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BOUNDS ON THE CHVÁTAL RANK OF POLYTOPES IN THE 0/1-CUBE*

FRIEDRICH EISENBRAND, ANDREAS S. SCHULZ

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Gomory's and Chvátal's cutting-plane procedure proves recursively the validity of linear inequalities for the integer hull of a given polyhedron. The Chvátal rank of the polyhedron is the number of rounds needed to obtain all valid inequalities. It is well known that the Chvátal rank can be arbitrarily large, even if the polyhedron is bounded, if it is 2-dimensional, and if its integer hull is a 0/1-polytope.

We show that the Chvátal rank of polyhedra featured in common relaxations of many combinatorial optimization problems is rather small; in fact, we prove that the rank of every polytope contained in the n-dimensional 0/1-cube is at most $n^2(1+\log n)$. Moreover, we also demonstrate that the rank of any polytope in the 0/1-cube whose integer hull is defined by inequalities with constant coefficients is $O(n)$.

Finally, we provide a family of polytopes contained in the 0/1-cube whose Chvátal rank is at least $(1+\epsilon)n$, for some $\epsilon > 0$.

1. Introduction

Chvátal [12] (and, implicitly, Gomory [25–27]) established cutting-plane proofs as a way to certify certain properties of combinatorial problems, e.g., to testify that there are no k pairwise non-adjacent nodes in a given graph, that there is no acyclic subdigraph with k arcs in a given digraph, or that there is no tour of length at most k in a prescribed instance of the traveling

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salesperson problem. In this paper, we discuss the length of such proofs. Let us first recall the notion of a cutting-plane proof. A sequence of inequalities

$$c_1 x \leqslant \delta_1, c_2 x \leqslant \delta_2, \ldots, c_m x \leqslant \delta_m \tag{1}$$

is called a *cutting-plane proof* of $cx \leqslant \delta$ from a given system of linear inequalities $Ax \leqslant b$, if $c_1, \ldots, c_m$ are integral, $c_m = c$, $\delta_m = \delta$, and if, for $i = 1, \ldots, m$, $c_i x \leqslant \delta_i'$ is a nonnegative linear combination of $Ax \leqslant b, c_1 x \leqslant \delta_1, \ldots, c_{i-1} x \leqslant \delta_{i-1}$ for some δ_i' with $\lfloor \delta_i' \rfloor \leqslant \delta_i$. Obviously, if there is a cutting-plane proof of $cx \leqslant \delta$ from $Ax \leqslant b$, then every integer solution of $Ax \leqslant b$ satisfies $cx \leqslant \delta$. Chvátal [12] showed that the converse holds as well. That is, if all integer points in a nonempty polytope $\{x \in \mathbb{R}^n : Ax \leqslant b\}$ satisfy an inequality $cx \leqslant \delta$, for some $c \in \mathbb{Z}^n$, then there is a cutting-plane proof of $cx \leqslant \delta$ from $Ax \leqslant b$. Schrijver extended this result to rational polyhedra [42].

In a way, the sequential order of the inequalities in (1) obscures the (recursive) structure of the cutting-plane proof; it is better revealed by a directed graph with vertices $0, 1, 2, \ldots, m$, in which an arc goes from node i to node j if the i-th inequality has a positive coefficient in the linear combination of the j-th inequality. Here, 0 serves as a representative for any inequality in $Ax \leqslant b$. The number of arcs in a longest simple path terminating at node i is commonly referred to as the *depth* of the i-th inequality $c_i x \leqslant \delta_i$ w.r.t. the cutting plane proof. The depth of the m-th inequality is called the *depth of the proof*, while m is the so-called *length* of the cutting-plane proof. We also say that an inequality $cx \leqslant \delta$ has *depth (at most) d relative to a polyhedron* $\{x : Ax \leqslant b\}$ if it has a cutting-plane proof from $Ax \leqslant b$ of depth less than or equal to d. The following theorem clarifies the relation between the depth and the length of a cutting-plane proof. It closely resembles the relation between the height and the number of nodes of a recursion tree in which every interior node has degree at most n. It can be proved with the help of Farkas' Lemma.

Theorem 1.1 **(Chvátal, Cook, and Hartmann [14]).** *Let $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$, let $Ax \leqslant b$ have an integer solution, and let $cx \leqslant \delta$ have depth at most d relative to $Ax \leqslant b$. Then there is a cutting-plane proof of $cx \leqslant \delta$ from $Ax \leqslant b$ of length at most $(n^{d+1} - 1)/(n - 1)$.*

