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# Integer-empty polytopes in the 0/1-cube with maximal Gomory–Chvátal rank

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ABSTRACT

We provide a complete characterization of all polytopes $P \subseteq [0, 1]^n$ with empty integer hulls, whose Gomory–Chvátal rank is $n$ (and, therefore, maximal). In particular, we show that the first Gomory–Chvátal closure of all these polytopes is identical.



## 1. Introduction

The Gomory–Chvátal procedure is a well-known technique to derive valid inequalities for the integral hull $P_I$ of a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. It was introduced by Chvátal [2] and, implicitly, by Gomory [6–8] as a means to establish certain combinatorial properties via cutting-plane proofs. Cutting planes and Gomory–Chvátal cuts, in particular, belong to today's standard toolbox in integer programming. However, despite significant progress in recent years (see, e.g., [1,3,5,9]), the Gomory–Chvátal procedure is still not fully understood from a theoretical standpoint, especially in the context of polytopes contained in the 0/1-cube. For example, the question whether the currently best known upper bound of $O(n^2 \log n)$ on the Gomory–Chvátal rank, established in [5], is tight, remains open. In [5], it was also shown that there is a class of polytopes contained in the $n$-dimensional 0/1-cube whose rank exceeds $n$. (See [11] for a more explicit construction.) However, no family of polytopes in the 0/1-cube is known that realizes super-linear rank, and thus there is a large gap between the best known upper bound and the largest realized rank.

We consider the special case of $P \subseteq [0, 1]^n$ with $P_I = \emptyset$ and Gomory–Chvátal rank $\text{rk}(P) = n$ (i.e., maximal rank, as $\text{rk}(P) \leq n$ holds for all $P \subseteq [0, 1]^n$ with $P_I = \emptyset$; see [1]). This case is of particular interest as, so far, all known proofs of polynomial upper bounds on the rank of polytopes in the 0/1-cube (cf., [1,5]) crucially depend on this special case. The improvement from $O(n^3 \log n)$ in [1] to $O(n^2 \log n)$ in [5] as an upper bound on the rank of polytopes in $[0, 1]^n$ is a direct consequence of a better upper bound on the rank of certain polytopes in the 0/1-cube that do not contain integral points. It can actually be shown that lower bounds on the rank of polytopes $P \subseteq [0, 1]^n$ with $P_I = \emptyset$ play a crucial role in understanding the rank of *any* (well-defined) cutting-plane procedure [10]. Moreover, in many cases, the rank of a face $F \subseteq P$ with $F_I = \emptyset$ induces a lower bound on the rank of $P$ itself. In fact, the construction of the aforementioned families of polytopes in $[0, 1]^n$ whose rank is strictly larger than $n$ exploits this connection.

In view of this, a thorough understanding of the Gomory–Chvátal rank of polytopes $P \subseteq [0, 1]^n$ with $P_I = \emptyset$ might help to derive better upper and lower bounds for the general case. In this paper, we characterize all polytopes $P \subseteq [0, 1]^n$ with $P_I = \emptyset$ and $\text{rk}(P) = n$. In particular, we show that after applying the Gomory–Chvátal procedure once, one always obtains the same polytope. Furthermore, we show that $P \subseteq [0, 1]^n$ with $P_I = \emptyset$ has $\text{rk}(P) = n$ if and only if $P \cap F \neq \emptyset$ for all one-dimensional faces $F$ of the 0/1-cube $[0, 1]^n$.

The paper is organized as follows. In Section 2, we introduce our notation and recall some basic facts about the Gomory–Chvátal procedure. Afterwards, in Section 3, we derive the characterization of all polytopes $P \subseteq [0, 1]^n$ with $P_I = \emptyset$ and $\text{rk}(P) = n$. In particular, in Section 3.2, we relate the rank of a polytope $P \subseteq [0, 1]^n$ with $P_I = \emptyset$ to the rank of its faces. We then prove the characterization for the two-dimensional case in Section 3.3, which is an essential ingredient for the subsequent generalization to arbitrary dimension in Section 3.4.

