(t-r/c)}{r}\right] 2\pi\sin^2\theta \quad (7.1.2.3)$$

and this is the flux function for such a dipole, in the sense defined in 1.3. If we define the dimensionless variables

$$t' = \frac{t}{T} \qquad r' = \frac{r}{cT} \quad (7.1.2.4)$$

we can rewrite (7.1.2.3) as

$$F(r,\theta,t) = \frac{1}{4\pi\varepsilon_o}\frac{1}{cT}\left[\dot{p}(t'-r') + \frac{p(t'-r')}{r'}\right] 2\pi\sin^2\theta \quad (7.1.2.5)$$

where now the "dot" above a variable indicates differentiation with respect to the dimensionless time variable.

7.1.3 An Oscillating Electric Dipole

Let us consider a particular case of the situation above. Suppose the dipole moment is a constant plus a sinusoidally varying function of time, with period T. That is,

$$p(t) = \left[p_o + p_1\cos\left(\frac{2\pi t}{T}\right)\right] \quad (7.1.3.1)$$

in which case (7.1.2.3) becomes

$$F(r,\theta,t) = \frac{1}{4\pi\varepsilon_o}\left[\frac{-\frac{2\pi}{T}p_1\sin\left(\frac{2\pi(t-r/c)}{T}\right)}{c} + \frac{\left(p_o + p_1\cos\left(\frac{2\pi(t-r/c)}{T}\right)\right)}{r}\right] 2\pi\sin^2\theta$$

(7.1.3.2)

or in terms of our dimensionless variables

$$F(r',\theta,t') = \frac{1}{4\pi\varepsilon_o}\left\{\frac{2\pi}{cT}\left[-2\pi p_1\sin\left(2\pi(t'-r')\right) + \frac{\left(p_o + p_1\cos\left(2\pi(t'-r')\right)\right)}{r'}\right]\sin^2\theta\right\}$$

(7.1.3.3)

Figure 7.1-1 below shows the field lines for at $t = 0$ as defined by (7.1.3.3), for two values of the flux function, -3 and +3.

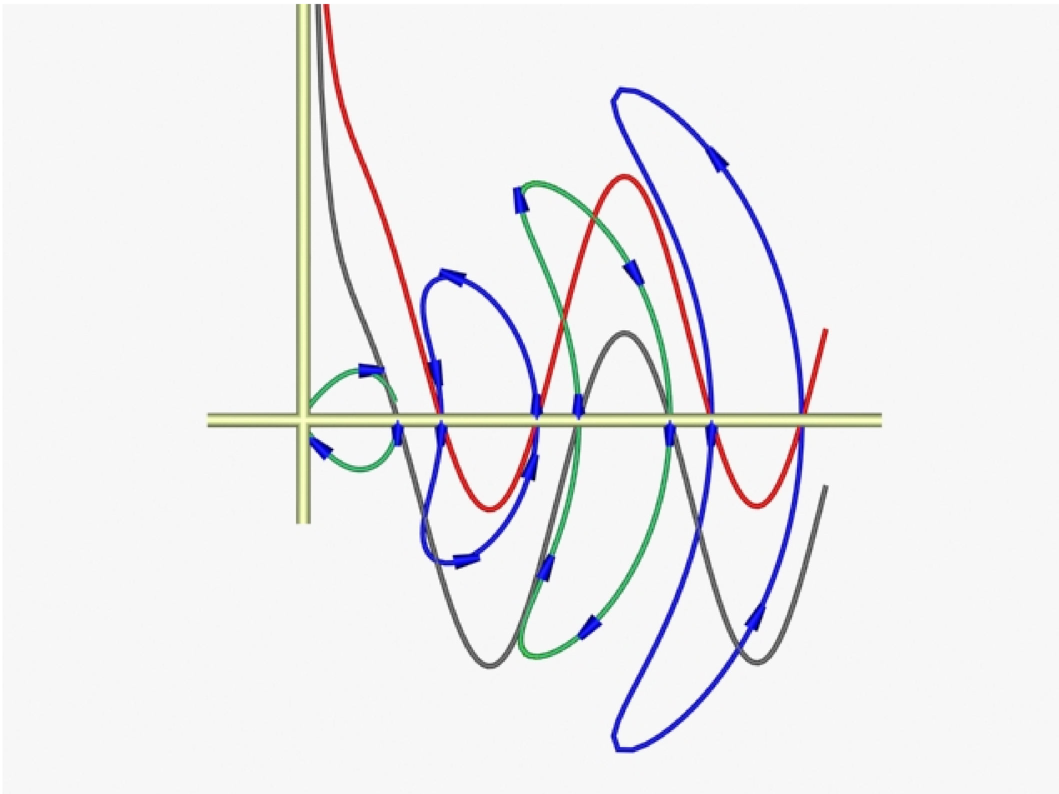

Figure 7.1-1: Field lines and flux function of an oscillating electric dipole.

7.1.4 A Rotating Electric Dipole

Let is look at the electric and magnetic fields of a rotating electric dipole which is oriented perpendicular to its axis of rotation. We have

$$\mathbf{p}(t) = p_o\left[\cos\omega t\,\hat{\mathbf{x}} + \sin\omega t\,\hat{\mathbf{y}}\right] \tag{7.1.4.1}$$

$$\dot{\mathbf{p}}(t) = \omega p_o\left[-\sin\omega t\,\hat{\mathbf{x}} + \cos\omega t\,\hat{\mathbf{y}}\right] \tag{7.1.4.2}$$

$$\ddot{\mathbf{p}}(t) = -\omega^2 p_o\left[\cos\omega t\,\hat{\mathbf{x}} + \sin\omega t\,\hat{\mathbf{y}}\right] \tag{7.1.4.3}$$

We are going to look at the fields only in the xy plane at $z = 0$. Thus

$$\hat{\mathbf{n}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \tag{7.1.4.4}$$