Gomory–Chvátal cutting-planes have gained importance for at least three reasons. First, the cutting-plane method is a (theoretical) tool to obtain a linear description of the integer hull of a polyhedron. In fact, as mentioned earlier, any valid inequality for the integer hull has a cutting-plane proof from the defining system of the polyhedron. The *Chvátal rank* of this polyhedron is the smallest number d such that all inequalities valid for its integer hull have depth at most d relative to the defining system. Hence, if we later state lower and upper bounds for the depth of inequalities they immediately apply to the Chvátal rank of the corresponding polyhedron as well. Second, despite the early computational disappointments with Gomory's cutting-plane method [25–27], it is of practical relevance. On the one hand, it has stimulated to a certain extent the search for problem-specific cutting planes, which became the basis of an own branch of combinatorial optimization, namely polyhedral combinatorics (see, e.g., [28,39,41]). On the other hand, Balas et al. [2] successfully incorporated Gomory's mixed integer cuts within a Branch-and-Cut framework. Third, cutting planes are of interest to mathematical logic and complexity theory. Cook, Coullard, and Turán [15] were the first to consider cutting-plane proofs as a propositional proof system. In particular, they pointed out that the cutting-plane proof system is a strengthening of resolution proofs. Since the work of Haken [30], exponential lower bounds are known for the latter. Results of Chvátal, Cook, and Hartmann [14], of Bonet, Pitassi, and Raz [8], of Impagliazzo, Pitassi, and Urquhart [35], and of Pudlák [40] imply exponential lower bounds on the length of cutting-plane proofs as well. On the other hand, there is no upper bound on the length of cutting-plane proofs in terms of the dimension of the corresponding polyhedron, as the following well-known example shows. The Chvátal rank of the polytope defined by

$$\begin{aligned} -t\,x_1 + x_2 &\leqslant 1 \\ t\,x_1 + x_2 &\leqslant t+1 \\ x_1 &\leqslant 1 \\ x_1, x_2 &\geqslant 0 \end{aligned}$$

grows with t. Notice that the integer hull of this 2-dimensional polytope is a 0/1-polytope, i.e., all its vertices have components 0 or 1 only. In contrast, we give a polynomial bound in the dimension for the Chvátal rank of any polytope contained in the 0/1-cube. Then, Theorem 1.1 implies the existence of exponentially long cutting-plane proofs, matching the known exponential lower bounds.

In polyhedral combinatorics, it has been custom to consider the depth of a class of inequalities if not as an indicator of quality at least as a measure of its complexity. Hartmann, Queyranne, and Wang [34] gave conditions under which an inequality has depth at most 1 and used them to establish that several classes of inequalities for the traveling salesperson polytopes have depth at least 2, as was claimed before in [3,9–11,22,24,29]. However, it follows from a recent result in [19] that deciding whether a given inequality $cx \leqslant \delta$ has depth at least 2 can in general not be conducted in polynomial time, unless P = NP. Chvátal, Cook, and Hartmann [14] (see also [32]) answered questions and proved conjectures of Schrijver, of Barahona, Grötschel, and Mahjoub [4], of Jünger, of Chvátal [13], and of Grötschel and Pulleyblank [29] on the behavior of the depth of certain inequalities relative to popular relaxations of the stable set polytope, the bipartite-subgraph polytope, the acyclic-subdigraph polytope, and the traveling salesperson polytope, resp. They obtained similar results for the set-covering and the set-partitioning polytope, the knapsack polytope, and the maximum-cut polytope, and so did Schulz [44] for the transitive packing, the clique partitioning, and the interval order polytope. The observed increase of the depth was never faster than a linear function of the dimension; we prove that this indeed has to be the case: The depth of any inequality with coefficients bounded by a constant is $O(n)$, relative to a polytope in the 0/1-cube. Naturally, most polytopes associated with combinatorial optimization problems are 0/1-polytopes.

Main Results. We present two new upper bounds on the depth of inequalities relative to polytopes in the 0/1-cube. For notational convenience, let P be an arbitrary polytope contained in the 0/1-cube, i.e., $P \subseteq [0,1]^n$, and let $cx \leqslant \delta, c \in \mathbb{Z}^n$ be an arbitrary inequality valid for the integer hull P_I of P.

We prove first that the depth of $cx \leqslant \delta$ relative to P is at most $n^2 + 2n \log \|c\|_\infty$. This yields an $O(n^2 \log n)$ bound on the Chvátal rank of P since any 0/1-polytope P_I can be represented by a system of inequalities $Ax \leqslant b$ with $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ such that the absolute value of each entry in A is bounded by $n^{n/2}$. Note that the latter bound is sharp, i.e., there exist 0/1-polytopes with facets for which any inducing inequality $ax \leqslant \beta$, $a \in \mathbb{Z}^n$ satisfies $\|a\|_\infty \in \Omega(n^{n/2})$ [1].

Second, we show that the depth of $cx \leqslant \delta$ relative to P is no more than $\|c\|_1 + n$. A similar result was previously known for monotone polyhedra [14]. In fact, we present a reduction to the monotone case that is of interest in its own right because of the smooth interplay of unimodular transformations and the rounding operator. Asymptotically, the second bound gives an improvement by a factor of n to the before-mentioned bound if the components of c are bounded by a constant.

Third, we construct a family of polytopes in the n-dimensional 0/1-cube whose Chvátal rank is at least $(1+\epsilon)n$, for some $\epsilon > 0$. In other words, if $r(n)$ denotes the maximal Chvátal rank over all polytopes that are contained in $[0,1]^n$, then it is one outcome of our study that $r(n) \geqslant (1+\epsilon)n$ for infinitely many $n \in \mathbb{N}$, and $r(n) \leqslant n^2(1+\log n)$ for all $n \in \mathbb{N}$.

