Faraday's Law

Equation Chapter (Next) Section 1
Equation Chapter (Next) Section 1
Equation Chapter (Next) Section 1
Equation Chapter (Next) Section 1

The falling magnet

We discuss in the last section below the appropriate equation if the motion of the magnet is given, rather than determined dynamically, as in the development below.

The equation of motion

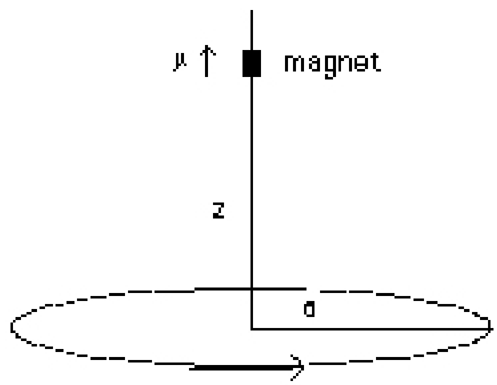

Figure Error! No text of specified style in document.**-1: Geometry of the falling ring**

We have a 3D dipole with dipole moment $\vec{\mu} = \mu\hat{\mathbf{z}}$. It moves on the axis of a circular loop of radius *a*, resistance *R*, inductance *L*, with inductive time constant *L/R*. It moves downward under the influence of gravity. The equation of motion is

$$m\frac{d^2z}{dt^2} = -mg + \mu\frac{dB_z}{dz} \tag{4.1.1}$$

where B_Z is the field due the current *I* in the ring (positive in the direction shown in **Figure** Error! No text of specified style in document.**-1**)[1]. The expression for B_Z is

$$B_z = \frac{\mu_o I a^2}{2(a^2+z^2)^{3/2}} \tag{4.1.2}$$

so that equation (4.1.1) is

[1]This is the appropriate equation for both the situation of the ring at rest and the magnet moving, or the magnet at rest and the ring moving--the mass *m* switches from the mass of the magnet to the mass of the ring, depending on the situation.

$$m\frac{d^2z}{dt^2} = -mg + \frac{\mu\mu_o I a^2}{2}\frac{d}{dz}\frac{1}{(a^2+z^2)^{3/2}} \quad (4.1.3)$$

or

$$m\frac{d^2z}{dt^2} = -mg - \frac{3\mu\mu_o I a^2}{2}\frac{z}{(a^2+z^2)^{5/2}} \quad (4.1.4)$$

Determining an equation for *I* from Faraday's Law

Faraday's Law is

$$\oint \mathbf{E}\cdot\mathbf{dl} = -\frac{d}{dt}\int \mathbf{B}\cdot\mathbf{dA} = -\frac{d}{dt}\int \mathbf{B}_{dipole}\cdot\mathbf{dA} - L\frac{dI}{dt} \quad (4.1.5)$$

and if $\mathbf{E} = \rho\mathbf{J}$, then

$$\oint \mathbf{E}\cdot\mathbf{dl} = \oint \rho\mathbf{J}\cdot\mathbf{dl} = I\oint \rho dl/A = IR, \text{ with } R = \oint \rho dl/A \quad (4.1.6)$$

so that

$$IR = -L\frac{dI}{dt} - \frac{d}{dt}\int \mathbf{B}_{dipole}\cdot\mathbf{dA} \quad (4.1.7)$$

We need to determine the magnetic flux through the ring due to the dipole field. To do this we calculate the flux through a spherical cap of radius $\sqrt{a^2+z^2}$ with an opening angle given θ given by $\sin\theta = a/\sqrt{a^2+z^2}$ (this is the same as the flux through the ring because $\nabla\cdot\mathbf{B} = 0$).

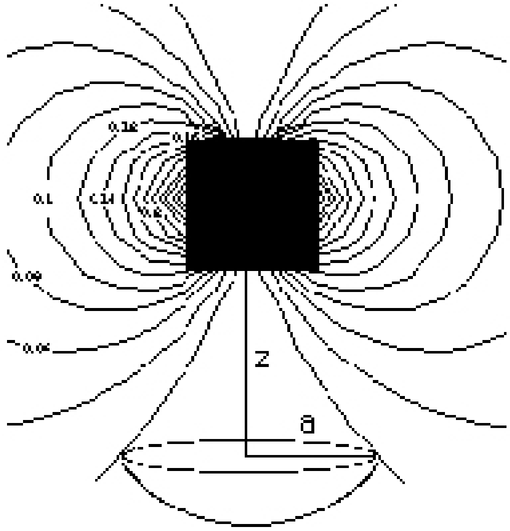

Figure Error! No text of specified style in document.**-2: Dipole flux through a polar cap**

