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The Gomory-Chvátal Closure of a Nonrational Polytope Is a Rational Polytope

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The question as to whether the Gomory-Chvátal closure of a nonrational polytope is a polytope has been a longstanding open problem in integer programming. In this paper, we answer this question in the affirmative by combining ideas from polyhedral theory and the geometry of numbers.

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1. Introduction. Cutting-plane methods, when combined with branch and bound, are among the most successful techniques for solving integer programming problems in practice; numerous types of cutting planes have been studied in the literature and several of them are used in commercial solvers (see, e.g., Cornuéjols [3] and the references therein). Cutting planes also give rise to a rich theory (see again Cornuéjols [3]). In general, a cutting plane for a polyhedron P is an inequality that is satisfied by all integer points in P and, when added to the polyhedron P, typically yields a stronger relaxation of its integer hull. A Gomory-Chvátal (GC) cutting plane (Gomory [8], Chvátal [2]) is an inequality of the form $cx \le \lfloor \delta \rfloor$, where c is an integral vector and $cx \le \delta$ is valid for P. The GC closure of P is the intersection of all half-spaces defined by such inequalities; it is usually denoted by P'. Even though the GC closure is defined as the intersection of an infinite number of half-spaces, the GC closure of a rational polyhedron is again a rational polyhedron. Namely, Schrijver [11] showed that for a rational polyhedron P, the GC cuts corresponding to a totally dual integral system of linear inequalities describing P specify its closure P' fully. For polyhedra that cannot be described by rational data, the situation is different. It is well known that the integer hull P_I of an unbounded nonrational polyhedron P may not be a polyhedron (see, e.g., Halfin [9]). In fact, it may not be a closed set, and the GC closure may not be a rational polyhedron. On the other hand, in the case of a nonrational polytope, P_I is the convex hull of a finite set of integer points and, therefore, a rational polytope. There is no notion of total dual integrality for nonrational systems of linear inequalities, and Schrijver [11] asked whether the GC closure of an arbitrary polytope is a rational polytope. In this paper, we show that this is indeed the case: the GC closure of a nonrational polytope is a rational polytope; that is, it can be described by a finite set of rational inequalities.

Even though GC cuts were originally introduced for polyhedra, they have lately been applied to other convex sets as well. Of particular relevance is the work by Dey and Vielma [6], who showed that the GC closure of a full-dimensional ellipsoid described by rational data is a polytope. Dadush et al. [4] recently extended this result to strictly convex bodies and to the intersection of strictly convex bodies with rational polyhedra. Because the original proof of Schrijver for rational polyhedra relies strongly on polyhedral properties, Dey and Vielma [6] and Dadush et al. [4] had to develop a new proof technique, which can roughly be described as follows: One first shows that there exists a finite set of GC cuts that separate every nonintegral point on the boundary of the strictly convex body. These cuts define a polytope that is contained in the convex body. Furthermore, the intersection of this polytope with the boundary of the body is contained in the body's closure. In a second step, one proves that only a finite set of additional inequalities is needed to fully describe the GC closure of the body.

Our general proof strategy for showing the polyhedrality of the GC closure of a nonrational polytope is inspired by the work of Dadush et al. [4]. Yet, the key argument is very different because their proof relies on properties of strictly convex bodies that do not extend to polytopes. More precisely, strictly convex bodies do not have any higher-dimensional *flat* faces, and therein lies the main difficulty in establishing the polyhedrality of the elementary closure for nonrational polytopes. Our proof is geometrically motivated and uses ideas from convex analysis, polyhedral theory, and the geometry of numbers. In particular, the underlying geometric idea relies on properties of integer lattices and reduced lattice bases.

Simultaneously, and independently from this work, Dadush et al. [5] proved that the GC closure of any compact convex set is a rational polytope.

This paper is organized as follows: After introducing our notation in §2, we provide a sketch of our proof in §3. Section 4 covers some required background material, and §5 contains the main part of the proof.

2. Basics and notations. For a closed and convex set $K \subseteq \mathbb{R}^n$ and a vector $a \in \mathbb{R}^n$, we define $a_K := \max\{ax \mid x \in K\}$. The hyperplane $\{x \in \mathbb{R}^n \mid ax = a_0\}$ is denoted by $(ax = a_0)$ and, similarly, $(ax \leq a_0)$ denotes the half-space $\{x \in \mathbb{R}^n \mid ax \leq a_0\}$. For $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we write $\gcd(a)$ to denote the greatest common divisor of the numbers $a_1, \ldots, a_n$. For any integer k, $[k] := \{1, \ldots, k\}$. For a subset U of $\mathbb{R}^n$, $\operatorname{aff}(U)$ denotes the smallest affine subspace containing U and $\operatorname{int}(U)$ the interior of U. The relative boundary and relative interior of U (that is, the boundary and interior of U considered as a subset of $\operatorname{aff}(U)$) are denoted by $\operatorname{rbd}(U)$ and $\operatorname{ri}(U)$, respectively. Moreover, $B(0, \varepsilon)$ denotes the full-dimensional ball in $\mathbb{R}^n$ around the origin with radius ε.

For any set $S \subseteq \mathbb{Z}^n$, we use

$$C_S(K) := \bigcap_{a \in S} (ax \leq \lfloor a_K \rfloor),$$

to denote the intersection of all half-spaces corresponding to GC cuts for K with normal vector in S. For $S = \mathbb{Z}^n$, one obtains the GC closure K' of the set K.

3. General proof idea. Our general strategy for proving that for any polytope a finite number of GC cuts is sufficient to describe the polytope's closure is a modification of the two-step technique of Dadush et al. [4] for a strictly convex body K: They first construct a finite set $S \subseteq \mathbb{Z}^n$ such that

$$C_S(K) \subseteq K, \tag{K1}$$

$$C_S(K) \cap \operatorname{bd}(K) \subseteq \mathbb{Z}^n, \tag{K2}$$

and then argue that S needs to be augmented by a finite set of vectors only. In particular, they demonstrate with property (K2) that every fractional point on the boundary of the strictly convex body is separated by a GC cut. Obviously, the same cannot be true for polytopes because this would otherwise imply that the GC procedure separates fractional points in the relative interior of the facets of an integral polytope. Furthermore, in the case of a general polytope P, we cannot assume full-dimensionality. This is because a unimodular transformation that maps P to a full-dimensional polytope in a lower-dimensional space may not exist if P is contained in some nonrational affine subspace. In particular, this observation forces us to consider the *relative* boundary of the polytope instead of its boundary. Hence, our general strategy for proving the polyhedrality of P' is as follows: First, we show that one can find a finite set S of integral vectors such that

$$C_S(P) \subseteq P, \tag{P1}$$

$$C_S(P) \cap \operatorname{rbd}(P) \subseteq P'. \tag{P2}$$

We then argue that given the polytope $C_S(P)$, no more than a finite number of additional GC cuts are necessary to describe the closure P'.

The main challenge of this proof strategy lies in showing the existence of a set S satisfying property (P1). The presence of higher-dimensional faces with nonrational affine hulls requires the development of new arguments compared to the proof for strictly convex bodies. The outlined general strategy is implemented in four main steps:

1. Show that there exists a finite set $S \subseteq \mathbb{Z}^n$ such that $C_S(P) \subseteq P$.
2. Show that for any face F of P, $F' = P' \cap F$. In particular, show that if $F = P \cap (ax = a_P)$, then for every GC cut for F there exists a GC cut for P that has the same impact on the maximal rational affine subspace of $(ax = a_P)$.
3. Show that if there exists a finite set S satisfying (P1) and (P2), then P' is a rational polytope.
4. Prove that P' is a rational polytope by induction on the dimension of $P \subseteq \mathbb{R}^n$.

In the remainder of this section, we describe the reasoning behind each step of the proof and sketch some of the applied techniques.

Step 1: *Constructing a subset of P from a finite number of GC cuts*. Suppose that we can find a finite set $S \subseteq \mathbb{Z}^n$ with $C_S(P) \subseteq P$ for some full-dimensional polytope $P \subseteq \mathbb{R}^n$ for which a nonrational inequality $ax \leq a_P$ is facet defining. Because this inequality cannot be facet defining for the *rational* polytope $C_S(P)$, there must exist a finite set of GC cuts that dominate $ax \leq a_P$. More formally, there must exist a subset $S_a \subseteq S$ such that $C_{S_a}(P) \subseteq (ax \leq a_P)$. If V_R denotes the maximal rational affine subspace of $(ax = a_P)$, that is, the affine hull

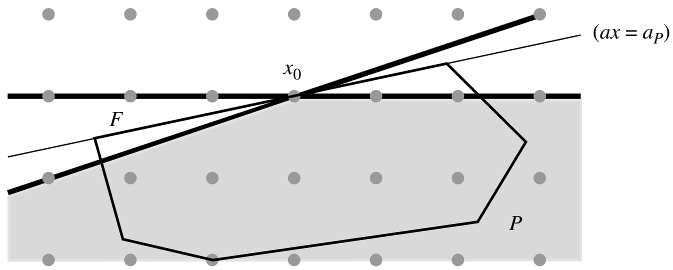

Figure 1. Construction of a finite set of GC cuts that dominate a nonrational facet-defining inequality $ax \le a_P$. Here, the hyperplane $(ax = a_P)$ contains only one rational, in fact, one integral point x_0 and has, therefore, one nonrational direction ($V_R = \{x_0\}$ and $\dim(V_R) = 0$). Two GC cuts separate every point in the hyperplane $(ax = a_P)$ that is not in V_R.

of all rational points in $(ax = a_P)$, then the GC cuts associated with the vectors in S_a have to separate every point in $(ax = a_P)\backslash V_R$. Indeed, we show that for each nonrational facet-defining inequality $ax \le a_P$, such a finite set S_a exists and, also, how it can be constructed. For this, we first establish the existence of a sequence of integral vectors satisfying a specific list of properties. These vectors give rise to GC cuts that separate all points in the nonrational facet $F = P \cap (ax = a_P)$ that are not contained in V_R.

The number of GC cuts needed in our construction for separating these points only depends on the dimension of V_R. If $\dim(V_R) = n - 2$, that is, the hyperplane $(ax = a_P)$ has a single nonrational direction, then only two cuts are necessary. One can visualize these cuts to form a kind of "tent" in the half-space $(ax \le a_P)$, with the ridge being V_R (see Figure 1 for an illustration). With each decrease in the dimension of V_R by 1, the number of necessary cuts is doubled. Hence, at most 2^{n-1} GC cuts suffice to separate the nonrational parts of a nonrational facet of the polytope.

The proof of Step 1 uses many classic results from convex and polyhedral theory as well as from number theory (e.g., Diophantine approximations, integral lattices, and reduced lattice bases).

Step 2: *A homogeneity property*: $F' = P' \cap F$. As the second step of the proof, we show a property of the GC closure that is well known for rational polytopes (see, e.g., Schrijver [12]): if one applies the closure operator to a face of a polytope, the result is the same as if one intersects the closure of the polytope with the face. As it turns out, the same is true for nonrational polytopes. The proof for the rational case is based on the observation that any GC cut for a face $F = P \cap (ax = a_P)$ can be "rotated" so that it becomes a valid GC cut for P. In particular, the rotated cut has the same impact on the hyperplane $(ax = a_P)$ as the original cut for F. Although the exact same property does not hold in the nonrational case, we show that there is a rotation of any cut for F that results in a GC cut for P, which has the same impact on the maximal rational affine subspace V_R of $(ax = a_P)$. Because Step 1 of our proof implies that the nonrational parts of a face are separated in the first round of the GC procedure in any event, this property suffices to show that $F' = P' \cap F$.

The insights gained in this second step will be useful for Step 4 of the proof, where we show the main result by induction on the dimension of the polytope. Knowing that the GC closure of a lower-dimensional facet F of P is a polytope, each of the finite number of cuts describing F' can be rotated in order to become a GC cut for P. We thereby establish the existence of a finite set $S_F \subseteq \mathbb{Z}^n$ with the property that $C_{S_F}(P) \cap F = F'$. Because the facets of P constitute the relative boundary of the polytope, the union of all these sets will give rise to a set $S \subseteq \mathbb{Z}^n$ that satisfies property (P2).

Step 3: *Finite augmentation property*. A statement similar to the one in Step 3 has been established by Dadush et al. [4] for full-dimensional convex bodies. Because a nonrational polytope P can be contained in some nonrational affine subspace and, thus, a unimodular transformation of P to a full-dimensional polytope in a lower-dimensional space is not possible, we need the extension to lower-dimensional polytopes (Lemma 5.5). However, the basic observation for proving this part is the same as in Dadush et al. [4]: Every additional undominated GC cut has to separate a point that is contained in the relative interior of P. Even though in the non-full-dimensional case there are infinitely many cuts with this property, we argue that only a finite number of them needs to be considered.

Step 4: *Proof of the main result*. As the final step of the proof, we establish the main result: the GC closure of any polytope can be described by a finite set of inequalities. The proof is by induction on the dimension of the polytope and uses the observations made in the steps above. Step 1 provides a finite set $S \subseteq \mathbb{Z}^n$ satisfying $C_S(P) \subseteq P$. Applying the induction assumption to the facets of P and using the homogeneity property of Step 2, we augment S for each facet F by a finite set $S_F \subseteq \mathbb{Z}^n$ such that the resulting set of integral vectors satisfies properties (P1) and (P2). From that it follows with the finite augmentation property proven in Step 3 that P' is a polytope.

We will elaborate on each of the four steps in a separate subsection of §5.

4. Preliminaries. We now state various results from the literature and derive some basic facts regarding Diophantine approximations and lattice bases that are utilized in the subsequent sections. The first lemma links the absolute value of the determinant of an integral non-singular square matrix to the number of integer points contained in the parallelepiped spanned by the columns of the matrix (see, e.g., Barvinok [1]).