Finally, we also show that the number of inequalities in any linear description of a polytope $P \subseteq [0,1]^n$ with empty integer hull is exponential in n, whenever there is an inequality of depth n.

Related Work. Bockmayr and Eisenbrand [5] derived the first polynomial upper bound of $6n^3 \log n$ on the Chvátal rank of polytopes in the n-dimensional 0/1-cube, via a geometric argument. Subsequently, Schulz [45] and Hartmann [33] independently obtained a simpler proof as well as a slightly better bound of $n^2 \log(n^{n/2})$, by using bit-scaling. The reader is referred to the joint journal version of their papers [7], where the authors actually proved that the depth of any inequality $cx \leqslant \delta$, $c \in \mathbb{Z}^n$, which is valid for P_I is at most $n^2 \log \|c\|_\infty$, relative to P. For monotone polytopes P, Chvátal, Cook, and Hartmann [14] showed that the depth of any inequality $cx \leqslant \delta$ that is valid for P_I is at most $\|c\|_1$. Moreover, they also identified polytopes stemming from relaxations of combinatorial optimization problems that have Chvátal rank at least n.

Ultimately, our study of $r(n)$ can also be seen as a continuation of the investigation of combinatorial properties of 0/1-polytopes, like their diameter [38], their number of facets [23], their number of vertices in a 2-dimensional projection [36], or their feature of admitting polynomial-time simplex-type algorithms for optimization [46].

The paper is organized as follows. We start with some preliminaries and introduce some notation in Section 2. We also show that any linear description of a polytope in the 0/1-cube that has empty integer hull and Chvátal rank n needs to contain at least 2^n inequalities. In Section 3, we prove the $O(n^2 \log n)$ upper bound on the Chvátal rank of polytopes in the 0/1-cube. Then, in Section 4, we utilize unimodular transformations as a key tool to derive an $O(n)$ bound on the depth of inequalities with small coefficients, relative to polytopes in the 0/1-cube. Finally, we present the new lower bound on the Chvátal rank in Section 5.

2. Preliminaries

A *polyhedron* P is a set of points of the form $P = \{x \in \mathbb{R}^n | Ax \leqslant b\}$, for some matrix $A \in \mathbb{R}^{m \times n}$ and some vector $b \in \mathbb{R}^m$. The polyhedron is *rational* if both A and b can be chosen to be rational. If P is bounded, then P is called a *polytope*. The *integer hull* P_I of a polyhedron P is the convex hull of the integer points in P. The *half space* $H = (cx \leqslant \delta)$ is the set $\{x \in \mathbb{R}^n \,|\, cx \leqslant \delta\}$, for some non-zero vector $c \in \mathbb{Q}^n$ and $\delta \in \mathbb{Q}$. It is called *valid* for a subset S of $\mathbb{R}^n$, if $S \subseteq H$. Sometimes we also say that the inequality $cx \leqslant \delta$ is valid for

S. If $cx \leqslant \delta$ is valid for a polyhedron P, then $\{x \in P \mid cx = \delta\}$ is called a *face* of P. If the components of c are relatively prime integers, i.e., $c \in \mathbb{Z}^n$ and $\gcd(c) = 1$, then $H_I = (cx \leqslant \lfloor \delta \rfloor)$, where $\lfloor \delta \rfloor$ is the largest integer number less than or equal to δ. The *elementary closure* P' of a polyhedron P is the set

$$P' = \bigcap_{H \supseteq P} H_I,$$

where the intersection ranges over all rational half spaces containing P. We refer to an application of the $'$ operator as one iteration of the Gomory–Chvátal procedure. If we set $P^{(0)} = P$ and $P^{(i+1)} = (P^{(i)})'$, for $i \geqslant 0$, then the Chvátal rank of P is the smallest number t such that $P^{(t)} = P_I$. We denote the Chvátal rank of a polyhedron P by $\operatorname{rank}(P)$. The depth of an inequality $cx \leqslant \delta$ with respect to P is the smallest k such that $cx \leqslant \delta$ is valid for $P^{(k)}$.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron. A polyhedron Q with $Q \supseteq P$ is called a *weakening* of P, if $Q_I = P_I$. If $cx \leqslant \delta$ is valid for P_I, then the depth of this inequality with respect to Q is an upper bound on the depth of this inequality with respect to P. It is easy to see that each polytope $P \subseteq [0,1]^n$ has a rational weakening in the 0/1-cube.

The following important lemma can be found in [43, p. 340] (see also [16, Lemma 6.33]). It allows to use induction on the dimension of the considered polyhedra and provides the key for the termination of the Gomory–Chvátal procedure, which was shown by Schrijver for rational polyhedra in [42].

Lemma 2.1. *Let F be a face of a rational polyhedron P. Then $F' = P' \cap F$.*

Lemma 2.1 yields the following upper bound on the Chvátal rank of rational polytopes in the 0/1-cube with empty integer hull (see [7] for details).

Lemma 2.2. *Let $P \subseteq [0,1]^n$ be a d-dimensional rational polytope in the 0/1-cube with $P_I = \emptyset$. If $d = 0$, then $P' = \emptyset$; if $d > 0$, then $P^{(d)} = \emptyset$.*

Thus, if $cx \leqslant \delta$ is valid for a rational polytope $P \subseteq [0,1]^n$ and $cx \leqslant \delta - 1$ is valid for P_I, then $cx \leqslant \delta - 1$ is valid for $P^{(n)}$.