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## 2. Preliminaries

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a polytope with $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$. The *Gomory–Chvátal closure* of $P$ is defined as

$$P' := \bigcap_{\lambda\in\mathbb{R}^m_+,\lambda A\in\mathbb{Z}^n} \{x : \lambda Ax \leq \lfloor \lambda b\rfloor\}.$$

The result $P'$ is again a polytope (see [2]), and one can apply the operator iteratively. We let $P^{(i+1)} := (P^{(i)})'$ for $i \geq 0$ and $P^{(0)} := P$. The resulting sequence $\{P^{(i)}\}_{i\geq 0}$ becomes stationary after finitely many steps [2], and the smallest $k$ such that $P^{(k+1)} = P^{(k)}$ is the *Gomory–Chvátal rank of* $P$ (in the following often *rank of* $P$), denoted by $\mathrm{rk}(P)$. In particular, $P^{(\mathrm{rk}(P))} = P_I$, where $P_I := \mathrm{conv}(P \cap \mathbb{Z}^n)$ denotes the *integral hull of* $P$.

We will make repeated use of the following well-known lemma:

**Lemma 2.1** (*[4, Lemma 6.33]*). *Let $P$ be a rational polytope and let $F$ be a face of $P$. Then $F' = P' \cap F$.*

If $P \subseteq [0, 1]^n$ and $P_I = \emptyset$, Lemma 2.1 can be used to derive an upper bound on $\mathrm{rk}(P)$.

**Lemma 2.2** (*[1, Lemma 3]*). *Let $P \subseteq [0, 1]^n$ be a polytope with $P_I = \emptyset$. Then $\mathrm{rk}(P) \leq n$.*

This bound is actually tight; a family of polytopes $A_n \subseteq [0, 1]^n$ with $(A_n)_I = \emptyset$ and $\mathrm{rk}(A_n) = n$ was described in [3, p. 481].

For $i \in [n]$, the $i$-th coordinate flip maps $x_i \mapsto 1-x_i$ and $x_j \mapsto x_j$ for $i \neq j$. Another property that we will extensively use is that the Gomory–Chvátal operator is commutative with unimodular transformations, in particular, coordinate flips.

**Lemma 2.3** (*[5, Lemma 4.3]*). *Let $P \subseteq [0, 1]^n$ be a polytope and let $u$ be a coordinate flip. Then $(u(P))' = u(P')$.*

Given polytopes $P \subseteq [0, 1]^n$, $Q \subseteq [0, 1]^k$, and a $k$-dimensional face $F$ of $[0, 1]^n$, we say that $P \cap F \cong Q$ if the canonical projection of $P \cap F$ onto $[0, 1]^k$ is equal to $Q$. We denote the *interior of* $P$ by $\mathrm{Int}(P)$ and, with $P, F$, and $Q$ as before, the *relative interior of $P$ with respect to $F$* is defined as $\mathrm{RInt}_F(P) := \mathrm{Int}(Q)$. We use $e$ to denote the all-one vector, and $\frac{1}{2}e$ to denote the all-one-half vector. If $I \subseteq [n] \times \{0, 1\}$, $\frac{1}{2}e^I$ has coordinates $\frac{1}{2}e^I_i = \frac{1}{2}$ whenever $(i, l) \notin I$, and $\frac{1}{2}e^I_i = l$ for $(i, l) \in I$. Similarly, if $F$ is a face of $[0, 1]^n$, we define $\frac{1}{2}e^F \in F$ to be $\frac{1}{2}$ in those coordinates not fixed by $F$. Moreover, we define $F_k$ to be the set of all vectors $x \in \left\{0, \frac{1}{2}, 1\right\}^n$ such that exactly $k$ coordinates are equal to $\frac{1}{2}$, and the remaining coordinates are in $\{0, 1\}$. For convenience, we use $[n] := \{1, \ldots, n\}$ for $n \in \mathbb{N}$.

## 3. Polytopes $P \subseteq [0, 1]^n$ with $P_I = \emptyset$ and maximal rank

For $n \in \mathbb{N}$, we define the polytope $B_n \subseteq [0, 1]^n$ by

$$B_n := \left\{ x \in [0, 1]^n \,\middle|\, \sum_{i\in S} x_i + \sum_{i\in[n]\setminus S} (1 - x_i) \geq 1 \text{ for all } S \subseteq [n] \right\}.$$

Note that, $(B_n)_I = \emptyset$. This family of polytopes will be essential to our subsequent discussion.