LEMMA 4.1. *Let $v_1, \ldots, v_n \in \mathbb{Z}^n$ be linearly independent vectors. Then the number of integer points in the semi-open parallelepiped*

$$\Pi(v_1, \ldots, v_n) = \left\{ \sum_{i=1}^{n} \lambda_i v_i \,\middle|\, 0 \le \lambda_i < 1 \text{ for } i \in [n] \right\}$$

is equal to the absolute value of the determinant of the matrix with columns $v_1, \ldots, v_n$.

For linearly independent vectors $b_1, \ldots, b_l$ in $\mathbb{R}^n$, the *lattice* generated by the basis $B = (b_1, \ldots, b_l)$ is the set

$$\Lambda(B) := \left\{ x \in \mathbb{R}^n \,\middle|\, x = \sum_{j=1}^{l} \lambda_j b_j,\ \lambda_j \in \mathbb{Z} \text{ for } j \in [l] \right\}.$$

For any lattice $\Lambda \subset \mathbb{R}^n$, the affine volume of the fundamental parallelepiped of a basis of the lattice (the parallelepiped spanned by the basis vectors) does not depend on the basis itself. It is denoted by $\det(\Lambda)$. If we define $L_0 := \{0\}$ and $L_k := \operatorname{span}(b_1, \ldots, b_k)$ for $k \in [l]$, and if $\tilde{b}_k$ denotes the orthogonal projection of b_k onto $L_{k-1}^{\perp}$, then

$$\det(\Lambda) = \prod_{k=1}^{l} \|\tilde{b}_k\|.$$

A famous result due to Lenstra et al. [10] states that for every lattice in $\mathbb{R}^n$, there exists a basis whose vectors are *almost* orthogonal to each other. Such basis is referred to as a *reduced basis* and its *orthogonality defect* can be bounded by a constant that only depends on the dimension n. One can slightly modify the lattice basis reduction algorithm of Lenstra et al. to obtain the following result (see Dunkel [7, p. 62, Theorem 4.5] for details).

THEOREM 4.1. *Let $(ax = 0)$ be an integral hyperplane in $\mathbb{R}^n$ and let $U \subseteq (ax = 0)$ be a k-dimensional linear vector space. Assume that U is generated by integral vectors $u_1, \ldots, u_k \in \mathbb{Z}^n$ that define a basis of the lattice $U \cap \mathbb{Z}^n$. If $k \ge 1$, assume that for any $v \in ((ax = 0) \cap \mathbb{Z}^n) \setminus U$,*

$$\|v\|^2 \ge \frac{1}{2} \left(\sum_{p=1}^{k} \|u_p\| \right)^2.$$

Then one can extend $u_1, \ldots, u_k$ by vectors $v_1, \ldots, v_l \in \mathbb{Z}^n$, $l = n - k - 1$, to a basis of $(ax = 0) \cap \mathbb{Z}^n$ such that there exists a constant c that only depends on l such that for $j \in [l]$,

$$\|\tilde{v}_j\| \ge c \|v_j\|,$$

where $\tilde{v}_j$ denotes the orthogonal projection of v_j onto $\operatorname{span}(u_1, \ldots, u_k, v_1, \ldots, v_{j-1})^{\perp}$.

Next, we show that if a point can be written as linear combination of an orthogonal basis $\tilde{w}_1, \ldots, \tilde{w}_l$ derived from vectors $w_1, \ldots, w_l$ with small multipliers and if the orthogonal projections are not too short, the point can also be written as a linear combination of $w_1, \ldots, w_l$ with multipliers of bounded size.

LEMMA 4.2. *Let $R > 0$ be a constant and let $u_1, \ldots, u_k$, $w_1, \ldots, w_l$ be linearly independent vectors in $\mathbb{R}^n$ with $\|w_j\| = R$, for $j \in [l]$. Furthermore, define $U_0 := \operatorname{span}(u_1, \ldots, u_k)$ and $U_j := \operatorname{span}(u_1, \ldots, u_k, w_1, \ldots, w_j)$, and let $\tilde{w}_j$ denote the orthogonal projection of w_j onto $U_{j-1}^{\perp}$. If there exists a constant $c > 0$ such that $\|\tilde{w}_j\| \ge cR$, then there exists a constant c_1 only depending on l and c such that*

$$U_0 + \left\{ \sum_{j=1}^{l} \tilde{\lambda}_j \tilde{w}_j \,\middle|\, \tilde{\lambda}_j \in [-1, 1] \text{ for } j \in [l] \right\} \subseteq U_0 + \left\{ \sum_{j=1}^{l} \lambda_j w_j \,\middle|\, \lambda_j \in [-c_1, c_1] \text{ for } j \in [l] \right\}.$$

Proof. The proof of the lemma is by induction on l. For $j \in [l]$, the orthogonal projection $\tilde{w}_j$ of w_j onto $U_{j-1}^{\perp}$ has a unique representation:

$$\tilde{w}_j = w_j - \sum_{p=1}^{k} \alpha_{jp} u_p - \sum_{t=1}^{j-1} \alpha_{jt} \tilde{w}_t, \tag{1}$$

where $\alpha_{jp} \in \mathbb{R}$ for $p \in [k]$, and for $t \in [j-1]$, $\alpha_{jt} = (w_j \tilde{w}_t)/\|\tilde{w}_t\|^2$. First, consider the case $l = 1$. Take an arbitrary $x = u + \tilde{\lambda}_1 \tilde{w}_1$, where $u \in U_0$ and $\tilde{\lambda}_1 \in [-1, 1]$. Then

$$x = u + \tilde{\lambda}_1 \tilde{w}_1 = u + \tilde{\lambda}_1 \left(w_1 - \sum_{p=1}^{k} \alpha_{1p} u_p \right) = \left(u - \tilde{\lambda}_1 \sum_{p=1}^{k} \alpha_{1p} u_p \right) + \tilde{\lambda}_1 w_1$$

and $c_1 = 1$ satisfies the conditions of the lemma. Therefore, assume that the statement of the lemma is true for some $l \geq 1$ with constant $c_1 = c_1(l, c)$. Now take an $x = u + \sum_{j=1}^{l+1} \tilde{\lambda}_j \tilde{w}_j$, where $u \in U_0$ and $\tilde{\lambda}_j \in [-1, 1]$ for $j \in [l+1]$. Using the induction assumption and (1), we get

$$\begin{aligned} x = u + \sum_{j=1}^{l} \tilde{\lambda}_j \tilde{w}_j + \tilde{\lambda}_{l+1} \tilde{w}_{l+1} &= u' + \sum_{j=1}^{l} \lambda_j w_j + \tilde{\lambda}_{l+1} \left(w_{l+1} - \sum_{p=1}^{k} \alpha_{l+1p} u_p - \sum_{t=1}^{l} \frac{w_{l+1} \tilde{w}_t}{\|\tilde{w}_t\|^2} \tilde{w}_t \right) \\ &= u'' + \sum_{j=1}^{l} \lambda_j w_j + \tilde{\lambda}_{l+1} \left(w_{l+1} - \sum_{j=1}^{l} \frac{w_{l+1} \tilde{w}_j}{\|\tilde{w}_j\|^2} \tilde{w}_j \right) \end{aligned}$$

for some u', $u'' \in U_0$ and numbers λ_j satisfying $|\lambda_j| \leq c_1(l, c)$ for $j \in [l]$. Let us define

$$y := \sum_{j=1}^{l} \frac{w_{l+1} \tilde{w}_j}{\|\tilde{w}_j\|^2} \tilde{w}_j = \sum_{j=1}^{l} \nu_j \tilde{w}_j.$$

Then

$$|\nu_j| = \frac{|w_{l+1} \tilde{w}_j|}{\|\tilde{w}_j\|^2} \leq \frac{\|w_{l+1}\| \|\tilde{w}_j\|}{\|\tilde{w}_j\|^2} = \frac{\|w_{l+1}\|}{\|\tilde{w}_j\|} \leq \frac{R}{Rc} = \frac{1}{c}.$$

By applying the induction assumption a second time, we get

$$\begin{aligned} y &\in U_0 + \left\{ \sum_{j=1}^{l} \nu_j \tilde{w}_j \,\middle|\, \nu_j \in [-1/c, 1/c] \text{ for } j \in [l] \right\} = U_0 + \frac{1}{c} \left\{ \sum_{j=1}^{l} \nu_j \tilde{w}_j \,\middle|\, \nu_j \in [-1, 1] \text{ for } j \in [l] \right\} \\ &\subseteq U_0 + \frac{1}{c} \left\{ \sum_{j=1}^{l} \gamma_j w_j \,\middle|\, \gamma_j \in [-c_1, c_1] \text{ for } j \in [l] \right\} = U_0 + \left\{ \sum_{j=1}^{l} \gamma_j w_j \,\middle|\, \gamma_j \in [-c_1/c, c_1/c] \text{ for } j \in [l] \right\}. \end{aligned}$$

In particular, there exists some $u''' \in U_0$ and numbers $\gamma_j \in [-c_1/c, c_1/c]$ for $j \in [l]$ such that

$$y = u''' + \sum_{j=1}^{l} \gamma_j w_j.$$

Hence, we obtain

$$x = u'' + \sum_{j=1}^{l} \lambda_j w_j + \tilde{\lambda}_{l+1} \left(w_{l+1} - u''' - \sum_{j=1}^{l} \gamma_j w_j \right) = \hat{u} + \sum_{j=1}^{l} (\lambda_j - \tilde{\lambda}_{l+1} \gamma_j) w_j + \tilde{\lambda}_{l+1} w_{l+1},$$

where $\hat{u} \in U_0$ and

$$|\lambda_j - \tilde{\lambda}_{l+1} \gamma_j| \leq |\lambda_j| + |\gamma_j| \leq c_1(l, c) + \frac{c_1(l, c)}{c}.$$

Thus, $c_1(l+1, c) := c_1(l, c)\,(1 + 1/c)$ is the desired constant for $l + 1$. □

Next, we state a famous result regarding simultaneous Diophantine approximations: a finite set of real numbers can be approximated by rational numbers with one common low denominator (see, e.g., Schrijver [12]).

Theorem 4.2 (Dirichlet). *For any $a \in \mathbb{R}^n$ and $\varepsilon \in (0, 1)$, there exist integers $p_1, \ldots, p_n$ and $q > 0$ such that for $i \in [n]$,*

$$\left| a_i - \frac{p_i}{q} \right| < \frac{\varepsilon}{q}.$$

We now extend the theorem to the case that there are rational linear dependencies between the components of the nonrational vector that also should be satisfied by its approximation.

LEMMA 4.3. *Let $a \in \mathbb{R}^n$ and $k \le n-1$. Let $u_1, \ldots, u_k \in \mathbb{Z}^n$ be linearly independent vectors such that $au_j = 0$ for $j \in [k]$. For any $\varepsilon \in (0,1)$, there exists an integer vector $p = (p_1, \ldots, p_n)$ and an integer $q > 0$ such that $pu_j = 0$ for $j \in [k]$ and such that for $i \in [n]$,*

$$\left| a_i - \frac{p_i}{q} \right| < \frac{\varepsilon}{q}.$$

PROOF. Let U denote the $k \times n$ matrix with rows $u_1, \ldots, u_k$; that is, $Ua = 0$. Since $\operatorname{rank}(U) = k$, there exists (after possibly reordering the indices) a rational $k \times (n-k)$ matrix $\tilde{U}$ such that the system of equalities $Ua = 0$ is equivalent to the system

$$\begin{bmatrix} a_{n-k+1} \\ \vdots \\ a_n \end{bmatrix} = \tilde{U} \begin{bmatrix} a_1 \\ \vdots \\ a_{n-k} \end{bmatrix}.$$

In particular, one can find a positive integer s and integers r_{it}, for $n-k+1 \le i \le n$ and $t \in [n-k]$, such that

$$a_i = \frac{1}{s} \sum_{t=1}^{n-k} r_{it} a_t.$$

Let us define the constants

$$K_1 := \min\left\{ \frac{1}{s}, \frac{s}{(n-k)\max_{i,t} |r_{it}|} \right\}$$

and $\varepsilon_1 := K_1 \varepsilon$. Let $\tilde{p}_1, \ldots, \tilde{p}_{n-k}$ and $\tilde{q}$ be integers according to Theorem 4.2 that satisfy

$$\left| a_i - \frac{\tilde{p}_i}{\tilde{q}} \right| < \frac{\varepsilon_1}{\tilde{q}}$$

for $i = 1, \ldots, n-k$. We define

$$q := s\tilde{q}$$

$$p_i := s\tilde{p}_i \quad \text{for } i = 1, \ldots, n-k$$

$$p_i := \sum_{t=1}^{n-k} r_{it} \tilde{p}_t \quad \text{for } i = n-k+1, \ldots, n.$$

Note that

$$\begin{bmatrix} p_{n-k+1} \\ \vdots \\ p_n \end{bmatrix} = \tilde{U} \begin{bmatrix} p_1 \\ \vdots \\ p_{n-k} \end{bmatrix},$$

implying $pu_l = 0$ for $l \in [k]$. Furthermore, for $i = 1, \ldots, n-k$, we have

$$\left| a_i - \frac{p_i}{q} \right| = \left| a_i - \frac{\tilde{p}_i}{\tilde{q}} \right| < \frac{\varepsilon_1}{q/s} \le \frac{\varepsilon}{q}.$$

Then we obtain for $i = n-k+1, \ldots, n$,

$$\left| a_i - \frac{p_i}{q} \right| = \left| a_i - \frac{1}{s} \sum_{t=1}^{n-k} r_{it} \frac{\tilde{p}_t}{\tilde{q}} \right| = \frac{1}{s} \left| \sum_{t=1}^{n-k} r_{it} a_t - \sum_{t=1}^{n-k} r_{it} \frac{\tilde{p}_t}{\tilde{q}} \right| \le \frac{1}{s} \sum_{t=1}^{n-k} |r_{it}| \left| a_t - \frac{\tilde{p}_t}{\tilde{q}} \right| \le \frac{\varepsilon}{q},$$

and the lemma follows. □

From the last lemma, we obtain the following corollary.