With these methods at hand one can prove the following result due to Hartmann [32].

Lemma 2.3. *If $P \subseteq [0,1]^n$ is a polytope and $\sum_{i \in I} x_i - \sum_{j \in J} x_j \leqslant r$ is valid for P_I for some subsets I and J of $\{1, \ldots, n\}$, then this inequality has depth at most n^2 with respect to P.*

A side-product of our result in Section 4.3 is a reduction of this bound to $2n$.

Chvátal, Cook, and Hartmann [14, p. 481] provided the following family of rational polytopes in the 0/1-cube with empty integer hull and Chvátal rank n:

$$P_n = \Big\{x \in \mathbb{R}^n \mid \sum_{j\in J} x_j + \sum_{j\notin J}(1-x_j) \geqslant \frac{1}{2}, \text{ for all } J \subseteq \{1,\ldots,n\}\Big\}. \tag{2}$$

The polytopes in this example have exponentially many inequalities, and this indeed has to be the case, as the following result shows.

Proposition 2.4. *Let $P \subseteq [0,1]^n$ be a polytope in the 0/1-cube with $P_I = \emptyset$ and rank$(P) = n$. Any inequality description of P has at least 2^n inequalities.*

Proof. For a polytope $P \subseteq \mathbb{R}^n$ and for some $i \in \{1,\ldots,n\}$ and $\ell \in \{0,1\}$, let $P_i^\ell \subseteq \mathbb{R}^{n-1}$ be the polytope defined by

$$P_i^\ell = \{x \in [0,1]^{n-1} \mid (x_1,\ldots,x_{i-1},\ell,x_{i+1},\ldots,x_n)^T \in P\}.$$

Notice that, if P is contained in a facet $(x_i = \ell)$ of $[0,1]^n$ for some $\ell \in \{0,1\}$ and some $i \in \{1,\ldots,n\}$, then the Chvátal rank of P is the Chvátal rank of P_i^ℓ. Let P be a polytope in the 0/1-cube with $P_I = \emptyset$ and rank$(P) = n$. We will prove next that every one-dimensional face F_1 of the cube satisfies $F_1 \cap P \neq \emptyset$. We proceed by induction on n.

If $n = 1$, this is definitely true since P is not empty and since F_1 is the cube itself.

For $n > 1$, observe that every one-dimensional face F_1 of the cube lies in a facet $(x_i = \ell)$ of the cube, for some $\ell \in \{0,1\}$ and for some $i \in \{1,\ldots,n\}$. Since P has Chvátal rank n it follows that $\tilde{P} = (x_i = \ell) \cap P$ has Chvátal rank $n-1$. This can be seen as follows. Suppose the Chvátal rank of $\tilde{P}$ was at most $n-2$ and suppose that $\ell = 1$. (The case $\ell = 0$ is similar.) Then, with Lemma 2.1, the inequality $x_i \leqslant 0$ is valid for $P^{(n-1)}$. Since the Chvátal rank of $P \cap (x_i = 0)$ is at most $n-1$, it follows again with Lemma 2.1 that the inequality $x_i > 0$ is valid for $P^{(n-1)}$. Thus the Chvátal rank of P is at most $n-1$, a contradiction. It then follows by induction that $(F_1)_i^\ell \cap \tilde{P}_i^\ell \neq \emptyset$, thus $F_1 \cap P \neq \emptyset$.

Each 0/1-point has to be cut off from P by some inequality, as $P_I = \emptyset$. If an inequality $cx \leqslant \delta$ cuts off two different 0/1-points simultaneously, then it must also cut off a 1-dimensional face of $[0,1]^n$. Because of our previous observation this is not possible, and hence there is at least one inequality for each 0/1-point which cuts off only this point. Since there are 2^n different 0/1-points in the cube, the claim follows. ∎

We conclude the first section by introducing some further notation. The ℓ_∞-*norm* $\|c\|_\infty$ of a vector $c \in \mathbb{R}^n$ is the largest absolute value of its entries,

$\|c\|_\infty = \max\{|c_i| \mid i = 1,\dots,n\}$. The ℓ_1-*norm* $\|c\|_1$ of c is the sum $\|c\|_1 = \sum_{i=1}^n |c_i|$. We define the function $\log : \mathbb{N} \to \mathbb{N}$ as

$$\log n = \begin{cases} 1 & \text{if } n = 0 \\ 1 + \lfloor \log_2(n) \rfloor & \text{if } n > 0. \end{cases}$$

Note that $\log n$ is the number of bits in the binary representation of n. For a vector $x \in \mathbb{R}^n$, $\lfloor x \rfloor$ denotes the vector obtained by component-wise application of $\lfloor \cdot \rfloor$.