Using the expressions for **p** and **n** above, we easily have that

$$\mathbf{p}\cdot\mathbf{n} = p_o\left[\cos\omega t\,\hat{\mathbf{x}} + \sin\omega t\,\hat{\mathbf{y}}\right]\cdot\left[\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}\right] = p_o\cos(\phi-\omega t) \tag{7.1.4.5}$$

$$\dot{\mathbf{p}}\cdot\mathbf{n} = \omega p_o\left[-\sin\omega t\,\hat{\mathbf{x}} + \cos\omega t\,\hat{\mathbf{y}}\right]\cdot\left[\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}\right] = \omega p_o\sin(\phi-\omega t) \tag{7.1.4.6}$$

$$\dot{\mathbf{p}}\times\mathbf{n} = \omega p_o\left[-\sin\omega t\,\hat{\mathbf{x}} + \cos\omega t\,\hat{\mathbf{y}}\right]\times\left[\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}\right] = -\omega p_o\,\hat{\mathbf{z}}\cos(\phi-\omega t) \tag{7.1.4.7}$$

$$\ddot{\mathbf{p}}\times\mathbf{n} = -\omega^2 p_o\left[\cos\omega t\,\hat{\mathbf{x}} + \sin\omega t\,\hat{\mathbf{y}}\right]\times\left[\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}\right] = -\omega^2 p_o\,\hat{\mathbf{z}}\sin(\phi-\omega t) \tag{7.1.4.8}$$

$$\left[3\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}})-\mathbf{p}\right] = \left[3p_o\cos(\phi-\omega t)\left(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}\right) - p_o\left(\cos\omega t\,\hat{\mathbf{x}} + \sin\omega t\,\hat{\mathbf{y}}\right)\right] \tag{7.1.4.9}$$

$$\left[3\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}})-\mathbf{p}\right] = p_o\hat{\mathbf{x}}\left[3\cos(\phi-\omega t)\cos\phi - \cos\omega t\right] + p_o\hat{\mathbf{y}}\left[3\cos(\phi-\omega t)\sin\phi - \sin\omega t\right] \tag{7.1.4.10}$$

$$\left[3\hat{\mathbf{n}}(\dot{\mathbf{p}}\cdot\hat{\mathbf{n}})-\dot{\mathbf{p}}\right] = \left[\omega p_o\sin(\phi-\omega t)\left(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}\right) - \omega p_o\left[-\sin\omega t\,\hat{\mathbf{x}} + \cos\omega t\,\hat{\mathbf{y}}\right]\right] \tag{7.1.4.11}$$

$$\left[3\hat{\mathbf{n}}(\dot{\mathbf{p}}\cdot\hat{\mathbf{n}})-\dot{\mathbf{p}}\right] = \omega p_o\mathbf{x}\left[3\sin(\phi-\omega t)\cos\phi + \sin\omega t\ \right] + \omega p_o\hat{\mathbf{y}}\left[3\sin(\phi-\omega t)\sin\phi - \cos\omega t\right] \tag{7.1.4.12}$$

$$\left[(\ddot{\mathbf{p}}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}\right] = -\omega^2 p_o\,\hat{\mathbf{z}}\sin(\phi-\omega t)\times\left[\left(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}\right)\right] \tag{7.1.4.13}$$

$$\left[(\ddot{\mathbf{p}}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}\right] = \omega^2 p_o\sin(\phi-\omega t)\left[\sin\phi\,\hat{\mathbf{x}} - \cos\phi\,\hat{\mathbf{y}}\right] \tag{7.1.4.14}$$

So we have for the electric field

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) = {} & \frac{p_o\hat{\mathbf{x}}\left[3\cos(\phi-\omega t)\cos\phi - \cos\omega t\right] + p_o\hat{\mathbf{y}}\left[3\cos(\phi-\omega t)\sin\phi - \sin\omega t\right]}{4\pi\,\varepsilon_o r^3} \\ & + \frac{\omega p_o\mathbf{x}\left[3\sin(\phi-\omega t)\cos\phi + \sin\omega t\ \right] + \omega p_o\hat{\mathbf{y}}\left[3\sin(\phi-\omega t)\sin\phi - \cos\omega t\right]}{4\pi\,\varepsilon_o c\, r^2} \\ & + \frac{\omega^2 p_o\sin(\phi-\omega t)\left[\sin\phi\,\hat{\mathbf{x}} - \cos\phi\,\hat{\mathbf{y}}\right]}{4\pi\,\varepsilon_o r c^2}\end{aligned} \tag{7.1.4.15}$$

$$\mathbf{B}(\mathbf{r},t) = -\frac{\mu_o}{4\pi}\hat{\mathbf{z}}\omega p_o\left[\frac{\omega\sin(\phi-\omega t)}{cr} + \frac{\cos(\phi-\omega t)}{r^2}\right] \tag{7.1.4.16}$$

7.2 Linear Antenna (from S. Olbert and N. Derby)

7.2.1 Notations, Definitions, Basics

Following common conventions let **E**, **B**, **A**, ϕ, ρ_c and **j** stand, respectively, for electric field, magnetic field, vector potential, scalar potential, charge density and electric current density. Let us also introduce **D** and **H** such that

$$\mathbf{D} = \varepsilon_\mathbf{o}\mathbf{E} \qquad \mathbf{H} = \frac{\mathbf{B}}{\mu_o} \tag{7.2.1.1}$$

Recall that the speed of light c is related to ε_o and μ_o by

$$c = \frac{1}{\sqrt{\varepsilon_\mathbf{o}\mu_o}}$$

With these notations Maxwell Equations acquire the compact form

$$\nabla \cdot \mathbf{D} = \rho_\mathbf{c} \qquad \nabla \cdot \mathbf{B} = 0 \tag{7.2.1.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j} \tag{7.2.1.3}$$

Let's define vector and scalar potentials **A** and ϕ such that

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \tag{7.1.1.4}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{7.1.1.5}$$

With the auxiliary condition

$$\nabla \cdot \mathbf{A} + \varepsilon_\mathbf{o}\mu_o \frac{\partial\ \phi}{\partial t} = 0 \tag{7.1.1.6}$$

imposed, one can then readily show that Maxwell's Equations lead to

$$\nabla^2\phi - \varepsilon_\mathbf{o}\mu_o \frac{\partial^2\phi}{\partial t^2} = -\frac{\rho_c}{\varepsilon_\mathbf{o}} \tag{7.1.1.7}$$

$$\nabla^2 A_i - \varepsilon_\mathbf{o}\mu_o \frac{\partial^2 A_i}{\partial t^2} = -\mu_o j_i \tag{7.1.1.8}$$

where the subscript i indicates one of the Cartesian components of **A** or **j**. To prove the above, one needs the identity

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) \equiv \nabla^2 \mathbf{A} \tag{7.1.1.9}$$

By virtue of equations (7.1.1.2), (7.1.1.3), (7.1.1.4) and (7.1.1.9), one has

$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \nabla(\nabla \cdot \mathbf{A}) - \frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{7.1.1.10}$$

In order to find the electric field for a given current system it is mathematically more expedient to solve first equation (7.1.1.5) for **A** and then use equation (7.1.1.10) to work out the expression of **E**. In fact, the general solution of equation (7.1.1.5) is known to be

$$\mathbf{A} = \frac{\mu_o}{4\pi} \int d^3x' \int dt' \frac{\mathbf{j}(\mathbf{x}', t')}{|\,\mathbf{x} - \mathbf{x}'\,|} \delta(t' - t_{ret}) \tag{7.1.1.11}$$

where

$$t_{ret} = t - \frac{|\,\mathbf{x} - \mathbf{x}'\,|}{c} \tag{7.1.1.12}$$

is the retarded time.