COROLLARY 4.1. *Let $a \in \mathbb{R}^n$ and $k \le n-1$. Let $u_1, \ldots, u_k \in \mathbb{Z}^n$ be linearly independent vectors satisfying $au_j = 0$ for $j \in [k]$. Then there exists a sequence $\{a^i\}_{i \in \mathbb{N}} \subseteq \mathbb{Z}^n$ such that $a^i \perp u_j$ for $j \in [k]$ and such that*

$$\|a^i\| \|\bar{a}^i - \bar{a}\| \longrightarrow 0, \tag{2}$$

where $\bar{a} = a/\|a\|$ and $\bar{a}^i = a^i/\|a^i\|$.

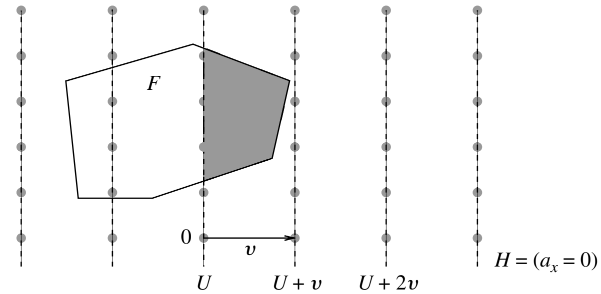

FIGURE 2. The lattice of integer points in H can be partitioned into layers that are parallel to U and obtained by shifting U by integral multiples of v. The facet $F = P \cap H$ does not intersect the affine space $U + v$. We can rotate the hyperplane H around U to obtain a GC cut for P that separates every point in the gray area.

5. The main proof.

In this section, we prove the main result of the paper, following the sequence of four steps outlined in §3.

5.1. Step 1.

The first and most difficult step of our proof is to show that for any polytope P, there exists a finite set of GC cuts that defines a subset of P. In fact, we prove that for each nonrational facet-defining inequality $ax \leq a_P$, for P one can construct a finite set S_a of integral vectors that satisfies $C_{S_a}(P) \subseteq (ax \leq a_P)$. In particular, with V_R denoting the maximal rational affine subspace of $(ax = a_P)$, we show that the set of points in $P \cap (ax = a_P) \backslash V_R$ can be partitioned into a finite number of segments such that for each segment there exists a single GC cut that separates all points in the segment. The number of segments will thereby depend only on the dimension of V_R.

Our proof technique has a clear geometric interpretation. It is motivated by an observation for rational polytopes that can be illustrated as follows: Suppose that $H = (ax = a_P)$ is an integral hyperplane in $\mathbb{R}^n$. We can assume without loss of generality (w.l.o.g.) that $a_p = 0$ and that the hyperplane is defined by integral vectors $u_1, \ldots, u_{n-2}$ and v, which span a parallelepiped that does not contain any interior integral points. In other words, these vectors form a basis of the lattice defined by the integer points in H. Let $U := \operatorname{span}(u_1, \ldots, u_{n-2})$. Then $U + \lambda v \subseteq H$, for any number λ. One can imagine that the set of integer points in H can be partitioned into subsets (or layers) associated with the parallel affine subspaces that are obtained by shifting U by some integral multiple of v (see Figure 2). Now consider a rational polytope P with facet $F = P \cap (ax = 0)$ such that F is a subset of $U + \{\lambda v \mid \lambda < 1\}$. Then F does not intersect the affine subspace spanned by the integer points in $U + v$ but lies completely on one side of this subspace in H. Given this setting, there is some "gap" between F and $U + v$. It therefore appears intuitive that a slight "rotation of the hyperplane H around U in direction of v" should result in some hyperplane $(hx = 0)$ that corresponds to a GC cut for P. Such a hyperplane would separate every point in $F \cap (U + \{\lambda v \mid \lambda > 0\})$ and imply that $P' \cap F \subseteq U + \{\lambda v \mid \lambda \leq 0\}$. In other words, one iteration of the GC procedure would guarantee that P' does not contain any points in H that lie strictly between the two affine subspaces U and $U + v$. Figure 3 illustrates the described situation in dimension two.

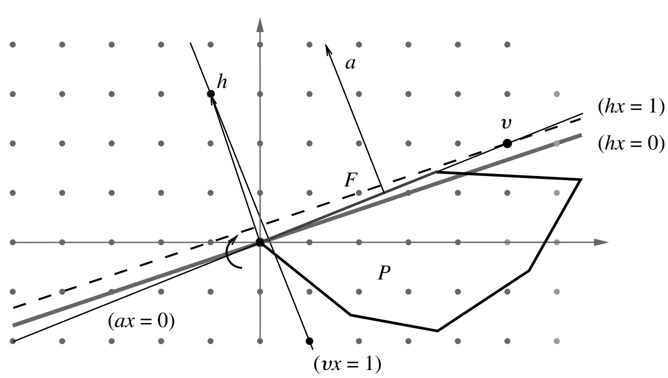

FIGURE 3. Geometry of GC cuts for rational polytopes for $n = 2$. The hyperplane $(ax = 0)$ with $a = (-2, 5)$ is spanned by $v = (5, 2)$. Here, $U = \{0\}$. The line segment $[0, v]$ (that is, the parallelepiped spanned by v) does not contain any interior integral points; that is, $\gcd(v) = 1$. There exists an integral vector $h_0 = (-1, 3)$ such that $h_0 v = 1$ and the same is true for any $h = h_0 + ka$ with $k \in \mathbb{Z}$. Hence, by choosing k large enough, we can find an integral h with $hv = 1$ such that hx is maximized over P by a vertex in F. Since $F \subseteq \{\lambda v \mid \lambda < 1\}$, we get that $\max\{hx \mid x \in P\} = \max\{hx \mid x \in F\} < hv = 1$, implying that $hx \leq 0$ is a GC cut for F that separates every point in $F \cap (0, v)$.

As we formally prove in Lemma 5.1 and Corollary 5.1, this intuition is justified. Most importantly, it will assist in constructing GC cuts that separate the points in the nonrational parts of facets with nonrational affine hulls. In the following, we illustrate the basic idea for the special case of a nonrational facet-defining hyperplane $(ax = a_P)$ for which the maximal rational affine subspace V_R is integral and has dimension $n-2$. (There is a natural generalization of this approach for the case that V_R is non-integral or of smaller dimension.) Let $F = P \cap (ax = a_P)$ be a facet of a polytope P and let us assume w.l.o.g. that $a_P = 0$. Furthermore, suppose that $(ax = 0)$ is spanned by integral vectors $u_1, \ldots, u_{n-2}$ and some nonrational vector v. Then we can approximate the hyperplane $(ax = 0)$ by a sequence of integral hyperplanes $(a^i x = 0)$ that are spanned by the vectors $u_1, \ldots, u_{n-2}$ together with an approximation $v^i \in \mathbb{Z}^n$ of the nonrational direction v. That is, the approximations also contain $U := \operatorname{span}(u_1, \ldots, u_{n-2}) = V_R$. It is intuitive that the norm of the integral vector v^i has to increase with the accuracy of the approximation because the distance of v^i to the nonrational hyperplane $(ax = 0)$ must become smaller. Now consider the perturbation P^i of P that is obtained by replacing the nonrational facet-defining inequality $ax \le 0$ by the approximation $a^i x \le 0$. For large enough norm of the vector v^i, the facet $F^i = P^i \cap (a^i x = 0)$ does not intersect the *integral* affine subspace $U + v^i$. Hence, with the earlier observation, there exists a GC cut $h^i x \le 0$ for P^i that separates every point in $U + \{\lambda v^i \mid \lambda > 0\}$. Our general strategy is to utilize this cut to derive a GC cut $hx \le 0$ for P that removes every point in $U + \{\lambda v \mid \lambda > 0\}$. Note that such h would need to have a strictly positive scalar product with v, and the maximum of hx over P would have to be attained at a vertex in F and be strictly smaller than 1. Ideally, we would want the vector h^i to satisfy these conditions. However, the modified Diophantine approximation that we use to generate the sequence of normal vectors a^i, and thus v^i, does not guarantee these properties for every h^i. One difficulty, for example, is that $h^i v^i > 0$ does not necessarily imply $hv > 0$ (see also Figure 4). Hence, the construction of the vector h has to balance the goal of making the scalar product hv strictly positive, but less than 1.

A rather complicated construction and analysis in Lemma 5.2 will show that an integral vector h with the desired properties always exists. It gives rise to a GC cut that separates every point in the set $U + \{\lambda v \mid \lambda > 0\}$. Similarly, one can construct a cut for the nonrational part on the "other side" of U, that is, for $U - \{\lambda v \mid \lambda > 0\}$. Geometrically, the nonrational part of $(ax = 0)$ is partitioned into two sets associated with the directions $\pm v$. The two corresponding cutting planes form a "tent" in the half-space $(ax \le 0)$ (see Figure 5). In the generalization to lower-dimensional subspaces U, the nonrational part of $(ax = 0)$ spanned by nonrational vectors $v_1, \ldots, v_l$, will be partitioned into 2^l disjoint sets that correspond to the vectors $(\pm v_1, \ldots, \pm v_l)$.

The first lemma and corollary of this subsection formalize the observations for rational polytopes described above, which can be regarded as the geometric foundation of the proof of Step 1.

LEMMA 5.1. *Let $u_1, \ldots, u_{n-2}$ and v be linearly independent vectors in $\mathbb{Z}^n$ such that*

$$\left\{ \sum_{i=1}^{n-2} \gamma_i u_i + \lambda v \;\middle|\; \gamma_i \in \mathbb{R} \text{ for } i \in [n-2],\ 0 < \lambda < 1 \right\} \cap \mathbb{Z}^n = \varnothing. \tag{3}$$

Then there exists a vector $y \in \mathbb{Z}^n$ such that $u_i y = 0$ for $i \in [n-2]$ and such that $vy = 1$.

PROOF. First, let us assume that the semi-open parallelepiped spanned by the vectors $u_1, \ldots, u_{n-2}$ does not contain any integral points apart from 0, that is,

$$\left\{ \sum_{i=1}^{n-2} \gamma_i u_i \;\middle|\; 0 \le \gamma_i < 1 \text{ for } i \in [n-2] \right\} \cap \mathbb{Z}^n = \{0\}. \tag{4}$$

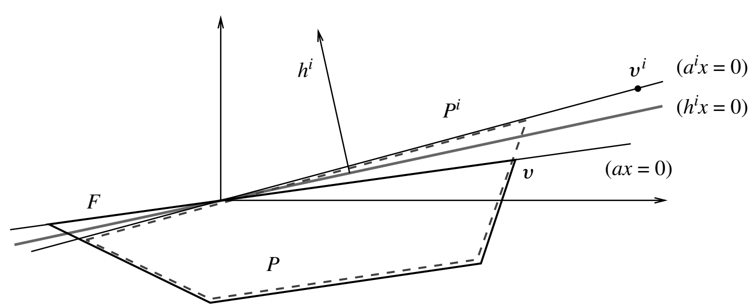

FIGURE 4. One of the difficulties in the construction of GC cuts separating nonrational parts of facets. $h^i x \le 0$ is a GC cut for the approximation P^i (drawn with a dashed line) that separates every point λv^i of $F^i = P^i \cap (a^i x = 0)$ with $\lambda > 0$. However, even if the cut $h^i x \le 0$ is also a valid GC cut for P, it does not separate any point λv with $\lambda > 0$, since $h^i v < 0$.

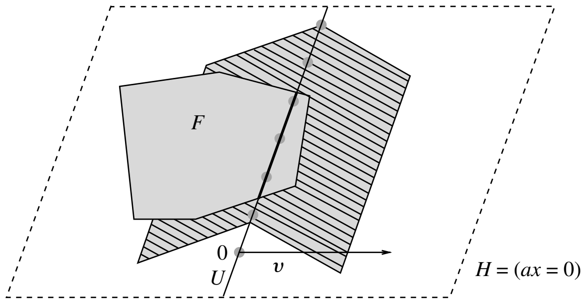

FIGURE 5. Separation of nonrational parts of facets for $n=3$. Here, $H=(ax=0)$ is a nonrational hyperplane with maximal rational affine subspace U of dimension $n-2$. The nonrational direction is given by the nonrational vector v. There exist two GC cuts that separate all points in $F\backslash U$; their hyperplanes form a "tent" below H with ridge U.

Together with (3), we have

$$\left\{\sum_{i=1}^{n-2}\gamma_i u_i+\lambda v \;\middle|\; 0\le\gamma_i<1 \text{ for } i\in[n-2],\ 0\le\lambda<1\right\}\cap\mathbb{Z}^n=\{0\};$$

that is, the semi-open parallelepiped spanned by all $n-1$ vectors also does not contain any integral points apart from 0. Now consider the system

$$Vy:=\begin{bmatrix}u_1\\u_2\\\vdots\\u_{n-2}\\v\end{bmatrix}y=\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}=:b.$$

Note that V has full row rank and column rank $n-1$. There exists a unimodular matrix $U\in\mathbb{Z}^{n\times n}$, that is, $|\det(U)|=1$, such that

$$VU=\begin{bmatrix}u_1\\u_2\\\vdots\\u_{n-2}\\v\end{bmatrix}U=\begin{bmatrix}\tilde{u}_1\\\tilde{u}_2\\\vdots\\\tilde{u}_{n-2}\\\tilde{v}\end{bmatrix}=:[\,\tilde{V}\mid 0\,],$$

where each $\tilde{u}_i=u_iU$ and $\tilde{v}=vU$ has its n-th component zero and where $\tilde{V}$ is a nonsingular integral $(n-1)\times(n-1)$ matrix. The semi-open parallelepiped spanned by the vectors $\tilde{u}_1,\ldots,\tilde{u}_{n-2}$ and $\tilde{v}$ in $(x_n=0)$ does not contain any integral points apart from 0. Indeed, suppose there was an integral point $z=\gamma_1\tilde{u}_1+\cdots+\gamma_{n-2}\tilde{u}_{n-2}+\lambda\tilde{v}$ with $0\le\gamma_i<1$, for $i\in[n-2]$, and $0\le\lambda<1$, such that not all of these coefficients are zero. Then

$$zU^{-1}=\gamma_1\tilde{u}_1U^{-1}+\cdots+\gamma_{n-2}\tilde{u}_{n-2}U^{-1}+\lambda\tilde{v}U^{-1}=\gamma_1u_1+\cdots+\gamma_{n-2}u_{n-2}+\lambda v$$

is an integral point different from 0 in the semi-open parallelepiped spanned by $u_1,\ldots,u_{n-2}$ and v, which is a contradiction. Now observe that Lemma 4.1 implies $|\det(\tilde{V})|=1$. Therefore, the system

$$\tilde{V}\tilde{y}=b$$

has an integral solution $\tilde{y}\in\mathbb{Z}^{n-1}$. The vector $\bar{y}=[\tilde{y}^T\,0]^T$ satisfies $VU\bar{y}=b$ and, consequently, $y=U\bar{y}$ is an integral solution to $Vy=b$.