The flux through a spherical cap only involves the radial component of the dipole field, given by

$$B_z = \frac{\mu_o}{2\pi}\frac{\mu\cos\theta}{r^3} \tag{4.1.8}$$

Our expression for the flux is thus

$$\int \mathbf{B}_{dipole}\cdot\mathbf{dA} = \int \frac{\mu_o}{2\pi}\frac{\mu\cos\theta}{r^3}2\pi r^2\sin\theta d\theta \tag{4.1.9}$$

which can be integrated to give

$$\int \mathbf{B}_{dipole}\cdot\mathbf{dA} = -\frac{\mu_o\mu}{r}\int\cos\theta d\cos\theta = -\frac{\mu_o\mu}{2r}\left(\cos^2\theta - 1\right) = \frac{\mu_o\mu}{2r}\sin^2\theta = \frac{\mu_o\mu}{2}\frac{a^2}{\left(a^2+z^2\right)^{3/2}} \tag{4.1.10}$$

Inserting (4.1.10) into (4.1.6) yields

$$IR = -L\frac{dI}{dt} - \frac{d}{dt}\frac{\mu_o\mu}{2}\frac{a^2}{\left(a^2+z^2\right)^{3/2}} \tag{4.1.11}$$

We assume that μ is constant, with $\mu = M_o$, so that (4.1.11) becomes

$$IR = -L\frac{dI}{dt} + \frac{3\mu_o a^2 M_o}{2}\frac{z}{\left(a^2+z^2\right)^{5/2}}\frac{dz}{dt} \tag{4.1.12}$$

Equations (4.1.4) and (4.1.12) are our coupled ordinary differential equations which determine the dynamics of the situation.

Dimensionless Form of the Equations

We now put these equations into dimensionless form. We measure all distances in terms of the distance *a*, and all times in terms of the time $\sqrt{a/g}$. Let

$$z' = \frac{z}{a} t' = \frac{t}{\sqrt{a/g}} I' = \frac{I}{I_o}, where I_o = \frac{mga^2}{\mu_o M_o} \tag{4.1.13}$$

The time $\sqrt{a/g}$ is roughly the time it would take the magnet to fall under the influence of gravity through a distance *a* starting from rest. The current I_O is roughly the current in the ring that is required to produce a force sufficient to offset gravity when the magnet is a distance *a* above the ring. In terms of these variables, our equations (4.1.4) and (4.1.11) are

$$\frac{d^2 z'}{dt'^2} = -1 - \frac{3}{2}\frac{z' I'}{(1+z'^2)^{5/2}} \tag{4.1.14}$$

$$I' = -\frac{L}{R}\sqrt{\frac{g}{a}}\frac{dI'}{dt'} + \frac{3(\mu_o M_o)^2}{2mga^3 R}\sqrt{\frac{g}{a}}\frac{z'}{\left(1+z'^2\right)^{5/2}}\frac{dz'}{dt'} \tag{4.1.15}$$

We introduce the four dimensionless parameters

$$\alpha = \frac{R}{L}\sqrt{\frac{a}{g}} \qquad \beta = \frac{\mu_o M_o}{L I_o a} = \frac{(\mu_o M_o)^2}{Lmga^3} \qquad \lambda = \frac{L}{\mu_o a} \qquad D = \sqrt{\frac{g}{a}}\frac{\mu_o a}{R} = \frac{1}{\lambda\alpha} \tag{4.1.16}$$

and we define the reduced flux rate function *F(z')* as

$$F(z') = \frac{3}{2}\frac{z'}{(1+z'^2)^{5/2}} \tag{4.1.17}$$

Note that we can write the reference current I_O as

$$\frac{I_o a^2}{M_o} = \frac{mga^4}{\mu_o M_o^{\,2}} = \frac{1}{\lambda\beta} \tag{4.1.18}$$