Center-fed Linear Antenna: Vector Potential

Lay a linear antenna of length l along the z-axis with its center at the origin of a Cartesian coordinate system. Let the charge be fed through center harmonically so that we can put for the current density in the antenna

$$\begin{aligned} j_x &= j_y = 0 \\ j_z &= I \sin\left(\frac{1}{2}kl - k\,|\,z'\,|\right) \delta(x') \delta(y') e^{-i\omega t} \end{aligned} \tag{7.2.1.13}$$

where ω is the circular frequency and k is the wave number so that

$$k = \frac{\omega}{c} \tag{7.2.1.14}$$

We choose the phase of the physical current density so that it is the imaginary part of (7.2.1.13).

Inserting equation (7.2.1.13) into equation (7.2.1.11) yields

$$\begin{aligned} A_x &= A_y = 0 \\ A_z &= \frac{\mu_o}{4\pi} I \int_{-l/2}^{+l/2} dz' \sin\left(\frac{1}{2}kl - k\,|\,z'\,|\right) \frac{\exp(-i\omega\, t_{ret})}{|\,\mathbf{x} - \mathbf{x}'\,|} \end{aligned} \tag{7.2.1.15}$$

To simplify notation it is convenient to replace time and space variables, t and $\mathbf{x}$, by corresponding dimensionless quantities, to wit:

$$a = \frac{1}{2}kl \tag{7.2.1.16}$$

$$\omega t \to t; \quad k\mathbf{x} \to \mathbf{x}; \quad k\mathbf{x}' \to \mathbf{x}' .$$

With the dimensionless cylindrical coordinates ρ and ϕ such that

$$x = \rho\cos(\phi) \qquad\qquad y = \rho\sin(\phi)$$

we then have for our geometrical setup

$$A_z = \frac{\mu_o}{4\pi} I \int_{-a}^{+a} dz' \, \sin(a - |z'|) \frac{\exp(-it_{ret})}{\sqrt{\rho^2 + (z - z')^2}} \tag{7.2.1.17}$$

Because of the mirror symmetry about the ($x'y'$)-plane, we can convert the integration in equation (7.2.1.17) to the positive half above this plane, ($z' \geq 0$). With proper adjustments of signs we thus can replace equation (7.2.1.17) by

$$A_z = \frac{\mu_o}{4\pi} I e^{it} \int_o^a dz' \, \sin(a - z') \left(\frac{\exp(iR_+)}{R_+} + \frac{\exp(iR_-)}{R_-} \right) \tag{7.2.1.18}$$

where we have put for short

$$R_\pm = \sqrt{\rho^2 + (z \pm z')^2}\,; \qquad\qquad \rho^2 = x^2 + y^2 \tag{7.2.1.19}$$

Center-fed Linear Antenna: Cylindrical Components of E

We are now ready to work out explicit expressions of E_Z and E_ρ.

Using equations (7.2.1.18) and (7.2.1.10) and taking advantage of axial symmetry, (E_ϕ = 0), we get

$$\frac{iE_z}{\omega} = -A_z - \frac{\partial^2 A_z}{\partial z^2} \tag{7.2.1.20}$$

$$\frac{iE_\rho}{\omega} = -\frac{\partial^2 A_z}{\partial z \partial \rho} \tag{7.2.1.21}$$

Consider some function K containing in its argument ($z \pm z'$); clearly, differentiating with respect to z can be exchanged with that with respect to z', viz.,

$$\frac{\partial K(z \pm z')}{\partial z} = \pm \frac{\partial K(z \pm z')}{\partial z'} \tag{7.2.1.22}$$

With this in mind, one can show that differentiating an integral expression of the form

$$F(z) = \int_o^a dz' \sin(a - z')\left[K(z + z') + K(z - z')\right] \qquad (7.2.1.23)$$

leads to the following results

$$\frac{\partial F}{\partial z} = \int_o^a dz' \cos(a - z')\left[K(z + z') - K(z - z')\right] \qquad (7.2.1.24)$$

$$F + \frac{\partial^2 F}{\partial z^2} = K(z + a) + K(z - a) - 2\cos a \; K(z) \qquad (7.2.1.25)$$

Applying this to equation (7.2.1.20) we find, with some surprise, that the z-component of **E** reduces to the elementary form

$$E_z = iE_o e^{-i\tau}\left(\frac{\exp[i(D_+ - r)]}{D_+} + \frac{\exp[i(D_- - r)]}{D_-} - \frac{2\cos a}{r}\right) \qquad (7.2.1.26)$$

where

$$E_o = \frac{\mu_o I\omega}{4\pi}; \quad \tau = t - r; \qquad D_\pm = \sqrt{\rho^2 + (z \pm a)^2}; \qquad r = \sqrt{\rho^2 + z^2} \qquad (7.2.1.27)$$

Unfortunately, the ρ-component of **E** is not reducible to an elementary expression.

Carrying out the differentiation in equation (7.2.1.21) first with respect to z, then making use of relation (7.2.1.24) and the fact that

$$\frac{\partial R_\pm}{\partial \rho} = \frac{\rho}{R_\pm}$$

we find the following integral expression for E_ρ

$$E_\rho = iE_o e^{-it}\rho\int_o^a dz' \cos(a - z')\left[G(R_+) - G(R_-)\right] \qquad (7.2.1.28)$$

where we have put for short

$$G(\xi) = \frac{\exp(i\xi)}{\xi^3}(i\xi - 1) \qquad (7.2.1.29)$$

In practical applications we need only the real (or imaginary) parts of equations (7.2.1.26) and (7.2.1.28). Doing the necessary algebra, we arrive at desired answer

$$E_z = E_o(Q_1 \sin\tau - Q_2 \cos\tau) + iE_o(Q_1 \cos\tau + Q_2 \sin\tau) \qquad (7.2.1.30)$$

$$E_\rho = -E_o(P_1 \sin\tau + P_2 \cos\tau) - iE_o(P_1 \cos\tau - P_2 \sin\tau) \qquad (7.2.1.31)$$

where

$$P_1 = \rho \int_o^a dz' \cos(a - z') \left[g_1(R_+, r) - g_1(R_-, r) \right] \tag{7.2.1.32}$$

