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TRANSITIVE PACKING: A UNIFYING CONCEPT IN COMBINATORIAL OPTIMIZATION*

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Abstract. This paper attempts to provide a better understanding of the facial structure of polyhedra previously investigated separately. It introduces the notion of transitive packing and the transitive packing polytope. Polytopes that turn out to be special cases of the transitive packing polytope include the node packing, acyclic subdigraph, bipartite subgraph, planar subgraph, clique partitioning, partition, transitive acyclic subdigraph, interval order, and relatively transitive subgraph polytopes. We give cutting plane proofs for several rich classes of valid inequalities of the transitive packing polytope, thereby introducing generalized cycle, generalized clique, generalized antihole, generalized antiweb, and odd partition inequalities. On the one hand, these classes subsume several known classes of valid inequalities for several special cases; on the other hand, they yield many new inequalities for several other special cases. For some of the classes we also prove a lower bound on their Gomory–Chvátal rank. Finally, we relate the concept of transitive packing to generalized (set) packing and covering, as well as to balanced and ideal matrices.

Key words. combinatorial optimization, polyhedral combinatorics, 0/1-polytope, Gomory–Chvátal cut, transitive packing, independence system

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1. Introduction.

Various types of packing problems and related polyhedra play a central role in combinatorial optimization. Due to both a large variety of practical applications and their interesting structural properties, they have received considerable attention in the literature; see, e.g., [3, 43] for an overview. One of the classic examples is the *node packing problem* in graphs and the associated *node packing polytope*. (Alternative names are *vertex packing*, *stable set*, *coclique*, *anticlique*, or *independent set* problem and polytope, respectively.) The node packing problem on a finite, undirected, loopless graph G with node weights is the problem of finding a subset of mutually nonadjacent nodes such that the total weight of the selected subset is maximal. If we denote by A the edge-node incidence matrix of the graph G, it can be formulated as

$$\text{(1.1)} \qquad \begin{aligned} \text{maximize} \quad & cx \\ \text{subject to} \quad & Ax \leqslant \mathbb{1}, \\ & x_u \in \{0,1\}, \end{aligned}$$

where c is an arbitrary vector of weights and $\mathbb{1}$ denotes (here and henceforth) the all-one vector of compatible dimension. The node packing polytope is defined as the

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convex hull of feasible solutions to (1.1) and has been studied in, among other works, [26, 37, 42, 52].

The node packing problem can be extended to hypergraphs, where it reads

$$\begin{aligned} &\text{maximize} && cx \\ &\text{subject to} && Ax \leqslant p_A - \mathbb{1}, \\ &&& x_u \in \{0,1\}, \end{aligned} \tag{1.2}$$

and A is now an arbitrary 0/1 matrix (the edge-node incidence matrix of the hypergraph), and the ith component of the vector p_A gives the number of positive entries in row i of the matrix A. If A does not contain a zero row, the undominated rows of A can be interpreted as the incidence vectors of the circuits of an *independence system*. Hence, problem (1.2) can be seen as the problem of finding an independent set of maximal weight. The convex hull of incidence vectors of independent sets (solutions to (1.2)) is known as the *independence system polytope*. Substantial work has been done to find classes of valid inequalities for the independence system polytope, mainly based on the study of special configurations of the family of circuits. Among these are, to name a few, the acyclic subdigraph polytope [25, 29], the bipartite subgraph polytope [4], and the planar subgraph polytope [30]. We refer the reader to [20, 32] and [1, 2, 16, 39, 45] for the study of the facial structure of the independence system polytope in general.

In section 2, we introduce an extension of the node packing problem in hypergraphs, called *transitive packing*, by taking transitive elements into account. The problems we consider can be described as

$$\begin{aligned} &\text{maximize} && cx \\ &\text{subject to} && Ax \leqslant p_A - \mathbb{1}, \\ &&& x_u \in \{0,1\}, \end{aligned} \tag{1.3}$$

where A is now an arbitrary $0/\pm1$ matrix, and the ith component of the vector p_A gives the number of positive entries in row i of the matrix A. Many combinatorial optimization problems can be modeled as *transitive packing problems*. We do not (and cannot) list all problems that fit with this novel framework, but we name a few of them that we are going to revisit later. Indeed, besides those that can be interpreted as finding an independent set of maximal weight, there are the clique partitioning problem [27, 28, 41], the partition problem [10], the transitive acyclic subdigraph problem [34], the interval graph completion problem [35, 49], and the relatively transitive subgraph problem [31, 50, 51].

One of our main purposes is to derive broad classes of valid inequalities for the *transitive packing polytope*, the convex hull of feasible solutions to (1.3). In section 4, we present *generalized cycle*, *generalized clique*, *generalized antihole*, *generalized antiweb*, and *odd partition inequalities*, which are valid for the transitive packing polytope. These classes explain and classify many known inequalities for polytopes that fit with this general framework. Thereby, we emphasize the relations between, and the common structure of (inequalities for), different polyhedra, formerly independently studied, and we provide new insights as well as new inequalities for some of the special polytopes that arise from certain hypergraphs and choices of transitive elements. We show how the knowledge of structural properties of the transitive packing polytope makes it possible to derive results for these special problems.

We derive most of the inequalities for the transitive packing polytope by *integer rounding*. This provides *cutting plane proofs* for many of the known inequalities for special polytopes that have not been observed before. It may also be seen as a guide for using certain patterns of the (initial) constraint matrix A to obtain new inequalities in a systematic way. The latter property might be of some importance for solving general 0/1 integer programs. Moreover, the derivation of the inequalities may be seen as a guideline for generalizing each valid inequality for the node packing polytope whose cutting plane proof is known.

Section 5 is concerned with an interesting subclass of the transitive packing polytopes, formed by those whose corresponding hypergraph is actually a graph. In section 6, we discuss the separation problem associated with the classes of inequalities introduced before. Finally, in section 8 we recall the strong relation between *set covering* and independence system polytopes, point out its extension to *generalized set covering* and transitive packing polytopes, translate our results into this context, and briefly discuss the relation of our work to $0/\pm1$ matrices that are *balanced* or *ideal*.

Subsequent to the original introduction of transitive packing [49, 36], Borndörfer and Weismantel [7, 8] introduced another scheme that also helps to explain and classify inequalities within the context of a packing polytope and to get cutting plane proofs. We refer to [48] for a discussion of similarities and differences between this scheme and transitive packing.

2. The transitive packing polytope. A *hypergraph* is an ordered pair $(N, \mathcal{H})$, where N is a finite ground set, the set of *nodes*, and $\mathcal{H}$ is a collection of distinct subsets of N, the set of (*hyper*)*edges*. We only deal with hypergraphs without loops, i.e., we always assume that $|H| \geqslant 2$ for all $H \in \mathcal{H}$. We refer to [6] for a thorough introduction to hypergraphs. Here, we are interested in hypergraphs with additional node subsets associated with each edge.

DEFINITION 2.1. *Let* $(N, \mathcal{H})$ *be a hypergraph, and let* $\mathrm{tr} : \mathcal{H} \to 2^N$ *be a mapping from the set of edges to the powerset of* N, *with the property that* $\mathrm{tr}(H) \subseteq N \setminus H$. *We call the ordered triple* $(N, \mathcal{H}, \mathrm{tr})$ *an* extended hypergraph, *and* $\mathrm{tr}(H)$ *the set of* transitive elements *associated with the edge* H.

In the special case that $\mathrm{tr}(H) = \emptyset$ for all $H \in \mathcal{H}$, we often simply write $(N, \mathcal{H})$ instead of $(N, \mathcal{H}, \mathrm{tr})$. We are interested in packing nodes of an extended hypergraph whereby the restrictions imposed by the edges may be compensated by picking transitive elements. This is made precise by the following definition.

DEFINITION 2.2. *Let* $(N, \mathcal{H}, \mathrm{tr})$ *be an extended hypergraph. A subset* S *of the nodes is a* transitive packing (*in* $(N, \mathcal{H}, \mathrm{tr})$) *if, for every* $H \in \mathcal{H}$ *such that* $H \subseteq S$, *there exists a node* $u \in S \cap \mathrm{tr}(H)$.

In other words, a transitive packing S is a set of nodes that contains an edge only if S contains at least one node from the set of transitive elements associated with that edge. Given, in addition to $(N, \mathcal{H}, \mathrm{tr})$, a weight function $c : N \to \mathbb{Q}$, the (*maximum weight*) *transitive packing problem* consists of finding a transitive packing $S \subseteq N$ of maximal weight $c(S)$. As indicated in the introduction, the maximum weight transitive packing problem is equivalent to the integer linear programming problem

$$\begin{aligned}
&\text{maximize} && cx \\
(2.1)\quad &\text{subject to} && x(H) - x(\mathrm{tr}(H)) \leqslant |H| - 1 \quad \text{for all } H \in \mathcal{H}, \\
(2.2)\quad &&& x \leqslant \mathbb{1}, \\
(2.3)\quad &&& x \geqslant 0, \\
(2.4)\quad &&& x \in \mathbb{Z}^N.
\end{aligned}$$

Note that the constraint matrix of the inequalities (2.1) is the edge-node incidence matrix of the hypergraph $(N, \mathcal{H})$, with additional -1's for the transitive elements of the edge represented by the particular row. We call the inequalities (2.1) *transitivity constraints*.

In the following, we study the transitive packing polytope $P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})$ of the extended hypergraph $(N, \mathcal{H}, \mathrm{tr})$, which is defined as the convex hull of the incidence vectors of transitive packings in $(N, \mathcal{H}, \mathrm{tr})$, i.e.,

$$P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr}) := \mathrm{conv}\{\chi^S \in \mathbb{R}^N : S \text{ transitive packing in } (N, \mathcal{H}, \mathrm{tr})\}.$$

In other words, $P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})$ is equal to the integer hull of the feasible solutions to (2.1)–(2.3). At this point, it seems reasonable to introduce a few examples to illustrate the applicability of the results to be presented. Of course, if $\mathrm{tr}(H) = \emptyset$ and $|H| = 2$ for all edges $H \in \mathcal{H}$, a transitive packing reduces to an ordinary node packing in the graph $(N, \mathcal{H})$. However, to motivate hypergraphs and transitive elements, we show now that the acyclic subdigraph polytope as well as the clique partitioning polytope and the partition polytope can be obtained by special choices of the hypergraph and the transitive elements. Other examples will be discussed in section 7.

The acyclic subdigraph polytope. An instance of the *acyclic subdigraph problem* consists of a directed graph $D = (V, A)$ and a weight function $c : A \to \mathbb{Q}$. The objective is to determine a set of arcs $B \subseteq A$ such that the digraph (V, B) is acyclic, i.e., does not contain a directed cycle, and such that $c(B)$ is as large as possible. The *acyclic subdigraph polytope* is the convex hull of incidence vectors of acyclic arc subsets of A. It was studied by Grötschel, Jünger, and Reinelt (see [24, 25, 29]) and Goemans and Hall [23]. If we choose the arc set A of the digraph D as the node set of the hypergraph, if we declare the directed cycles in D as the edges of this hypergraph, and if we let $\mathrm{tr}(H) = \emptyset$ for all $H \in \mathcal{H}$, the acyclic subdigraph polytope appears as a special transitive packing polytope.

The clique partitioning polytope. An instance of the *clique partitioning problem* consists of an undirected graph $G = (V, E)$ and a weight function $c : E \to \mathbb{Q}$. A set $F \subseteq E$ of edges is called a *clique partitioning* of G if there is a partition of V into nonempty, disjoint sets $W_1, W_2, \ldots, W_k$ such that the subgraph induced by each W_i is a clique and such that $F = \bigcup_{i=1}^{k} \{\{u, v\} : u, v \in W_i, u \neq v\}$. Equivalently, a clique partitioning is a subrelation of the symmetric relation represented by G that is an equivalence relation, i.e., in particular transitive. The weight of such a clique partitioning F is $c(F)$. The task is to determine a clique partitioning of minimal weight. (Of course, since we do not restrict the objective function, we could have written that we want to find a clique partitioning of maximal weight as we always do in the context of transitive packing. However, for historical reasons we chose this variant.) The *clique partitioning polytope* is the convex hull of the incidence vectors of all clique partitionings in G. It was introduced and studied by Grötschel and Wakabayashi [27, 28] and has recently been further investigated by Oosten, Rutten, and Spieksma [41]. To show that it is an instance of a transitive packing polytope, it is sufficient to deal with a graph instead of a hypergraph. Indeed, we take as the set N of nodes the edges of G, and two nodes are adjacent (form a hyperedge) if and only if the associated edges are incident in the original graph G. That is, the extended hypergraph we consider is precisely the *line graph* of G, and the transitive element that we attach to a pair of incident edges $\{u, v\}$, $\{v, w\}$ in G is the edge $\{u, w\}$ if it exists.

The partition polytope. An instance of the *graph partitioning problem* consists of an undirected, connected graph $G = (V, E)$, a weight function $c : E \to \mathbb{Q}$, and an integer $r \leqslant |V|$. An r-partition of the node set V is a set of node subsets $N_1, N_2, \ldots, N_r$ such that $N_i \cap N_j = \emptyset$ (for all $i \neq j$) and $\cup_{i=1}^{r} N_i = V$. Some of the subsets N_i may be empty. The weight of an r-partition is the total weight of the edges with end points in two different subsets. The goal is to determine an r-partition of minimal weight. Chopra and Rao [10] have studied polytopes for several variations of this problem. We consider one of them here. This case arises when $r = |V|$. For a complete graph G, this problem is equivalent to the clique partitioning problem. For arbitrary graphs G, Chopra and Rao define the *partition polytope* as the convex hull of the incidence vectors of all sets of edges in G which are not cut by an r-partition. It follows from [10, Lemma 2.2] that the partition polytope arises as a transitive packing polytope by taking the edges of G as the set N and by letting every $(|C|-1)$-cardinality subset of edges of a cycle C in G be the edges of the hypergraph $\mathcal{H}$. The transitive set related to such a hyperedge contains exactly the missing edge from the cycle C.

Before studying the transitive packing polytope, we shall discuss an algorithmic aspect of the concept of transitive packings. How is $(N, \mathcal{H}, \mathrm{tr})$ given? Having in mind problems like the acyclic subdigraph problem, it does not seem to be satisfactory to assume that it is given as a list of hyperedges and their transitive elements. Indeed, the number of directed cycles in a digraph can be exponential in the number of nodes. From the point of view of polyhedral combinatorics, it rather seems to be reasonable to assume that the linear programming problem arising from (2.1)–(2.4) by dropping the integrality constraint (2.4) is solvable in time polynomially bounded in $|N|$ and the input size of c. This means, given a point $x \in \mathbb{Q}^N$ contained in the unit hypercube, we assume that the separation problem formed by x and the class of inequalities (2.1) is solvable in polynomial time. In particular, this guarantees that the decision version of the transitive packing problem belongs to the class NP. Since the node packing problem on graphs is NP-hard, the same holds for the transitive packing problem.

Let us continue with the study of the transitive packing polytope. Since the empty set as well as all singletons of N are transitive packings, we immediately obtain the following result.