The parameters have the following physical meanings. The quantity α is the ratio of the free fall time to the inductive time constant--if α is very large, inductive effects are negligible. The quantity β is roughly the ratio of the current due to induction alone, assuming the resistance is zero (Φ_{dipole} / L with $\Phi_{dipole} \approx \dfrac{\mu_o M_o}{a}$), to the reference current I_O. With these definitions, our equations become

$$\frac{d^2 z'}{dt'^2} = -1 - F(z')I' \tag{4.1.19}$$

$$\frac{dI'}{dt'} = -\alpha I' + \beta F(z')\frac{dz'}{dt'} \tag{4.1.20}$$

If we define the speed $\mathrm{v}' = dz' / dt'$, then we can write three coupled first-order ordinary differential equations for the triplet (z', I', v'), as

$$dz' / dt' = \mathrm{v}' \tag{4.1.21}$$

$$\frac{dI'}{dt'} = -\alpha I' + \beta F(z')\frac{dz'}{dt'} \tag{4.1.22}$$

$$\frac{d\mathrm{v}'}{dt'} = -1 - F(z')\, I' \tag{4.1.23}$$

Conservation of Energy

We assume that μ is constant. If we multiply (4.1.4) by $\mathrm{v} = \dfrac{dz}{dt}$ and (4.1.12) by *I*, after some algebra, we find that

$$\frac{d}{dt}\left[\tfrac{1}{2} m\mathrm{v}^2 + mgz + \tfrac{1}{2} L\, I^2\right] = -I^2 R \tag{4.1.24}$$

which expresses conservation of energy for the falling magnet plus the magnetic field of the ring. The dimensionless form of this equation is

$$\frac{d}{dt'}\left[\tfrac{1}{2}\mathrm{v}'^2 + z' + \frac{1}{4\beta} I'^2\right] = -\frac{\alpha}{2\beta} I'^2 \tag{4.1.25}$$

Suppose the resistance of the ring (the superconducting case) is zero (i.e., $\alpha = 0$). In this case, (4.1.25) becomes, with one integration

$$I' = C - \frac{\beta}{\left(1+z'^2\right)^{3/2}} \tag{4.1.26}$$

If we impose boundary conditions that $I' = 0$ when $t' = 0$, with $z' = z_o'$ and $v' = v_o'$ at $t' = 0$, then this is

$$I' = \beta\left[\frac{1}{\left(1+z_o'^2\right)^{3/2}} - \frac{1}{\left(1+z'^2\right)^{3/2}}\right] \tag{4.1.27}$$

Using this equation, (4.1.25) for the conservation of energy becomes

$$\tfrac{1}{2}v'^2 + \left\{\left(z' - z_o'\right) + \frac{\beta}{4}\left[\frac{1}{\left(1+z_o'^2\right)^{3/2}} - \frac{1}{\left(1+z'^2\right)^{3/2}}\right]^2\right\} = 0 \tag{4.1.28}$$

Numerical Solutions

A magnet falls through a copper ring. At the ring, the speed of the magnet decreases. When the magnet is through the ring, the magnet resumes free fall. We show a numerical solution to equations (4.1.21) through (4.1.23) above, appropriate to this case. The initial conditions (z', I', v') for the solution plotted below are (2,0,0). The values of (α, β) are (10,100). Below we plot position as a function of time and current as a function of time (using our dimensionless parameters).

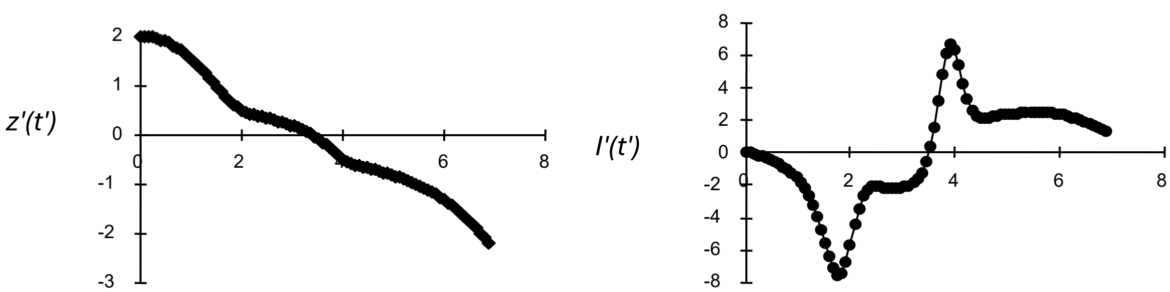

Figure Error! No text of specified style in document.-3: Numerical Solutions for the Falling Ring Equations

The behavior of these solutions is what we expect. When the magnet reaches a distance of about *a* from the ring, it slows down, because of the increasing current in the ring, which repels the magnet. As the magnet passes through the ring, the current reverses direction, now attracting the magnet from above, which also slows the magnet. Finally the magnet falls far enough that the current in the ring goes to zero, and the magnet is again in free fall.