with

$$g_1(\xi, r) = \frac{\xi \sin(\xi - r) + \cos(\xi - r)}{\xi^3} \tag{7.2.1.33}$$

and

$$P_2 = \rho \int_o^a dz' \cos(a - z') \left[g_2(R_+, r) - g_2(R_-, r) \right] \tag{7.2.1.34}$$

with

$$g_2(\xi, r) = \frac{\xi \cos(\xi - r) - \sin(\xi - r)}{\xi^3} \tag{7.2.1.35}$$

Note in passing that $g_1(\xi, r) = \frac{\partial g_2(\xi, r)}{\partial r}$ and $g_2(\xi, r) = \frac{-\partial g_1(\xi, r)}{\partial r}$. Finally

$$Q_1 = \frac{\cos(D_+ - r)}{D_+} + \frac{\cos(D_- - r)}{D_-} - 2\frac{\cos a}{r} \tag{7.2.1.36}$$

$$Q_2 = \frac{\sin(D_+ - r)}{D_+} + \frac{\sin(D_- - r)}{D_-} \tag{7.2.1.37}$$

Center-fed Linear Antenna: Asymptotic Expressions

It is of some interest to work out asymptotic expressions for equations (7.2.1.30) and (7.2.1.31). We have for

$$r \to \infty: \qquad R_\pm - r \approx \pm z' \cos\theta, \quad D_\pm \approx \pm a \cos\theta \quad ;$$

where

$$\cos\theta = \frac{z}{r}; \qquad \sin\theta = \frac{\rho}{r}.$$

Retaining only the leading term in powers of (1/*r*) we find

$$P_1 \approx \frac{2\sin\theta}{r} \int_0^a dz' \cos(a - z') \sin(z' \cos\theta) \qquad P_2 \approx 0 .$$

The integral is elementary; integrating by parts one gets

$$\int_0^a dz' \cos(a-z')\sin(z'\cos\theta) = \frac{\cos\theta}{\sin^2\theta}\left[\cos(a\cos\theta) - \cos a\right]$$

Next, we find that

$$Q_1 \approx \frac{2}{r}\left[\cos(a\cos\theta) - \cos a\right]; \qquad Q_2 \approx 0$$

and, therefore,

$$E_\rho = -E_z \cot\theta; \qquad E_z = 2E_o\left[\cos(a\cos\theta) - \cos a\right]\frac{\sin(t-r)}{r} \quad .$$

We can convert these results to spherical coordinates r and θ. The components of **E**-field in these coordinates are related to those given in cylindrical coordinates by

$$E_r = E_\rho \sin\theta + E_z \cos\theta$$
$$E_\theta = E_\rho \cos\theta - E_z \sin\theta$$

so that

$$E_\theta = -2E_o\left[\frac{\cos(a\cos\theta) - \cos a}{\sin\theta}\right]\frac{\sin(t-r)}{r} \qquad E_r = 0\,,$$

which agrees with formulas found in the literature (see, e.g., J. D. Jackson, page 402).

7.2.2 Flux Function for Linear Antenna

Recall that one of the Maxwell equations in a current-free region is

$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \nabla \times \mathbf{B} \qquad or \qquad \mathbf{E} = c^2 \nabla \times \int dt \mathbf{B} \tag{7.2.2.1}$$

Axial symmetry of the antenna makes **B** a toroidal vector; hence, according to equation(1.3.1.4), the flux of the **E**-field, F, is

$$F(\rho, z, t) = 2\pi c^2 \rho \int dt B_\phi(\rho, z, t) \tag{7.2.2.2}$$

Furthermore, recalling that the **B**-field is derivable from the vector potential **A**

$$\mathbf{B} = \nabla \times \mathbf{A}$$

and that for a linear antenna

$$\mathbf{A} = A_z \hat{\mathbf{z}}$$

we have

$$B_\varphi = -\frac{\partial A_z}{\partial \rho} \qquad B_z = B_\rho = 0 \tag{7.2.2.3}$$

where, in terms of reduced (dimensionless) variables defined in (7.2.1.16),

$$A_z = \frac{\mu_o}{4\pi} I e^{-it} \int_o^a dz' \sin(a - z') \left(\frac{\exp(iR_+)}{R_+} + \frac{\exp(iR_-)}{R_-} \right) \tag{7.2.2.4}$$

Introducing Hertz's *Superpotential* Z defined by

$$Z(\rho, z, t) = -\int dt A_z(\rho, z, t) = -i \ A_z(\rho, z, t) \tag{7.2.2.5}$$

we thus have for the flux of the electric field of the antenna

$$F = c^2 2\pi\rho \frac{\partial Z}{\partial \rho} \tag{7.2.2.6}$$

and for the components of the electric field

$$E_\rho = -c^2 \frac{\partial F}{2\pi\rho\partial z} \tag{7.2.2.7}$$

$$E_z = c^2 \frac{\partial F}{2\pi\rho\partial\rho} \tag{7.2.2.8}$$

Note that

$$\mathbf{E} \cdot \nabla F = 0$$

which shows that the lines of the **E**-field are defined by the equation

$$F(\rho,z,t) = \text{constant.}$$

7.2.3 Singular Points of E-Field of Linear Antenna

Using cylindrical coordinates and dimensionless variables (ρ, z) with the antenna along the z-axis, we have for the z-component of the electric field

$$E_z = Q_1 \cos\tau + Q_2 \sin\tau \tag{7.2.3.1}$$

where

$$\tau = t - r; \ \ r = \sqrt{\rho^2 + z^2} \tag{7.2.3.2}$$

$$Q_1 = \frac{\cos(D_+ - r)}{D_+} + \frac{\cos(D_- - r)}{D_-} - 2\frac{\cos a}{r} \tag{7.2.3.3}$$

$$Q_2 = \frac{\sin(D_+ - r)}{D_+} + \frac{\sin(D_- - r)}{D_-} \tag{7.2.3.4}$$

$$D_\pm = \sqrt{\rho^2 + (z \pm a)^2} \tag{7.2.3.5}$$

and again $a = \frac{2\pi}{\lambda} \times \left(\frac{\textit{antenna length}}{2} \right)$. In the equatorial plane of the antenna, (z = 0),

$$\begin{aligned} \frac{1}{2}Q_1 &= \frac{\cos(D-\rho)}{D} - \frac{\cos a}{\rho} \\ \frac{1}{2}Q_2 &= \frac{\sin(D-\rho)}{D} \end{aligned} \tag{7.2.3.6}$$

where $D = \sqrt{\rho^2 + a^2}$. (7.2.3.7)

Note that E_z may be re-written as

$$E_z = \sqrt{Q_1^{\,2} + Q_2^{\,2}}\left(\sin\delta\cos\tau + \cos\delta\sin\tau\right) = \sqrt{Q_1^{\,2} + Q_2^{\,2}}\,\sin(\tau + \delta) \tag{7.2.3.8}$$

where $\delta = \arctan\left(\frac{Q_1}{Q_2}\right)$. (7.2.3.9)