### 3.1. Properties of $B_n$

In the following section, we will characterize $B^{(k)}_n$ and show, specifically, that $B^{(n-2)}_n = \left\{\frac{1}{2}e\right\}$. Moreover, we will show that $\left\{0, \frac{1}{2}\right\}$-cuts, i.e., Gomory–Chvátal cuts with $\lambda \in \left\{0, \frac{1}{2}\right\}^m$, suffice to deduce $(B_n)_I = \emptyset$, and the rank with respect to the classical Gomory–Chvátal procedure coincides with the rank if one were to use $\left\{0, \frac{1}{2}\right\}$-cuts only. Clearly, with $B_n$ as above and $F$ being a $k$-dimensional face of $[0, 1]^n$, we have $B_n \cap F \cong B_k$. As a direct consequence of the proof of [3, Lemma 7.2] one obtains:

**Lemma 3.1.** *Let $P \subseteq [0, 1]^n$ be a polytope with $F_k \subseteq P$ for some $k < n$. Then $F_{k+1} \subseteq P'$.*

**Proof.** We include a proof for completeness. Let $P$ as above and let $ax < b + 1$ with $a \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$ be valid for $P$. We have to show that $ap \leq b$ for every $p \in F_{k+1}$. Let $p \in F_{k+1}$ be arbitrary. If $ap \in \mathbb{Z}$, we are done. So assume that $ap \notin \mathbb{Z}$. Then there exists $i \in [n]$ such that $a_i \neq 0$ and $p_i = \frac{1}{2}$. We define the points $p^0, p^1$ by setting $p^0_j = p^1_j = p_j$ for all $j \neq i$, $p^0_i = 0$, and $p^1_i = 1$. Hence, $p = \frac{1}{2}p^0 + \frac{1}{2}p^1$. Note that, $p^0, p^1 \in F_k \subseteq P$ and, therefore, $ap^l < b + 1$ holds for $l \in \{0, 1\}$. We derive $ap + \frac{1}{2} \leq \max\{ap^0, ap^1\} < b + 1$ and thus $ap < b + \frac{1}{2}$. Since $ap \in \frac{1}{2}\mathbb{Z}$, it follows that $ap \leq b$, hence $p \in P'$. As the choice of $p \in F_{k+1}$ was arbitrary, we obtain $F_{k+1} \subseteq P'$. □

Note that, $F_2 \subseteq B_n$. Thus, by Lemma 3.1, we have:

**Corollary 3.2.** $F_k \subseteq B^{(k-2)}_n$.

The following theorem specifies a family of valid inequalities for $B^{(k)}_n$.

**Theorem 3.3.** *Let $B_n$ be defined as above and $k \leq n$. Then*

$$\sum_{i\in I} x_i + \sum_{i\in\tilde{I}\setminus I} (1 - x_i) \geq 1$$

*is valid for $B^{(k)}_n$ for all $I \subseteq \tilde{I} \subseteq [n]$ with $|\tilde{I}| = n - k$. Moreover, these inequalities can be derived as iterated $\left\{0, \frac{1}{2}\right\}$-cuts.*

**Proof.** The proof is by induction on $k$. First, let us look at the case $k = 0$. By definition, $\sum_{i\in I} x_i + \sum_{i\in\tilde{I}\setminus I}(1 - x_i) \geq 1$ with $\tilde{I} = [n]$ is valid for $B_n$. Now consider $0 < k \leq n$, and assume that the claim holds for $k - 1$. Let $\tilde{I} \subseteq [n]$ with $|\tilde{I}| = n - k$ be arbitrary. We have to prove that $\sum_{i\in I} x_i + \sum_{i\in\tilde{I}\setminus I}(1 - x_i) \geq 1$ with $I \subseteq \tilde{I}$ is valid for $B^{(k)}_n$. Let $I_0 = \tilde{I} \cup \{h\}$ for some $h \notin \tilde{I}$. Note that, such an $h$ exists as $k > 0$. Then

$$x_h + \sum_{i\in I} x_i + \sum_{i\in\tilde{I}\setminus I} (1 - x_i) = \sum_{i\in I\cup\{h\}} x_i + \sum_{i\in I_0\setminus(I\cup\{h\})} (1 - x_i) \geq 1$$

and

$$(1 - x_h) + \sum_{i\in I} x_i + \sum_{i\in\tilde{I}\setminus I} (1 - x_i) = \sum_{i\in I} x_i + \sum_{i\in I_0\setminus I} (1 - x_i) \geq 1$$

are valid for $B^{(k-1)}_n$, by induction hypothesis. By adding the two inequalities, we obtain

$$2\sum_{i\in I} x_i + 2\sum_{i\in\tilde{I}\setminus I} (1 - x_i) \geq 1$$

and, therefore, $\sum_{i\in I} x_i + \sum_{i\in\tilde{I}\setminus I}(1 - x_i) \geq \left\lceil\frac{1}{2}\right\rceil = 1$ is valid for $B^{(k)}_n$. □

We immediately obtain the following corollary.