If assumption (4) is not satisfied, then we can find a set of $n-2$ integral vectors $u'_1,\ldots,u'_{n-2}$ spanning the same linear vector space as $u_1,\ldots,u_{n-2}$ such that (4) holds. Consequently, there is a vector $y\in\mathbb{Z}^n$ such that $u'_iy=0$ for $i\in[n-2]$ and $vy=1$. Because every u_i can be written as a linear combination of the u'_i, we have $u_iy=0$ as well. □

In the following corollary, we apply the above lemma and characterize how rational faces that satisfy a certain property behave under the GC procedure. More precisely, suppose that $H=(ax=a_P)$ is an integral supporting hyperplane for a rational polytope P. Furthermore, assume that the face $F=P\cap H$ does not share any points with an $(n-2)$-dimensional affine subspace $\bar{U}$ spanned by some set of $n-1$ integral points in H. Then all points of F that lie strictly between $\bar{U}$ and the parallel affine subspace $\bar{U}'$ that is obtained by shifting $\bar{U}$ in H

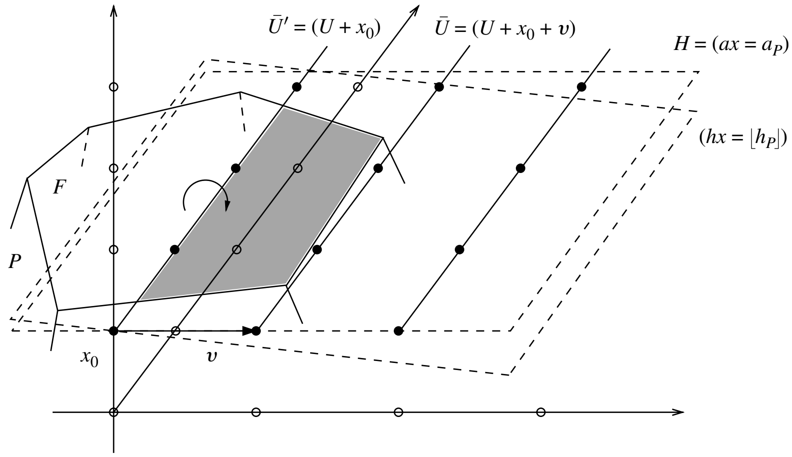

FIGURE 6. Illustration of Corollary 5.1 for $n = 3$. $H = (ax = a_P)$ is an integral hyperplane and $F = P \cap H$ is a face of the rational polytope P. Note that the integer points in H are drawn as filled black points. Since $F \cap (U + x_0 + v) = \varnothing$, there is a GC cut that separates all points in the shaded area between $(U + x_0)$ and $(U + x_0 + v)$ in H.

toward F until the next layer of integer points is touched, will be separated by a single GC cut (see Figure 6 for an illustration). Note that the normal vector h of such a cut has to be perpendicular to every vector in $\bar{U}$. Put differently, the hyperplane $(hx = \lfloor h_P \rfloor)$ has to be parallel to $\bar{U}$.

COROLLARY 5.1. *Let P be a polytope in $\mathbb{R}^n$ and let $H = (ax = a_p)$ be a supporting hyperplane such that $P \subseteq (ax \leq a_P)$. Assume that we can write $H = x_0 + \operatorname{span}(u_1, \ldots, u_{n-2}, v)$ for integral vectors $x_0, u_1, \ldots, u_{n-2}, v$. Let $U = \operatorname{span}(u_1, \ldots, u_{n-2})$ and $F = P \cap (ax = a_P)$. If*
(i) $\{u + \lambda v \mid u \in U, 0 < \lambda < 1\} \cap \mathbb{Z}^n = \varnothing$ *and*
(ii) $F \subseteq \{x_0 + u + \lambda v \mid u \in U, \lambda < 1\}$,
then $P' \cap F \subseteq \{x_0 + u + \lambda v \mid u \in U,\ \lambda \leq 0\}$.

PROOF. We can assume w.l.o.g. that $x_0 = 0$. Because of assumption (ii), there exists an $\varepsilon \in [0, 1)$ such that

$$F \subseteq \{u + \lambda v \mid u \in U, \lambda \leq \varepsilon\}. \tag{5}$$

With assumption (i), Lemma 5.1 implies the existence of a vector $y \in \mathbb{Z}^n$ such that $u_i y = 0$ for $i \in [n-2]$ and $vy = 1$, and the same is true for any integral vector $y + ka$, where $k \in \mathbb{N}$. Now let $r_1, \ldots, r_m$ denote the set of edge directions emanating from the vertices of F to vertices of P that are not in F. Then $r_s a < 0$ for $s \in [m]$. We can choose k large enough, so that the maximum of $(y + ka)$ over P is attained at a point in F. Then with (5) and $u_i(y + ka) = 0$ for $i \in [n-2]$, we get for arbitrary $u \in U$,

$$\max\{(y + ka)x \mid x \in P\} = \max\{(y + ka)x \mid x \in F\} \leq (y + ka)(u + \varepsilon v) = \varepsilon.$$

Hence, $(y + ka)x \leq 0$ is a GC cut for P. Now consider any point $x = u + \lambda v \in F$ such that $u \in U$ and $\lambda > 0$. Then $(y + ka)x = \lambda(y + ka)v = \lambda > 0$. Hence, the point x violates the GC cut $(y + ka)x \leq 0$ and, therefore, $x \notin P'$. □

Although the above lemma and corollary concern integral hyperplanes, in the remainder of this subsection we will focus on affine spaces that cannot be described by rational data. Lemma 5.2 below can be seen as the core of the proof of Step 1. Therein, we establish for every nonrational hyperplane $V = (ax = 0)$ the existence of sequences of vectors and numbers that satisfy a distinct list of properties. The sequences are associated with integral approximations of the hyperplane V. The starting point in the construction of these sequences is the special Diophantine approximation $\{a^i\}$ of the nonrational normal vector a from Corollary 4.1. If $u_1, \ldots, u_k$ denote a maximal set of integral and linearly independent vectors in V, then the normal vectors $a^i \in \mathbb{Z}^n$ are perpendicular to each of the vectors $u_1, \ldots, u_k$. As a result, the approximations $(a^i x = 0)$ of the hyperplane V contain the maximal rational subspace $V_R = \operatorname{span}(u_1, \ldots, u_k)$ of V. In particular, $(ax = 0) \cap (a^i x = 0) = V_R$. Each integral hyperplane $(a^i x = 0)$ is spanned by the vectors $u_1, \ldots, u_k$ together with $l = n - 1 - k$ additional integral vectors, denoted by $v_1^i, \ldots, v_l^i$, which can be regarded as approximations of the nonrational directions of V. These vectors will be chosen very carefully among the infinite number of possible sets of vectors spanning $(a^i x = 0)$ because not all choices will guarantee the properties that we require for the other sequences and

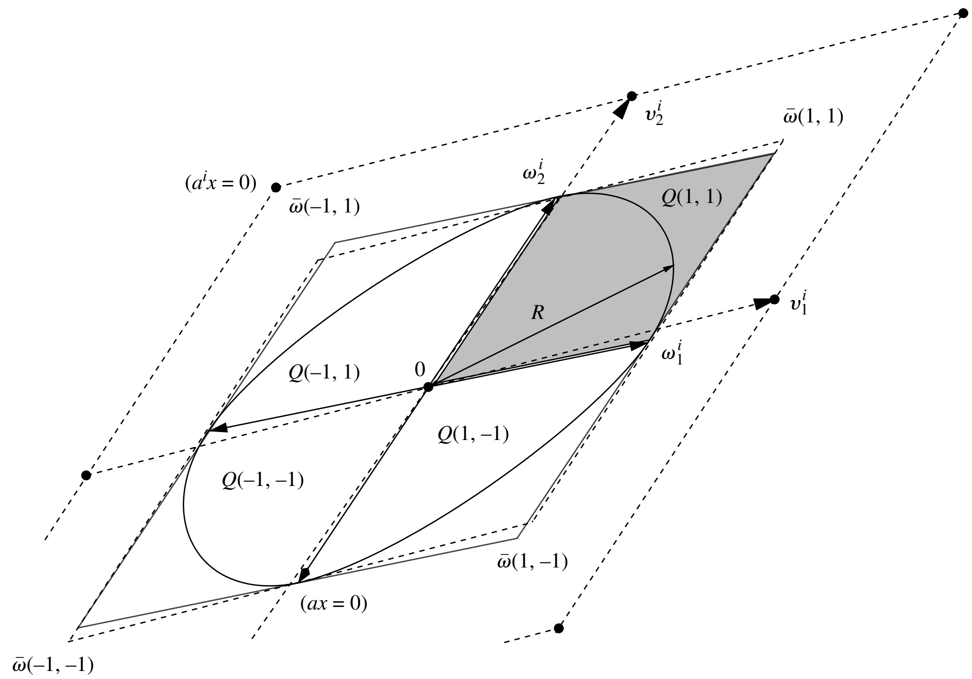

FIGURE 7. Illustration of Lemma 5.2 for $n = 3$ and $\dim(V_R) = 0$. The nonrational hyperplane $(ax = 0)$ is approximated by integral hyperplanes $(a^i x = 0)$. The integral vectors v_1^i and v_2^i span$(a^i x = 0)$ and are almost orthogonal to each other. Their directions give rise to nonrational vectors w_1^i and w_2^i of a given length R in $(ax = 0)$. For each parallelepiped $Q(\delta)$ spanned by vectors $\delta_1 w_1^i$ and $\delta_2 w_2^i$, with $\delta \in \{-1, 1\}^2$, there exists a GC cut that separates every point in $Q(\delta)\setminus V_R$.

numbers derived from them. Most importantly, they will be *almost* orthogonal to one another. The vectors v_j^i give rise to nonrational vectors w_j^i that span the nonrational part of $(ax = 0)$. More precisely, each w_j^i is obtained as projection of the vector v_j^i onto $(ax = 0)$, scaled by a factor, so that all w_j^i have a same given length. We refer to Figure 7 for an illustration. As the quality of the approximations of V increases with the index i, the w_j^i's will, at some point, also be almost orthogonal to one another. This property of the w_j^i's turns out to be material in the subsequent proof of Step 1. Apart from the mentioned sequences a^i, v_j^i, and w_j^i, which have very natural geometric interpretations, we also establish a sequence of integral vectors $h^i(\delta)$, for each $\delta \in \{-1, 1\}^l$, whose construction is more involved. They arise as integral linear combinations of the integral vectors found in Lemma 5.1, which were the basis for GC cuts separating points in rational facets between affine layers of integral points (see Figure 6 and Corollary 5.1). Some of the properties that these vectors satisfy are as follows: Each $h^i(\delta)$ is perpendicular to the vectors $u_1, \ldots, u_k$ and, therefore, the hyperplane $(h^i x = 0)$ is parallel to V_R. Moreover, the scalar product of $h^i(\delta)$ with each nonrational vector $\delta_j w_j^i$ is strictly positive, but very small.

To understand the motivation behind these properties, let us consider the nonrational parallelepiped $Q(\delta)$ that is spanned by $u_1, \ldots, u_k$ and the nonrational vectors $\delta_1 w_1^i, \ldots, \delta_l w_l^i$. When maximizing $h^i(\delta)$ over $Q(\delta)$, the maximum is attained at $\bar{w}(\delta) = \delta_1 w_1^i + \cdots + \delta_l w_l^i$ or any other point in $Q(\delta)$ that can be written as $\bar{w}(\delta) + u$ for some $u \in V_R$. Moreover, the properties of $h^i(\delta)$ guarantee that $0 < h^i(\delta)\bar{w}(\delta) < 1$. As a consequence, $h^i(\delta)x \leq 0$ is a GC cut for $Q(\delta)$ that separates every point of $Q(\delta)\setminus V_R$. Thus, for the special case that the nonrational polytope is the $(n-1)$-dimensional parallelepiped $Q(\delta)$ or contained in it, the single integral vector $h^i(\delta)$ implies a finite set S with the properties that we are looking for in Step 1 of the proof.

For a general polytope P with facet $F = P \cap (ax = 0)$, the goal is to cover F with the 2^l parallelepipeds associated with $(\pm w_1^i, \ldots, \pm w_l^i)$. Then every vector $h^i(\delta)$ will give rise to a GC cut that separates all the points in corresponding parallelepiped that do not belong to V_R. Note that for this, we also need the property that, when $h^i(\delta)$ is maximized over P, the maximum is attained at a vertex in F. In other words, every vector $h^i(\delta)$ must have a nonpositive scalar product with the directions of edges connecting a vertex in F and a vertex outside of F. Indeed, we construct the $h^i(\delta)$ in Lemma 5.2 with the requirement that for an arbitrary given set of vectors $r_1, \ldots, r_m$, their scalar product with these vectors is nonpositive. The proof of Lemma 5.2 strongly relies on properties of reduced bases of integral lattices.