In the equatorial plane, z = 0, E_ρ vanishes everywhere, so singular points occur at values of ρ for which E_z = 0 since $\frac{E_z}{E_\rho} \Rightarrow \frac{0}{0}$ there. The condition E_z = 0 occurs when the phase of E_z is a multiple of π, that is, when $\tau + \delta = n\pi$ for any integer n. Thus, the locations of the of the singular points satisfy the equation

$$t = \rho - \arctan\left(\frac{Q_1}{Q_2}\right) + n\pi \tag{7.2.3.10}$$

Typical graphs of t (for n=0) as a function of ρ for values of a between 0 and $\pi/2$ are shown in Figure 7.2-1. Note that in general, t has one maximum and one minimum. Since t depends only on the ratio of Q_1 to Q_2, the Q_i's can be replaced by $\overline{Q_i} = UQ_i$, where U is any non-zero function of ρ. We choose $U = \frac{\rho D}{2}$ so that

$$\begin{aligned} \overline{Q_1} &= \rho\cos(D-\rho) - D\cos a \\ \overline{Q_2} &= \rho\sin(D-\rho) \end{aligned} \tag{7.2.3.11}$$

The resonant antenna (a=$\pi/2$) deserves special attention. Since cos a = 0, we have

$$t = \rho - \arctan\left(\frac{\overline{Q}_1}{\overline{Q}_2}\right) + n\pi = \rho - \arctan(\cot(D-\rho)) + n\pi = \rho - \left[\frac{\pi}{2} - (D-\rho)\right] + n\pi$$

(7.2.3.12)

so that

$$t = D - \frac{\pi}{2} + n\pi = \sqrt{\rho^2 + \left(\frac{\pi}{2}\right)^2} - \frac{\pi}{2} + n\pi \tag{7.2.3.13}$$

or inverting this to get ρ as a function of t,

$$\rho = \sqrt{\bar{t}(\bar{t} + \pi)} \tag{7.2.3.14}$$

where $\bar{t} \equiv t - n\pi$. So, at t = 0, the singular points are at ρ = 0, $\sqrt{2}\pi$, etc.

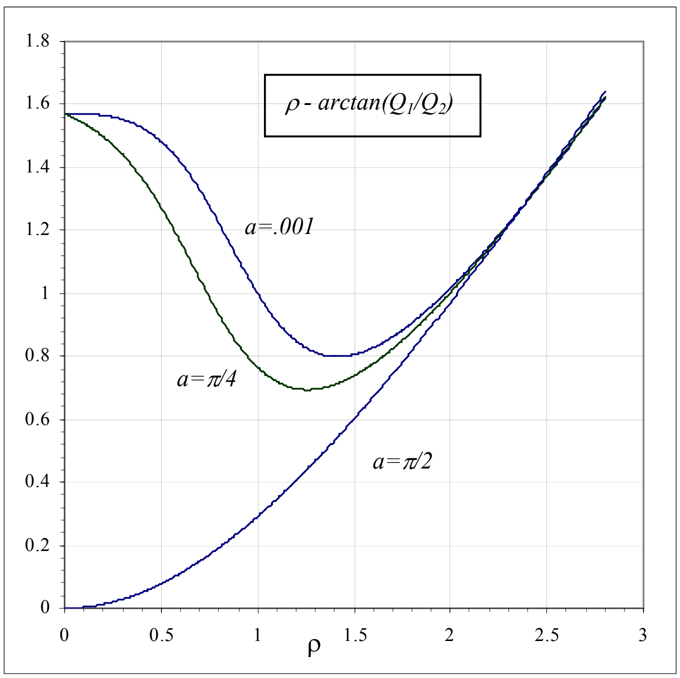

Figure 7.2-1 t vs ρ (a= .001, π/4, π /2)

Computing the phase velocity

Consider some constant value of the phase $(\tau + \delta)$. Then,

$$\frac{d}{d\rho}(\tau + \delta) = 0 \tag{7.2.3.15}$$

yields the expression for the phase velocity. In detail, we have

$$t = \rho - \arctan\left(\frac{\overline{Q}_1}{\overline{Q}_2}\right) \tag{7.2.3.16}$$

and its total derivative with respect to t:

$$1 = \dot{\rho}\left(1 - \frac{d}{d\rho}\arctan\left(\frac{\overline{Q}_1}{\overline{Q}_2}\right)\right) \tag{7.2.3.17}$$

or

$$v_{phase} = \dot{\rho} = \frac{1}{1 - \frac{\overline{Q}_2\overline{Q}_1{}' - \overline{Q}_1\overline{Q}_2{}'}{\overline{Q}_1{}^2 + \overline{Q}_2{}^2}} \tag{7.2.3.18}$$

For the special case of the resonant antenna ($a=\pi/2$) we can resort to the formula $\rho = \sqrt{\bar{t}(\bar{t}+\pi)}$ which yields

$$v_{phase} = \dot{\rho} = \sqrt{1 + \left(\frac{\pi}{2\rho}\right)^2} \ . \tag{7.2.3.19}$$

Note that in general when

$$W(\overline{Q}_1, \overline{Q}_2) \equiv \overline{Q}_2\overline{Q}_1{}' - \overline{Q}_1\overline{Q}_2{}' \ = \ \overline{Q}_1{}^2 + \overline{Q}_2{}^2 \tag{7.2.3.20}$$

the phase velocity $v_{phase} \to \infty$ (or $dt/d\rho = 0$). This happens at a well-defined place $\rho = \rho_c$ and time $t = t_c$. In Figure 7.2-1, the graph of t reaches its minimum value t_c at $\rho = \rho_c$.

The sign of v_{phase} can be used to define two regions, an outer region ($\rho > \rho_c$) where v_{phase} is positive and an inner region ($\rho < \rho_c$) where v_{phase} is negative (so that points of constant phase move toward the antenna). The topology of the electric field near the singular points in the inner region ($\rho < \rho_c$) is different from the outside region ($\rho > \rho_c$). By analyzing the "slope" equation of a singular point, viz.

$$\frac{E_z}{E_\rho} = \frac{dz}{d\rho} = \frac{Q_1\cos\tau + Q_2\sin\tau}{-P_1\cos\tau + P_2\sin\tau} \to \frac{0}{0} \tag{7.2.3.21}$$

one can show with the help of L'Hospital's rule that the singular points in the inner region are X-points and those in the outer region are O-points.