**Corollary 3.4.** $B^{(n-2)}_n = \left\{\frac{1}{2}e\right\}$.

**Proof.** First note that, $\frac{1}{2}e \in B^{(n-2)}_n$ by Corollary 3.2. By Theorem 3.3 we know that $\sum_{i\in I} x_i + \sum_{i\in\tilde{I}\setminus I}(1 - x_i) \geq 1$ with $I \subseteq \tilde{I} = \{u, v\} \subseteq I$ is valid for $B^{(n-2)}_n$, for any pair $u, v \in [n]$, $u \neq v$. Therefore $x_u + x_v \geq 1$, $x_u + (1 - x_v) \geq 1$, $(1 - x_u) + x_v \geq 1$, and $(1 - x_u) + (1 - x_v) \geq 1$ are valid for $B^{(n-2)}_n$, which implies $x_u = x_v = \frac{1}{2}$. □

The following lemma characterizes the vertices of $B_n$.

**Lemma 3.5.** $B_n = \text{conv}(F_2)$.

**Proof.** Note that, $\text{conv}(F_2) \subseteq B_n$. We will show that every vertex $\tilde{x}$ of $B_n$ belongs to $F_2$, which would complete the proof. So let $\tilde{x}$ be an arbitrary vertex of $B_n$.

First, we prove that $\tilde{x}$ is half-integral. Suppose not. Let $D = \left\{i \in [n] \mid \tilde{x}_i \notin \left\{0, \frac{1}{2}, 1\right\}\right\}$. By applying appropriate coordinate flips, we may assume, without loss of generality, that $\tilde{x}_i < \frac{1}{2}$ for all $i \in D$. Since $\tilde{x}$ is a vertex of $B_n$, there exists an index set $I \subseteq [n]$ such that $\sum_{i\in I} \tilde{x}_i + \sum_{i\in[n]\setminus I}(1 - \tilde{x}_i) = 1$. Note that this implies $D \subseteq I$. If there exists $d \in D$ such that $d \notin I$, then $\sum_{i\in I\cup\{d\}} \tilde{x}_i + \sum_{i\in[n]\setminus(I\cup\{d\})}(1 - \tilde{x}_i) < 1$—a contradiction. We also obtain $|D| > 1$; otherwise the inequality cannot hold at equality. Let $s_I = \sum_{i\in I} \tilde{x}_i + \sum_{i\in[n]\setminus I}(1 - \tilde{x}_i) - 1$ for all $I \subseteq [n]$. As $|D| \geq 2$, there exists $I \subseteq [n]$ with $s_I > 0$. (Just choose any $I$ with $I \cap D = \emptyset$). Let $s = \min_{I\subseteq[n],s_I>0} s_I$, and let $j, k \in D, j \neq k$. For some sufficiently small $0 < \delta < \frac{1}{2}s$, we define $y, z \in [0, 1]^n$ with $y_i = \tilde{x}_i = z_i$ for all $j \neq i \neq k$ and $y_j = \tilde{x}_j + \delta, y_k = \tilde{x}_k - \delta, z_j = \tilde{x}_j - \delta$, and $z_k = \tilde{x}_k + \delta$. Note that $\tilde{x} = \frac{1}{2}(y + z)$. It remains to show that $y, z \in B_n$, which would contradict that $\tilde{x}$ is a vertex of $B_n$. We have earlier seen that whenever $\sum_{i\in I} \tilde{x}_i + \sum_{i\in[n]\setminus I}(1 - \tilde{x}_i) = 1$ holds for some $I \subseteq [n]$, then $D \subseteq I$. Therefore, $\sum_{i\in I} y_i + \sum_{i\in[n]\setminus I}(1 - y_i) = \sum_{i\in I} \tilde{x}_i + \sum_{i\in[n]\setminus I}(1 - \tilde{x}_i) + \delta - \delta = 1$ as $D \subseteq I$. Moreover, whenever $\sum_{i\in I} \tilde{x}_i + \sum_{i\in[n]\setminus I}(1 - \tilde{x}_i) > 1$ holds for $I \subseteq [n]$, then

$$\sum_{i\in I} y_i + \sum_{i\in[n]\setminus I}(1 - y_i) \geq \sum_{i\in I} \tilde{x}_i + \sum_{i\in[n]\setminus I}(1 - \tilde{x}_i) - 2\delta$$
$$\geq \sum_{i\in I} \tilde{x}_i + \sum_{i\in[n]\setminus I}(1 - \tilde{x}_i) - s \geq 1.$$

Thus, $y \in B_n$, and $z \in B_n$ follows similarly. Consequently, $\tilde{x}$ is half-integral.