3. A New Upper Bound on the Chvátal Rank

We call a vector c *saturated* with respect to a polytope P, if $\max\{cx \mid x \in P\} = \max\{cx \mid x \in P_I\}$. If $Ax \leqslant b$ is an inequality description of P_I, then $P = P_I$ if and only if each row vector of A is saturated w.r.t. P. It was shown in [7] that an integral vector $c \in \mathbb{Z}^n$ is saturated after at most $n^2 \log \|c\|_\infty$ steps of the Gomory–Chvátal procedure. Since each 0/1-polytope has a representation $Ax \leqslant b$ with $A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m$, such that each absolute value of an entry in A is bounded by $n^{n/2}$ (see, e.g., [39]), the known bound of $O(n^3 \log n)$ follows. One drawback in this proof is that faces of P that do not contain 0/1-points are taken to have worst-case behavior n. The following observation is crucial to derive a better bound.

Lemma 3.1. *Let $cx \leqslant \alpha$ be valid for P_I and $cx \leqslant \gamma$ be valid for P, where $\alpha \leqslant \gamma$, $\alpha, \gamma \in \mathbb{Z}$ and $c \in \mathbb{Z}^n$. If, for each $\beta \in \mathbb{R}$, $\beta > \alpha$, the polytope $F_\beta = P \cap (cx = \beta)$ does not intersect two opposite facets of the 0/1-cube, then the depth of $cx \leqslant \alpha$ relative to P is at most $2(\gamma - \alpha)$.*

Proof. Notice that $F'_\beta = \emptyset$ for each $\beta > \alpha$, since there exists some $i \in \{1,\dots,n\}$ and some $\varepsilon > 0$ such that $x_i \geqslant \epsilon$ and $x_i \leqslant 1 - \epsilon$ are valid for F_β. Thus, $x_i \geqslant 1$ and $x_i \leqslant 0$ are valid for F'_β. The proof is by induction on $\gamma - \alpha$. If $\alpha = \gamma$, there is nothing to prove. So let $\gamma - \alpha > 0$. Since $F'_\gamma = \emptyset$, Lemma 2.1 implies that $cx \leqslant \gamma - \epsilon$ is valid for P' for some $\epsilon > 0$, Hence, the inequality $cx \leqslant \gamma - 1$ is valid for $P^{(2)}$. ∎

Proposition 3.2. *Let P be a rational polytope in the n-dimensional 0/1-cube. An integral vector $c \in \mathbb{Z}^n$ is saturated w.r.t. $P^{(t)}$ if $t \geqslant n^2 + 2n \log \|c\|_\infty$.*

Proof. We can assume that $c \geqslant 0$ and that $P_I \neq \emptyset$. (It was already shown in [7] that polytopes with empty integer hull have Chvátal rank at most n, see Lemma 2.2.) The proof is by induction on n and $\log \|c\|_\infty$. The claim holds for $n=1,2$ since the Chvátal rank of a polytope in the 1- or 2-dimensional 0/1-cube is at most 1 and 4, resp.

So let $n>2$. If $\log\|c\|_\infty=1$, the claim follows, e.g., from Theorem 4.6 below. So let $\log\|c\|_\infty>1$. Write $c=2c^{(1)}+c^{(2)}$, where $c^{(1)}=\lfloor c/2\rfloor$ and $c^{(2)}\in\{0,1\}^n$. By induction, it takes at most $n^2+2n\log\|c^{(1)}\|_\infty=n^2+2n\log\|c\|_\infty-2n$ iterations of the Gomory–Chvátal procedure until $c^{(1)}$ is saturated. Let $k=n^2+2n\log\|c\|_\infty-2n$.

Let $\alpha=\max\{cx\mid x\in P_I\}$ and $\gamma=\max\{cx\mid x\in P^{(k)}\}$. The "integrality gap" $\gamma-\alpha$ is at most n. This can be seen as follows. Choose $\hat{x}\in P^{(k)}$ with $c\hat{x}=\gamma$ and let $x_I\in P_I$ satisfy $c^{(1)}x_I=\max\{c^{(1)}x\mid x\in P^{(k)}\}$. One can choose x_I out of P_I since $c^{(1)}$ is saturated w.r.t. $P^{(k)}$. It follows that

$$\gamma-\alpha\leqslant c(\hat{x}-x_I)=2c^{(1)}(\hat{x}-x_I)+c^{(2)}(\hat{x}-x_I)\leqslant n\,,$$

where the last inequality follows from $c^{(1)}\hat{x}\leqslant c^{(1)}x_I$ and $c^{(2)}\in\{0,1\}^n$.

Consider now the fixing of an arbitrary variable x_i to an arbitrary value $\ell\in\{0,1\}$. The result is the polytope

$$P_i^\ell=\{x\in[0,1]^{n-1}\mid(x_1,\dots,x_{i-1},\ell,x_{i+1},\dots,x_n)^T\in P\}$$

in the $(n-1)$-dimensional 0/1-cube for which, by the induction hypothesis, the vector

$$\widetilde{c_i}=(c_1,\dots,c_{i-1},c_{i+1},\dots,c_n)$$

is saturated after at most

$$(n-1)^2+2(n-1)\log\|\widetilde{c_i}\|_\infty\leqslant n^2+2n\log\|c\|_\infty-2n$$

iterations.