General Discussion of Singular Points

The field line equation for a two-variable case is

$$\frac{dz}{d\rho} = \frac{E_z(\rho, z)}{E_\rho(\rho, z)} \tag{7.2.3.22}$$

Assume that there are points S in the (ρ, z) 'plane' where both E_z and E_ρ vanish at the same time making equation (7.2.3.22) an indeterminate form. To ascertain the shape of the field lines near these singular points, expand E_z and E_ρ in power series in $d\rho$ and dz about a given singular point $S = (\rho_s, z_s)$. The increments

$$\begin{aligned} d\rho &= \rho - \rho_s \\ dz &= z - z_s \end{aligned} \tag{7.2.3.23}$$

are taken to be of the first order. Thus,

$$\left(\frac{dz}{d\rho}\right)_S = \frac{\left(\frac{\partial E_z}{\partial \rho}\right)_S d\rho + \left(\frac{\partial E_z}{\partial z}\right)_S dz + \ldots}{\left(\frac{\partial E_\rho}{\partial \rho}\right)_S d\rho + \left(\frac{\partial E_\rho}{\partial z}\right)_S dz + \ldots} \tag{7.2.3.24}$$

so to first order

$$\left(\frac{dz}{d\rho}\right)_S = \frac{E_{z,\rho} + E_{z,z}\left(\frac{dz}{d\rho}\right)_S}{E_{\rho,\rho} + E_{\rho,z}\left(\frac{dz}{d\rho}\right)_S} \tag{7.2.3.25}$$

where, for brevity,

$$E_{i,j} \equiv \left(\frac{\partial E_i}{\partial x_j}\right)_S \tag{7.2.3.26}$$

Solving (7.2.3.25) for $z' = \left(\frac{dz}{d\rho}\right)_S$, we get

$$E_{\rho,z} z'^2 + \left(E_{\rho,\rho} - E_{z,z}\right) z' - E_{z,\rho} = 0 \tag{7.2.3.27}$$

or

$$z' = \frac{1}{E_{\rho,z}}\left(\frac{E_{\rho,\rho} - E_{z,z}}{2} \pm \sqrt{\left(\frac{E_{\rho,\rho} - E_{z,z}}{2}\right)^2 + E_{\rho,z}E_{z,\rho}}\right) \tag{7.2.3.28}$$

Assuming that all the derivatives of $\boldsymbol{E}$ are well-behaved, then, if the discriminant

$$\Delta = \left(\frac{E_{\rho,\rho} - E_{z,z}}{2}\right)^2 + E_{\rho,z}E_{z,\rho} \tag{7.2.3.29}$$

is positive, the singular point *S* is called an *X*-point since the field lines appear to cross each other in the limit of small $d\rho$ and dz; otherwise it is called an *O*-point since the lines form infinitesimal loops (or possibly spirals) encircling the point *S*.

Singular Points of the Linear Antenna

The above discussion is quite general for a two-variable (two-'dimensional') topology. Turning to the specific case of a linear antenna, we immediately realize that there are an infinite series of singular points, both along the *z*-axis (the polar axis of the antenna) and along the ρ-axis (the equatorial plane of the antenna).

Along the Polar (z) Axis of the Antenna

Inspection of the expression for E_ρ shows that E_ρ is an *odd* function in both ρ and z, so that we may put

$$E_\rho = \rho z\, \Psi(\rho, z, t) \tag{7.2.3.30}$$

where Ψ is *even* in both ρ and z and is well-behaved everywhere outside the antenna. More explicitly, we have

$$\Psi(\rho, z, t) = \Psi_1(\rho, z)\cos\tau + \Psi_2(\rho, z)\sin\tau \tag{7.2.3.31}$$

Moreover, inspection of E_z shows that it is an *even* function of both ρ and z

$$E_z = Q_1(\rho, z)\cos\tau + Q_2(\rho, z)\sin\tau \tag{7.2.3.32}$$

Now, along the z-axis, $E_\rho \Rightarrow 0$ for $\rho \Rightarrow 0$. Thus, wherever E_z=0, that is, at all values of z for which

$$\tan\left(t - |z|\right) = \frac{-Q_1(0, z)}{Q_2(0, z)} \tag{7.2.3.33}$$

there is a singular point.

Examine (7.2.3.28) near such a singular point on the z-axis. That is, let $\rho = \varepsilon$ where ε is assumed to be of *second* order so that $\varepsilon << d\rho$. Since E_ρ is even in ρ, $E_{z,\rho}$ goes to zero as ρ goes to zero. Furthermore,

$$E_{\rho,z} = \varepsilon\, \Psi(0, z_s, t) \tag{7.2.3.34}$$

so that $E_{\rho,z}E_{z,\rho}$ in equation (7.2.3.29) is of second order compared to

$$\eta^2 \equiv \left(\frac{E_{\rho,\rho} - E_{z,z}}{2}\right)^2 \tag{7.2.3.35}$$

provided η is finite and well-behaved as $\rho \Rightarrow 0$. This is so for the linear antenna. Thus, equation (7.2.3.29) reduces to

$$z' = \frac{\eta \pm |\eta|}{\varepsilon\, \Psi(0, z, t)} \tag{7.2.3.36}$$

which implies that in the limit $\varepsilon \Rightarrow 0$, $z' \Rightarrow 0$ or ∞. In other words, the singular points along the z-axis are X-points with two separatrix lines crossing at 90°, one line horizontal and the other vertical.

Along the Equatorial Plane (z = 0)

For finite values of ρ, as $z \Rightarrow 0$, $E_\rho \Rightarrow 0$ so, whenever $E_z = 0$, there is a singular point. This will happen whenever

$$\tan(t-\rho) = \frac{-Q_1(\rho,0)}{Q_2(\rho,0)} \qquad (7.2.3.37)$$

Now, for $z \Rightarrow 0$, $E_{\rho,\rho} \Rightarrow 0$ and $E_{z,z} \Rightarrow 0$ (because E_z is even in z). Therefore, equation (7.2.3.28) reduces to

$$z' = \pm\left(\sqrt{\frac{E_{z,\rho}}{E_{\rho,z}}}\right)_{z\Rightarrow 0} \qquad (7.2.3.38)$$

It remains to work out in detail the two derivatives at $z = 0$.

Using equation (7.2.3.32) we have for constant t:

$$E_{z,\rho} = \left[\left(Q_1' - Q_2\frac{\partial r}{\partial \rho}\right)\cos\tau + \left(Q_1\frac{\partial r}{\partial \rho} - Q_2'\right)\sin\tau\right]_{z=0} \qquad (7.2.3.39)$$

with $Q_i' \equiv \left(\frac{\partial Q_i}{\partial \rho}\right)_{z=0}$.