PROPOSITION 2.3. *Let $(N, \mathcal{H}, \mathrm{tr})$ be an extended hypergraph.*

(i) *The transitive packing polytope $P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr}) \subseteq \mathbb{R}^N$ is full dimensional, i.e., $\dim(P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})) = |N|$.*

(ii) *The nonnegativity constraint $x_u \geqslant 0$ defines a facet of $P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})$ for each node $u \in N$.*

Because of the transitive elements, it is more difficult to characterize the facet defining inequalities of type $x_u \leqslant 1$ for $u \in N$. Clearly, all these inequalities are facet defining if $|H| \geqslant 3$ for all edges $H \in \mathcal{H}$. But as soon as $\{u, v\} \in \mathcal{H}$ and $\mathrm{tr}(\{u, v\}) = \emptyset$, for instance, the face induced by $x_u \leqslant 1$ is properly contained in the facet defined by $x_v \geqslant 0$. But even if $\mathrm{tr}(\{u, v\}) \neq \emptyset$, it may happen that whenever u is chosen, we cannot choose another element. While it is possible to give a concise characterization in the absence of transitive elements, we are content with a sufficient condition in the general case.

LEMMA 2.4. *Let $P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})$ be the transitive packing polytope associated with the extended hypergraph $(N, \mathcal{H}, \mathrm{tr})$.*

(i) *If $\mathrm{tr}(H)$ is the empty set for all edges $H \in \mathcal{H}$ such that $|H| = 2$, then an inequality $x_u \leqslant 1$ with $u \in N$ defines a facet of $P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})$ if and only if $|H| \geqslant 3$ for all edges $H \in \mathcal{H}$ that contain u.*

(ii) *Let* $u \in N$. *If there exists for all edges* $\{u,v\} \in \mathcal{H}$ *a node* $w \in \mathrm{tr}(\{u,v\})$ *such that neither* $\{u,w\} \in \mathcal{H}$, $\{v,w\} \in \mathcal{H}$, *nor* $\{u,v,w\} \in \mathcal{H}$, *then the inequality* $x_u \leqslant 1$ *defines a facet of* $P_{\mathrm{TP}}(N,\mathcal{H},\mathrm{tr})$.

Proof. In case (i), the incidence vectors of the transitive packings $\{u\}$ and $\{u,v\}$ for all $v \in N \setminus \{u\}$ provide the needed set of linearly independent vectors. In case (ii), we proceed as follows. Besides $\{u\}$, we first choose a set $\{u,w\}$ such that $\{u,w\} \notin \mathcal{H}$. (Notice that our assumptions imply the existence of such a node w.) Then, by taking $\{u,v,w\}$, we collect all nodes $v \in N$ such that $\{u,v\} \in \mathcal{H}$, $w \in \mathrm{tr}(\{u,v\})$, $\{v,w\} \notin \mathcal{H}$, and $\{u,v,w\} \notin \mathcal{H}$. Now, we may forget these nodes v and the node w and continue with the remaining nodes in the same manner. Since $\{u,v\} \in \mathcal{H}$ for the nodes v above, they cannot occur in the role of w. Hence, the incidence vectors of the constructed transitive packings are linearly independent. □

We illuminate Lemma 2.4 by applying it to the node packing, the acyclic subdigraph, the clique partitioning, and the partition polytopes. For the node packing polytope of a graph G, (i) says that an inequality $x_u \leqslant 1$ is facet defining for a node u if and only if u is isolated, i.e., if G does not contain an edge incident to u. This is a special case of the well-known fact that a clique inequality defines a facet if and only if the clique is maximal [42]. Given a digraph $D = (V,A)$ and an arc $(u,v) \in A$, Lemma 2.4(i) implies that $x_{uv} \leqslant 1$ defines a facet of the acyclic subdigraph polytope of D if and only if $(v,u) \notin A$. This was shown before by Grötschel, Jünger, and Reinelt [25]. If G is a graph without isolated edges, the assumption of Lemma 2.4(ii) is never met by an edge of the clique partitioning polytope of G. Indeed, Grötschel and Wakabayashi [28] proved that no upper bound constraint defines a facet of this polytope. Finally, Lemma 2.4(ii) also tells us that $x_e \leqslant 1$ defines a facet of the partition polytope if the edge e does not belong to any cycle of length 3.

We conclude this first section on the transitive packing polytope by observing that a transitivity constraint $x(H') - x(\mathrm{tr}(H')) \leqslant |H'| - 1$ is dominated by $x(H) - x(\mathrm{tr}(H)) \leqslant |H| - 1$ if $H \subseteq H'$ and $\mathrm{tr}(H) \subseteq \mathrm{tr}(H')$.

3. The independence system polytope. So far we have mentioned only in the introduction that the transitive packing problem subsumes independent set problems. This section is intended to recall the needed definitions and to explain the relation in detail. An *independence system* is a pair $(N,\mathcal{I})$, with ground set N and a family $\mathcal{I}$ of subsets of N, that contains the empty set and is closed under set inclusion; i.e., for any set $I \in \mathcal{I}$ every subset $I' \subseteq I$ belongs also to $\mathcal{I}$. The elements of $\mathcal{I}$ are called *independent sets*. A subset of N that does not belong to $\mathcal{I}$ is called *dependent*, and the minimal dependent sets (with respect to set inclusion) are the *circuits* of the independence system. The collection of circuits forms a *clutter*, i.e., a family of sets such that no two of them are comparable with respect to set inclusion. Since a subset of N is independent if and only if it does not contain a circuit, an independence system is fully characterized by the family of its circuits. Conversely, every clutter $\mathcal{C} \subseteq 2^N$ determines a unique independence system with ground set N and $\{I \subseteq N : C \nsubseteq I$ for all $C \in \mathcal{C}\}$ as the family of its independent sets. The *independence system polytope* is defined as the convex hull of all incidence vectors of independent sets. It coincides with the transitive packing polytope $P_{\mathrm{TP}}(N,\mathcal{H})$, where $\mathrm{tr}(H) = \emptyset$ for all $H \in \mathcal{H}$, and $\mathcal{H}$ is the set of circuits. (To be accurate, this is only true when we make the standard assumption that all singletons are independent. Remember that we have defined the transitive packing polytope only for hypergraphs without loops.) In the following we will sometimes speak of independent sets instead of transitive packings, of circuits instead of edges, and of circuit constraints instead of transitivity (or packing) constraints when dealing with the special case formed by transitive packing problem instances without transitive elements. As an example of an independence system, we may consider the one defined by the acyclic arc subsets of a digraph. The dicycles are one-to-one with the circuits, and the independence system polytope is the acyclic subdigraph polytope.

Given a hypergraph $(N, \mathcal{H})$, we define its *upper closure* $\mathcal{H}^+$ and its *reduction* $\mathcal{H}^-$ as $\mathcal{H}^+ := \{H' \subseteq N : \text{there exists an } H \in \mathcal{H} \text{ such that } H \subseteq H'\}$ and $\mathcal{H}^- := \{H \in \mathcal{H} : \text{there exists no } H' \in \mathcal{H} \text{ such that } H' \subset H\}$, respectively. Notice that $P_{\text{TP}}(N, \mathcal{H}^+) = P_{\text{TP}}(N, \mathcal{H}) = P_{\text{TP}}(N, \mathcal{H}^-)$. These notions prove useful for characterizing the facet defining packing constraints. Observe that for clutters, for instance the circuits of independence systems, we have $\mathcal{H} = \mathcal{H}^-$.

THEOREM 3.1. *Let $(N, \mathcal{H})$ be a hypergraph. For $H \in \mathcal{H}$, the inequality $x(H) \leqslant |H| - 1$ defines a facet of $P_{\text{TP}}(N, \mathcal{H})$ if and only if $H \in \mathcal{H}^-$ and for all $u \in N \setminus H$ there exists an $H' \subset H$ with $|H'| = |H| - 1$ such that $H' \cup \{u\} \notin \mathcal{H}^+$.*

Proof. Necessity of the stated condition is obvious; otherwise, the face under consideration would be the intersection of some other faces. To show sufficiency we take first the incidence vectors of all $|H|$ subsets of H of size $|H| - 1$. According to the assumption, for each node $u \in N \setminus H$ there exists a subset H' of H of size $|H| - 1$ such that $H' \cup \{u\}$ is independent. Adding the corresponding incidence vectors to our former set completes the proof. □

Theorem 3.1 implies, in particular, that all dicycle inequalities of the acyclic subdigraph polytope are facet defining. A direct proof of this result is given in [25].

Subclasses of the classes of valid inequalities that we introduce in the next section for the transitive packing polytope have been presented earlier for the independence system polytope; generalized cycle, generalized clique, and generalized antihole inequalities by Euler, Jünger, and Reinelt [20], and generalized antiweb inequalities by Laurent [32]. It will turn out that our inequalities are more general, even if we restrict ourselves to the independence system polytope. Nevertheless, in order to keep the terminology simple, we will give the new inequalities the same names and point out the restrictions that lead to the known inequalities, respectively. So far, no cutting plane proofs have been presented for the formerly known inequalities.

4. Valid inequalities. Let $P \subseteq \mathbb{R}^N$ be a rational polyhedron, for instance the initial relaxation of $P_{\text{TP}}(N, \mathcal{H}, \text{tr})$ defined by (2.1)–(2.3). One way to produce a characterization of the integer hull P_{I} of P by means of linear inequalities is integer rounding. For a thorough discussion of this topic, its history, and its applications to integer programming and combinatorial optimization, we refer the reader to the textbooks of Cook, Cunningham, Pulleyblank, and Schrijver [15, Chapter 6.7] and of Nemhauser and Wolsey [38, Chapter II.1] and to Schrijver [47, Chapter 23]. Here, we briefly review the basic definitions that will be needed later on.

If we set

$$P' := \{x \in P : ax \leqslant \beta \text{ for all } a \in \mathbb{Z}^N, \beta \in \mathbb{Z} \text{ with } \max\{ax : x \in P\} < \beta + 1\},$$

then P' can be seen as obtained from P by one step of rounding. In particular, if $P = \{x \in \mathbb{R}^N : Ax \leqslant b\}$ for an integer matrix A and integer right-hand side b, then

$$P' = \{x \in \mathbb{R}^N : \lambda Ax \leqslant \lfloor \lambda b \rfloor \text{ for all vectors } \lambda \geqslant 0 \text{ with } \lambda A \in \mathbb{Z}^N\}.$$

Obviously, the integer hull P_{I} of P, i.e., the convex hull of the integral points in P, is contained in P'. Furthermore $P' = P$ if and only if $P = P_{\text{I}}$. If we define $P^{(0)} := P$

and, recursively, $P^{(t+1)} := (P^{(t)})'$ for all nonnegative integers t, then $P_{\mathrm{I}} \subseteq P^{(t)}$ for all nonnegative integers t. Schrijver [46] showed that P' is again a polyhedron and that there is a nonnegative integer t such that $P^{(t)} = P_{\mathrm{I}}$. The (*Gomory–Chvátal*) *rank* of P is the smallest t such that $P^{(t)} = P_{\mathrm{I}}$. Let $ax \leqslant \beta$ be a valid inequality for P_{I}. Its *depth* relative to P is the smallest d such that $ax \leqslant \beta$ is valid for $P^{(d)}$. Therefore the rank of P equals the maximal depth, relative to P, of an inequality valid for P_{I}.

Let $Ax \leqslant b$ be a system of linear inequalities, and let $cx \leqslant \delta$ be an inequality. Moreover, let $c_1x \leqslant \delta_1, c_2x \leqslant \delta_2, \ldots, c_mx \leqslant \delta_m$ be a sequence of linear inequalities such that each vector c_i, $i = 1, \ldots, m$, is integral, $c_m = c$, $\delta_m = \delta$, and for $i = 1, \ldots, m$ the inequality $c_ix \leqslant \delta_i'$ is a nonnegative linear combination of the inequalities $Ax \leqslant b$, $c_1x \leqslant \delta_1, \ldots, c_{i-1}x \leqslant \delta_{i-1}$ for some δ_i' with $\lfloor \delta_i' \rfloor \leqslant \delta_i$. Such a sequence is called a *cutting plane proof* of $cx \leqslant \delta$ from $Ax \leqslant b$, and m is the *length* of this proof. The depth of the final inequality $cx \leqslant \delta$ is the *depth* of the proof. Every integer solution of $Ax \leqslant b$ satisfies $cx \leqslant \delta$. Let $P = \{x : Ax \leqslant b\}$. Since $P^{(t)} = P_{\mathrm{I}}$ for some t, the converse is true as soon as P_{I} is nonempty. That is, every inequality $cx \leqslant \delta$ with c integral and valid for P_{I} has a cutting plane proof from $Ax \leqslant b$. Clearly, the length of a cutting plane proof of a valid inequality for P_{I} is at least its depth; however, the length can be significantly bigger (see, e.g., [12]).

The idea of deriving cutting planes by rounding based on the exploitation of problem structure can, in particular, be used to obtain valid inequalities for the transitive packing polytope. Thereby, we also show that many inequalities valid for the polytopes which arise from $P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})$ by certain choices of $(N, \mathcal{H}, \mathrm{tr})$ have short and insightful cutting plane proofs from the initial relaxation (2.1)–(2.3).

4.1. Generalized cycle inequalities. We first use cycles of the hypergraph $(N, \mathcal{H})$ to obtain a class of valid inequalities for the transitive packing polytope, each of which has a cutting plane proof from (2.1)–(2.3) of length 1. Recall that a *cycle* in a hypergraph is a sequence of vertices and of edges of the form $(u_1, H_1, u_2, H_2, \ldots, u_k, H_k, u_{k+1})$ such that the vertices $u_1, \ldots, u_k$ are distinct, $u_{k+1} = u_1$, the edges $H_1, \ldots, H_k$ are distinct, and for $i = 1, \ldots, k$ both u_i and u_{i+1} are contained in H_i. We start, however, with a few more assumptions.

Definition 4.1. *Let $(N, \mathcal{H})$ be a hypergraph, and let q, s, and r be positive integers such that $q \geqslant 2$ and $1 \leqslant r \leqslant q-1$. For convenience, we set $k := sq + r$. Let $N_1, \ldots, N_k$ be a sequence of pairwise disjoint nonempty subsets of N. For $i = 1, \ldots, k$, let $H_i \in \mathcal{H}$ be an edge such that $\bigcup_{j=i}^{i+q-1} N_j \subseteq H_i$. (Indices greater than k are taken modulo $k+1$ and shifted by $+1$.) We denote by C the union of all these edges H_i, $C := \bigcup_{i=1}^{k} H_i$, and by $m(u)$ the multiplicity of a node $u \in C$ in this edge collection, i.e., $m(u) := |\{i \in \{1, \ldots, k\} : u \in H_i\}|$. We assume that $m(u) \leqslant q$ for all nodes $u \in C$. Then we call the hypergraph $(C, \{H_i : i = 1, 2, \ldots, k\})$ a* generalized (k, q)-cycle (*contained in* $(N, \mathcal{H})$).