Lemma 5.2. *Let $R > 0$ be a constant and let $V = (ax = 0) \subseteq \mathbb{R}^n$ be a nonrational hyperplane through the origin; that is, $a \in \mathbb{R}^n \backslash \mathbb{Q}^n$. Let U be the maximal rational subspace of V and assume that U is spanned by vectors $u_1, \ldots, u_k \in \mathbb{Z}^n$; that is, $\dim(U) = k$, $0 \le k \le n-2$. Furthermore, let $r_1, \ldots, r_m \in \mathbb{R}^n$ such that for $s \in [m]$,*

$$r_s a < 0. \tag{6}$$

Then there exists a constant $c > 0$ only depending on $l := n - k - 1$ and a constant $C > 0$ such that there exist sequences

$$\{a^i\} \subseteq \mathbb{Z}^n, \qquad \{v_1^i\}, \ldots, \{v_l^i\} \subseteq \mathbb{Z}^n, \qquad \{q_1^i\}, \ldots, \{q_l^i\} \subseteq \mathbb{R}, \qquad \{w_1^i\}, \ldots, \{w_l^i\} \subseteq \mathbb{R}^n$$

that satisfy the following properties:

(i) $\gcd(a^i) = 1$.

(ii) $r_s a^i \le 0$ *for* $s \in [m]$.

(iii) $\|a^i\| \|\bar{a}^i - \bar{a}\| \longrightarrow 0$, *where* $\bar{a}^i = a^i / \|a^i\|$ *and* $\bar{a} = a / \|a\|$.

(iv) $(a^i x = 0) = \operatorname{span}(u_1, \ldots, u_k, v_1^i, \ldots, v_l^i)$.

(v) $\|v_j^i\| \longrightarrow \infty$ *for* $j \in [l]$.

(vi) $\|v_j^i / q_j^i - w_j^i\| \longrightarrow 0$ *for* $j \in [l]$.

(vii) $\|w_j^i\| = R$ *for* $j \in [l]$.

(viii) $V = \operatorname{span}(u_1, \ldots, u_k, w_1^i, \ldots, w_l^i)$.

(ix) $\|\tilde{w}_j^i\| \ge cR$ *for* $j \in [l]$, *where* $\tilde{w}_j^i$ *is the orthogonal projection of* w_j^i *onto* $\operatorname{span}(u_1, \ldots, u_k, w_1^i, \ldots, w_{j-1}^i)^{\perp}$.

(x) *For every $\varepsilon > 0$, there is an index $i_0(\varepsilon)$ such that for all $i \ge i_0(\varepsilon)$ and for all $\alpha \in \mathbb{R}_+^l$ with $\|\alpha\|_\infty \le 1$, there exist vectors $\{h_\alpha^i(\delta)\} \subseteq \mathbb{Z}^n$ for all $\delta \in \{-1, 1\}^l$ such that*

$$\begin{aligned} h_\alpha^i(\delta) &\perp u_p \quad \text{for } p \in [k] \\ |h_\alpha^i(\delta)(\delta_j w_j^i) - \alpha_j| &\le \varepsilon \quad \text{for } j \in [l] \\ h_\alpha^i(\delta)(\delta_j v_j^i) &= \lfloor \alpha_j q_j^i \rfloor \quad \text{for } j \in [l] \\ h_\alpha^i(\delta) r_s &\le 0 \quad \text{for } s \in [m] \\ |h_\alpha^i(\delta) a^i| &\le C \|a^i\|^2. \end{aligned}$$

Proof. Let us assume w.l.o.g. that the vectors $u_1, \ldots, u_k$ form a basis of the lattice $U \cap \mathbb{Z}^n$. If this is not the case, we can replace the original vectors by another set of vectors in U that has this property. Let V_{IR} denote the set of points in V that are not contained in the maximal rational subspace of V; that is, $V_{IR} := V \backslash U$. Let $\{a^i\} \subseteq \mathbb{Z}^n$ be a sequence of vectors according to Corollary 4.1 such that for $p \in [k]$, $a^i \perp u_p$ and

$$\|a^i\| \|\bar{a}^i - \bar{a}\| \longrightarrow 0. \tag{7}$$

We can assume w.l.o.g. that $\gcd(a^i) = 1$ because the same properties hold if we divide a^i by some positive integer. Thus, the sequence $\{a^i\}$ satisfies properties (i) and (iii). Furthermore, (7) implies for $s \in [m]$,

$$|r_s \bar{a}^i - r_s \bar{a}| \longrightarrow 0.$$

As $r_s \bar{a} < 0$ by assumption (6), there exists some constant $\beta > 0$ such that $r_s \bar{a}^i \le -\beta$ for large enough i. Hence, noting that $\|a^i\| \longrightarrow \infty$ because of $a \in \mathbb{R}^n \backslash \mathbb{Q}^n$, it also holds that for $s \in [m]$ and large enough i,

$$r_s a^i \le -\beta. \tag{8}$$

In particular, property (ii) is guaranteed for a large enough i.

Let $\Lambda^i = (a^i x = 0) \cap \mathbb{Z}^n$ denote the lattice defined by the integer points in the integral hyperplane $(a^i x = 0)$. In the following claim, we show that norm of the shortest vector in $\Lambda^i \backslash U$ grows with i.

Claim 5.1. *Let z^i denote a shortest vector in $\Lambda^i \backslash U$. Then $\|z^i\| \longrightarrow \infty$.*

Proof of Claim 5.1. Suppose that there exists some positive constant K such that for all i, one can find a point $z^i \in \Lambda^i \backslash U$ with $\|z^i\| \le K$. Let $\text{proj}(z^i)$ denote the projection of z^i onto the hyperplane $(ax = 0)$; that is, $\text{proj}(z^i) + \lambda a = z^i$, where $\lambda = (az^i)/\|a\|^2$. As $z^i \notin (ax = 0)$, we have $\|z^i - \text{proj}(z^i)\| > 0$. Furthermore, because the number of integer points in $B(0, K)$ is finite, there must exist some positive number D such that $\|z^i - \text{proj}(z^i)\| \ge D$ for every i. However, using $\bar{a}^i z^i = 0$ and (7), we get

$$\|z^i - \text{proj}(z^i)\| = |\lambda|\|a\| = \frac{|az^i|}{\|a\|} = |\bar{a}z^i| = |\bar{a}z^i - \bar{a}^i z^i| \le \|\bar{a} - \bar{a}^i\| K \longrightarrow 0,$$

which is a contradiction. □

Claim 5.1 implies that, for sufficiently large i, we can assume for every $v \in \Lambda^i \backslash U$,

$$\|v\|^2 \ge \frac{1}{2}\left(\sum_{p=1}^{k} \|u_p\|\right)^2.$$

Since $(a^i x = 0)$ is an integral hyperplane and $U \subseteq (a^i x = 0)$, we can find integral vectors $v_1^i, \ldots, v_l^i$ according to Theorem 4.1. That is,

$$(a^i x = 0) = \text{span}(u_1, \ldots, u_k, v_1^i, \ldots, v_l^i),$$

and $u_1, \ldots, u_k, v_1^i, \ldots, v_l^i$ form a basis of the lattice Λ^i. Let $\tilde{v}_1^i$ be the orthogonal projection of v_1^i onto $U^\perp$ and let $\tilde{v}_j^i$ denote the orthogonal projection of v_j^i onto $\text{span}(u_1, \ldots, u_k, v_1^i, \ldots, v_{j-1}^i)^\perp$, for $j = 2, \ldots, l$. Then it also holds by Theorem 4.1 that for $j \in [l]$,

$$\|\tilde{v}_j^i\| \ge c_1 \|v_j^i\|, \tag{9}$$

where c_1 is a constant that only depends on l. With this, property (iv) of the lemma follows. Furthermore, observe that $v_j^i \in \Lambda^i \backslash U$ for $j \in [l]$. Hence, Claim 5.1 implies property (v).

Since $u_1, \ldots, u_k, v_1^i, \ldots, v_l^i$ form a basis of Λ^i, we have for every $s \in [l]$,

$$\left\{ \sum_{p=1}^{k} \gamma_p u_p + \sum_{j=1}^{l} \lambda_j v_j^i \,\middle|\, \gamma_p \in \mathbb{R} \text{ for } p \in [k], \lambda_j \in \mathbb{R} \text{ for } j \in [l], 0 < \lambda_s < 1 \right\} \cap \mathbb{Z}^n = \varnothing. \tag{10}$$

Indeed, if this were not the case and there was a point $z \in \mathbb{Z}^n$ such that $z = \sum_{p=1}^{k} \gamma_p u_p + \sum_{j=1}^{l} \lambda_j v_j^i$ and such that $0 < \lambda_s < 1$, then

$$z' = \sum_{p=1}^{k} (\gamma_p - \lfloor \gamma_p \rfloor) u_p + \sum_{j=1}^{l} (\lambda_j - \lfloor \lambda_j \rfloor) v_j^i \in (\mathbb{Z}^n \cap \Pi(u_1, \ldots, u_k, v_1^i, \ldots, v_l^i)).$$

That is, z' is an integral point in the semi-open parallelepiped spanned by the basis vectors. Because of $0 < \lambda_s < 1$, it holds that $z' \ne 0$, and this cannot be true. Now, let us define for every $j \in [l]$,

$$(w_j^i, q_j^i) := \arg\min \left\{ \left\| \frac{1}{q} v_j^i - w \right\| \,\middle|\, w \in V_{IR}, \|w\| = R, \; q \in \mathbb{R}_+ \right\}. \tag{11}$$

Intuitively, w_j^i is the closest point in the intersection of V_{IR} with the ball $B(0, R)$ to the line spanned by v_j^i. The definition of w_j^i immediately implies property (vii) of the lemma. In the following claim, we show property (vi).

Claim 5.2. *For $j \in [l]$, we have $q_j^i \longrightarrow \infty$ and $\|v_j^i / q_j^i - w_j^i\| \longrightarrow 0$.*

Proof of Claim 5.2. We first show the second part. Let w denote the projection of the point $(R\bar{v}_j^i)$ onto the nonrational hyperplane $(ax = 0)$, where $\bar{v}_j^i = v_j^i / \|v_j^i\|$. We have $w = R\bar{v}_j^i - \lambda a$, where $\lambda = (aR\bar{v}_j^i)/\|a\|^2$. Furthermore, let $q = \|v_j^i\|/R > 0$. Note that for $\bar{w} = w/\|w\|$, it holds that $R\bar{w} \in V_{IR}$ and $\|R\bar{w}\| = R$. Therefore, $(R\bar{w}, q)$ is a feasible pair in the minimization (11) that defines (w_j^i, q_j^i). Consequently,

$$\left\| \frac{v_j^i}{q_j^i} - w_j^i \right\| \le \left\| \frac{v_j^i}{q} - R\bar{w} \right\| = \left\| \frac{v_j^i}{\|v_j^i\|/R} - R\bar{w} \right\| = \|R\bar{v}_j^i - R\bar{w} + (w - w)\| \le \|R\bar{v}_j^i - w\| + \|w - R\bar{w}\|.$$

We get, using $\bar{a}^i \bar{v}_j^i = 0$ and (7), that

$$\|R\bar{v}_j^i - w\| = |\lambda|\|a\| = |R\bar{a}\bar{v}_j^i| = R|\bar{a}\bar{v}_j^i - \bar{a}^i \bar{v}_j^i| \le R\|\bar{a} - \bar{a}^i\|\|\bar{v}_j^i\| = R\|\bar{a} - \bar{a}^i\| \longrightarrow 0.$$

This also implies that $\|w\| \longrightarrow R$ and, therefore, the second part of the claim holds. The first part, $q_j^i \longrightarrow \infty$, follows from $\|v_j^i\| \longrightarrow \infty$, $\|w_j^i\| = R$, and $\|v_j^i / q_j^i - w_j^i\| \longrightarrow 0$. □

Next, we prove property (ix). By (9), we have for $j \in [l]$,

$$\frac{1}{q_j^i}\|\tilde{v}_j^i\| \geq \frac{1}{q_j^i} c_1 \|v_j^i\|. \tag{12}$$

Let $\tilde{w}_j^i$ denote the orthogonal projection of w_j^i onto $\operatorname{span}(u_1, \ldots, u_k, w_1^i, \ldots, w_{j-1}^i)^\perp$ for $j \in [l]$. Because of Claim 5.2, there is for every $\tau > 0$ a number $N(\tau)$, such that for all $i \geq N(\tau)$,

$$\frac{\|v_j^i\|}{q_j^i} - \tau \leq \|w_j^i\| \leq \frac{\|v_j^i\|}{q_j^i} + \tau$$

and

$$\frac{\|\tilde{v}_j^i\|}{q_j^i} - \tau \leq \|\tilde{w}_j^i\| \leq \frac{\|\tilde{v}_j^i\|}{q_j^i} + \tau.$$

Now let γ be some small constant such that $c_1 > \gamma > 0$. By (12),

$$\frac{\|\tilde{v}_j^i\|}{q_j^i} - (c_1 - \gamma)\frac{\|v_j^i\|}{q_j^i} \geq \gamma \frac{\|v_j^i\|}{q_j^i}.$$

Using this observation and $R = \|w_j^i\|$, we obtain

$$\begin{aligned}\|\tilde{w}_j^i\| - (c_1 - \gamma)R &\geq \frac{\|\tilde{v}_j^i\|}{q_j^i} - \tau - (c_1 - \gamma)\left(\frac{\|v_j^i\|}{q_j^i} + \tau\right) \geq \gamma \frac{\|v_j^i\|}{q_j^i} - \tau - (c_1 - \gamma)\tau \\ &\geq \gamma(R - \tau) - \tau - (c_1 - \gamma)\tau.\end{aligned}$$

Note that we can choose τ small enough such that the last expression is nonnegative. Hence, $c = (c_1 - \delta) > 0$ is the desired constant for property (ix). Since c_1 only depends on l, the same is true for c.