Since $\frac{\partial r}{\partial \rho} = 1$ and, by virtue of equation (7.2.3.37),

$$\cos\tau = \frac{Q_2}{\sqrt{Q_1^2 + Q_2^2}} \qquad \sin\tau = \frac{-Q_1}{\sqrt{Q_1^2 + Q_2^2}}, \qquad (7.2.3.40)$$

equation (7.2.3.39) becomes

$$E_{z,\rho} = \frac{Q_1Q_2' - Q_1'Q_2}{\sqrt{Q_1^2 + Q_2^2}} - \sqrt{Q_1^2 + Q_2^2} \quad . \qquad (7.2.3.41)$$

Numerical calculations show that for small ρ, $E_{z,\rho}$ is negative and becomes positive for $\rho > \rho_c$, where ρ_c is determined from the condition $E_{z,\rho} = 0$. Moreover, one finds that $E_{\rho,z}$ is negative for all $\rho < \rho_c$ and beyond. Thus, equation (7.2.3.41) implies that for all $\rho < \rho_c$, the singular points in the equatorial plane of the antenna are of the *X*-type and become *O*-points for $\rho > \rho_c$.

Figure 7.2-2 shows these two classes of singular points for the resonant antenna case.

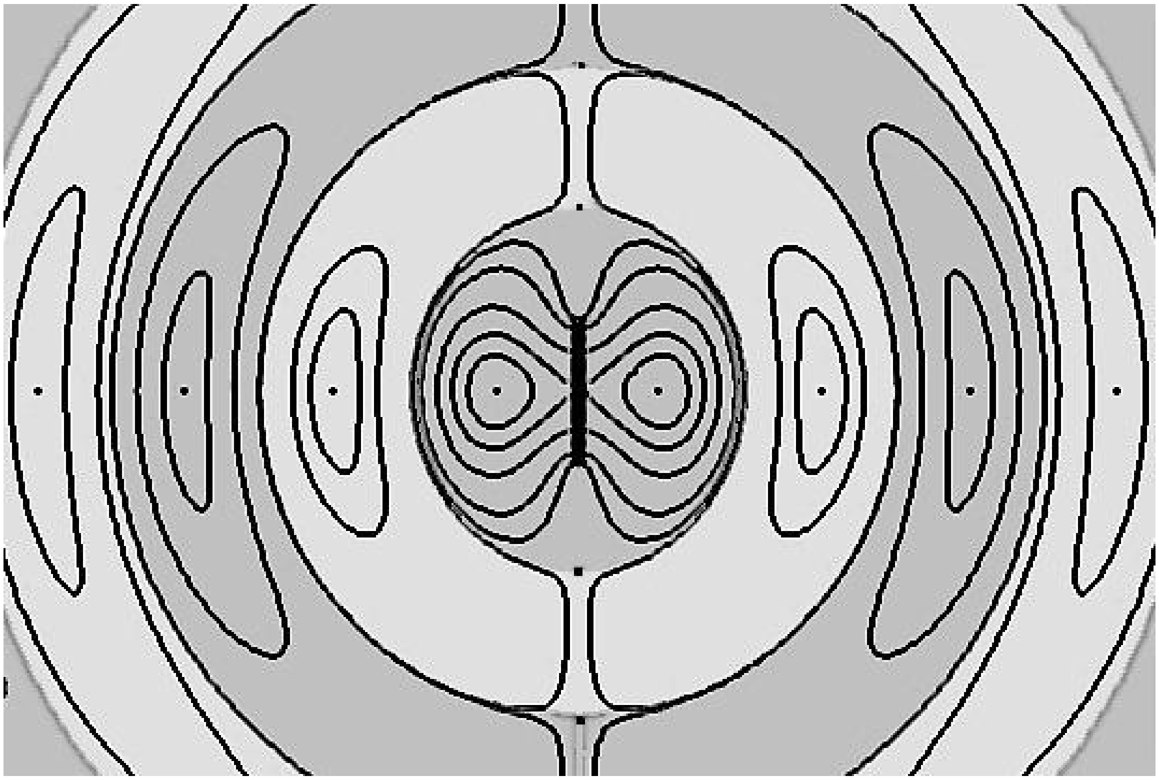

Figure 7.2-2: Singular points for the resonant antenna.

8 Appendices

8.1 What is new in Version 1.1 compared to Version 1.0?

We have added an extensive discussion of the mathematics of the linear antenna, see Section 7.2.

8.2 Time Evolution of Field Lines Using Flux Functions

This appendix extends considerations in 1.3.2 above. We use spherical coordinates here, instead of cylindrical coordinates as in the main text, with no loss of generality. Suppose that we have a scalar function $A(r,\theta,t)$ such that the components of the magnetic field are given at all times *t* by

$$\mathbf{B}(r,\theta,t) = \nabla \times (A(r,\theta,t)\hat{\phi}) \tag{9.2.1}$$

or

$$B_r(r,\theta) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta A(r,\theta)\right] and B_\theta(r,\theta) = -\frac{1}{r}\frac{\partial}{\partial r}\left[rA(r,\theta)\right] \tag{9.2.2}$$

If we define the flux function $F(r,\theta,t) = 2\pi\ r\sin\theta A(r,\theta,t)$ (cf. 1.3.1.4), then

$$\frac{1}{2\pi}\frac{\partial F(r,\theta,t)}{\partial\theta} = r^2\sin\theta B_r(r,\theta,t) \qquad \frac{1}{2\pi}\frac{\partial F(r,\theta,t)}{\partial r} = -r\sin\theta B_\theta(r,\theta,t) \tag{9.2.3}$$

We want to determine the time evolution of field lines in this case. That is, how does one make the correspondence between a field line at one time and the "same" field line at a different time?

First of all, suppose that the curve $R(\theta,t_o)$ at time t_o satisfies $F(R(\theta,t_o),\theta,t_o) = F_o$. Then we can show that $R(\theta,t_o)$ is a field line at that time, as follows. If we look at the variation in the flux function as we vary the spatial values at a given time, we have

$$F(r+\delta r,\theta+\delta\theta,t_0) = F_o + \frac{\partial F}{\partial r}\delta r + \frac{\partial F}{\partial\theta}\delta\theta \tag{9.2.4}$$

If $R(\theta,t_o)$ is such that it exhibits constant flux levels at time t_o along its length, we must have that

$$\frac{\partial F}{\partial r}\delta R + \frac{\partial F}{\partial\theta}\delta\theta = 0 \tag{9.2.5}$$

but this is simply

$$\frac{\delta R}{r\delta\theta} = -\frac{1}{r}\frac{\partial F}{\partial \theta} / \frac{\partial F}{\partial r} = \frac{B_r}{B_\theta} \tag{9.2.6}$$

This is the equation defining a field line, so that at t_o $R(\theta, t_o)$ is a field line.