To illuminate this definition, Figures 4.1 and 4.2 show a generalized $(10, 4)$-cycle and two generalized $(5, 2)$-cycles, respectively. Observe that every generalized cycle is a cycle of the hypergraph, but not vice versa. In fact, the name is a concession to the literature, where already a substructure of the generalized cycles just introduced got this name; see [20]. We now develop an inequality supported by a generalized cycle and its set of transitive elements. So let $(C, \{H_i : i = 1, 2, \ldots, k\})$ be a generalized (k, q)-cycle in $(N, \mathcal{H}, \mathrm{tr})$, and assume that the set $\mathrm{tr}(C) := \bigcup_{i=1}^{k} \mathrm{tr}(H_i)$ of transitive elements does not interact with C itself, i.e., $\mathrm{tr}(H_i) \cap C = \emptyset$ for $i = 1, \ldots, k$. To simplify the notation, we denote by $n(u) := |\{i \in \{1, \ldots, k\} : u \in \mathrm{tr}(H_i)\}|$ the multiplicity of a node $u \in N \setminus C$ with respect to the transitive sets of the edges of the

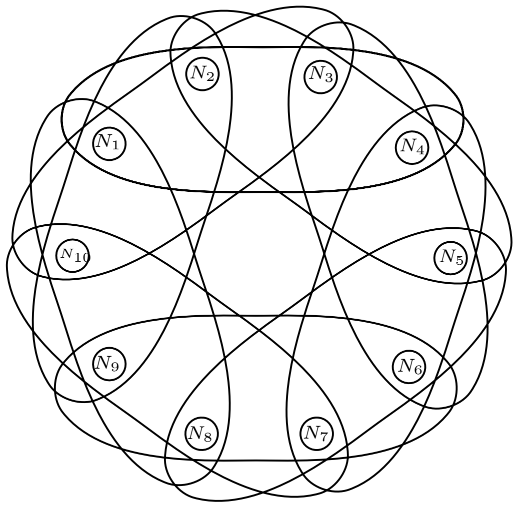

FIG. 4.1. *A generalized* $(10, 4)$*-cycle with* $C = \bigcup_{i=1}^{k} N_i$.

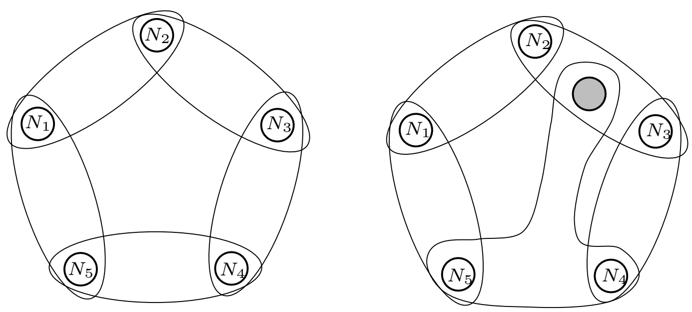

FIG. 4.2. *Two generalized* $(5, 2)$*-cycles. The second one illustrates the case* $C \supset \bigcup_{i=1}^{k} N_i$.

cycle. Furthermore, we let $\lceil \alpha \rceil_q$ be the smallest integer that is bigger than or equal to the scalar α as well as divisible by q.

Adding the transitivity constraints associated with the edges of the generalized (k, q)-cycle,

$$\sum_{u \in H_i} x_u - \sum_{u \in \mathrm{tr}(H_i)} x_u \leqslant |H_i| - 1 \quad \text{for } i = 1, \ldots, k,$$

an appropriate multiple of upper bound constraints,

$$(q - m(u))x_u \leqslant q - m(u) \quad \text{for } u \in C \setminus \bigcup_{i=1}^{k} N_i,$$

as well as an appropriate multiple of nonnegativity constraints,

$$-(\lceil n(u) \rceil_q - n(u))x_u \leqslant 0 \quad \text{for } u \in \mathrm{tr}(C) \text{ with } n(u) \not\equiv 0 \bmod q,$$

and dividing the result by q, we obtain

$$\sum_{u \in C} x_u - \sum_{u \in \mathrm{tr}(C)} \frac{\lceil n(u) \rceil_q}{q} x_u \leqslant \frac{q|C| - k}{q}.$$

Rounding down the right-hand side completes the proof of the following result.

THEOREM 4.2. *Let* $(N,\mathcal{H},\mathrm{tr})$ *be an extended hypergraph, and let, for* $k > q$, $k \not\equiv 0 \bmod q$, *the hypergraph* $(C, \{H_i : i = 1,2,\ldots,k\})$ *be a generalized* (k,q)*-cycle in* $(N,\mathcal{H})$ *such that* $\mathrm{tr}(H_i) \cap C = \emptyset$ *for* $i = 1,\ldots,k$. *Then, the generalized* (k,q)*-cycle inequality*

$$\sum_{u\in C} x_u - \sum_{u\in \mathrm{tr}(C)} \frac{\lceil n(u)\rceil_q}{q} x_u \leqslant |C| - \left\lceil \frac{k}{q} \right\rceil \tag{4.1}$$

is valid for the transitive packing polytope $P_{\mathrm{TP}}(N,\mathcal{H},\mathrm{tr})$.

We now relate this first class of inequalities for the transitive packing polytope $P_{\mathrm{TP}}(N,\mathcal{H},\mathrm{tr})$ to the four selected examples. For the node packing polytope, we obtain exactly the *odd cycle inequalities* introduced by Padberg [42]. This is true because all edges of the (hyper)graph have size 2, and hence all sets N_i have to be singletons. If C is the set of nodes of an odd cycle in a graph G, then the associated odd cycle inequality reads

$$x(C) \leqslant \frac{|C|-1}{2}.$$

The *Möbius ladder inequalities* form a quite prominent class of facet defining inequalities for the acyclic subdigraph polytope. The support of any of these inequalities is defined as follows.

DEFINITION 4.3 (see [25]). *Let* $C_1, C_2, \ldots, C_k$ *be a sequence of different dicycles in a digraph* $D = (V,A)$ *such that the following hold:*

(1) $k \geqslant 3$ *and* k *odd.*

(2) C_i *and* C_{i+1}, $i \in \{1,2,\ldots,k-1\}$, *have a directed path* P_i *in common;* C_1 *and* C_k *have a directed path* P_k *in common.*

(3) *Given any dicycle* C_j, $j \in \{1,2,\ldots,k\}$, *set* $I_j := \{1,2,\ldots,k\} \cap (\{j-2, j-4, j-6,\ldots\} \cup \{j+1, j+3, j+5, \ldots\})$. (*Indices greater than* k *are taken modulo* $k+1$ *and shifted by* $+1$; *indices less than* 0 *are first shifted by* -1 *and then taken modulo* $k+1$.) *Then every set* $(\bigcup_{i=1}^k C_i) \setminus \{a_i : i \in I_j\}$ *contains exactly one dicycle* (*namely,* C_j), *where* a_i, $i \in I_j$, *is any arc contained in the dipath* P_i.

(4) *The largest acyclic arc set in* $\bigcup_{i=1}^k C_i$ *has cardinality* $|\bigcup_{i=1}^k C_i| - \frac{k+1}{2}$.

Then the arc set $M := \bigcup_{i=1}^k C_i$ *is called a* (k-)Möbius ladder.

From Definition 4.3(4) it follows that for any k-Möbius ladder M contained in a digraph D the Möbius ladder inequality

$$x(M) \leqslant |M| - \frac{k+1}{2} \tag{4.2}$$

is valid for the acyclic subdigraph polytope of D. Definition 4.3(3)–(4) seem to be rather unhandy. There exists a large subclass, however, where these conditions are naturally satisfied. Let $C_1, C_2, \ldots, C_k$, $k \geqslant 5$, be a sequence of directed cycles satisfying (1) and (2). If no two different dicycles C_i and C_j, with $j \neq i-1, i+1$, share a node, Grötschel, Jünger, and Reinelt [25] observed that the union of these dicycles forms a Möbius ladder. Such a situation is depicted in Figure 4.3. We now prove that this subclass is contained in the class of generalized cycle inequalities, as has essentially been shown in the context of the independence system polytope by Euler, Jünger, and Reinelt [20].

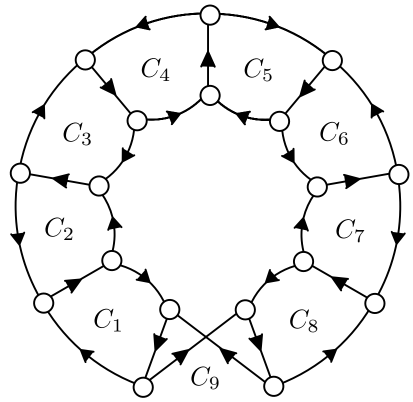

FIG. 4.3. *A 9-Möbius ladder.*

THEOREM 4.4. *Let D be a digraph, and let, for $k \geqslant 5$, $C_1, C_2, \ldots, C_k$ be a sequence of different dicycles in D satisfying Definition* 4.3(1)–(2). *If no two different dicycles C_i and C_j, with $j \neq i-1, i+1$, have a node in common $(i, j = 1, 2, \ldots, k)$, the Möbius ladder inequality* (4.2) *is contained in the class of generalized $(k, 2)$-cycle inequalities for the acyclic subdigraph polytope of D.*

Proof. We make use of the notation introduced in the discussion of the generalized cycle inequalities. We choose $q = 2$ and let k be the number of dicycles. The sets N_i, $i = 1, 2, \ldots, k$, are defined by the arcs forming the dipaths P_i, respectively. For $i = 1, 2, \ldots, k$, the arc sets N_i and N_{i+1} are contained in the hyperedge given by the dicycle C_{i+1}. Observe that no arc in $M = \bigcup_{i=1}^{k} C_i$ occurs in more than two dicycles. The claim now follows from Theorem 4.2. □

Theorem 4.4 throws some light on the Möbius ladder inequalities. The way we derived the generalized cycle inequalities explains, in particular, why the sequence of dicycles should be odd, as was already observed by Grötschel, Jünger, and Reinelt: "For even k, the construction does not give anything interesting" [25, p. 34]. Notice that Theorem 4.4 remains true for those Möbius ladders where each triple of the dicycles $C_1, C_2, \ldots, C_k$ does not have a common arc.

In the case of the clique partitioning polytope, we are obviously restricted to generalized $(k, 2)$-cycles, as the underlying hypergraph is actually a graph, the line graph of the given graph $G = (V, E)$. Nevertheless, this class contains two known classes of valid inequalities. Both are facet defining if G is a complete graph. The first class is formed by the 2-*chorded odd cycle inequalities* introduced by Grötschel and Wakabayashi [28]. Let $C = \{e_1, e_2, \ldots, e_k\}$ be the set of edges of an odd cycle in G, say $e_i = \{u_i, u_{i+1}\}$, and let $\operatorname{tr}(C) = \{\{u_i, u_{i+2}\} \in E : i = 1, 2, \ldots, k\}$ be its set of 2-chords (transitive elements). (As before, indices greater than k are taken modulo $k+1$ and shifted by $+1$.) By observing that $C \cap \operatorname{tr}(C) = \emptyset$, we may apply Theorem 4.2 and obtain the 2-chorded odd cycle inequality

$$\sum_{i=1}^{k} x_{\{u_i, u_{i+1}\}} - \sum_{\substack{i=1 \\ \{u_i,\, u_{i+2}\} \in E}}^{k} x_{\{u_i, u_{i+2}\}} \leqslant \frac{k-1}{2}.$$

However, even structures that are not cycles in G lead to generalized $(k, 2)$-cycle inequalities. For $k \geqslant 3$ odd, assume that G contains the star formed by the sequence

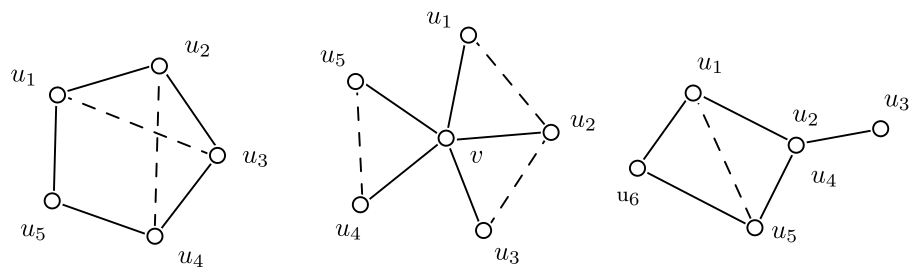

FIG. 4.4. *Generalized* $(5,2)$*-cycles for the clique partitioning polytope. The first is a* 2*-chorded odd cycle, the second is an odd wheel. The third is neither a* 2*-chorded odd cycle nor an odd wheel. The dotted edges indicate existing transitive edges* (*i.e., coefficient* -1 *in the associated inequalities*).

$\{v, u_i\}$, $i = 1, 2, \ldots, k$, of incident edges. Let $\mathrm{tr}(C)$ denote the associated set of 2-chords, i.e., $\mathrm{tr}(C) = \{\{u_i, u_{i+1}\} \in E : i = 1, 2, \ldots, k\}$. Again we have $\mathrm{tr}(C) \cap C = \emptyset$, and Theorem 4.2 implies that the *odd wheel inequality*

$$\sum_{i=1}^{k} x_{\{v,u_i\}} - \sum_{\substack{i=1 \\ \{u_i,\, u_{i+1}\} \in E}}^{k} x_{\{u_i,u_{i+1}\}} \leqslant \frac{k-1}{2}$$

is valid for the clique partitioning polytope. It was introduced and shown to be facet defining if G is complete by Chopra and Rao [10].

There are other structures that may form generalized $(k, 2)$-cycles in the line graph of G; see, for instance, Figure 4.4. We can summarize our observations as follows.

THEOREM 4.5. *The class of generalized* $(k, 2)$*-cycle inequalities for the clique partitioning polytope properly contains all* 2*-chorded odd cycle inequalities and all odd wheel inequalities.*

The odd wheel inequalities remain valid and facet defining for the partition polytope [10], where they also form a subclass of the generalized $(k, 2)$-cycle inequalities. In fact, it is immediate that they can be generalized such that the spokes of the wheel are paths instead of single edges. Moreover, from the class of generalized cycle inequalities we get what we may call *q-chorded cycle inequalities*, a generalization of the 2-chorded odd cycle inequalities of the clique partitioning polytope. Consider a cycle of length k in G, with nodes $1, \ldots, k$. Assume that G also contains the edges $\{i, i+q\}, i = 1, \ldots, k$. Then we define the *q-chorded cycle inequality* as

$$\sum_{i=1}^{k} x_{\{i,i+1\}} - \sum_{i=1}^{k} x_{\{i,i+q\}} \leqslant k - \left\lceil \frac{k}{q} \right\rceil .$$

Again, the edges $\{i, i+1\}$ may be replaced by paths.

We return to the study of the transitive packing polytope in general. Under different types of weak assumptions it is possible to show that the generalized cycle inequality (4.1) has depth 1 relative to (2.1)–(2.3). We present one condition that turns out to be widely applicable. We still use the notation introduced during the definition of a generalized cycle.

LEMMA 4.6. *Let* $(N, \mathcal{H}, \mathrm{tr})$ *be an extended hypergraph; let* $k > q$, $k \not\equiv 0 \bmod q$*; and let* $H_1, \ldots, H_k$ *be the sequence of edges of a generalized* (k, q)*-cycle with node set*

C in $(N,\mathcal{H})$. Assume that $\mathrm{tr}(H_i)\cap C=\emptyset$ *for* $i=1,2,\dots,k$. *If one of the following two conditions is satisfied, then the depth of the generalized (k,q)-cycle inequality* (4.1) *relative to* (2.1)–(2.3) *is* 1.