Now observe that property (ix) implies that the vectors $u_1, \ldots, u_k, w_1^i, \ldots, w_l^i$ are linearly independent. This is because $\|\tilde{w}_j^i\| > 0$ for $j \in [l]$ and

$$\operatorname{span}(u_1, \ldots, u_k, w_1^i, \ldots, w_l^i) = \operatorname{span}(\tilde{u}_1, \ldots, \tilde{u}_k, \tilde{w}_1^i, \ldots, \tilde{w}_l^i),$$

where $\tilde{u}_1 = u_1$ and where for $p = 2, \ldots, k$, the vector $\tilde{u}_p$ denotes the orthogonal projection of u_p onto $\operatorname{span}(u_1, \ldots, u_{p-1})^\perp$. Consequently, property (viii) is satisfied.

In the remainder of the proof, we show property (x). Because of (10) and Lemma 5.1, there exists for each $s \in [l]$ a vector $y_s^i \in \mathbb{Z}^n$ such that

$$y_s^i \in (u_1 x = 0) \cap \cdots \cap (u_k x = 0) \cap \bigcap_{j \neq s} (v_j^i x = 0) \cap (v_s^i x = 1) =: L_s^i. \tag{13}$$

Since L_s^i is the intersection of $n-1$ linearly independent hyperplanes in $\mathbb{R}^n$, it is a line. Because $a^i \perp u_j$ and $a^i \perp v_j^i$, the direction of the line is a^i. Let us assume w.l.o.g. that $a_1 \neq 0$, and therefore $a_1^i \neq 0$ for large enough i. Let $\bar{y}_s^i$ denote the intersection of L_s^i with the hyperplane $(x_1 = 0)$. Note that $\bar{y}_s^i \neq \pm\infty$ because of the assumption $a_1^i \neq 0$. That is, $\bar{y}_s^i$ is the unique solution to the system

$$\begin{bmatrix} e_1 \\ u_1 \\ \vdots \\ u_k \\ v_1^i \\ \vdots \\ v_{s-1}^i \\ v_s^i \\ v_{s+1}^i \\ \vdots \\ v_l^i \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

For convenience, we introduce some additional notation: Let U denote the matrix with rows u_p, $p \in [k]$, and let V^i_{-s} denote the matrix with rows v^i_j for all $j \in [l]$ such that $j \neq s$. Similarly, let W^i_{-s} denote the matrix with rows w^i_j for all $j \in [l]$ with $j \neq s$. Finally, let V^i and W^i denote the matrices with rows v^i_j and w^i_j, for $j \in [l]$, respectively. Then the above system becomes

$$\begin{bmatrix} e_1 \\ U \\ V^i_{-s} \\ v^i_s \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Claim 5.3. *For every $s \in [l]$, $\bar{y}^i_s q^i_s \longrightarrow \bar{x}^i_s$, where $\bar{x}^i_s$ denotes the unique solution to the linear system of equations*

$$\begin{bmatrix} e_1 \\ U \\ W^i_{-s} \\ w^i_s \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Proof of Claim 5.3. Let v^i_j/q^i_j denote the vector obtained by dividing every component of v^i_j with the scalar q^i_j. Furthermore, let V^i_{-s}/q^i_{-s} be the matrix with rows v^i_j/q^i_j for all $j \neq s$. Then

$$\begin{bmatrix} e_1 \\ U \\ V^i_{-s} \\ v^i_s \end{bmatrix} \bar{y}^i_s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} e_1 \\ U \\ V^i_{-s} \\ v^i_s/q^i_s \end{bmatrix} \bar{y}^i_s q^i_s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} e_1 \\ U \\ V^i_{-s}/q^i_{-s} \\ v^i_s/q^i_s \end{bmatrix} \bar{y}^i_s q^i_s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

where Claim 5.2 implies

$$\begin{bmatrix} e_1 \\ U \\ V^i_{-s}/q^i_{-s} \\ v^i_s/q^i_s \end{bmatrix} \longrightarrow \begin{bmatrix} e_1 \\ U \\ W^i_{-s} \\ w^i_s \end{bmatrix}. \quad \square$$

Now we will show that the entries of $\bar{x}^i_s$ cannot become arbitrarily large.

Claim 5.4. *There exists a constant $K_1 > 0$ such that for sufficiently large i, $\|\bar{x}^i_s\|_\infty \le K_1$.*

Proof of Claim 5.4. By definition,

$$\bar{x}^i_s = \begin{bmatrix} e_1 \\ U \\ W^i_{-s} \\ w^i_s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, it suffices to show that the entries of the inverse matrix in the above equation cannot be arbitrarily large. We have

$$(A^i)^{-1} := \begin{bmatrix} e_1 \\ U \\ W^i \end{bmatrix}^{-1} = \frac{1}{\det(A^i)} \operatorname{adj}(A^i),$$

where $\operatorname{adj}(A^i)$ denotes the adjugate matrix of A^i. Because all entries of A^i are bounded (note that $\|w^i_j\| = R$), every entry of $\operatorname{adj}(A^i)$ is bounded as well. Hence, it is sufficient to show that $|\det(A^i)|$ can be bounded from below for large enough i. The absolute value of the determinant of A^i corresponds to the volume of the parallelepiped spanned by the vectors $u_1, \ldots, u_k, w^i_1, \ldots, w^i_l, e_1$. Therefore, it holds that

$$|\det(A^i)| = \|\tilde{u}_1\| \cdots \|\tilde{u}_k\| \|\tilde{w}^i_1\| \cdots \|\tilde{w}^i_l\| \|\tilde{e}_1\|.$$

Here, $\tilde{e}_1$ denotes the orthogonal projection of e_1 onto $\operatorname{span}(u_1, \ldots, u_k, w^i_1, \ldots, w^i_l)^\perp$. Hence, by property (viii), the vector $\tilde{e}_1$ is the orthogonal projection of e_1 onto $V^\perp$. Because of the assumption $a_1 \neq 0$, it follows that $\|\tilde{e}_1\| > 0$. With property (ix), we obtain for sufficiently large i,

$$|\det(A^i)| \ge (cR)^l \|\tilde{u}_1\| \cdots \|\tilde{u}_k\| \|\tilde{e}_1\|.$$

The expression on the right is a strictly positive constant, and the claim follows. $\square$

Now let us define for any vector $M^i = (M_1^i, \ldots, M_l^i) \in \mathbb{N}^l$, the set

$$L^i(M^i) := (u_1 x = 0) \cap \cdots \cap (u_k x = 0) \cap (v_1^i x = M_1^i) \cap \cdots \cap (v_l^i x = M_l^i).$$

Note that $L^i(M^i)$ is a line with direction a^i and by virtue of (13),

$$(M_1^i y_1^i + \cdots + M_l^i y_l^i) \in L^i(M^i) \cap \mathbb{Z}^n;$$

that is, $L^i(M^i) \cap \mathbb{Z}^n \neq \varnothing$. Furthermore, $L^i(M^i)$ intersects $(x_1 = 0)$ in $\bar{y}^i(M^i) := M_1^i \bar{y}_1^i + \cdots + M_l^i \bar{y}_l^i$. We can write

$$L^i(M^i) = \{x \in \mathbb{R}^n \mid x = \bar{y}^i(M^i) + \mu a^i, \mu \in \mathbb{R}\}.$$

Observe that every line segment of length $\|a^i\|$ of $L^i(M^i)$ must contain an integral point. In the remainder of the proof we show that there exists a constant C such that for every $\alpha \in \mathbb{R}_+^l$ with $\|\alpha\|_\infty \leq 1$, there is a vector $M^i = M^i(\alpha) \in \mathbb{N}^l$ and a number $\mu_0^i = \mu_0^i(\alpha) \in \mathbb{R}$ such that for each $\mu \in [\mu_0^i, \mu_0^i + 1]$ and each $\delta \in \{-1, 1\}^l$ the sequence of vectors

$$h^i(\delta M^i, \mu) := \bar{y}^i(\delta M^i) + \mu a^i = \delta_1 M_1^i \bar{y}_1^i + \cdots + \delta_l M_l^i \bar{y}_l^i + \mu a^i \tag{14}$$

satisfies

$$h^i(\delta M^i, \mu) \perp u_p \quad \text{for } p \in [k] \tag{15}$$

$$h^i(\delta M^i, \mu)(\delta_j w_j^i) \longrightarrow \alpha_j \quad \text{for } j \in [l] \tag{16}$$

$$h^i(\delta M^i, \mu)(\delta_j v_j^i) = \lfloor \alpha_j q_j^i \rfloor \quad \text{for } j \in [l] \tag{17}$$

$$h^i(\delta M^i, \mu) r_s \leq 0 \quad \text{for } s \in [m] \tag{18}$$

$$|h^i(\delta M^i, \mu) a^i| \leq C\|a^i\|^2. \tag{19}$$

Here, the notation δM^i means $(\delta_1 M_1^i, \ldots, \delta_l M_l^i)$. Because every line segment of length $\|a^i\|$ contains an integral point, there must exist some $\mu^* \in [\mu_0^i, \mu_0^i + 1]$ such that $h^i(\delta M^i, \mu^*)$ is an integral vector. Consequently, this would imply property (x) of the lemma.

First, observe that condition (15) always holds because any h of the form (14) is a linear combination of vectors that are perpendicular to the vectors u_p, $p \in [k]$. Using definition (14) of $h^i(\delta M^i, \mu)$, condition (18) becomes

$$\delta_1 M_1^i r_s \bar{y}_1^i + \cdots + \delta_l M_l^i r_s \bar{y}_l^i + \mu r_s a^i \leq 0.$$

Now let $\beta > 0$ be the constant from (8), that is, $r_s a^i \leq -\beta$, for $s \in [m]$. Then (18) becomes

$$\mu \geq \frac{\delta_1 M_1^i r_s \bar{y}_1^i + \cdots + \delta_l M_l^i r_s \bar{y}_l^i}{-r_s a^i},$$

and for

$$\mu_0^i := \max_{s=1,\ldots,m} \left\{ \frac{\delta_1 M_1^i r_s \bar{y}_1^i + \cdots + \delta_l M_l^i r_s \bar{y}_l^i}{\beta} \right\},$$

this condition is satisfied for all $\mu \geq \mu_0^i$. Let $r^i \in \{r_1, \ldots, r_m\}$ such that

$$\mu_0^i = \frac{\delta_1 M_1^i r^i \bar{y}_1^i + \cdots + \delta_l M_l^i r^i \bar{y}_l^i}{\beta}.$$

Then by (14),

$$h^i(\delta M^i, \mu_0^i) = \delta_1 M_1^i \left(\bar{y}_1^i + \frac{1}{\beta} r^i \bar{y}_1^i a^i \right) + \cdots + \delta_l M_l^i \left(\bar{y}_l^i + \frac{1}{\beta} r^i \bar{y}_l^i a^i \right),$$

and (16) becomes for $\mu = \mu_0^i$ and $j = s$,

$$\delta_s \left(\sum_{j=1}^{l} \delta_j M_j^i \left(w_s^i \bar{y}_j^i + \frac{1}{\beta} r^i \bar{y}_j^i w_s^i a^i \right) \right) \longrightarrow \alpha_s. \tag{20}$$

Now let us define $M^i_j := \lfloor \alpha_j q^i_j \rfloor$. Note that $M^i \in \mathbb{N}^l$. In the following, we will show that this choice for M^i satisfies (20). For this, we consider the terms in (20) separately. We start with the terms $\delta_s \delta_j M^i_j w^i_s \bar{y}^i_j$. If $j = s$, then we get with Claims 5.2 ($q^i_s \to \infty$) and 5.3 ($w^i_s \bar{x}^i_s = 1$),

$$\begin{aligned}|\delta_s\delta_s M^i_s w^i_s \bar{y}^i_s - \alpha_s| &= |\lfloor \alpha_s q^i_s \rfloor w^i_s \bar{y}^i_s - \alpha_s| \le |\alpha_s w^i_s \bar{y}^i_s q^i_s - \alpha_s| + |w^i_s \bar{y}^i_s| \\ &= |\alpha_s w^i_s \bar{y}^i_s q^i_s - \alpha_s w^i_s \bar{x}^i_s| + \frac{1}{q^i_s}|w^i_s \bar{y}^i_s q^i_s - w^i_s \bar{x}^i_s + w^i_s \bar{x}^i_s| \\ &\le |w^i_s(\bar{y}^i_s q^i_s - \bar{x}^i_s)| + \frac{1}{q^i_s}|w^i_s(\bar{y}^i_s q^i_s - \bar{x}^i_s)| + \frac{1}{q^i_s}|w^i_s \bar{x}^i_s| \\ &\le \left(1 + \frac{1}{q^i_s}\right)\|w^i_s\|\|\bar{y}^i_s q^i_s - \bar{x}^i_s\| + \frac{1}{q^i_s} = R\left(1 + \frac{1}{q^i_s}\right)\|\bar{y}^i_s q^i_s - \bar{x}^i_s\| + \frac{1}{q^i_s} \longrightarrow 0.\end{aligned}$$