It follows that

$$\alpha-\ell c_i\geqslant\max\{\widetilde{c_i}\,x\mid x\in(P_i^\ell)^{(k)}\}=\max\{\widetilde{c_i}\,x\mid x\in(P_i^\ell)_I\}.$$

If $\beta>\alpha$, then $(cx=\beta)\cap P^{(k)}$ cannot intersect a facet of the cube, since a point in $P^{(k)}\cap(x_i=\ell)$, $\ell\in\{0,1\}$, has to satisfy $cx\leqslant\alpha$.

With Lemma 3.1, after $2n$ more iterations of the Gomory–Chvátal procedure, c is saturated, which altogether happens after $n^2+2n\log\|c\|_\infty$ iterations. ∎

We conclude this section with a new upper bound on the Chvátal rank.

Theorem 3.3. *The Chvátal rank of a polytope in the n-dimensional 0/1-cube is at most $n^2(1+\log n)$.*

Proof. Each polytope Q in the 0/1-cube has a rational weakening P. The integral 0/1-polytope P_I can be described by a system of integral inequalities $P_I = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ with $A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m$ such that each absolute value of an entry in A is bounded by $n^{n/2}$. We count the number of Gomory–Chvátal steps until all row-vectors of A are saturated. Proposition 3.2 implies that these row-vectors are saturated after at most $n^2 + 2n \log n^{n/2} \leqslant n^2(1+\log n)$ steps. ∎

4. A Different Upper Bound on the Depth

In this section we show that any inequality $cx \leqslant \delta$ that is valid for the integer hull of a polytope P in the n-dimensional 0/1-cube has depth at most $n + \|c\|_1$ w.r.t. P.

We start by recalling some useful properties of monotone polyhedra, prove then that the Gomory–Chvátal operator complies with unimodular transformations, and eventually reduce the general case to monotone polytopes via a special unimodular transformation.

4.1. Monotone Polyhedra

A nonempty polyhedron $P \subseteq \mathbb{R}^n_{\geqslant 0}$ is called *monotone* if $x \in P$ and $0 \leqslant y \leqslant x$ imply $y \in P$. Hammer, Johnson, and Peled [31] observed that a polyhedron P is monotone if and only if P can be described by a system $x \geqslant 0$, $Ax \leqslant b$ with $A, b \geqslant 0$. The next statements are proved in [32] and [14, p. 494]. We include a proof of Lemma 4.2 for the sake of completeness.

Lemma 4.1. *If P is a monotone polyhedron, then P' is monotone as well.*

Lemma 4.2. *Let P be a monotone polytope in the 0/1-cube and let $cx \leqslant \delta$, $c \in \mathbb{Z}^n$, be valid for P_I. Then $cx \leqslant \delta$ has depth at most $\|c\|_1 - \delta$.*

Proof. The proof is by induction on $\|c\|_1$. If $\|c\|_1 = 0$, the claim follows trivially. W.l.o.g., we can assume that $c \geqslant 0$ holds. Let $\gamma = \max\{cx \mid x \in P\}$ and let $J = \{j \mid c_j > 0\}$. If $\max\{\sum_{j\in J} x_j \mid x \in P\} = |J|$, then, since P is monotone, $\hat{x}$ with

$$\hat{x}_i = \begin{cases} 1 & \text{if } i \in J, \\ 0 & \text{otherwise} \end{cases}$$

is in P. Also $c\hat{x} = \gamma$ must hold. So $\gamma = \delta$ and the claim follows trivially. If $\max\{\sum_{j\in J} x_j \mid x \in P\} < |J|$, then $\sum_{j\in J} x_j \leqslant |J| - 1$ has depth at most 1. If

$\|c\|_1 = 1$ this also implies the claim, so assume $\|c\|_1 \geqslant 2$. By induction the valid inequalities $cx - x_j \leqslant \delta$, $j \in J$, have depth at most $\|c\|_1 - \delta - 1$. Adding up the inequalities $cx - x_j \leqslant \delta$, $j \in J$, and $\sum_{j \in J} x_j \leqslant |J| - 1$ results in

$$cx \leqslant \delta + (|J| - 1)/|J|.$$

Rounding down yields $cx \leqslant \delta$ and the claim follows. ∎

4.2. Unimodular Transformations

Unimodular transformations and in particular flipping operations will play a crucial role in relating the Chvátal rank of arbitrary polytopes in the 0/1-cube to the Chvátal rank of monotone polytopes. In this section, we show that unimodular transformations and the Gomory–Chvátal operator commute.

A *unimodular transformation* is a mapping

$$\begin{aligned} u : \mathbb{R}^n &\to \mathbb{R}^n \\ x &\mapsto Ux + v, \end{aligned}$$

where $U \in \mathbb{Z}^{n \times n}$ is a unimodular matrix, i.e., $\det(U) = \pm 1$, and $v \in \mathbb{Z}^n$.

Note that u is a bijection. Its inverse is the unimodular transformation $u^{-1}(x) = U^{-1}x - U^{-1}v$. Since $U^{-1} \in \mathbb{Z}^{n \times n}$, u is also a bijection of $\mathbb{Z}^n$.