(i) *Every edge $H\in\mathcal{H}\setminus\{H_1,\dots,H_k\}$ with $H\subseteq C$ satisfies $|\mathrm{tr}(H)\cap C|\geqslant 2$.*

(ii) *The generalized cycle satisfies $C=\bigcup_{i=1}^k N_i$ and $|N_i|=1$ for $i=1,2,\dots,k$, and every edge $H\in\mathcal{H}\setminus\{H_1,\dots,H_k\}$ with $H\subseteq C$ satisfies $|H|=q$.*

Proof. The same proof works for both cases. For $i=1,\dots,k$ we let u_i be an arbitrary representative of the node subset N_i, i.e., $u_i\in N_i$. We define the point $x\in\mathbb{R}^N$ as follows:

$$x_u:=\begin{cases}(q-1)/q & \text{if } u\in\{u_1,\dots,u_k\},\\ \quad 1 & \text{if } u\in C\setminus\{u_1,\dots,u_k\},\\ \quad 0 & \text{otherwise.}\end{cases}$$

Whereas x belongs to the initial linear relaxation of $P_{\mathrm{TP}}(N,\mathcal{H},\mathrm{tr})$, i.e., satisfies the inequalities (2.1)–(2.3), it violates inequality (4.1). Hence this inequality is not implied by the initial system. ∎

Notice that Lemma 4.6(ii) is satisfied in the case of the node packing and the clique partitioning polytopes.

Euler, Jünger, and Reinelt [20] introduced generalized cycle inequalities for the independence system polytope and showed that they are facet defining for the independence system induced by the edges of the generalized cycle. The generalized cycles presented here, restricted to independence systems, extend theirs, since they assumed that the nodes of $C\setminus\bigcup_{i=1}^k N_i$ are arranged in a certain sequence corresponding to that of the sets N_i.

Finally, we introduce a class of inequalities also supported by generalized cycles, which are in general weaker than the generalized cycle inequalities. This class arises from the class of generalized cycle inequalities when we pay no attention to repetitions of transitive elements. We call this class of valid inequalities *weak* generalized cycle inequalities. For ease of referencing, we state this as a lemma.

LEMMA 4.7. *Let $(N,\mathcal{H},\mathrm{tr})$ be an extended hypergraph, and let, for $k>q$, $k\not\equiv 0 \bmod q$, the hypergraph $(C,\{H_i : i=1,2,\dots,k\})$ be a generalized (k,q)-cycle in $(N,\mathcal{H})$ such that $\mathrm{tr}(H_i)\cap C=\emptyset$ for $i=1,\dots,k$. Then, the weak generalized (k,q)-cycle inequality*

$$\sum_{u\in C}x_u-\sum_{u\in\mathrm{tr}(C)}n(u)x_u\leqslant |C|-\left\lceil\frac{k}{q}\right\rceil$$

is valid for the transitive packing polytope $P_{\mathrm{TP}}(N,\mathcal{H},\mathrm{tr})$.

Clearly, in the case $n(u)\leqslant 1$ for all nodes $u\in N$, a generalized (k,q)-cycle inequality and its weak version coincide.

4.2. Generalized clique inequalities. A second well-known class of valid inequalities for the node packing polytope are *clique inequalities*; see, e.g., [42]. Such an inequality is supported by a clique C in the given graph and is of the form

$$x(C)\leqslant 1.$$

It defines a facet if and only if the clique is maximal (with respect to set inclusion). We now describe how the clique inequalities can be extended to the transitive packing polytope.

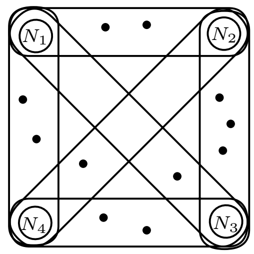

FIG. 4.5. *A* $(4,2)$*-clique. The points indicate other nodes of the clique.*

DEFINITION 4.8. *Let* $(N,\mathcal{H})$ *be a hypergraph, and let* $N_1,\ldots,N_k$*, for integers* $k \geqslant q \geqslant 2$*, be a collection of mutually disjoint nonempty subsets of the node set* N*. For each* q*-element subset* $\{i_1,\ldots,i_q\} \subseteq \{1,\ldots,k\}$ *of indices, we let* $H_{i_1,\ldots,i_q} \in \mathcal{H}$ *be an edge such that* $\bigcup_{j=1}^q N_{i_j} \subseteq H_{i_1,\ldots,i_q}$*. We assume that the edges in any collection of intersecting edges all have one common index. Let* C *be the union of these edges,* $C := \bigcup_{1\leqslant i_1<i_2<\cdots<i_q\leqslant k} H_{i_1,\ldots,i_q}$*. Then, we call the hypergraph*

$$(C, \{H_{i_1,\ldots,i_q} : 1 \leqslant i_1 < i_2 < \cdots < i_q \leqslant k\})$$

a generalized (k,q)-clique (*contained in* $(N,\mathcal{H})$).

Figure 4.5 depicts a generalized $(4,2)$-clique. Observe that the class of generalized $(3,2)$-cliques coincides with that of generalized $(3,2)$-cycles. Whenever we deal with generalized cliques in the context of extended hypergraphs, we assume that C and its set $\mathrm{tr}(C) := \bigcup_{1\leqslant i_1<i_2<\cdots<i_q\leqslant k} \mathrm{tr}(H_{i_1,\ldots,i_q})$ of transitive elements are disjoint, i.e., $\mathrm{tr}(H_{i_1,\ldots,i_q}) \cap C = \emptyset$ for all $1 \leqslant i_1 < i_2 < \cdots < i_q \leqslant k$. We denote by $\mathrm{mtr}(C)$ the multiset that arises from the union of the transitive elements $\mathrm{tr}(H_{i_1,\ldots,i_q})$. In other words, the multiplicity of a node $u \in \mathrm{mtr}(C)$ is precisely the number of edges $H_{i_1,\ldots,i_q}$ of which u is a transitive element.

THEOREM 4.9. *Let* $(N,\mathcal{H},\mathrm{tr})$ *be an extended hypergraph, and let, for* $k \geqslant q \geqslant 2$*, the hypergraph* $(C, \{H_{i_1,\ldots,i_q} : 1 \leqslant i_1 < i_2 < \cdots < i_q \leqslant k\})$ *be a generalized* (k,q)*-clique in* $(N,\mathcal{H})$ *such that* $\mathrm{tr}(H_{i_1,\ldots,i_q}) \cap C = \emptyset$ *for* $1 \leqslant i_1 < i_2 < \cdots < i_q \leqslant k$*. Then the generalized* (k,q)*-clique inequality*

$$x(C) - x(\mathrm{mtr}(C)) \leqslant |C| - k + q - 1 \tag{4.3}$$

is valid for $P_{\mathrm{TP}}(N,\mathcal{H},\mathrm{tr})$*.*

Proof. The proof is by induction on the size k of the generalized clique. Observe that for $k = q$ inequality (4.3) coincides with a transitivity constraint. In order to show its validity for $k > q$, we consider all $\binom{k}{\ell}$ generalized (ℓ,q)-cliques that are induced by the ℓ-element subsets of $\{N_1,\ldots,N_k\}$ for $\ell := \lfloor k(q-1)/q \rfloor + 1$. If we take the sum of their corresponding generalized (ℓ,q)-clique inequalities, we obtain an inequality whose support coincides with $C \cup \mathrm{tr}(C)$. Due to the assumptions on the relation of edges, a node $u \in N_i$ for some $i \in \{1,\ldots,k\}$ has coefficient $\binom{k-1}{\ell-1}$. The coefficient of a node $u \in C \setminus \bigcup_{i=1}^k N_i$ is less than or equal to $\binom{k-1}{\ell-1}$. The coefficient of each element in the multiset $\mathrm{mtr}(C)$ is $\binom{k-q}{\ell-q}$. In order to bring these coefficients into a line, we add suitable multiples of the upper bound inequalities $x_u \leqslant 1$ for nodes $u \in C \setminus \bigcup_{i=1}^k N_i$, and of the nonnegativity constraints $x_u \geqslant 0$ for $u \in \mathrm{mtr}(C)$. The resulting inequality then becomes

$$\binom{k-1}{\ell-1}\big(x(C)-x(\mathrm{mtr}(C))\big) \leqslant \binom{k-1}{\ell-1}|C| + \binom{k}{\ell}(q-\ell-1).$$

Dividing this new inequality by $\binom{k-1}{\ell-1}$ results in

$$x(C)-x(\mathrm{mtr}(C)) \leqslant |C|-k+q-1+\frac{k-\ell}{\ell}(q-1),$$

and by the choice of ℓ we can truncate the last term of the right-hand side to 0. □

Observe that in the case $q=2$, the size ℓ of the generalized cliques to be considered in the proof of Theorem 4.9 is $\ell = \lceil \frac{k+1}{2} \rceil$. This implies that the depth of the presented cutting plane proof is at most $\lceil \log(k-1) \rceil$. After drawing some conclusions from Theorem 4.9 for the acyclic subdigraph polytope and the clique partitioning polytope, we show that this bound is almost the best possible.

Again, if we consider the case of independence systems, the definition of generalized cliques given above is slightly more general than that of Euler, Jünger, and Reinelt [20]. They assumed that a node $u \in C \setminus \bigcup_{i=1}^{k} N_i$ cannot be contained in more than $\binom{k-1}{q-1} - 1$ edges (with common subindex) of the generalized (k,q)-clique. They showed that the corresponding generalized clique inequalities are facet inducing for the independence system with ground set C and circuits $H_{i_1,\dots,i_q}$.

Euler, Jünger, and Reinelt also observed that in the case of the acyclic subdigraph polytope the simple k-fence inequalities are contained in the class of generalized clique inequalities. We now show that even the k-fence inequalities (not necessarily simple) are contained in the class of generalized $(k,2)$-clique inequalities.

A *simple k-fence* ($k \geqslant 3$) is a digraph that is isomorphic to the digraph $F = (U, B_1 \cup B_2)$ on $2k$ nodes $U = \{u_1, u_2, \dots, u_{2k}\}$, where

$$B_1 = \{(u_i, u_{k+i}) : i = 1, \dots, k\},$$
$$B_2 = \bigcup_{i=1}^{k} \{(u_{k+i}, v) : v \in \{u_1, \dots, u_k\} \setminus \{u_i\}\}.$$

Adopting the notation of [25], we call the arcs in B_1 *pales* and the arcs in B_2 *pickets*. A *k-fence* is a digraph that arises from a simple k-fence by repeated subdivision of arcs; i.e., an arc (u,v) may be replaced by (u,w) and (w,v), where w is a new node, and so on. To keep the notation simple, we assume that $F = (U, B_1 \cup B_2)$ is a k-fence and call the arcs on the directed paths from u_i to u_{k+i} pales and those on the directed paths from u_{k+i} to v, $v \neq u_i$, pickets as well. If D is a digraph that contains the k-fence F, the *k-fence inequality*

$$x(B_1 \cup B_2) \leqslant |B_1 \cup B_2| - k + 1 \tag{4.4}$$

defines a facet of the acyclic subdigraph polytope of D; see [25].

THEOREM 4.10. *Let D be a digraph, and let $F = (U, B_1 \cup B_2)$ be a k-fence contained in D. Then the k-fence inequality* (4.4) *is contained in the class of generalized $(k,2)$-clique inequalities for the acyclic subdigraph polytope of D.*

Proof. We continue to use the notation introduced when we defined generalized cliques. We set N_i to be the set of pales on the path from u_i to u_{k+i} for $i = 1, 2, \dots, k$. Furthermore, for $1 \leqslant i < j \leqslant k$, we define H_{ij} to be the dicycle in F formed by the set of pales on the paths from u_i to u_{k+i} and from u_j to u_{k+j} as well as the pickets on the paths from u_{k+i} to u_j and u_{k+j} to u_i. Thus the k-fence F defines a generalized $(k,2)$-clique, and its k-fence inequality coincides with the corresponding generalized $(k,2)$-clique inequality. □

Whereas the class of generalized (k,q)-clique inequalities for the acyclic subdigraph polytope is richer than the class of k-fence inequalities, the class of generalized $(k,2)$-clique inequalities turns out to be precisely the class of $(1,k)$-2-*partition inequalities* for the clique partitioning polytope of a graph $G=(V,E)$. (Here, $q>2$ is not possible.) The latter inequalities are due to Grötschel and Wakabayashi [28] and are of the following form. Let $v,u_1,u_2,\dots,u_k \in V$ be a set of $k+1$ vertices such that $\{u_i,v\}\in E$ for $i=1,2,\dots,k$. Then the inequality

$$\sum_{i=1}^{k} x_{\{u_i,v\}} - \sum_{\substack{1\leqslant i<j\leqslant k\\ \{u_i,u_j\}\in E}} x_{\{u_i,u_j\}} \leqslant 1 \tag{4.5}$$

is valid for the clique partitioning polytope. It is facet defining if G is complete; see [28].

THEOREM 4.11. *The class of generalized* $(k,2)$*-clique inequalities for the clique partitioning polytope of a graph G coincides with the class of* $(1,k)$*-2-partition inequalities.*

Proof. Let us first consider a $(1,k)$-2-partition inequality (4.5). Since the edges $\{u_i,v\}$ and $\{u_j,v\}$ for $i,j=1,2,\dots,k$, $i\neq j$, form a hyperedge and since the transitive edges of these hyperedges are distinct from the edges $\{u_i,v\}\in E$ for $i=1,2,\dots,k$, this inequality is a generalized $(k,2)$-clique inequality. On the other hand, a generalized $(k,2)$-clique of the line graph of G always leads to the support of a $(1,k)$-2-partition inequality: since all participating edges in G have to be pairwise incident, either they share one common node or we have $k=3$. In the former case they form the support of a $(1,k)$-2-partition inequality. The latter case contradicts the assumption that the generalized clique and its transitive elements do not intersect. □

For the partition polytope, there can exist generalized (k,q)-clique inequalities for any q.

We are now about to show that the depth of the generalized $(k,2)$-clique inequalities tends to infinity with k.

THEOREM 4.12. *Let* $(C,\{H_{ij}:1\leqslant i<j\leqslant k\})$ *be a generalized* $(k,2)$*-clique of the extended hypergraph* $(N,\mathcal{H},\mathrm{tr})$*. Assume that* $N_i=\big(\bigcap_{i<j\leqslant k} H_{ij}\big)\bigcap\big(\bigcap_{1\leqslant j<i} H_{ji}\big)$*, for* $i=1,2,\dots,k$*, and that each edge* $H\in\mathcal{H}$ *such that* $H\subseteq C$ *satisfies* $N_i\cup N_j\subseteq H$ *for some* $i,j\in\{1,2,\dots,k\}$*,* $i\neq j$*. Then the depth of the generalized* $(k,2)$*-clique inequality* (4.3) *relative to* (2.1)–(2.3) *is at least* $\log k-1$*.*

In order to prove this theorem we make use of the following lemma of Chvátal, Cook, and Hartmann [12].

LEMMA 4.13 (see [12]). *Let P be a rational polyhedron in $\mathbb{R}^N$. Let y and z be points in $\mathbb{R}^N$, and let $\mu_1,\mu_2,\dots,\mu_d$ be positive numbers. Furthermore, for $t=0,1,\dots,d$ set*

$$x^{(t)} := y - \sum_{i=1}^{t}\frac{1}{\mu_i}z.$$

If $y\in P$ and if, for all $t=1,\dots,d$, every inequality $ax\leqslant\beta$ valid for $P\cap\mathbb{Z}^N$ with $a\in\mathbb{Z}^N$ and $az<\mu_t$ satisfies $ax^{(t)}\leqslant\beta$, then $x^{(t)}\in P^{(t)}$ for all $t=0,1,\dots,d$.