Hence,

$$\delta_s \delta_s M^i_s w^i_s \bar{y}^i_s \longrightarrow \alpha_s. \tag{21}$$

For $j \neq s$, it similarly follows by Claims 5.2 ($q^i_j \to \infty$) and 5.3 ($w^i_s \bar{x}^i_j = 0$) that

$$\begin{aligned}|\delta_s\delta_j M^i_j w^i_s \bar{y}^i_j| &= |\lfloor \alpha_j q^i_j \rfloor w^i_s \bar{y}^i_j| \le |\alpha_j w^i_s \bar{y}^i_j q^i_j - \alpha_j w^i_s \bar{x}^i_j| + |w^i_s \bar{y}^i_j| \\ &\le |w^i_s(\bar{y}^i_j q^i_j - \bar{x}^i_j)| + \frac{1}{q^i_j}|w^i_s(\bar{y}^i_j q^i_j - \bar{x}^i_j)| \le R\left(1 + \frac{1}{q^i_j}\right)\|\bar{y}^i_j q^i_j - \bar{x}^i_j\| \longrightarrow 0,\end{aligned}$$

that is,

$$\delta_s \delta_j M^i_j w^i_s \bar{y}^i_j \longrightarrow 0. \tag{22}$$

Now consider the terms $\delta_s \delta_j M^i_j (1/\beta) r^i \bar{y}^i_j w^i_s a^i$. With Claim 5.3, we obtain

$$\begin{aligned}\left|\delta_s\delta_j M^i_j \frac{1}{\beta} r^i \bar{y}^i_j w^i_s a^i\right| &= \frac{1}{\beta}|\lfloor \alpha_j q^i_j \rfloor (r^i \bar{y}^i_j)(w^i_s a^i)| \le \frac{1}{\beta}|\alpha_j (r^i \bar{y}^i_j q^i_j)(w^i_s a^i)| + \frac{1}{\beta}|(r^i \bar{y}^i_j)(w^i_s a^i)| \\ &\le \frac{1}{\beta}|r^i \bar{y}^i_j q^i_j||w^i_s a^i| + \frac{1}{\beta}|r^i \bar{y}^i_j||w^i_s a^i| = \frac{1}{\beta}\left(1 + \frac{1}{q^i_j}\right)|r^i \bar{y}^i_j q^i_j||w^i_s a^i| \\ &= \frac{1}{\beta}\left(1 + \frac{1}{q^i_j}\right)|r^i \bar{y}^i_j q^i_j - r^i \bar{x}^i_j + r^i \bar{x}^i_j||w^i_s a^i| \\ &\le \frac{1}{\beta}\left(1 + \frac{1}{q^i_j}\right)\|r^i\|(\|\bar{y}^i_j q^i_j - \bar{x}^i_j\| + \|\bar{x}^i_j\|)|w^i_s a^i|.\end{aligned}$$

We can bound

$$\frac{1}{\beta}\left(1 + \frac{1}{q^i_j}\right)\|r^i\|(\|\bar{y}^i_j q^i_j - \bar{x}^i_j\| + \|\bar{x}^i_j\|)$$

from above by Claims 5.2, 5.3, and 5.4 for sufficiently large i. Furthermore, using (7) and $\bar{a} w^i_s = 0$, we get for each $s \in [l]$,

$$|w^i_s a^i| = \|a^i\||\bar{a}^i w^i_s| = \|a^i\||\bar{a}^i w^i_s - \bar{a} w^i_s| \le \|a^i\|\|\bar{a}^i - \bar{a}\|\|w^i_s\| = R\|a^i\|\|\bar{a}^i - \bar{a}\| \longrightarrow 0. \tag{23}$$

It follows that

$$\delta_s \delta_j M^i_j \frac{1}{\beta} r^i \bar{y}^i_j w^i_s a^i \longrightarrow 0. \tag{24}$$

Observations (21), (22), and (24) imply (20), that is,

$$h^i(\delta M^i, \mu^i_0)(\delta_s w^i_s) \longrightarrow \alpha_s.$$

With (23), we obtain for all $\mu \in [\mu^i_0, \mu^i_0 + 1]$,

$$h^i(\delta M^i, \mu)(\delta_s w^i_s) \longrightarrow \alpha_s.$$

Note that this convergence is essentially independent of α; that is, for every $\varepsilon > 0$, there is some $N(\varepsilon)$ such that for all $i \ge N$ and for all $\alpha \in [0, 1]^l$, $|h^i(\delta M^i, \mu)(\delta_s w^i_s) - \alpha_s| \le \varepsilon$. In particular, there must exist sequences of integral $h^i_\alpha(\delta)$ with this property. This proves condition (16). For condition (17), observe that $hv^i_j = M^i_j$ for every $h \in L^i(M^i)$ and every $j \in [l]$. We thus get

$$h^i(\delta M^i, \mu)(\delta_j v^i_j) = \delta_j^2 M^i_j = \lfloor \alpha_j q^i_j \rfloor.$$

Finally, consider condition (19). For every $\mu \in [\mu^i_0, \mu^i_0 + 1]$, we have

$$\begin{aligned} |h^i(\delta M^i, \mu)a^i| &\le |h^i(\delta M^i, \mu^i_0)a^i| + |a^i a^i| = \left|\sum_{j=1}^{l} \delta_j \lfloor a_j q^i_j \rfloor \left(\bar{y}^i_j + \frac{1}{\beta}(r^i \bar{y}^i_j)a^i\right)a^i\right| + \|a^i\|^2 \\ &\le \sum_{j=1}^{l}\left(\left|\alpha_j q^i_j\left(\bar{y}^i_j a^i + \frac{1}{\beta} r^i \bar{y}^i_j a^i a^i\right)\right| + \left|\bar{y}^i_j a^i + \frac{1}{\beta} r^i \bar{y}^i_j a^i a^i\right|\right) + \|a^i\|^2 \\ &\le \sum_{j=1}^{l}\left(1 + \frac{1}{q^i_j}\right)\|\bar{y}^i_j q^i_j\|\left(\|a^i\| + \frac{1}{\beta}\|r^i\|\|a^i\|^2\right) + \|a^i\|^2. \end{aligned}$$

Since $\|\bar{y}^i_j q^i_j\|$ is bounded because of Claims 5.3 and 5.4, since $q^i_j \to \infty$ by Claim 5.2, and since $\|r^i\|$ is bounded as well, there exists some constant $C > 0$ such that condition (19) is satisfied for sufficiently large i. Note that this constant does not depend on α. □

In the proof above, we chose the vectors $v^i_1, \ldots, v^i_l$, which span together with the vectors $u_1, \ldots, u_k$ the integral approximation $(a^i x = 0)$ of $(ax = 0)$, in a specific way. The vectors $u_1, \ldots, u_k, v^i_1, \ldots, v^i_l$ form a basis of the lattice of integer points in $(a^i x = 0)$ and they satisfy properties that are characteristic for reduced bases. In other words, the vectors $v^i_1, \ldots, v^i_l$ are almost perpendicular to each other. We already leveraged this special property when we showed property (ix) of Lemma 5.2 and also in the proof of Claim 5.4, which was required for the final analysis in the lemma. However, there is a second reason why an arbitrary choice of these vectors would not allow us to prove the main result of this section. Recall that our goal is to show that for each nonrational facet-defining inequality $ax \le a_P$ of a polytope P, there exists a finite set $S_a \subseteq \mathbb{Z}^n$ such that $C_{S_a}(P) \subseteq (ax \le a_P)$. To illustrate why reduced bases are crucial, consider the special case that $a_P = 0$ and $(ax = 0) \cap \mathbb{Q}^n = \{0\}$. The basic geometric motivation behind the construction in Lemma 5.2 arose from the objective to cover $F = P \cap (ax = 0)$ with at most 2^{n-1} parallelepipeds, spanned by the vectors $\delta_1 w^i_1, \ldots, \delta_{n-1} w^i_{n-1}$, where $\delta \in \{-1, 1\}^{n-1}$. Indeed, if these parallelepipeds covered F, then the vectors $h^i_\alpha(\delta)$ from Lemma 5.2 gave rise to GC cuts that separate every point in F apart from 0: This is because we can choose the vectors r_s for Lemma 5.2 such that $h^i x$ is maximized over P by a point in F. Then for an appropriate choice of the parameters α and ε in (x), we can achieve for $h^i = h^i_\alpha$,

$$\max\{h^i(\delta)x \mid x \in P\} = \max\{h^i(\delta)x \mid x \in F\} \le h^i(\delta)(\delta_1 w^i_1 + \cdots + \delta_{n-1} w^i_{n-1}) < 1$$

and, consequently, $h^i(\delta)x \le 0$ is a GC cut for P. As this is true for every $\delta \in \{-1, 1\}^{n-1}$, these 2^{n-1} GC cuts imply $P' \cap F = \{0\}$.

Since P is bounded, F is contained in some ball of radius R around the origin. Clearly, if the $w^i_1, \ldots, w^i_{n-1}$ are orthogonal to each other and of length R, then the parallelepipeds cover this ball and therefore F (see Figure 8). However, the smaller the angles between the vectors $w^i_1, \ldots, w^i_{n-1}$, the longer the w^i_j's have to be

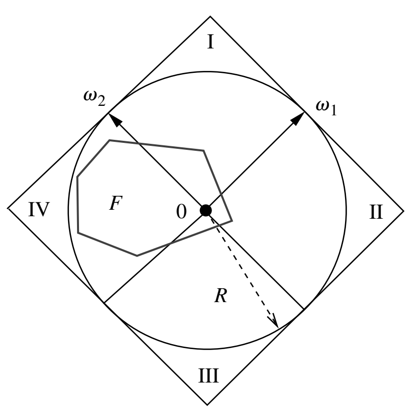

FIGURE 8. Illustration of why reduced bases play a crucial role. The facet $F = P \cap (ax = 0)$ is contained in $B(0, R)$. If w^i_1 and w^i_2 are orthogonal to each other and of length R, the four parallelepipeds spanned by the vectors $\pm w^i_1$ and $\pm w^i_2$ cover F.

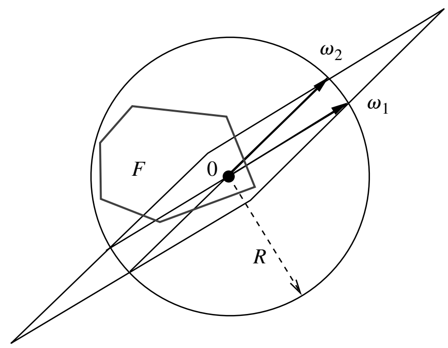

FIGURE 9. Illustration of why reduced bases play a crucial role. The facet $F = P \cap (ax = 0)$ is contained in $B(0, R)$. If w_1^i and w_2^i are of length R, but the angle between them is very small, the four parallelepipeds spanned by the vectors $\pm w_1^i$ and $\pm w_2^i$ do not cover F.

to guarantee that F is completely covered (see Figure 9). The analysis in Lemma 5.2 required that the w_j^i's have a fixed length, which is chosen at the beginning of the construction. Because the lattices of integer points in $\text{span}(v_1^i, \dots, v_{n-1}^i)$ change with every index i, arbitrary bases of the lattices would result in arbitrary angles between the w_j^i's. Therefore, it is not certain that any fixed length R would guarantee the covering property that is needed. By choosing reduced bases, we make sure that the v_j^i's and, hence, the w_j^i's are almost orthogonal to each other. Moreover, their orthogonality defect only depends on the dimension. As a consequence, we can choose a certain fixed length R for the vectors w_j^i that only depends on the radius of the ball that fits F and the dimension n.

In the next lemma, we utilize the sequences from Lemma 5.2 to prove that for every nonrational facet-defining inequality $ax \le a_P$ of a polytope P, there exists a finite set S_a of integral vectors such that $C_{S_a}(P) \subseteq (ax \le a_P)$. This property immediately implies the existence of a finite set $S \subset \mathbb{Z}^n$ with $C_S(P) \subseteq P$.

LEMMA 5.3. *Let P be a polytope in $\mathbb{R}^n$ and let $(ax = a_p)$ be a nonrational supporting hyperplane with $P \subseteq (ax \le a_P)$. Then there exists a finite set $S \subseteq \mathbb{Z}^n$ such that $C_S(P) \subseteq (ax \le a_P)$.*

PROOF. There are three possible types of a nonrational inequality $ax \le a_P$:
(a) $a \in \mathbb{Q}^n$ and $a_P \in \mathbb{R}\backslash\mathbb{Q}$.
(b) $a \in \mathbb{R}^n\backslash\mathbb{Q}^n$ and $(ax = a_P) \cap \mathbb{Q}^n \neq \varnothing$.
(c) $a \in \mathbb{R}^n\backslash\mathbb{Q}^n$ and $(ax = a_P) \cap \mathbb{Q}^n = \varnothing$.

Case (a): If $a \in \mathbb{Q}^n$, then we can assume w.l.o.g. that $a \in \mathbb{Z}^n$ by scaling (a, a_P) by some rational number, if necessary. Consequently, $ax \le \lfloor a_P \rfloor$ is a GC cut for P and $(ax \le \lfloor a_P \rfloor) \subseteq (ax \le a_P)$. Then $S = \{a\}$ has the desired property and we are done.

In the following, let us assume that $a \in \mathbb{R}^n\backslash\mathbb{Q}^n$ and that the same is true for every λa with $\lambda \in \mathbb{R}$. Let $F = P \cap (ax = a_P)$ and let $r_1, \dots, r_m \in \mathbb{R}^n$ denote the set of edge directions emanating from the vertices of F to vertices of P that are not in F. Note that $r_s a < 0$ for $s \in [m]$.

Case (b): Let V_R denote the maximal rational affine subspace contained in $(ax = a_P)$ and let $u_1, \dots, u_k \in \mathbb{Z}^n$ and $x_0 \in (ax = a_P) \cap \mathbb{Q}^n$ such that $V_R = x_0 + \text{span}(u_1, \dots, u_k)$. Define $l := n - k - 1$ and $U := \text{span}(u_1, \dots, u_k)$. Note that $U = \{0\}$ is possible. Since P is bounded, there exists an $R_1 > 0$ such that for every $x \in F$ there is an $u \in U$ with

$$x \in x_0 + u + B(0, R_1). \tag{25}$$

Let $p_0 \in \mathbb{Z}^n$ and let $q_0 \ge 1$ be an integer such that $x_0 = p_0/q_0$. Furthermore, let c be the constant from property (ix) in Lemma 5.2 and let c_1 be the constant from Lemma 4.2. Let us fix a constant R such that $R \ge R_1 c_1/c$ and consider the sequences that exist according to Lemma 5.2 for $V = (ax = 0)$ and R.