Consider the rational halfspace $(cx \leqslant \delta)$, $c \in \mathbb{Z}^n, \delta \in \mathbb{Q}$. The set $u(cx \leqslant \delta)$ is the rational halfspace

$$\begin{aligned} \{x \in \mathbb{R}^n \mid c\,u^{-1}(x) \leqslant \delta\} &= \{x \in \mathbb{R}^n \mid cU^{-1}x \leqslant \delta + cU^{-1}v\} \\ &= (cU^{-1}x \leqslant \delta + cU^{-1}v). \end{aligned}$$

Notice that the vector cU^{-1} is also integral. Let S be some subset of $\mathbb{R}^n$. It follows that $(cx \leqslant \delta) \supseteq S$ if and only if $(cU^{-1}x \leqslant \delta + cU^{-1}v) \supseteq u(S)$.

Consider now the first elementary closure P' of some polyhedron P,

$$P' = \bigcap_{\substack{(cx \leqslant \delta) \supseteq P \\ c \in \mathbb{Z}^n}} (cx \leqslant \lfloor \delta \rfloor).$$

It follows that

$$u(P') = \bigcap_{\substack{(cx \leqslant \delta) \supseteq P \\ c \in \mathbb{Z}^n}} (cU^{-1}x \leqslant \lfloor \delta \rfloor + cU^{-1}v).$$

From this one can derive the next lemma.

Lemma 4.3. *Let P be a polyhedron and u be a unimodular transformation. Then*

$$u(P') = (u(P))'.$$

Corollary 4.4. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron and let $cx \leqslant \delta$ be a valid inequality for P_I. Let u be a unimodular transformation. The inequality $cx \leqslant \delta$ is valid for $P^{(k)}$ if and only if $u(cx \leqslant \delta)$ is valid for $(u(P))^{(k)}$.*

The *i-th flipping operation* is the unimodular transformation

$$\begin{aligned} \pi_i : \qquad & \mathbb{R}^n \to \mathbb{R}^n \\ & (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{i-1}, 1 - x_i, x_{i+1}, \dots, x_n). \end{aligned}$$

It has the representation

$$\begin{aligned} \pi_i : \mathbb{R}^n &\to \mathbb{R}^n \\ x &\mapsto Ux + e_i, \end{aligned}$$

where U coincides with the identity matrix I_n except for $U_{(i,i)}$, which is -1. Note that the flipping operation is a bijection of $[0,1]^n$. For the set $(cx \leqslant \delta)$ one has $\pi_i(cx \leqslant \delta) = \widetilde{c}x \leqslant \delta - c_i$. Here $\widetilde{c}$ coincides with c except for a change of sign in the i-th component.

4.3. The Reduction to Monotone Weakenings

If one wants to examine the depth of a particular inequality with respect to a polytope $P \subseteq [0,1]^n$, one can apply a series of flipping operations until all its coefficients are nonnegative. An inequality with nonnegative coefficients defines a (fractional) 0/1-knapsack polytope K. The depth of this inequality with respect to the convex hull of P and K is then an upper bound on its depth with respect to P. We will show that $\mathrm{conv}(P, K)^{(n)}$ has a monotone weakening in the 0/1-cube.

Lemma 4.5. *Let $P \subseteq [0,1]^n$ be a polytope in the 0/1-cube, with $P_I = K_I$, where $K = \{x \mid cx \leqslant \delta,\ 0 \leqslant x \leqslant 1\}$ and $c \geqslant 0$. Then, $P^{(n)}$ has a rational, monotone weakening Q in the 0/1-cube.*

Proof. We can assume that P is rational. Let $\hat{x}$ be a 0/1-point which is not contained in P, i.e., $c\hat{x} > \delta$. Let $I = \{i \mid \hat{x}_i = 1\}$. The inequality $\sum_{i \in I} x_i \leqslant |I|$ is valid for the cube and thus for P. Since $c \geqslant 0$, the corresponding face $F = \{x \mid \sum_{i \in I} x_i = |I|,\ x \in P\}$ of P does not contain any 0/1-points. Lemma 2.2 implies that $\sum_{i \in I} x_i \leqslant |I| - 1$ is valid for $P^{(n)}$.

Thus, for each 0/1-point $\hat{x}$ which is not in P, there exists a nonnegative rational inequality $a_{\hat{x}}x \leqslant \gamma_{\hat{x}}$ which is valid for $P^{(n)}$ and which cuts $\hat{x}$ off. Thus

$$\begin{aligned} &0 \leqslant x_i \leqslant 1, \;\; i \in \{1,\dots,n\} \\ &a_{\hat{x}}\, x \leqslant \gamma_{\hat{x}}, \quad \hat{x} \in \{0,1\}^n,\; \hat{x} \notin P \end{aligned}$$

is the desired weakening. ∎

Theorem 4.6. *Let $P \subseteq [0,1]^n$, $P \neq \emptyset$ be a nonempty polytope in the 0/1-cube and let $cx \leqslant \delta$ be a valid inequality for P_I with $c \in \mathbb{Z}^n$. Then $cx \leqslant \delta$ has depth at most $n + \|c\|_1$ with respect to P.*