Proof of Theorem 4.12. For $i = 1, \ldots, k$ let u_i be an arbitrary representative of the node subset N_i, i.e., $u_i \in N_i$. Let C_1 be the union of these nodes u_i, $C_1 := \bigcup_{i=1}^k \{u_i\}$. Moreover, denote by C_2 the rest of the generalized $(k,2)$-clique C, that is, $C_2 := C \backslash C_1$. For a nonnegative integer t we define

$$x^{(t)} := \chi^{C_2} + 2^{-(t+1)} \chi^{C_1}.$$

If $t < \log k - 1$, then

$$x^{(t)}(C) - x^{(t)}(\mathrm{mtr}(C)) = \chi^{C_2}\chi^{C_2} + 2^{-(t+1)}\chi^{C_1}\chi^{C_1} = |C| - k + 2^{-(t+1)}k > |C| - k + 1,$$

and so $x^{(t)}$ fails to satisfy the generalized $(k,2)$-clique inequality (4.3). It remains to show that $x^{(t)} \in P^{(t)}$ for all t. For this we use Lemma 4.13 with $y := \chi^{C_2} + \frac{1}{2}\chi^{C_1}$, $z := \chi^{C_1}$, and $\mu_t := 2^{t+1}$. Observe that y is a solution to (2.1)–(2.3). Now consider an arbitrary inequality $ax \leqslant \beta$, valid for $P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})$ and such that $a \in \mathbb{Z}^N$ and $a\chi^{C_1} < \mu_t$. We need to verify that $ax^{(t)} \leqslant \beta$. Whereas this is obvious if $a\chi^{C_1} \leqslant 0$, in the case $a\chi^{C_1} > 0$ we have

$$ax^{(t)} = a\chi^{C_2} + \frac{1}{\mu_t} a\chi^{C_1} < a\chi^{C_2} + 1 \leqslant a(\chi^{C_2} + \chi^{\{u_i\}}) \leqslant \beta$$

for a representative u_i such that $a_{u_i} \geqslant 1$. The last inequality follows from $\chi^{C_2} + \chi^{\{u_i\}} \in P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})$. □

Theorem 4.12 was proved before for the special instances formed by the clique inequalities of the node packing polytope [11] and by the simple k-fence inequalities of the acyclic subdigraph polytope [12]. Notice that the assumption of Theorem 4.12 is also satisfied by the k-fence inequalities since each dicycle contained in a fence uses pales between at least two different pairs of nodes. Moreover, Theorem 4.12 also applies to the $(1,k)$-2-partition inequalities of the clique partitioning polytope.

4.3. Generalized antihole inequalities. Another class of valid inequalities for the node packing polytope is supported by odd antiholes. An *odd antihole* in a graph is the complement of an odd cycle of length at least five without a chord. Let O denote the set of vertices of an odd antihole. Then the *odd antihole inequality* associated with O is

$$x(O) \leqslant 2.$$

Again, it turns out that these inequalities form a special case of a more general principle.

DEFINITION 4.14. *Let $(N, \mathcal{H})$ be a hypergraph, and let q and s be integers such that $s \geqslant q \geqslant 2$. For convenience, we set $k := qs + 1$. Let $N_1, N_2, \ldots, N_k$ be a sequence of mutually disjoint nonempty subsets of the node set N. Moreover, for each $\ell \in \{1, 2, \ldots, k\}$ and for every q-element set of indices $\{i_1, i_2, \ldots, i_q\} \subseteq \{\ell, \ell+1, \ldots, \ell+s-1\}$ (where indices greater than k are taken modulo $k+1$ and shifted by $+1$) we let the set $H^\ell_{i_1,i_2,\ldots,i_q}$ be an edge such that $\bigcup_{j=1}^q N_{i_j} \subseteq H^\ell_{i_1,i_2,\ldots,i_q}$. In addition, we assume, for each $\ell \in \{1, 2, \ldots, k\}$, that the edges in any collection of intersecting edges of type $H^\ell_{i_1,i_2,\ldots,i_q}$ all have one common (sub)index. We denote by O^ℓ the union of these edges, $O^\ell := \bigcup_{\ell \leqslant i_1 < i_2 < \cdots < i_q \leqslant \ell+s-1} H^\ell_{i_1,i_2,\ldots,i_q}$, and by O the union of all these edges, $O := \bigcup_{\ell=1}^k O^\ell$. Moreover, let $\widetilde{m}(u) := |\{\ell \in \{1, 2, \ldots, k\} : u \in O^\ell\}|$ for a node $u \in O$. We assume that $\widetilde{m}(u) \leqslant s$ for all nodes $u \in O$. Then the hypergraph*

$$(O, \{H^\ell_{i_1,i_2,\ldots,i_q} : \ell \leqslant i_1 < i_2 < \cdots < i_q \leqslant \ell + s - 1 \textit{ for some } \ell \in \{1, 2, \ldots, k\}\})$$

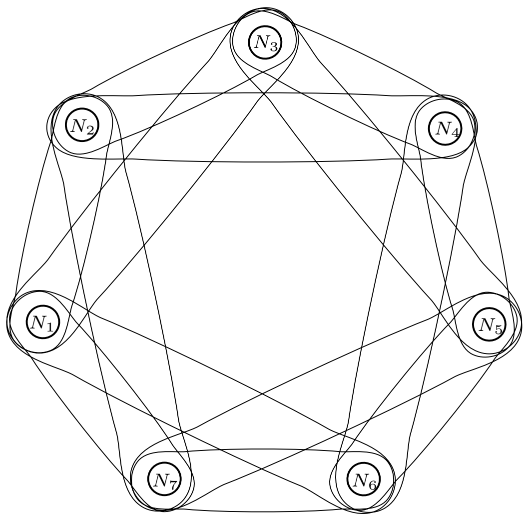

FIG. 4.6. *A generalized* $(3,2)$*-antihole* (*with* $O = \bigcup_{i=1}^{7} N_i$).

is called a generalized (s,q)-antihole (*contained in* $(N,\mathcal{H})$).

Figure 4.6 depicts a generalized $(3,2)$-antihole. Notice that it may happen that the same edge wears different names. For instance, if $O = \bigcup_{i=1}^{k} N_i$ and $q < s$, then $H^{\ell}_{\ell+s-q,\dots,\ell+s-1} = H^{\ell+1}_{\ell+s-q,\dots,\ell+s-1}$. Given a generalized antihole that is contained in a given extended hypergraph, we define $\widetilde{n}(u)$ to be the multiplicity of a node u contained in the transitive sets associated with that generalized antihole, i.e., $\widetilde{n}(u) := |\{H^{\ell}_{i_1,i_2,\dots,i_q} : u \in \mathrm{tr}(H^{\ell}_{i_1,i_2,\dots,i_q})$ for some $\ell \in \{1,2,\dots,k\}, \ell \leqslant i_1 < i_2 < \cdots < i_q \leqslant \ell+s-1\}|$. Thus, if the same edge occurs more often under different names, we count the number of names. We set $\mathrm{tr}(O) := \bigcup_{\ell=1}^{k}(\bigcup_{\ell \leqslant i_1 < i_2 < \cdots < i_q \leqslant \ell+s-1} \mathrm{tr}(H^{\ell}_{i_1,i_2,\dots,i_q}))$.

THEOREM 4.15. *Let* $(N,\mathcal{H},\mathrm{tr})$ *be an extended hypergraph, and let the hypergraph* $(O, \{H^{\ell}_{i_1,i_2,\dots,i_q} : \ell \leqslant i_1 < i_2 < \cdots < i_q \leqslant \ell+s-1$ *for some* $\ell \in \{1,2,\dots,k\}\})$ *be a generalized* (s,q)*-antihole in* $(N,\mathcal{H})$ *such that* $\mathrm{tr}(O) \cap O = \emptyset$. *Then, the generalized* (s,q)*-antihole inequality*

$$\sum_{u\in O} x_u - \sum_{u\in \mathrm{tr}(O)} \frac{\lceil \widetilde{n}(u)\rceil_s}{s} x_u \leqslant |O| - q(s-q+1) - 1 \tag{4.6}$$

is valid for $P_{\mathrm{TP}}(N,\mathcal{H},\mathrm{tr})$. *It has a cutting plane proof from* (2.1)–(2.3) *of depth at most* $\lceil \log(s-1)\rceil + 1$.

Proof. Let $N_1, N_2, \dots, N_k$ be the sequence of nodes underlying the generalized (s,q)-antihole. Notice that for every $\ell \in \{1,2,\dots,k\}$ the edges $\{H^{\ell}_{i_1,i_2,\dots,i_q} : \ell \leqslant i_1 < i_2 < \cdots < i_q \leqslant \ell+s-1\}$ induce a generalized (s,q)-clique. Each set N_i is contained in precisely s of these k cliques. By adding up the k associated (s,q)-clique inequalities, the appropriate number of upper bound constraints $x_u \leqslant 1$ for $u \in O \setminus \bigcup_{i=1}^{k} N_i$ (namely, $s - \widetilde{m}(u)$ many), as well as the appropriate number of nonnegativity constraints for each element $u \in \mathrm{tr}(O)$ (namely, $\lceil \widetilde{n}(u)\rceil_s - \widetilde{n}(u)$), we obtain that

$$s\sum_{u\in O} x_u - \sum_{u\in \mathrm{tr}(O)} \lceil \widetilde{n}(u)\rceil_s x_u \leqslant s|O| - k(s-q+1)$$

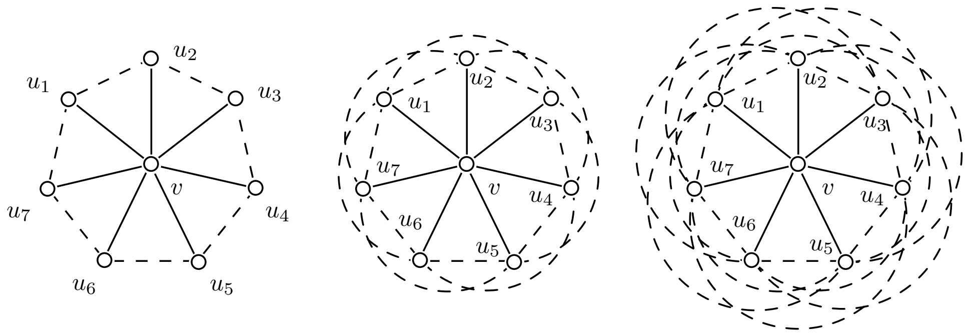

FIG. 4.7. *From left to right: a generalized cycle, a generalized antihole, and a generalized clique with associated transitive elements in the case of the clique partitioning problem.*

is valid for $P_{\rm TP}(N, \mathcal{H}, {\rm tr})$. Division by s and taking the floor of the right-hand side gives the desired inequality. The bound on the depth of its cutting plane proof follows immediately from that for the generalized clique inequalities. ∎

To see that we indeed derive from Theorem 4.15 the usual odd antihole inequalities for the node packing polytope of a graph G, we proceed as follows. Let O be the node set of an odd antihole in G, $O = \{u_1, u_2, \ldots, u_k\}$, and assume that u_ℓ and $u_{\ell+s}$ as well as u_ℓ and $u_{\ell+s+1}$ are not adjacent, for $\ell = 1, 2, \ldots, k$. We now relate this to a generalized antihole. Clearly, $q = 2$, and hence $|O| = k = 2s + 1$. It remains to identify the edges. For $\ell \in \{1, 2, \ldots, k\}$ we take as edges H^ℓ_{ij} the edges of the clique induced by the nodes $u_\ell, u_{\ell+1}, \ldots, u_{\ell+s-1}$. Notice that several edges in G are taken more than once but under different names. Finally, observe that the right-hand side of (4.6) simplifies to 2.

Since line graphs do not contain odd antiholes (with more than five nodes), there do not exist generalized antihole inequalities for the clique partitioning polytope when we assume that $s \geqslant 3$ and that u_ℓ and $u_{\ell+s}$ as well as u_ℓ and $u_{\ell+s+1}$ are not linked by a hyperedge, for each $\ell = 1, 2, \ldots, k$. Others may well exist; see, for instance, Figure 4.7. We record this as a lemma.

LEMMA 4.16. *Let $G = (V, E)$ be a graph, and let $v, u_1, u_2, \ldots, u_k \in V$, be distinct nodes such that $\{v, u_i\} \in E$, for $i = 1, 2, \ldots, k$, for $k = 2s + 1$, and $s \geqslant 2$. Define the set $T := \{\{u_i, u_j\} \in E : \ell \leqslant i < j \leqslant \ell + s - 1$ for some $\ell \in \{1, 2, \ldots, k\}\}$. The inequality*

$$\sum_{i=1}^{k} x_{\{v,u_i\}} - \sum_{\{u_i,u_j\} \in T} x_{\{u_i,u_j\}} \leqslant 2$$

is valid for the clique partitioning polytope of G. (Again, indices greater than k are taken modulo $k + 1$ and shifted by $+1$.)

We note that generalized $(2, 2)$-antihole inequalities of the transitive packing polytope coincide with generalized $(5, 2)$-cycle inequalities. So far, antihole inequalities have not been exploited for the acyclic subdigraph polytope or the partition polytope.

4.4. Generalized antiweb inequalities. The main idea in the derivation of the generalized antihole inequalities was to combine generalized clique inequalities in a manner oriented on the cutting plane proof of generalized cycle inequalities. This can be generalized and leads for the node packing polytope to the antiweb inequalities [52].

For integers $1 \leqslant s \leqslant k$, a (k,s)-*antiweb* is a graph with node set $W = \{u_1, u_2, \dots, u_k\}$ such that each node u_i is adjacent to all other nodes but not to the $\max\{0, k-2s+1\}$ nodes $u_{i+s}, u_{i+s+1}, \dots, u_{i+k-s}$. (Again, indices greater than k are taken modulo $k+1$ and shifted by $+1$.) The associated *antiweb inequality* is

$$x(W) \leqslant \left\lfloor \frac{k}{s} \right\rfloor.$$

We proceed by introducing special hypergraphs that we call generalized antiwebs.