Now, let that field line evolve in time in the manner we have defined in 1.3.2. We then assert that at time t this "same" field line is given by $R(\theta, t)$, where $R(\theta, t)$ satisfies the equation $F(R(\theta,t),\theta,t) = F_o$. That is, the evolution of the field line in time can be traced by solving $F(R(\theta,t),\theta,t) = F_o$, where the same constant F_O characterizes the "same" field line at all times t.

To show this, we first review how we define the motion of field lines. For magnetic field lines, we follow the evolution of a field line by following the motion of a low energy particle gyrating about the field line as time progresses. Physically, we trace our field lines by tracing the motion of particles attached to the field lines. We know that this motion is given by the $\mathbf{E}\times\mathbf{B}$ drift. That is, low energy particles of either sign will move in a time-changing magnetic field at a velocity given by $\mathbf{v} = \mathbf{E}\times\mathbf{B}/B^2$.

To calculate this drift velocity, we need **E**. How do we calculate **E** in this situation? Faraday's Law and (9.2.1) tell us that

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = \nabla\times\left(-\frac{\partial A}{\partial t}\hat{\phi}\right) \tag{9.2.7}$$

so that we have

$$\mathbf{E} = -\frac{\partial A}{\partial t}\hat{\phi} \tag{9.2.8}$$

and therefore, using (9.2.8) in $\mathbf{v} = \mathbf{E}\times\mathbf{B}/B^2$, we have

$$\mathbf{v} = -\frac{\partial A}{\partial t}\hat{\boldsymbol{\theta}}\times\frac{\left[\hat{\mathbf{r}}B_r + \hat{\boldsymbol{\theta}}B_\theta\right]}{\left[B_r^{\,2} + B_\theta^{\,2}\right]} = -\frac{\partial A}{\partial t}\frac{\left[-\hat{\mathbf{r}}B_\theta + \hat{\boldsymbol{\theta}}B_r\right]}{\left[B_r^{\,2} + B_\theta^{\,2}\right]} \tag{9.2.9}$$

or, using (9.2.3)

$$\mathbf{v} = -\frac{\partial A}{\partial t}\frac{\left[-\hat{\mathbf{r}}B_\theta + \hat{\boldsymbol{\theta}}B_r\right]}{\left[B_r^{\,2} + B_\theta^{\,2}\right]} = -\frac{\partial A}{\partial t}\frac{\left[\hat{\mathbf{r}}\frac{1}{2\pi}\frac{1}{r\sin\theta}\frac{\partial F}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{2\pi}\frac{1}{r^2\sin\theta}\frac{\partial F}{\partial \theta}\right]}{\left[\left(\frac{1}{2\pi}\frac{1}{r^2\sin\theta}\frac{\partial F}{\partial \theta}\right)^2 + \left(\frac{1}{2\pi}\frac{1}{r\sin\theta}\frac{\partial F}{\partial r}\right)^2\right]} \tag{9.2.10}$$

\

Now we know what our drift velocity is in terms of the flux function. We can thus define the way a given point *(x,y)* on a field line "drifts" or evolves in time. Let $\delta\mathbf{r}$ be the distance a point (x,y) on a given field line moves in time δt. Then we want (using $F(r,\theta,t) = 2\pi\; r\sin\theta A(r,\theta,t)$)

$$\delta\mathbf{r} = \mathbf{v}\delta t = -\delta t \frac{\partial F}{\partial t} \frac{\left[\hat{\mathbf{r}} \frac{\partial F}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial F}{\partial \theta} \right]}{\left[\frac{1}{r^2} \left(\frac{\partial F}{\partial \theta} \right)^2 + \left(\frac{\partial F}{\partial r} \right)^2 \right]} \tag{9.2.11}$$

Note that $\delta\mathbf{r} \cdot \mathbf{B}$ is zero, meaning that a field line moves perpendicular to itself in time. This is not a problem, since there is no physical meaning to a field line moving parallel to itself, so that we may take any parallel motion to be zero.

Now, we want to show that as we follow a field line in time using (9.2.11), then that field line always has the same value of the flux function $F(R(\theta,t),\theta,t)$. Let F_O be the value of F at (r,θ,t). Then at $(r+\delta r, \theta+\delta\theta, t+\delta t)$, we have

$$F(r+\delta r, \theta+\delta\theta, t+\delta t) = F_o + \frac{\partial F}{\partial r}\delta r + \frac{\partial F}{\partial \theta}\delta\theta + \frac{\partial F}{\partial t}\delta t \tag{9.2.12}$$

which means that for those points $(r+\delta r, \theta+\delta\theta, t+\delta t)$ that preserve F_O,

$$\frac{\partial F}{\partial r}\delta r + \frac{\partial F}{\partial \theta}\delta\theta + \frac{\partial F}{\partial t}\delta t = 0 \tag{9.2.13}$$

With no loss of generality, we can assume that our displacement $\delta\mathbf{r} = \delta r\hat{\mathbf{r}} + r\delta\theta\,\hat{}$ is perpendicular to the field (see above), which means that

$$\frac{1}{r}\frac{\delta r}{\delta\theta} = -\frac{B_\theta}{B_r} = -\frac{-\frac{1}{2\pi}\frac{1}{r\sin\theta}\frac{\partial F}{\partial r}}{\frac{1}{2\pi}\frac{1}{r^2\sin\theta}\frac{\partial F}{\partial \theta}} = r\frac{\frac{\partial F}{\partial r}}{\frac{\partial F}{\partial \theta}} \tag{9.2.14}$$

where we have used (9.2.3). Now, (9.2.13) and (9.2.14) can be considered as two equations for the two unknowns δr and $\delta\theta$. Solving for these gives

$$\delta\mathbf{r} = \delta r\hat{\mathbf{r}} + r\delta\theta\hat{\boldsymbol{\theta}} = -\delta t \frac{\partial F}{\partial t} \frac{\left[\hat{\mathbf{r}} \frac{\partial F}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{\mathrm{r}} \frac{\partial F}{\partial \theta} \right]}{\left[\left(\frac{\partial F}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial F}{\partial \theta} \right)^2 \right]} \tag{9.2.15}$$

But this equation, which preserves the value of F_O as the field line evolves in time, is the same as equation (9.2.11) above, which describes the $\mathbf{E}\times\mathbf{B}$ drift of the field line. Therefore, the $\mathbf{E}\times\mathbf{B}$ drift of the field line points also conserves the value of the flux function, as was to be demonstrated.

9 References

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