To finish the proof, we show that $\tilde{x}$ has exactly two coordinates that are equal to $\frac{1}{2}$. Suppose that there are more than two entries equal to $\frac{1}{2}$. Then $\sum_{i\in I} \tilde{x}_i + \sum_{i\in[n]\setminus I}(1 - \tilde{x}_i) \geq \frac{3}{2}$ for all $I \subseteq [n]$. Similarly, less than two entries equal to $\frac{1}{2}$ is not possible as we would obtain $\sum_{i\in I} \tilde{x}_i + \sum_{i\in[n]\setminus I}(1 - \tilde{x}_i) = \frac{1}{2} < 1$ for $I = \{i \in [n] \mid \tilde{x}_i = 0\}$. Hence, $\tilde{x} \in F_2$. □

We conclude this section by relating $B_n$ to arbitrary polytopes $P \subseteq [0, 1]^n$ with $P_I = \emptyset$.

**Theorem 3.6.** *Let $P \subseteq [0, 1]^n$ with $P_I = \emptyset$. Then $P^{(l)} \subseteq B_n^{(l-1)}$.*

**Proof.** Let $p \in \{0, 1\}^n$ be arbitrary, and let $I := \{i \in [n] \mid p_i = 0\}$. As $P_I = \emptyset$, we can find $\epsilon_p > 0$ such that $\sum_{i\in I} x_i + \sum_{i\in[n]\setminus I}(1 - x_i) \geq \epsilon_p$ is valid for $P$, whereas $\sum_{i\in I} p_i + \sum_{i\in[n]\setminus I}(1 - p_i) = 0$; the inequality separates $p$ from $P$. In particular, we know that $\sum_{i\in I} x_i + \sum_{i\in[n]\setminus I}(1 - x_i) \geq 1$ is valid for $P'$. Since $p \in \{0, 1\}^n$ was chosen arbitrarily, we obtain that $\sum_{i\in I} x_i + \sum_{i\in[n]\setminus I}(1 - x_i) \geq 1$ is valid for $P'$ for every $I \subseteq [n]$, which implies $P' \subseteq B_n$. The claim follows from the fact that the Gomory–Chvátal procedure maintains inclusions. □

### 3.2. The sandwich theorem

In this section, we will derive bounds on the growth of the rank of a polytope $P \subseteq [0, 1]^n$ with $P_I = \emptyset$.

**Theorem 3.7** (*Sandwich Theorem*)**.** *Let $P \subseteq [0, 1]^n$ with $P_I = \emptyset$. Then*

$$k \leq \text{rk}(P) \leq k + 1$$

*where $k = \max_{(i,l)\in[n]\times\{0,1\}} \text{rk}(P \cap \{x_i = l\})$. Moreover, if there exist $i \in [n]$ and $l \in \{0, 1\}$ such that $\text{rk}(P \cap \{x_i = l\}) < k$, then $\text{rk}(P) = k$.*

**Proof.** Clearly, $k \leq \text{rk}(P)$ as there exists $(i, l) \in [n] \times \{0, 1\}$ such that $\text{rk}(P \cap \{x_i = l\}) = k$. For the other inequality, observe that $P^{(k)} \cap \{x_i = l\} = (P \cap \{x_i = l\})^{(k)} = \emptyset$, by Lemma 2.1. It follows that $x_i < 1$ and $x_i > 0$ are valid for $P^{(k)}$ for all $i \in [n]$. Hence $x_i \leq 0$ and $x_i \geq 1$ are valid for $P^{(k+1)}$ for all $i \in [n]$, and, therefore, $P^{(k+1)} = \emptyset$, i.e., $\text{rk}(P) \leq k + 1$.