DEFINITION 4.17. *Let $(N, \mathcal{H})$ be a hypergraph, and let k, s, and q be integers such that $k \geqslant s \geqslant q \geqslant 2$. Let $N_1, N_2, \dots, N_k$ be a sequence of mutually disjoint nonempty subsets of the node set N. For each $\ell \in \{1, 2, \dots, k\}$ and each q-element set of indices $\{i_1, i_2, \dots, i_q\} \subseteq \{\ell, \ell+1, \dots, \ell+s-1\}$ (where indices are taken modulo $k+1$ and shifted by $+1$), we let $H^\ell_{i_1,i_2,\dots,i_q} \in \mathcal{H}$ be an edge such that $\bigcup_{j=1}^q N_{i_j} \subseteq H^\ell_{i_1,i_2,\dots,i_q}$. In addition, we assume, for each $\ell \in \{1, 2, \dots, k\}$, that the edges in any collection of intersecting edges of type $H^\ell_{i_1,i_2,\dots,i_q}$ all have one common (sub)index. For each ℓ, we denote by W^ℓ the union of the associated edges, $W^\ell := \bigcup_{\ell \leqslant i_1 < i_2 < \cdots < i_q \leqslant \ell+s-1} H^\ell_{i_1,i_2,\dots,i_q}$. Moreover, we let W denote the union of all these edges, $W := \bigcup_{\ell=1}^k W^\ell$. Again, for $u \in W$ we let $\widetilde{m}(u)$ be the multiplicity of u with respect to its occurrence in W^ℓ, $\ell = 1, 2, \dots, k$, i.e., $\widetilde{m}(u) := |\{\ell \in \{1, 2, \dots, k\} : u \in W^\ell\}|$. If $\widetilde{m}(u) \leqslant s$ for all $u \in W$, then we call the hypergraph*

$$(W, \{H^\ell_{i_1,i_2,\dots,i_q} : \ell \leqslant i_1 < i_2 < \cdots < i_q \leqslant \ell + s - 1 \textit{ for some } \ell \in \{1, 2, \dots, k\}\})$$

a generalized (k, s, q)-antiweb (*contained in* $(N, \mathcal{H})$).

THEOREM 4.18. *Let $(N, \mathcal{H}, \mathrm{tr})$ be an extended hypergraph, and let the hypergraph $(W, \{H^\ell_{i_1,i_2,\dots,i_q} : \ell \leqslant i_1 < i_2 < \cdots < i_q \leqslant \ell + s - 1 \textit{ for some } \ell \in \{1, 2, \dots, k\}\})$ be a generalized (k, s, q)-antiweb in $(N, \mathcal{H})$ such that $\mathrm{tr}(W) \cap W = \emptyset$. Then, the generalized (k, s, q)-antiweb inequality*

$$\sum_{u \in W} x_u - \sum_{u \in \mathrm{tr}(W)} \frac{\lceil \widetilde{n}(u) \rceil_s}{s} x_u \leqslant \left\lfloor \frac{s|W| - k(s-q+1)}{s} \right\rfloor \tag{4.7}$$

is valid for $P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})$. It has a cutting plane proof from (2.1)–(2.3) *of depth at most $\lceil \log(s-1) \rceil + 1$. Here, $\mathrm{tr}(W) := \bigcup_{\ell=1}^k (\bigcup_{\ell \leqslant i_1 < i_2 < \cdots < i_q \leqslant \ell+s-1} \mathrm{tr}(H^\ell_{i_1,i_2,\dots,i_q}))$ and $\widetilde{n}(u) := |\{H^\ell_{i_1,i_2,\dots,i_q} : u \in \mathrm{tr}(H^\ell_{i_1,i_2,\dots,i_q})$ for some $\ell \in \{1, 2, \dots, k\}$, $\ell \leqslant i_1 < i_2 < \cdots < i_q \leqslant \ell + s - 1\}|$.*

Proof. The cutting plane proof goes along the line of the proof of the validity of generalized antihole inequalities (Theorem 4.15) and is therefore omitted. ∎

It follows from their construction that generalized (k, s, q)-antiweb inequalities subsume all the former classes of inequalities for the transitive packing polytope $P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})$. In fact,

- if $q = s$ and if s does not divide k, we obtain the class of generalized (k, q)-cycle inequalities;
- if $s = k$, the class of generalized antiweb inequalities contains the class of generalized (k, q)-clique inequalities;
- if $k = qs + 1$, we have the class of generalized (s, q)-antihole inequalities.

Laurent [32] previously extended antiwebs to the independence system polytope; however, the inequalities (4.7) restricted to this setting are more general. Laurent used one-element sets N_i and edges that are precisely the union of q of these. She showed that such an inequality is facet defining for the polytope associated with the independence system defined by the circuits of her antiweb.

4.5. Odd partition inequalities. In this section, we introduce another new class of inequalities for the transitive packing polytope. It is an extension of a class of inequalities recently proposed by Caprara and Fischetti [9] for the acyclic subdigraph polytope.

Assume that we are given an extended hypergraph $(N, \mathcal{H}, \mathrm{tr})$. Let $H_1, \dots, H_k$ be a collection of distinct edges of $\mathcal{H}$, and let $m(u)$ and $n(u)$ denote the multiplicity of a node $u \in N$ in this collection and the associated set of transitive elements, respectively. That is, $m(u) := |\{i \in \{1, \dots, k\} : u \in H_i\}|$ and $n(u) := |\{i \in \{1, \dots, k\} : u \in \mathrm{tr}(H_i)\}|$. We denote the difference of these two numbers by $d(u)$, $d(u) := m(u) - n(u)$. Let W be the union of all the nodes involved, $W := \bigcup_{i=1}^{k}(H_i \cup \mathrm{tr}(H_i))$, and let W^{odd} be the set of those nodes that occur either in an odd number of edges H_i or in an odd number of transitive sets $\mathrm{tr}(H_i)$ but not both, $W^{\mathrm{odd}} := \{u \in W : d(u) \text{ odd}\}$. Furthermore, let $(W_1^{\mathrm{odd}}, W_2^{\mathrm{odd}})$ be a partition of W^{odd} such that $\sum_{i=1}^{k}|H_i| + |W_1^{\mathrm{odd}}| - k$ is odd. ($W_1^{\mathrm{odd}} = \emptyset$ or $W_2^{\mathrm{odd}} = \emptyset$ is possible.)

Taking the sum of the constraints

$$\begin{aligned} \sum_{u \in H_i} x_u - \sum_{u \in \mathrm{tr}(H_i)} x_u &\leqslant |H_i| - 1 && \text{for } i = 1, \dots, k, \\ x_u &\leqslant 1 && \text{for } u \in W_1^{\mathrm{odd}}, \\ -x_u &\leqslant 0 && \text{for } u \in W_2^{\mathrm{odd}}, \end{aligned}$$

and dividing the result by 2, we obtain

$$\begin{aligned} \sum_{u \in W \setminus W^{\mathrm{odd}}} \frac{d(u)}{2} x_u + \sum_{u \in W_1^{\mathrm{odd}}} \frac{d(u)+1}{2} x_u \qquad & \\ + \sum_{u \in W_2^{\mathrm{odd}}} \frac{d(u)-1}{2} x_u \leqslant \frac{\sum_{i=1}^{k}|H_i| + |W_1^{\mathrm{odd}}| - k}{2}. \end{aligned} \tag{4.8}$$

Rounding down the right-hand side gives the following inequality that is valid for the transitive packing polytope $P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})$,

$$\begin{aligned} \sum_{u \in W \setminus W^{\mathrm{odd}}} \frac{d(u)}{2} x_u + \sum_{u \in W_1^{\mathrm{odd}}} \frac{d(u)+1}{2} x_u \qquad & \\ + \sum_{u \in W_2^{\mathrm{odd}}} \frac{d(u)-1}{2} x_u \leqslant \frac{\sum_{i=1}^{k}|H_i| + |W_1^{\mathrm{odd}}| - k - 1}{2}. \end{aligned} \tag{4.9}$$

We call inequalities of type (4.9) *odd partition inequalities*. We continue by pointing out some special cases in which inequality (4.9) is dominated by other inequalities, as well as some other cases in which it has depth 1 relative to (2.1)–(2.3) and is therefore interesting.

LEMMA 4.19. *Let $(N, \mathcal{H}, \mathrm{tr})$ be a hypergraph with associated transitive elements, and let $H_1, \dots, H_k$ be a collection of distinct edges of $\mathcal{H}$. If $(H_k \cup \mathrm{tr}(H_k)) \cap \bigcup_{i=1}^{k-1}(H_i \cup \mathrm{tr}(H_i)) = \emptyset$, then the odd partition inequality* (4.9) *for $H_1, \dots, H_k$ is implied by the initial inequalities* (2.1)–(2.3) *and inequality* (4.9) *for $H_1, \dots, H_{k-1}$.*

Proof. Observe first that $H_k \cup \operatorname{tr}(H_k) \subseteq W^{\text{odd}}$. Thus the left-hand side of inequality (4.9) can be expressed as follows:

$$\sum_{u \in W \setminus W^{\text{odd}}} \frac{d(u)}{2} x_u + \sum_{u \in W_1^{\text{odd}} \setminus (H_k \cup \operatorname{tr}(H_k))} \frac{d(u)+1}{2} x_u$$
$$+ \sum_{u \in W_2^{\text{odd}} \setminus (H_k \cup \operatorname{tr}(H_k))} \frac{d(u)-1}{2} x_u + \sum_{u \in H_k \cap W_1^{\text{odd}}} x_u - \sum_{u \in \operatorname{tr}(H_k) \cap W_2^{\text{odd}}} x_u.$$

Notice that the first three terms precisely form the left-hand side of inequality (4.8) for $H_1, \ldots, H_{k-1}$ (where we use the natural restriction of W_1^{odd} and W_2^{odd}). We continue by distinguishing three cases, namely,

(i) $|(H_k \cap W_2^{\text{odd}}) \cup (\operatorname{tr}(H_k) \cap W_1^{\text{odd}})| \geqslant 2$,

(ii) $|(H_k \cap W_2^{\text{odd}}) \cup (\operatorname{tr}(H_k) \cap W_1^{\text{odd}})| = 1$, and finally,

(iii) $|(H_k \cap W_2^{\text{odd}}) \cup (\operatorname{tr}(H_k) \cap W_1^{\text{odd}})| = 0$.

In case (i), we add to inequality (4.8) for $H_1, \ldots, H_{k-1}$ the inequalities

$$x_u \leqslant 1 \text{ for } u \in H_k \cap W_1^{\text{odd}} \quad \text{and} \quad -x_u \leqslant 0 \text{ for } u \in \operatorname{tr}(H_k) \cap W_2^{\text{odd}}.$$

Then the left-hand side of the resulting inequality coincides with that of inequality (4.9). The numerator of the right-hand side is

$$\sum_{i=1}^{k-1} |H_i| + |W_1^{\text{odd}} \setminus (H_k \cup \operatorname{tr}(H_k))| - k + 1 + 2|H_k \cap W_1^{\text{odd}}|$$
$$= \sum_{i=1}^{k} |H_i| + |W_1^{\text{odd}}| - k + 1 - (|H_k \cap W_2^{\text{odd}}| + |\operatorname{tr}(H_k) \cap W_1^{\text{odd}}|),$$

which is, because of assumption (i), less than or equal to

$$\sum_{i=1}^{k} |H_i| + |W_1^{\text{odd}}| - k - 1,$$

which is the numerator of the right-hand side of inequality (4.9) for $H_1, \ldots, H_k$. Hence in this case inequality (4.9) has depth 0 relative to (2.1)–(2.3).

Since we assumed $\sum_{i=1}^{k} |H_i| + |W_1^{\text{odd}}| - k$ to be odd in order to derive inequality (4.9), the assumption in case (ii) guarantees that the numerator of the right-hand side of inequality (4.8) for $H_1, \ldots, H_{k-1}$ will be odd, too. Hence, the following inequality is valid for $P_{\text{TP}}(N, \mathcal{H}, \operatorname{tr})$, which is inequality (4.9) for $H_1, \ldots, H_{k-1}$:

$$\sum_{u \in W \setminus W^{\text{odd}}} \frac{d(u)}{2} x_u + \sum_{u \in W_1^{\text{odd}} \setminus (H_k \cup \operatorname{tr}(H_k))} \frac{d(u)+1}{2} x_u$$
$$+ \sum_{u \in W_2^{\text{odd}} \setminus (H_k \cup \operatorname{tr}(H_k))} \frac{d(u)-1}{2} x_u \leqslant \frac{\sum_{i=1}^{k-1} |H_i| + |W_1^{\text{odd}} \setminus (H_k \cup \operatorname{tr}(H_k))| - k}{2}.$$

By adding to this inequality the inequalities

$$x_u \leqslant 1 \text{ for } u \in H_k \cap W_1^{\text{odd}} \quad \text{and} \quad -x_u \leqslant 0 \text{ for } u \in \operatorname{tr}(H_k) \cap W_2^{\text{odd}},$$

we obtain inequality (4.9), which is therefore implied by (4.9) for $H_1, \ldots, H_{k-1}$ and the bound constraints (2.2) and (2.3).

In case (iii), we simply add the transitivity constraint (2.1) for H_k to inequality (4.8) for $H_1, \ldots, H_{k-1}$. It follows that inequality (4.9) again has depth 0 relative to system (2.1)–(2.3). □

Lemma 4.19 reflects, in particular, the trivial fact that we cannot hope to obtain a stronger inequality by adding inequalities with mutually disjoint support. We now present a condition that is sufficient to ensure that inequality (4.9) has depth 1, which leads us back to cycles in the hypergraph $(N, \mathcal{H})$.

LEMMA 4.20. *Let $(N, \mathcal{H}, \mathrm{tr})$ be an extended hypergraph, and let $H_1, \ldots, H_k$ be a collection of distinct edges in $\mathcal{H}$, $k \geqslant 2$. Let the sets W^{odd}, W_1^{odd}, and W_2^{odd} be defined as before. Assume that $\mathrm{tr}(H_j) \cap \bigcup_{i=1}^k H_i = \emptyset$ for $j = 1, \ldots, k$. If*

- *there exist k distinct nodes $u_1, \ldots, u_k \in N$ such that $u_i \in H_i \cap H_{i+1}$ but $u_i \notin H_j$ for $j \neq i, i+1$,*
- *the transitive set $\mathrm{tr}(H)$ of an edge $H \neq H_i$ $(i = 1, \ldots, k)$ that satisfies $H \subseteq \bigcup_{i=1}^k H_i$ intersects $\bigcup_{i=1}^k H_i$ either in at least one node different from $u_1, \ldots, u_k$ or in at least two nodes from $u_1, \ldots, u_k$, and*
- *$W_1^{\mathrm{odd}} \subseteq \left(\bigcup_{i=1}^k H_i\right) \setminus \{u_1, u_2, \ldots, u_k\}$,*

then the depth of the odd partition inequality (4.9) *relative to* (2.1)–(2.3) *is* 1.

Proof. Define the point $x \in \mathbb{R}^N$ as follows:

$$x_u := \begin{cases} 1/2 & \text{if } u \in \{u_1, \ldots, u_k\}, \\ 1 & \text{if } u \in \left(\bigcup_{i=1}^k H_i\right) \setminus \{u_1, \ldots, u_k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Whereas x belongs to the initial linear relaxation of $P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})$, i.e., satisfies inequalities (2.1)–(2.3), it violates inequality (4.9). Hence this inequality is not implied by the initial system. □

As mentioned before, Caprara and Fischetti [9] introduced the odd partition inequalities for the acyclic subdigraph polytope in order to show that a subclass of the Möbius ladder inequalities can be derived from the initial relaxation by a cutting plane proof of length 1, where all coefficients used are either 0 or $\frac{1}{2}$. Indeed, if $(C, \{H_i : i = 1, 2, \ldots, k\})$ is a generalized $(k, 2)$-cycle, we obtain the associated generalized $(k, 2)$-cycle inequality as an odd partition inequality by setting $W_1^{\mathrm{odd}} := \{u \in C : m(u) \text{ odd}\}$ and $W_2^{\mathrm{odd}} := \{u \in \mathrm{tr}(C) : n(u) \text{ odd}\}$. In section 4.1, we showed that the subclass of Möbius ladder inequalities where each triple of participating dicycles has an empty intersection is contained in the class of generalized $(k, 2)$-cycle inequalities for the acyclic subdigraph polytope. This implies Caprara and Fischetti's result.