These are approximate solutions only, using an Excel spreadsheet with an Euler integration scheme. In the final animations, we use a fourth order Runge-Kutta scheme to integrate the equations with high accuracy.

The topology of the field

How do we plot the field configuration given solutions for z' and I'? The absolute current is given by (cf. equations (4.1.15) and (4.1.16))

$$I = I_o I' = \frac{1}{\lambda\beta}\frac{M_o}{a^2}I' \tag{4.1.29}$$

How much freedom do we have in choosing the absolute value of the current once we have solved our dimensionless equations? And in particular how does that freedom affect the topology of the magnetic field? One measure of the shape of the total field is the ratio of the field at the center of the ring due to the ring to the field at the center of the ring due to the magnet when the magnet is a distance a above the ring, i.e.,

$$\frac{B_{ring}\Big|^{\text{center of ring}}}{B_{dipole}\Big|^{\text{at center of ring when dipole at a}}} = \frac{\dfrac{\mu_o I}{2a}}{\dfrac{\mu_o}{2\pi}\dfrac{M_o}{a^3}} \approx \frac{Ia^2}{M_o} \tag{4.1.30}$$

where we are dropping numerical factors. Clearly when this ratio varies the overall shape of the total field must vary. If we use (4.1.6.1) in (4.1.6.2) we have

$$\frac{B_{ring}\Big|^{\text{center of ring}}}{B_{dipole}\Big|^{\text{at center of ring when dipole at a}}} \approx \frac{a^2}{M_o}I = \frac{a^2}{M_o}\frac{1}{\lambda\beta}\frac{M_o}{a^2}I' = \frac{I'}{\lambda\beta} \tag{4.1.31}$$

The meaning of equation (4.1.31) is that the overall shape of the magnetic field topology is totally determined once we make the one remaining choice of the dimensionless constant λ, defined in equation (4.1.16) which up to this point we have not chosen (we have only picked values of α and β to solve our dimensionless equation). Once that choice is made, we have no additional freedom to affect the field topology.

The case when the motion of the magnet is given

A simpler case is the situation when we have a magnet on the axis of a ring which moves with a specified position and velocity. In that case we no longer must solve three coupled differential equations. We need only solve Eq. (4.1.12) for the current as a function of time, or in dimensionless form, Eq. (4.1.22). This is a first order differential equation for the current, if we assume that the position and velocity of the magnet is given. That is, we are solving this equation, where the terms on the right-hand-side are known functions of time.

$$\frac{dI'}{dt'} + \alpha I' = \beta F(z') \,\mathrm{v}(t') \tag{4.1.32}$$

We multiply by the "integrating factor" $e^{-\alpha t'}$ to obtain

$$e^{-\alpha t'}\left(\frac{dI'}{dt'} + \alpha I'\right) = \frac{d}{dt'}\left(e^{-\alpha t'} I'\right) = e^{-\alpha t'} \beta F(z'(t'))\,\mathrm{v}(t') \tag{4.1.33}$$

Integrating from 0 to time *T*, we have

$$e^{-\alpha T} I'(T) - I'(0) = \int_0^T dt' e^{-\alpha t'} \beta F(z'(t'))\,\mathrm{v}(t') \tag{4.1.34}$$

Assuming that the current initially is zero, we have that

$$I'(T) = \int_0^T dt' e^{-\alpha (T-t')} \beta F(z'(t'))\,\mathrm{v}(t') \tag{4.1.35}$$

Thus, we see that the current at time *T* depends on the history of the motion of the magnet, as we expect. However, the further in the past the motion is, in time units of $1/\alpha$, the less it contributes to the current value of the current. If the motion of the magnet stops, the current will simply decay from its stopping value exponentially as $e^{-\alpha t'}$.

In Faraday's Law application, we keep track of the history of the motion of the magnet or ring, as input by the user, so that we can perform the integration in the above equation.