It remains to show that $\text{rk}(P) = k$ if there exist $i \in [n]$ and $l \in \{0, 1\}$ such that $m := \text{rk}(P \cap \{x_i = l\}) < k$. Without loss of generality, we may assume that $l = 1$; otherwise we can apply the corresponding coordinate flip. Then $P^{(m)} \cap \{x_i = l\} = \emptyset$ and thus $x_i < 1$ is valid for $P^{(m)}$. Hence, $x_i \leq 0$ is valid for $P^{(k)}$. It follows that $P^{(k)} = P^{(k)} \cap \{x_i = 0\} = (P \cap \{x_i = 0\})^{(k)} = \emptyset$, which implies $\text{rk}(P) \leq k$. □

The upper bound in Theorem 3.7 is tight, as can be seen by considering the polytope $A_n$, introduced in [3, p. 481], whose definition is identical to that of $B_n$ except for the right-hand side, which is $\frac{1}{2}$. Then $\text{rk}(A_n) = n$ and $A_n$ satisfies the assumptions of the theorem. As $A_n \cap \{x_i = l\} \cong A_{n-1}$, we obtain that $\text{rk}(A_n \cap \{x_i = l\}) = n - 1$ for all $i \in [n]$ and $l \in \{0, 1\}$.

However, it is important to note that $\text{rk}(P \cap \{x_i = l\}) = k$ for all $(i, l) \in [n] \times \{0, 1\}$ is not sufficient for $\text{rk}(P) = k + 1$. By induction, we immediately obtain a necessary condition for $\text{rk}(P) = n$.

**Corollary 3.8.** *Let $P \subseteq [0, 1]^n$ be a polytope with $P_I = \emptyset$ and $\text{rk}(P) = n$. Then*

$$\text{rk}(P \cap F) = k$$

*for all $k$-dimensional faces $F$ of $[0, 1]^n$, $1 \leq k \leq n$.*

For the special case of $k = 1$, Corollary 3.8 was known before [5, Proof of Proposition 2.4].

### 3.3. The two-dimensional case

In this section, we will provide a full characterization of polytopes $P \subseteq [0, 1]^2$ with $P_I = \emptyset$ and $\text{rk}(P) = 2$. We will prove that $P \subseteq [0, 1]^2$ with $P_I = \emptyset$ has rank 2 if and only if $P\cap\{x_i = l\} \neq \emptyset$ for all $(i, l) \in [2] \times \{0, 1\}$, which happens if and only if $\frac{1}{2}e \in P'$. In case $P$ is a half-integral polytope, the latter condition is equivalent to $\frac{1}{2}e \in \text{Int}(P)$. The following theorem establishes the first part.

**Theorem 3.9.** *Let $P \subseteq [0, 1]^2$ be a polytope with $P_I = \emptyset$. Then $P \cap \{x_i = l\} \neq \emptyset$ for all $(i, l) \in [2] \times \{0, 1\}$ if and only if $\text{rk}(P) = 2$.*

**Proof.** First, we assume that $P$ contains points $x^0 = (c_0, 0)$, $x^1 = (0, c_1)$, $x^2 = (c_2, 1)$, and $x^3 = (1, c_3)$. As the rank is monotone, we may assume that these are the only intersections of $P$ with the boundary of the unit cube. Note that, $c_i \in (0, 1)$ for $0 \leq i \leq 3$. Let $ax < b + 1$ with $a \in \mathbb{Z}^2$ and $b \in \mathbb{Z}$ be valid for $P$. It is sufficient to prove that $a\left(\frac{1}{2}e\right) \leq b$ as this implies that $\frac{1}{2}e \in P' \neq \emptyset$. By using coordinate flips if necessary, we may assume that $a \geq 0$. Consequently, either $x^2$ or $x^3$ is maximizing $a$ over $P$. We claim that $ax^m - a\left(\frac{1}{2}e\right) \geq \frac{1}{2}$ for some $m \in \{2, 3\}$. This is sufficient to prove our hypothesis as $a\left(\frac{1}{2}e\right) \leq ax^m - \frac{1}{2} < b + 1 - \frac{1}{2} = b + \frac{1}{2}$ and as $a\left(\frac{1}{2}e\right) \in \frac{1}{2}\mathbb{Z}$, we obtain $a\left(\frac{1}{2}e\right) \leq b$. We distinguish three cases.

Case $a_2 = a_1$. We obtain that $a\left(\frac{1}{2}e\right) \in \mathbb{Z}$ and, therefore, $a\left(\frac{1}{2}e\right) \leq b$.