5. Transitive packing in graphs. An important subproblem of the transitive packing problem is formed by the instances where the given hypergraph is actually a graph. This section is devoted to discussing the polytopes associated with these instances in more detail. To avoid confusion, we still use the notation $(N, \mathcal{H}, \mathrm{tr})$ but assume throughout this section that $|H| = 2$ for all $H \in \mathcal{H}$. We call the triple $(N, \mathcal{H}, \mathrm{tr})$ an *extended graph*. The transitive packing polytope is then given as

$$P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr}) = \mathrm{conv}\left\{ x \in \{0,1\}^N : x_u + x_v - \sum_{w \in \mathrm{tr}(\{u,v\})} x_w \leqslant 1 \text{ for } \{u, v\} \in \mathcal{H} \right\}.$$

Recall that both the node packing polytope and the clique partitioning polytope are of this flavor. For the node packing polytope, it is known that all facet defining inequalities with right-hand side 1 are clique inequalities; see [42]. This remains true for the transitive packing polytope of the following extended graphs.

THEOREM 5.1. *Let* $(N, \mathcal{H}, \mathrm{tr})$ *be an extended graph such that for every clique* C *in* $(N, \mathcal{H})$ *the following condition is satisfied:*

Each node $u \in \mathrm{tr}(C)$ *belongs to* $\mathrm{tr}(\{v, w\})$ *for a unique edge* $\{v, w\}$ *induced by* C *and satisfies either*

- $\{u, v\}, \{u, w\} \notin \mathcal{H}$, *or*
- $\{u, v\} \notin \mathcal{H}$, $\{u, w\} \in \mathcal{H}$, *and* $v \in \mathrm{tr}(\{u, w\})$, *or*
- $\{u, w\} \notin \mathcal{H}$, $\{u, v\} \in \mathcal{H}$, *and* $w \in \mathrm{tr}(\{u, v\})$, *or*
- $\{u, v\}, \{u, w\} \in \mathcal{H}$, *and* $v \in \mathrm{tr}(\{u, w\})$ *and* $w \in \mathrm{tr}(\{u, v\})$.

Then, any facet defining inequality $cx \leqslant 1$ (*with* c *integral*) *of the transitive packing polytope* $P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})$ *either is of the form* $x_u \leqslant 1$ *or is a generalized* $(k, 2)$*-clique inequality.*

Proof. Since every singleton is a transitive packing, the coefficients of the vector c have value at most 1. If c has exactly one coefficient with value 1, indexed by, say, $u \in N$, then $c = \chi^{\{u\}}$. Otherwise, $cx \leqslant 1$ would be dominated by $x_u \leqslant 1$. So we may assume from now on that the number of coefficients of c with value 1 is at least two. Let C be the set of nodes u such that $c_u = 1$. Since $cx \leqslant 1$ is valid, the nodes in C have to be pairwise adjacent, i.e., they induce a clique in $(N, \mathcal{H})$. From this validity it also follows that $\mathrm{tr}(C) \cap C = \emptyset$. It remains to be observed that the coefficient c_u of a transitive element $u \in \mathrm{tr}(C)$ is not zero. This follows from the assumptions with respect to transitive elements and the validity of $cx \leqslant 1$ for $P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})$. We just need to observe that the node set formed by u and the pair of nodes $v, w \in C$ such that $u \in \mathrm{tr}(\{v, w\})$ is a transitive packing in $(N, \mathcal{H}, \mathrm{tr})$. □

The assumptions made in Theorem 5.1 are satisfied, for instance, by the extended graphs corresponding to instances of the clique partitioning problem. Hence, if a graph G has no isolated edges, $(1, k)$-2-partition inequalities are the only facet defining inequalities with right-hand side 1 of the clique partitioning polytope of G. The latter observation was independently made in [41].

Notice that the assumptions of Theorem 4.12 are always satisfied for transitive packing problems in graphs. Consequently, the generalized $(k, 2)$-clique inequalities have depth at least $\log k - 1$, relative to (2.1)–(2.3).

If the transitive elements of a clique C do not interact with C itself, the clique and its transitive elements form the support of valid inequalities, where the nodes of the cliques have coefficients greater than one.

THEOREM 5.2. *Let* $(N, \mathcal{H}, \mathrm{tr})$ *be an extended graph, and let* C *be the node set of a generalized* $(k, 2)$*-clique in* $(N, \mathcal{H})$ *such that* $\mathrm{tr}(C) \cap C = \emptyset$. *Moreover, let* $t \geqslant 1$ *be an integer. Then, the* t*-reinforced generalized* $(k, 2)$*-clique inequality*

$$tx(C) - x(\mathrm{mtr}(C)) \leqslant \frac{t(t+1)}{2} \tag{5.1}$$

is valid for the transitive packing polytope $P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})$.

Proof. Let x be the incidence vector of a transitive packing in $(N, \mathcal{H}, \mathrm{tr})$, and assume that $x(C) = \mu$. Consequently, $x(\mathrm{mtr}(C)) \geqslant \mu(\mu - 1)/2$. Thus the left-hand side of inequality (5.1) is less than or equal to $t\mu - \mu(\mu - 1)/2$. Since

$$t\mu = \frac{\mu(\mu-1)}{2} + \frac{t(t+1)}{2} - \frac{(t-\mu)(t-\mu+1)}{2}$$

and the last term is nonnegative, x satisfies inequality (5.1). $\square$

The proof of Theorem 5.2 implies immediately that the faces of two nonempty face defining t-reinforced generalized $(k, 2)$-clique inequalities with the same support but different values of t in general contain different sets of incidence vectors of transitive packings. The proof also implies a range on t in order to ensure that the intersection of the transitive packing polytope and the hyperplane defined by a t-reinforced generalized $(k, 2)$-clique inequality is nonempty.

COROLLARY 5.3. *Let* $(N, \mathcal{H}, \mathrm{tr})$ *be an extended graph, and let* C *be the node set of a generalized* $(k, 2)$*-clique in* $(N, \mathcal{H})$ *such that* $\mathrm{tr}(C) \cap C = \emptyset$*. Let* $t \geqslant 1$ *be an integer. If the* t*-reinforced generalized* $(k, 2)$*-clique inequality* (5.1) *defines a nonempty face of the transitive packing polytope* $P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})$*, then* $t \leqslant |C|$*.*

The bound on t can be strengthened if we assume that the t-reinforced generalized $(k, 2)$-clique inequality is facet defining.

LEMMA 5.4. *Let* $(N, \mathcal{H}, \mathrm{tr})$ *be an extended graph, and let* C *be the node set of a generalized* $(k, 2)$*-clique in* $(N, \mathcal{H})$ *such that* $\mathrm{tr}(C) \cap C = \emptyset$*. Let* $t \geqslant 1$ *be an integer. If the* t*-reinforced generalized* $(k, 2)$*-clique inequality* (5.1) *induces a facet of the transitive packing polytope* $P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})$*, then* $t \leqslant |C| - 2$*.*

Proof. The proof is by contradiction. Because of Corollary 5.3, we are left with the cases $t = |C|$ and $t = |C| - 1$. In the former case each point x contained in the facet under consideration would satisfy $x(C) = |C|$. Hence this facet would be contained in all faces induced by the upper bound constraints $x_u \leqslant 1$ for $u \in C$, a contradiction. In the latter case the $(|C| - 1)$-reinforced generalized $(k, 2)$-clique inequality (5.1) turns out to be the sum of all the transitivity constraints induced by pairs of nodes of the clique C, again a contradiction. $\square$

One might ask whether there exist transitive packing polytopes of extended graphs such that the t-reinforced generalized $(k, 2)$-clique inequalities are facet defining. This is indeed the case. Oosten, Rutten, and Spieksma [41] showed that the t-reinforced generalized $(k, 2)$-clique inequalities define facets of the clique partitioning polytope of a complete graph, for $t \leqslant k - 2$ of course.

One appealing aspect of our suggestion to treat suitable problems in the transitive packing context is the opportunity to use knowledge that is available, not only for the transitive packing polytope itself but also for some of its special cases. We ellucidate this by considering a simple example. Let us assume that the underlying graph G of a clique partitioning problem is bipartite. This implies for the associated extended graph $(N, \mathcal{H}, \mathrm{tr})$ that $\mathrm{tr}(H) = \emptyset$ for all edges $H \in \mathcal{H}$. In other words, the transitive packing (clique partitioning) polytope of G coincides with the node packing polytope of its line graph $(N, \mathcal{H})$. Since node packings in line graphs correspond one-to-one with matchings in the original graphs, we obtain the following result.

LEMMA 5.5. *Let* $G = (V, E)$ *be a bipartite graph. The clique partitioning polytope of* G *is completely characterized by the following linear inequalities:*

$$\begin{aligned} x_e &\geqslant 0 \quad \textit{for all edges } e \in E, \\ x(C) &\leqslant 1 \quad \textit{for all sets } C \subseteq E \textit{ of pairwise incident edges.} \end{aligned}$$

It also follows that the clique partitioning problem on bipartite graphs reduces to a matching problem and can hence be solved in polynomial time. This example is, as already indicated, an instance of a more general point of view. Whenever we can interpret a given problem as a transitive packing problem, and whenever the extended graph (or even hypergraph) of an instance of this problem does not have transitive elements but does have a structure such that the corresponding node packing (independence system) polytope can explicitly be described by linear inequalities, the same holds for the polytope associated with the original problem.

6. Separation. After introducing several classes of valid inequalities for the transitive packing polytope, one question that arises is whether we can use these inequalities efficiently in cutting plane algorithms for attacking the transitive packing problem. This topic is discussed in this section. We concentrate on generalized cycle and odd partition inequalities.

Given an integer polyhedron $P_{\mathrm{I}} = \mathrm{conv}\{x \in \mathbb{Z}^n : Ax \leqslant b\}$, where $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$, a $\{0, \frac{1}{2}\}$-*Gomory–Chvátal cut* is a valid inequality for P_{I} of the form $\lambda Ax \leqslant \lfloor \lambda b \rfloor$, with $\lambda \in \{0, \frac{1}{2}\}^m$ and $\lambda A \in \mathbb{Z}^n$. In other words, a $\{0, \frac{1}{2}\}$-Gomory–Chvátal cut has a cutting plane proof of length 1 from $Ax \leqslant b$, and the coefficients in the corresponding linear combination belong to $\{0, \frac{1}{2}\}$ only. Caprara and Fischetti [9] showed that the separation problem for any point $y \in \mathbb{Q}^n$ and the class of $\{0, \frac{1}{2}\}$-Gomory–Chvátal cuts is solvable in time polynomially bounded in the input size of A, b, and y, assuming that A has, at most, two odd coefficients in each row. For 0/1 polytopes P_{I} this remains true for a relaxation $\{x \in \mathbb{R}^n : A'x \leqslant b'\}$ of $\{x \in \mathbb{R}^n : Ax \leqslant b\}$, where $A'x \leqslant b'$ is obtained from $Ax \leqslant b$ by adding systematically lower bound constraints $x_u \geqslant 0$ and upper bound constraints $x_u \leqslant 1$ such that A' has, at most, two odd coefficients in each row. More precisely, we may replace each inequality $\sum_u a_{iu}x_u \leqslant b_i$ with more than three odd coefficients by

$$a_{iv}x_v + a_{iw}x_w + \sum_{u:a_{iu}\text{ even}} a_{iu}x_u + \sum_{u\in L_i}(a_{iu}-1)x_u + \sum_{u\in U_i}(a_{iu}+1)x_u \leqslant b_i + |U_i|$$

for all elements v, w with odd coefficients and for all (including trivial) partitions (L_i, U_i) of $\{u \in \{1, 2, \ldots, n\}\setminus\{v, w\} : a_{iu} \text{ odd}\}$ for $i = 1, 2, \ldots, m$. Although this leads in general to an exponential number of rows, the separation problem associated with the $\{0, \frac{1}{2}\}$-Gomory–Chvátal cuts of this relaxation can still be solved in polynomial time; see [9]. Observe that a weak generalized $(k, 2)$-cycle inequality can be derived as a $\{0, \frac{1}{2}\}$-Gomory–Chvátal cut of such a relaxation when $|H_i| = 2$ for all edges H_i of the supporting cycle $(C, \{H_i : i = 1, 2, \ldots, k\})$. (Indeed, we do not need the upper bound constraints here.)

THEOREM 6.1. *There exists a polynomial time algorithm that, for any extended hypergraph* $(N, \mathcal{H}, \mathrm{tr})$ *and for any point* $y \in \mathbb{Q}^N$*, either asserts that* y *satisfies all weak generalized* $(k, 2)$*-cycle inequalities supported by cycles* $(C, \{H_i : i = 1, 2, \ldots, k\})$ *such that* $|H_i| = 2$*,* $i = 1, 2, \ldots, k$*, or finds an inequality violated by* y *from a class of valid inequalities for* $P_{\mathrm{TP}}(N, \mathcal{H}, \mathrm{tr})$ *that contains all weak generalized* $(k, 2)$*-cycle inequalities supported by cycles* $(C, \{H_i : i = 1, 2, \ldots, k\})$ *such that* $|H_i| = 2$*,* $i = 1, 2, \ldots, k$.

Notice that this captures, in particular, all transitive packing problems in graphs. It covers, for instance, the 2-chorded odd cycle inequalities and the odd wheel inequalities for the clique partitioning and the partition polytope. The separation problem for the former class has previously been solved in [9, 33], the latter one in [17].

For the odd partition inequalities, we make use of both lower and upper bound constraints. Let us assume that $H_1, H_2, \ldots, H_k$ is the underlying collection of edges and that $d(u)$ odd implies that either $m(u) = 1$ and $n(u) = 0$ or $m(u) = 0$ and $n(u) = 1$ for all nodes $u \in N$. For a given partition $(W_1^{\mathrm{odd}}, W_2^{\mathrm{odd}})$ of W^{odd} the corresponding odd partition inequality can be obtained as a $\{0, \frac{1}{2}\}$-Gomory–Chvátal cut from the relaxed system

$$\sum_{u\in H_i} x_u - \sum_{u\in \operatorname{tr}(H_i)} x_u + \sum_{u\in (H_i\cup \operatorname{tr}(H_i))\cap W_1^{\text{odd}}} x_u - \sum_{u\in (H_i\cup \operatorname{tr}(H_i))\cap W_2^{\text{odd}}} x_u$$
$$\leqslant |H_i| + |(H_i \cup \operatorname{tr}(H_i)) \cap W_1^{\text{odd}}| - 1$$

for $i = 1, 2, \ldots, k$. For fixed κ, we denote by $\mathcal{C}_\kappa$ the class of odd partition inequalities such that $|H_i| \leqslant \kappa$, such that $d(u)$ odd implies that either $m(u) = 1$ and $n(u) = 0$ or $m(u) = 0$ and $n(u) = 1$ for all nodes $u \in N$, and such that $|(H_i \cup \operatorname{tr}(H_i)) \setminus W^{\text{odd}}| \leqslant 2$ for $i = 1, 2, \ldots, k$. The next observation follows again from Caprara and Fischetti's result.