Proof. One can assume that c is nonnegative, since one can apply a series of flipping operations. Notice that this can change the right hand side δ, but in the end δ has to be nonnegative since $P \neq \emptyset$. Let $K = \{x \in [0,1]^n \mid cx \leqslant \delta\}$ and consider the polytope $Q = \mathrm{conv}(K, P)$. The inequality $cx \leqslant \delta$ is valid for Q_I and the depth of $cx \leqslant \delta$ with respect to P is at most the depth of $cx \leqslant \delta$ with respect to Q. By Lemma 4.5, $Q^{(n)}$ has a monotone weakening S. The depth of $cx \leqslant \delta$ with respect to $Q^{(n)}$ is at most the depth of $cx \leqslant \delta$ with respect to S. But it follows from Lemma 4.2 that the depth of $cx \leqslant \delta$ with respect to S is at most $\|c\|_1 - \delta \leqslant \|c\|_1$. ∎

5. A New Lower Bound on the Chvátal Rank

To the best of the authors' knowledge, no example of a polytope P in the n-dimensional 0/1-cube with $\mathrm{rank}(P) > n$ has been provided in the literature so far. We show next that $r(n) > (1+\epsilon)n$, for infinitely many n, where $\epsilon > 0$ and where $r(n)$ denotes the maximal Chvátal rank over all polytopes that are contained in $[0,1]^n$. The construction relies on a lower bound for the fractional stable-set polytope due to Chvátal, Cook, and Hartmann [14].

Let $G = (V, E)$ be a graph on n vertices, $\mathcal{C}$ be the family of all cliques of G, and let $Q \subseteq \mathbb{R}^n$ be the fractional stable set polytope of G defined by the inequalities

$$\begin{aligned} x(C) &\leqslant 1 \text{ for all } C \in \mathcal{C}, \\ x_v &\geqslant 0 \text{ for all } v \in V. \end{aligned} \tag{3}$$

Let $\mathbf{e}$ be the vector of all ones. The following lemma is proved in [14, Proof of Lemma 3.1].

Lemma 5.1. *Let $k < s$ be positive integers and let G be a graph with n vertices such that every subgraph of G with s vertices is k-colorable. If P is a polyhedron that contains Q_I and the point $u = \frac{1}{k}\mathbf{e}$, then $P^{(j)}$ contains the point $x^j = (\frac{s}{s+k})^j u$.*

Let $\alpha(G)$ be the size of the largest independent subset of the nodes of G. It follows that $\mathbf{e}x \leqslant \alpha(G)$ is valid for Q_I. One has

$$\mathbf{e}\,x^j = \frac{n}{k}(\frac{s}{s+k})^j \geqslant \frac{n}{k}e^{-jk/s} \quad ,$$

and thus x^j does not satisfy the inequality $\mathbf{e}x \leqslant \alpha(G)$ for all $j < (s/k)\ln\frac{n}{k\alpha(G)}$.

Erdős proved in [21] that for every positive t there exist a positive integer c, a positive number δ and arbitrarily large graphs G with n vertices, cn edges, $\alpha(G) < tn$ and every subgraph of G with at most δn vertices is 3-colorable. One wants that $\ln\frac{n}{k\alpha(G)} \geqslant 1$ and that s/k grows linearly, so by chosing some $t < 1/(3e)$, $k = 3$ and $s = \lfloor \delta n \rfloor$ one has that x^j does not satisfy the inequality $\mathbf{e}x \leqslant \alpha(G)$ for all $j < (s/k)$.

We now give the construction. Let P be the convex hull of P_n defined in (2) and Q. $P_n \subseteq P$ contributes to the fact that $1/2\mathbf{e}$ is in $P^{(n-1)}$ [14, Lemma 7.2]. Thus $x_0 = 1/3\mathbf{e}$ is in $P^{(n-1)}$. Since the integer hull of P is equal to Q_I, it follows from the discussion above that the depth of $\mathbf{e}x \leqslant \alpha(G)$ with respect to $P^{(n-1)}$ is $\Omega(n)$. Thus the depth of $\mathbf{e}x \leqslant \alpha(G)$ is at least $(n-1) + \Omega(n) \geqslant (1+\epsilon)n$ for infinitely many n, where $\epsilon > 0$.

6. Concluding Remarks

Subsequently, Li has refined some of our results and especially extended them to polytopes in the $0/k$-cube [37]. Cornuéjols and Li [18] proved that the Gomory mixed-integer rank of polytopes in the n-dimensional 0/1-cube is at least n. (An upper bound of n follows from existing results in the literature.) Cook and Dash [17] presented a lower bound of n for the matrix-cut rank of polytopes contained in the n-dimensional 0/1-cube. Moreover, they adapted the proof of Proposition 2.4 to show that a polytope with empty integer hull and matrix-cut rank n has to have at least 2^n inequalities in any defining system of linear inequalities. Bockmayr and Eisenbrand [6] showed that the number of inequalities needed to describe the elementary closure P' of a rational polyhedron P is polynomially bounded in fixed dimension.

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Friedrich Eisenbrand

Max-Planck-Institut für Informatik
Stuhlsatzenhausweg 85
D-66123 Saarbrücken
Germany
eisen@mpi-sb.mpg.de

Andreas S. Schulz

M.I.T., Sloan School of Management
77 Massachusetts Avenue, E53-361
Cambridge, MA 02139-4307
USA
schulz@mit.edu