Case $a_2 \geq a_1 + 1$. It suffices to show that

$$ax^2 - a\left(\frac{1}{2}e\right) \geq \frac{1}{2} \Leftrightarrow a_1c_2 + a_2 - \frac{1}{2}a_1 - \frac{1}{2}a_2 \geq \frac{1}{2}$$
$$\Leftrightarrow \left(c_2 - \frac{1}{2}\right)a_1 + \frac{1}{2}a_2 \geq \frac{1}{2}.$$

This is true because $\left(c_2 - \frac{1}{2}\right) a_1 + \frac{1}{2} a_2 \geq (c_2 - \frac{1}{2}) a_1 + \frac{1}{2}(a_1 + 1) = c_2 a_1 - \frac{1}{2} a_1 + \frac{1}{2} a_1 + \frac{1}{2} = c_2 a_1 + \frac{1}{2} \geq \frac{1}{2}$.

Case $a_1 \geq a_2 + 1$. It suffices to show that $ax^3 - a\left(\frac{1}{2}e\right) \geq \frac{1}{2}$, which follows similarly.

For the other direction, observe that if there exists $(i, l) \in [2] \times \{0, 1\}$ such that $P \cap \{x_i = l\} = \emptyset$, then $\text{rk}(P) \leq 1$ follows with Corollary 3.8. □

The following theorem is our main result for the two-dimensional case:

**Theorem 3.10.** *Let $P \subseteq [0, 1]^2$ be a polytope with $P_I = \emptyset$. Then the following are equivalent:*

(a) $\text{rk}(P) = 2$;
(b) $P \cap \{x_i = l\} \neq \emptyset$ *for all* $(i, l) \in [2] \times \{0, 1\}$;
(c) $P' = \left\{\frac{1}{2}e\right\}$.

**Proof.** By Theorem 3.9, (a) ⇔ (b). Clearly, if $P' = \left\{\frac{1}{2}e\right\}$, then $\text{rk}(P) = 2$. For the other direction, observe that, by Theorem 3.6, $P' \subseteq B_2 = \left\{\frac{1}{2}e\right\}$ and thus, if $\text{rk}(P) = 2$, it follows that $P' = \left\{\frac{1}{2}e\right\}$. □

We conclude this section with the following lemma showing that whenever $\text{rk}(P) = 2$, then $\frac{1}{2}e \in \text{Int}(P)$.

**Lemma 3.11.** *If $P \subseteq [0, 1]^2$ is a polytope with $P_I = \emptyset$ and $\frac{1}{2}e \notin \text{Int}(P)$, then there exists $(i, l) \in [2] \times \{0, 1\}$ such that $P \cap \{x_i = l\} = \emptyset$. In particular, if $\text{rk}(P) = 2$ then $\frac{1}{2}e \in \text{Int}(P)$.*

**Proof.** The proof of the first part is by contradiction. So let $P \subseteq [0, 1]^2$ be a polytope with $P_I = \emptyset$ and $\frac{1}{2}e \notin \text{Int}(P)$. Suppose $P \cap \{x_i = l\} \neq \emptyset$ for all $(i, l) \in [2] \times \{0, 1\}$. Then there exists $\tilde{x} \in P \cap \{x_{\tilde{i}} = \tilde{l}\}$ with $(\tilde{i}, \tilde{l}) \in [2] \times \{0, 1\}$ and $a \in \mathbb{R}^2$ such that $ax \leq a\left(\frac{1}{2}e\right)$ is valid for $P$ and $a\tilde{x} = a\left(\frac{1}{2}e\right)$ (i.e., $ax = a\left(\frac{1}{2}e\right)$ is the hyperplane defined by the points $\tilde{x}$ and $\frac{1}{2}e$). Without loss of generality, we may assume that $\tilde{i} = 1$ and $\tilde{l} = 0$; otherwise we can apply coordinate permutations and flips. Then $\tilde{x}$ is of the form $\tilde{x} = (0, c)$ with $c \in (0, 1)$, as $P_I = \emptyset$. It is easy to see that the hyperplanes $ax = a\left(\frac{1}{2}e\right)$ and $x_1 = 1$ intersect in the point $\tilde{y} = (1, 1 - c)$. Note that, $\tilde{y}$ is not necessarily in $P$. Let $Q = [0, 1]^2 \cap \left\{ax \leq a\left(\frac{1}{2}e\right)\right\}$, and note that $P \subseteq Q$. If we maximize $x_2$ over $P$, we get $\max_{x \in P} x_2 \leq \max_{x \in Q} x_2 = \max_{x \in \{(1,1-c),(0,c)\}} x_2 < 1$, contradicting our assumption that $P \cap \{x_i = l\} \neq \emptyset$ for all $(i, l) \in [2] \times \{0, 1\}$. The second claim follows from Theorem 3.10. □