THEOREM 6.2. *There exists a polynomial time algorithm that, for any extended hypergraph* $(N, \mathcal{H}, \operatorname{tr})$, *for any fixed constant* κ, *and for any point* $y \in \mathbb{Q}^N$, *either asserts that* y *satisfies all odd partition inequalities in* $\mathcal{C}_\kappa$ *or finds an inequality violated by* y *from a class of valid inequalities for* $P_{\text{TP}}(N, \mathcal{H}, \operatorname{tr})$ *that contains the class* $\mathcal{C}_\kappa$ *of certain odd partition inequalities.*

7. Special polytopes. In this section, we discuss two more polytopes that arise from the transitive packing polytope by special choices of hypergraphs and transitive elements. The detailed discussion of a third one, the interval order polytope, which inspired the introduction and the study of the transitive packing polytope, is the subject of another paper; see [49, Chapter 5]. The insights obtained for the acyclic subdigraph polytope as well as for the clique partitioning and the partition polytope have been stated during the treatment above. We will not repeat them here. We also do not review special independence system polytopes since this model has been known for years. Instead we concentrate on two recently introduced polytopes that deal with transitive elements.

7.1. The transitive acyclic subdigraph polytope. An instance of the *transitive acyclic subdigraph problem* (or *poset problem*) consists of a directed graph $D = (V, A)$ and a weight function $c : A \to \mathbb{Q}$. The goal is to determine a set of arcs $B \subseteq A$ such that the digraph (V, B) is acyclic and transitively closed, i.e., such that it represents a partially ordered set and such that $c(B)$ is as large as possible. The *transitive acyclic subdigraph polytope* (or *partial order polytope*) of D is the convex hull of 0/1 incidence vectors of all transitive and acyclic arc sets of D. Equivalently, it is the integer hull of the polytope defined by

(7.1) $\quad x_{uv} \geqslant 0 \quad$ for all arcs $(u, v) \in A$,

(7.2) $\quad x_{uv} \leqslant 1 \quad$ for all arcs $(u, v) \in A$,

(7.3) $\quad x_{uv} + x_{vu} \leqslant 1 \quad$ for all pairs $(u, v), (v, u) \in A$,

(7.4) $\quad x_{uv} + x_{vw} \leqslant 1 \quad$ for all $(u, v), (v, w) \in A$ such that $(u, w) \notin A$,

(7.5) $\quad x_{uv} + x_{vw} - x_{uw} \leqslant 1 \quad$ for $(u, v), (v, w), (u, w) \in A$.

The transitive acyclic subdigraph polytope was introduced by Müller [33]. It arises as a transitive packing polytope of an extended graph $(N, \mathcal{H}, \operatorname{tr})$ defined as follows: the arc set A of the digraph D forms the node set N, and two nodes (u_1, v_1), $(u_2, v_2) \in A$ are said to be *adjacent* if $v_1 = u_2$ or $u_1 = v_2$ (or both). The transitive element that we associate with a pair of adjacent arcs $(u, v), (v, w) \in A$ is the arc (u, w), if it exists.

It has already been shown in [33] that the transitive acyclic subdigraph polytope is full dimensional, that the nonnegativity constraints (7.1) are facet defining, and

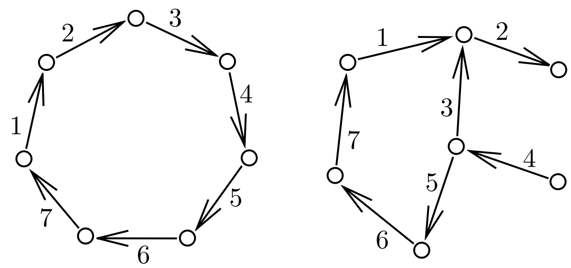

FIG. 7.1. *Two digraphs which are generalized $(7,2)$-cycles in the extended graphs corresponding to the transitive acyclic subdigraph problem. The numbers indicate the chosen sequence, respectively.*

that an upper bound constraint $x_{uv} \leqslant 1$ defines a facet if and only if for all $w \in V$ with $(w,u) \in A$ (or $(v,w) \in A$) also $(w,v) \in A$ (respectively, $(u,w) \in A$). The latter condition is precisely the translation of the assumption made in Lemma 2.4(ii). The only known nontrivial class of facet defining inequalities is associated with odd dicycles in D [33]. If $(u_1,u_2),(u_2,u_3),\ldots,(u_{k-1},u_k),(u_k,u_1)$ forms an odd dicycle in D, its *cycle inequality* is

$$\sum_{i=1}^{k} x_{u_i u_{i+1}} - \sum_{\substack{i=1\\(u_i,\,u_{i+2})\in A}}^{k} x_{u_i u_{i+2}} \leqslant \frac{k-1}{2}.$$

These cycle inequalities obviously belong to the class of generalized $(k,2)$-cycle inequalities. However, there is no reason to restrict ourselves to cycles in the digraph D. Figure 7.1 shows an arc configuration that defines a generalized cycle in the extended graph defined above but is no dicycle in D. Hence, we can present a much larger class of valid inequalities for the transitive acyclic subdigraph polytope.

LEMMA 7.1. *Let $D=(V,A)$ be a digraph. For $k \geqslant 3$ odd, let $a_1,a_2,\ldots,a_k$ be a sequence of arcs in A such that a_i,a_{i+1} are adjacent, $i=1,2,\ldots,k$. The inequalities*

$$\sum_{i=1}^{k} x_{a_i} - \sum_{\substack{a\,\in\,\mathrm{tr}(\{a_i,\,a_{i+1}\})\\ \text{for some } i}} \frac{\lceil n(a)\rceil_2}{2} x_a \leqslant \frac{k-1}{2} \quad \textit{and} \quad \sum_{i=1}^{k} x_{a_i} - \sum_{i=1}^{k} x_{\mathrm{tr}(\{a_i,a_{i+1}\})} \leqslant \frac{k-1}{2}$$

are valid for the transitive acyclic subdigraph polytope of D. Here, $n(a) = |\{i \in \{1,2,\ldots,k\} : a \in \mathrm{tr}(\{a_i,a_{i+1}\})\}|$. The latter class of inequalities is contained in a class of valid inequalities for the transitive acyclic subdigraph polytope of D for which the corresponding separation problem is solvable in polynomial time.

We note that there do not exist generalized $(k,2)$-cliques in the case of the transitive acyclic subdigraph polytope for $k \geqslant 4$. We close this section on the transitive acyclic subdigraph polytope with the observation that the transitive acyclic subdigraph polytope of a digraph D whose underlying graph is bipartite is completely described by (7.1)–(7.4). We may argue as follows. First observe that there do not exist transitive arcs. Let black and white be the two color classes of the underlying bipartite graph. The extended graph induced by D is also bipartite. Its color classes are the arcs directed from black to white and the arcs from white to black, respectively. Since it is known that the node packing polytope of a bipartite graph is completely described by the nonnegativity, the upper bound, and the edge constraints, our claim follows.

7.2. The relatively transitive subdigraph polytope. A digraph $D = (V, A)$ is said to be transitively closed, or just transitive, whenever the presence of two arcs (u, v), $(v, w) \in A$ implies the presence of the arc (u, w) in A. A subdigraph (V, B) of a digraph $D = (V, A)$ is called *relatively transitive* if for every dipath from u to v in (V, B) either $(u, v) \in B$ or (u, v) is not in A. We define the *relatively transitive subdigraph polytope* of D as the convex hull of the incidence vectors of all relatively transitive subdigraphs of D or, equivalently, as the integer hull of the polytope defined by

$$x_{uv} \geqslant 0 \quad \text{for all arcs } (u, v) \in A, \tag{7.6}$$

$$x_{uv} \leqslant 1 \quad \text{for all arcs } (u, v) \in A, \tag{7.7}$$

$$\sum_{a \in p} x_a - x_{uv} \leqslant |p| - 1 \quad \text{for all } (u, v) \in A \text{ and for all dipaths } p \in \mathcal{P}_{uv}^D, \tag{7.8}$$

where $\mathcal{P}_{uv}^D$ is the set of dipaths from u to v in D. The size $|p|$ of such a dipath p is the number of its arcs. Shallcross and Bland [51] (see also [50]) studied the convex hull of 0/1 points x whose complements $\overline{x} = \mathbb{1} - x$ satisfy (7.6)–(7.8). If D is transitively closed, these points represent the independent sets of the transitivity antimatroid of D. Shallcross and Bland were motivated by a question raised by Korte and Lovász [31] of whether the convex hull of these incidence vectors has a (computationally) nice description. Shallcross and Bland present some conditions on D such that their polytope, and therefore the relatively transitive subdigraph polytope, is completely described by (7.6)–(7.8). They also point out that maximizing a linear function over the relatively transitive subdigraph polytope is NP-hard in general, thereby answering Korte and Lovász's question to the negative.

The way we introduced the relatively transitive subdigraph polytope makes it likely to be a certain transitive packing polytope. To be precise, let the arc set A of the given digraph $D = (V, A)$ be the node set N of the extended hypergraph to be defined. The hyperedges are formed by the arcs of dipaths from node u to node v for all $u, v \in V$ such that $(u, v) \in A$. Finally, the transitive element associated with such a hyperedge is clearly the arc (u, v). Now, we may translate all the inequalities presented for the transitive packing polytope into this context, thus answering a question of Shallcross and Bland for other valid inequalities for the (complement of the) relatively transitive subdigraph polytope.

8. Concluding remarks. Notice that the inequalities presented above remain valid when we allow for hypergraphs with loops. Then, we cover, for instance, the *cut polytope* (see, e.g., [5, 18]) and the Boolean quadric polytope (e.g., [44]) as well.

It is well known (see [19]) that every *set packing problem*

$$\begin{aligned} \text{maximize} \quad & cx \\ \text{subject to} \quad & Ax \leqslant \mathbb{1}, \\ & x_u \in \{0, 1\}, \end{aligned} \tag{8.1}$$

where A is a matrix of zeros and ones, can be transformed into an equivalent node packing problem on the *intersection graph* of A. Every column becomes a node, and two nodes u and v are joined by an edge if and only if the matrix A contains a row with entry 1 in columns u and v. In other words, the convex hull of feasible solutions to (8.1) (the *set packing polytope* of A) is identical to the node packing polytope of the intersection graph of A. Hence transitive packing covers set packing as well since it subsumes node packing. However, generalized set packing polytopes [13] do not immediately occur as special instances of transitive packing polytopes. In fact, given a $0/\pm 1$ matrix A and the vector n_A whose components count the number of negative entries in the corresponding rows of A, Conforti and Cornuéjols defined (the integer hull of) $\{x : Ax \leqslant \mathbb{1} - n_A, 0 \leqslant x \leqslant \mathbb{1}\}$ as a *generalized set packing polytope*.

On the other hand, as already pointed out, the transitive packing polytope of an extended hypergraph with no transitive elements reduces to an independence system polytope. There is a close relation between independence system polytopes and set covering polytopes (see, e.g., [32, 39]). A *set covering polytope* is of the form $\text{conv}\{y \in \{0,1\}^n : Ay \geqslant \mathbb{1}\}$, where A is a 0/1 matrix. The points y in the set covering polytope and the points x in the independence system polytope of the circuit system defined by the undominated rows of A are related by the affine transformation $x = \mathbb{1} - y$. Explicitly, $x \in \text{conv}\{x \in \{0,1\}^n : Ax \leqslant p_A - \mathbb{1}\}$ if and only if $\mathbb{1} - x \in \text{conv}\{y \in \{0,1\}^n : Ay \geqslant \mathbb{1}\}$. Consequently, set covering polytopes and independence system polytopes are equivalent, modulo the above transformation. An implication of this is that any result stated for the independence system polytope can be translated to the set covering polytope and vice versa. Thus the work of Balas and Ng [1, 2], Cornuéjols and Sassano [16], Euler and Mahjoub [21], Nobili and Sassano [39], and Sassano [45] as well as others on the set covering polytope can be seen as contributions to the knowledge concerning the independence system polytope. For instance, the inequalities for the set covering polytope associated with *complete* (q,s)*-roses of order* k [45] turn out to be equivalent to the generalized (k,s,q)-antiweb inequalities of Laurent [32]. This implies especially that our extension of the class of antiweb inequalities for the independence system polytope extends the known rose inequalities for the set covering polytope, too.

If we apply the complementing of variables to the transitive packing polytope $P_{\text{TP}}(N,\mathcal{H},\text{tr}) = \text{conv}\{x \in \{0,1\}^N : Ax \leqslant p_A - \mathbb{1}\}$, where the $0/\pm 1$ matrix A is the extended edge-node incidence matrix of the extended hypergraph $(N,\mathcal{H},\text{tr})$, it turns out to be equivalent (modulo this affine transformation) to the polytope $Q(A) := \text{conv}\{x \in \{0,1\}^N : Ax \geqslant \mathbb{1} - n_A\}$. The natural linear relaxation of the polytope $Q(A)$ has been introduced by Conforti and Cornuéjols [13] in the context of balanced $0/\pm 1$ matrices as the (*fractional*) *generalized set covering polytope*. Conforti and Cornuéjols [13] as well as Nobili and Sassano [40] characterize when the fractional generalized set covering polytope is integral, i.e., when it coincides with the generalized set covering polytope. Our work can be seen as a contribution to the study of the generalized set covering polytope when it is properly contained in the corresponding fractional one. Recall that a $0/\pm 1$ matrix is *balanced* if, in every submatrix with exactly two nonzero entries per row and per column, the sum of the entries is a multiple of four [53]. We refer to Conforti, Cornuéjols, Kapoor, Vusǩović, and Rao [14] for a survey of balanced matrices and related concepts. Conforti and Cornuéjols [13] showed that a $0/\pm 1$ matrix A is balanced if and only if the fractional generalized set covering (or packing) polytope is integral for each submatrix of A. An extension of the concept of balanced $0/\pm 1$ matrices is ideal matrices. A $0/\pm 1$ matrix A is *ideal* if its fractional generalized set covering polytope is integral or, equivalently, if its fractional transitive packing polytope is integral. It would be very interesting, for problems that can be interpreted as transitive packing problems, to characterize when the extended edge-node incidence matrices of their associated extended hypergraphs are ideal. Little is known so far about ideal $0/\pm 1$ matrices; see [14, 40].

The way we introduced the transitive packing model and the name we gave to it reflect how we discovered it [49, Chapter 4] but may hide its full generality. To highlight and to slightly extend the generality of our model, we finally provide another presentation. A *directed hypergraph* is a pair $(N, \mathcal{H})$ consisting of a finite set N of nodes and of a set of directed hyperedges (hyperarcs). A *hyperarc* $(H^+, H^-) \in \mathcal{H}$ consists of two (possibly empty) disjoint subsets of N. For a survey of directed hypergraphs the reader is referred to [22]. Now, consider for $x \in \{0,1\}^N$ the following "directed hypergraph covering" constraints:

$$\overline{x}(H^+) + x(H^-) \geqslant 1 \quad \text{for all hyperarcs } (H^+, H^-) \in \mathcal{H},$$

where $\overline{x} = \mathbb{1} - x$ is the complement of the 0/1 vector x. Observe that this is equivalent to the transitivity constraints (2.1), with $H^+ = H$ and $H^- = \mathrm{tr}(H)$. In particular, this form emphasizes the symmetry of the role of hyperedges and their associated transitive sets. For example, reversing the direction of the hyperarcs simply amounts to exchanging x and $\overline{x}$.

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