Clearly, whenever $P$ is half-integral, then $\frac{1}{2}e \in \text{Int}(P)$ if and only if $P \cap \{x_i = l\} \neq \emptyset$ for all $(i, l) \in [2] \times \{0, 1\}$. In this case, we therefore obtain $\frac{1}{2}e \in \text{Int}(P)$ if and only if $\text{rk}(P) = 2$. If $P$ is not half-integral, however, then this may not be true. Namely, consider $P$ with $|P \cap \{x_i = l\}| = 1$ for all $(i, l) \in [2] \times \{0, 1\}$, and move the vertex of the form $(p, 1)$ inwards to $(p, 1 - \epsilon)$, for some $\epsilon > 0$. It is easy to see that $\epsilon$ can be chosen such that $\frac{1}{2}e$ remains in the interior, however the rank of the resulting polytope is 1.

### 3.4. The general case

In this section, we provide a complete characterization of all polytopes $P \subseteq [0, 1]^n$ with $P_I = \emptyset$ and $\text{rk}(P) = n$. The following is the main theorem of this paper.

**Theorem 3.12.** *Let $P \subseteq [0, 1]^n$ be a polytope with $P_I = \emptyset$. Then the following statements are equivalent:*

(a) $\text{rk}(P) = n$;
(b) $P' = B_n$;
(c) $F \cap P \neq \emptyset$ *for all one-dimensional faces* $F$ *of* $[0, 1]^n$;
(d) $\text{rk}(P \cap F) = k$ *for all* $k$*-dimensional faces* $F$ *of* $[0, 1]^n$.

**Proof.** First, we show that (c) implies (b). So let us assume that $H \cap P \neq \emptyset$ for all one-dimensional faces $H$ of $[0, 1]^n$. Consider $Q = P \cap F$ for some arbitrary two-dimensional face $F$ of $[0, 1]^n$. Then $F = \bigcap_{(i,l) \in I} \{x_i = l\}$ for some $I \subseteq [n] \times \{0, 1\}$ with $|I| = n - 2$. Let $J = [n] \setminus I$. Then $Q \cap \{x_i = l\} \neq \emptyset$ for all $(i, l) \in J \times \{0, 1\}$ as $F \cap \{x_i = l\}$ is a one-dimensional face of $[0, 1]^n$. Theorem 3.10 implies that $\tilde{Q}' = \left\{\frac{1}{2}e\right\}$, where $\tilde{Q} \cong Q$ and $\tilde{Q} \subseteq [0, 1]^2$. Thus, $Q' = \left\{\frac{1}{2}e^I\right\}$. As the choice of $I$ was arbitrary, we get $F_2 \subseteq P'$. By Lemma 3.5, $B_n \subseteq P'$ follows. Theorem 3.6 yields $P' \subseteq B_n$, which completes the proof of (b).

Now assume that $P' = B_n$. Corollary 3.4 gives $\left\{\frac{1}{2}e\right\} = B_n^{(n-2)} = P^{(n-1)}$. Together with Lemma 2.2, we obtain that $\text{rk}(P) = n$. So (b) implies (a).

By Corollary 3.8, $\text{rk}(P) = n$ implies $F \cap P \neq \emptyset$ for all $k$-dimensional faces $F$ of $[0, 1]^n$. That is, (d) follows from (a).

The missing implication, (d) to (c), is trivial. □

It is a direct consequence of Theorem 3.12 that, for any $n \in \mathbb{N}$, the only half-integral polytope $P \subseteq [0, 1]^n$ with maximal rank and $P_I = \emptyset$ is $A_n$. Theorem 3.12 also implies that optimizing a linear function $c$ over $P'$ can be done in polynomial time for polytopes $P \subseteq [0, 1]^n$ with $P_I = \emptyset$ and $\text{rk}(P) = n$. It suffices to apply coordinate flips so that $c \geq 0$, to then permute the coordinates such that $c_1 \geq c_2 \geq \cdots \geq c_n$, and to finally choose the optimal vertex from $F_2$.

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