First, observe that we can choose i large enough such that $a^i x \le \lfloor a^i x_0 \rfloor$ is a GC cut for P: property (ii) in Lemma 5.2 implies

$$\max\{a^i x \mid x \in P\} = \max\{a^i x \mid x \in F\} = a^i x_0 + \max\{a^i (x - x_0) \mid x \in F\},$$

and by property (iii) and boundedness of P, we have for all $x \in F$,

$$\|a^i (x - x_0)\| = \|a^i\| \|\bar{a}^i (x - x_0)\| = \|a^i\| \|(\bar{a}^i - \bar{a})(x - x_0)\| \le \|a^i\| \|\bar{a}^i - \bar{a}\| \|x - x_0\| \longrightarrow 0.$$

Hence, we can choose i large enough such that

$$\max\{a^i x \mid x \in P\} \le a^i x_0 + \frac{1}{2q_0},$$

which implies that $a^i x \le \lfloor a^i x_0 \rfloor$ is a GC cut for P.

Now let $\alpha = (1/(2q_0(l+1)))(1, \dots, 1)$. Also by Lemma 5.2, there exists an index i such that the vectors $v_j := v_j^i$ and $w_j := w_j^i$, for $j \in [l]$, and the integral vectors $h(\delta) := h_\alpha^i(\delta)$, $\delta \in \{-1, 1\}^l$, satisfy

$$h(\delta) \perp u_p \quad \text{for } p \in [k] \tag{26}$$

$$0 < \delta_j w_j h(\delta) \le (q_0(l+1))^{-1} \quad \text{for } j \in [l] \tag{27}$$

$$\delta_j h(\delta) v_j \ge 1 \quad \text{for } j \in [l] \tag{28}$$

$$0 \ge r_s h(\delta) \quad \text{for } s \in [m]. \tag{29}$$

Moreover, it holds that $\|\tilde{w}_j\| \ge cR$ for every $j \in [l]$, where $\tilde{w}_j$ denotes the orthogonal projection of w_j onto $\operatorname{span}(u_1, \dots, u_k, w_1, \dots, w_{j-1})^\perp$. Using (25), every point $x \in F$ can be written as $x = x_0 + u' + \sum_{j=1}^{l} \tilde{\lambda}_j \tilde{w}_j$, where $u' \in U$ and $|\tilde{\lambda}_j| \le R_1/(cR)$, for $j \in [l]$. Then it follows by Lemma 4.2 that every $x \in F$ can be expressed as

$$x = x_0 + u + \sum_{j=1}^{l} \lambda_j w_j, \tag{30}$$

where $u \in U$ and $|\lambda_j| \le c_1 R_1/(cR) \le 1$, $j \in [l]$. For any $\delta \in \{-1, 1\}^l$, we get with (26)–(29) and (30),

$$\begin{aligned}\max\{h(\delta)x \mid x \in P\} = \max\{h(\delta)x \mid x \in F\} &\le h(\delta)x_0 + \sum_{j=1}^{l} \max_{\lambda_j \in [-1, 1]} \{\lambda_j h(\delta) w_j\}\\ &= h(\delta)x_0 + \sum_{j=1}^{l} \delta_j h(\delta) w_j \le h(\delta)x_0 + l(q_0(l+1))^{-1} < \lfloor h(\delta)x_0 \rfloor + 1.\end{aligned}$$

Hence, $h(\delta)x \le \lfloor h(\delta)x_0 \rfloor$ is a GC cut for P for every $\delta \in \{-1, 1\}^l$. Now consider an arbitrary $x \in (ax = a_P) \backslash V_R$. By (30), there exists an $u \in U$ and $\lambda_j \in \mathbb{R}_+$ and $\delta_j \in \{-1, 1\}$ for $j \in [l]$ such that $x = x_0 + u + \sum_{j=1}^{l} \lambda_j \delta_j w_j$. Note that $\sum_{j=1}^{l} \lambda_j > 0$, as $x \notin V_R$. Consequently,

$$h(\delta)x = h(\delta)x_0 + \sum_{j=1}^{l} \lambda_j \delta_j h(\delta) w_j > h(\delta)x_0 \ge \lfloor h(\delta)x_0 \rfloor;$$

that is, x violates the GC cut $h(\delta)x \le \lfloor h(\delta)x_0 \rfloor$. Now let H denote the polyhedron defined by the intersection of the 2^l half-spaces associated with the GC cuts $h(\delta)x \le \lfloor h(\delta)x_0 \rfloor$, with $\delta \in \{-1, 1\}^l$. Then by the last observation,

$$((ax = a_P) \cap H) = \left((ax = a_P) \cap \bigcap_{\delta \in \{-1, 1\}^l} (h(\delta)x \le \lfloor h(\delta)x_0 \rfloor)\right) \subseteq V_R. \tag{31}$$

Similarly, let us consider the integral hyperplane $(a^i x = a^i x_0)$. Any $x \in (a^i x = a^i x_0) \backslash V_R$ can be written as

$$x = x_0 + u + \sum_{j=1}^{l} \lambda_j \delta_j v_j^i,$$

for some $u \in U$, and $\lambda_j \in \mathbb{R}_+$ and $\delta_j \in \{-1, 1\}$, $j \in [l]$; and in this representation it must also hold that $\sum_{j=1}^{l} \lambda_j > 0$. Then with (28)

$$h(\delta)x = h(\delta)x_0 + \sum_{j=1}^{l} \lambda_j \delta_j h(\delta) v_j > h(\delta)x_0 \ge \lfloor h(\delta)x_0 \rfloor.$$

This implies that also every point in $(a^i x = a^i x_0) \backslash V_R$ is separated by some GC cut $h(\delta)x \le \lfloor h(\delta)x_0 \rfloor$ and, thus,

$$((a^i x = a^i x_0) \cap H) \subseteq V_R. \tag{32}$$

Because every hyperplane $(h(\delta)x = \lfloor h(\delta)x_0 \rfloor)$ is parallel to V_R, either every point in V_R satisfies the corresponding inequality or every point in V_R violates it. Therefore,

$$((ax = a_P) \cap H) = ((a^i x = a^i x_0) \cap H) \in \{\varnothing, V_R\}.$$

Observe furthermore that every minimal face of $((a^i x \leq a^i x_0) \cap H)$ is also a minimal face of $((ax \leq a_P) \cap H)$ and vice versa. Consequently,

$$((a^i x \leq \lfloor a^i x_0 \rfloor) \cap H) \subseteq ((a^i x \leq a^i x_0) \cap H) = ((ax \leq a_P) \cap H) \subseteq (ax \leq a_P).$$

It follows that a^i and the vectors $h(\delta)$, for $\delta \in \{-1, 1\}^l$, form the desired set S of the lemma.

Case (c): In the remainder of the proof, we consider the case $(ax = a_P) \cap \mathbb{Q}^n = \varnothing$. Let $u_1, \ldots, u_k \in \mathbb{Z}^n$ be a maximal set of linearly independent integral vectors such that $au_i = 0$ for $i \in [k]$. Let $U := \text{span}(u_1, \ldots, u_k)$ and note that $U = \{0\}$ is possible. Furthermore, take an arbitrary point $x_0 \in F$. Since P is bounded, there exists a constant $R_1 > 0$ such that for every $x \in F$ there is an $u \in U$ such that

$$x \in x_0 + u + B(0, R_1). \tag{33}$$

Let us fix an $R \geq R_1 c_1 / c$, where c and c_1 are the constants from property (ix) in Lemma 5.2 and Lemma 4.2, respectively. Now consider the sequences that exist according to Lemma 5.2 for $V = (ax = 0)$ and R. property (ii) from Lemma 5.2 implies that if there exists an index i and an integer a_0^i such that

$$a_0^i + 1 > \max\{a^i x \mid x \in P\} = \max\{a^i x \mid x \in F\} \geq \min\{a^i x \mid x \in F\} > a_0^i,$$

then $a^i x \leq a_0^i$ is a GC cut for P with the property that every point in F violates the cut such that $(a^i x \leq a_0^i) \cap F = \varnothing$. In particular, one can then find an $\varepsilon_1 > 0$ such that $(P \cap (a^i x \leq a_0^i)) \subseteq (ax \leq a_P - \varepsilon_1)$. This implies that there exists a rational polyhedron $Q \supseteq P$ such that $(a^i x \leq a_0^i)$ is also a GC cut for Q and such that $Q \cap (a^i x \leq a_0^i) \subseteq (ax \leq a_P)$. The facet normals of Q together with a^i imply the desired set S of the lemma.

Let us assume in the remainder of the proof of part (c) that for every i, there exists an integer a_0^i such that

$$F \cap (a^i x = a_0^i) \neq \varnothing.$$

Let $y^i \in F \cap (a^i x = a_0^i)$. Since $\gcd(a^i) = 1$ according to property (i) of Lemma 5.2, there exists an $z_0^i \in (a^i x = a_0^i) \cap \mathbb{Z}^n$. We have

$$(a^i x = a_0^i) = z_0^i + \text{span}(u_1, \ldots, u_k, v_1^i, \ldots, v_l^i).$$

Let $\tilde{x}_0^i$ denote the projection of x_0 onto the hyperplane $(a^i x = a_0^i)$, that is,

$$\tilde{x}_0^i = x_0 + \frac{a_0^i - a^i x_0}{\|a^i\|^2} a^i. \tag{34}$$

Note that because of property (iii) in Lemma 5.2 and boundedness of P,

$$|a_0^i - a^i x_0| = |a^i y^i - a^i x_0| = \|a^i\| |\bar{a}^i y^i - \bar{a}^i x_0 + (\bar{a} x_0 - \bar{a} y^i)| \leq \|a^i\| \|\bar{a}^i - \bar{a}\| \|x_0 - y^i\| \longrightarrow 0. \tag{35}$$

We can assume w.l.o.g. that the point $z_0^i \in (a^i x = a_0^i) \cap \mathbb{Z}^n$ is chosen such that there exist numbers $\gamma_1^i, \ldots, \gamma_k^i$, $\mu_1^i, \ldots, \mu_l^i \in [0, 1]$ such that

$$\tilde{x}_0^i = z_0^i + \gamma_1^i u_1 + \cdots + \gamma_k^i u_k + \mu_1^i v_1^i + \cdots + \mu_l^i v_l^i. \tag{36}$$

Figure 10 illustrates the described situation. Next, we show that $\tilde{x}_0^i$, and therefore also x_0, is far away from any integer point in the hyperplane $(a^i x = a_0^i)$.

Claim 5.5. *Any vertex f^i of the parallelepiped $z_0^i + \bar{\Pi}(u_1, \ldots, u_k, v_1^i, \ldots, v_l^i)$ satisfies $\|x_0 - f^i\| \longrightarrow \infty$.*

Proof of Claim 5.5. Because F is bounded and because x_0 and y^i are points in F, there exists a constant K_1 such that for all i, $\|x_0 - y^i\| \leq K_1$. Then

$$\|f^i - y^i\| = \|f^i - x_0 + x_0 - y^i\| \leq \|f^i - x_0\| + \|x_0 - y^i\| \leq \|f^i - x_0\| + K_1$$

implies $\|f^i - x_0\| \geq \|f^i - y^i\| - K_1$. Hence, to show the claim it suffices to prove $\|f^i - y^i\| \longrightarrow \infty$. Suppose that there exists some positive constant $K_2 > 0$ such that for all i we have $\|f^i - y^i\| \leq K_2$. Note that then

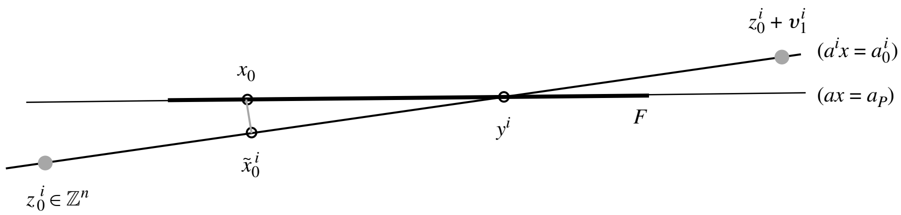

Figure 10. Situation in part (c) of the proof of Lemma 5.3 in the special case that for every i there exists an integer a_0^i such that $F \cap (a^i x = a_0^i) \neq \varnothing$.

$f^i \in B(x_0, K_1 + K_2) \cap \mathbb{Z}^n$. Let $\tilde{f}^i$ denote the projection of f^i onto the hyperplane $(ax = a_P)$; that is, $\tilde{f}^i + \lambda a = f^i$, where $\lambda = (af^i - a_P)/\|a\|^2 = (af^i - ay^i)/\|a\|^2$. Because $f^i \in \mathbb{Z}^n$ and $f^i \notin (ax = a_P)$ (remember that $(ax = a_P) \cap \mathbb{Q}^n = \varnothing$) and because the number of integer points in $B(x_0, K_1 + K_2)$ is finite, there must exist some positive number D such that $\|f^i - \tilde{f}^i\| \geq D$, for every i. However, with property (iii) from Lemma 5.2 and using $\bar{a}^i(f^i - y^i) = 0$, we get

$$\|f^i - \tilde{f}^i\| = \|\lambda a\| = \frac{|af^i - ay^i|}{\|a\|} = |\bar{a}(f^i - y^i) - \bar{a}^i(f^i - y^i)| \leq \|\bar{a} - \bar{a}^i\| \|f^i - y^i\| \leq K_2 \|\bar{a} - \bar{a}^i\| \longrightarrow 0,$$

which is a contradiction. □

Because the above claim implies that $\tilde{x}_0^i$ is far away from any integer point in the hyperplane $(a^i x = a_0^i)$, it is intuitive that not all the coefficients μ_j^i in the representation (36) can be close to 0 or 1. We formally prove this observation in the next claim.

Claim 5.6. *Let $K > 1$ be a constant. There exists an integer $N_1 = N_1(K)$ such that for every $i \geq N_1$, there exists an index $j \in [l]$ such that the coefficient μ_j^i in (36) satisfies*

$$\frac{K}{q_j^i} \leq \mu_j^i \leq 1 - \frac{K}{q_j^i}.$$

Proof of Claim 5.6. By Claim 5.5, any vertex f^i of the parallelepiped $z_0^i + \bar{\Pi}(u_1, \ldots, u_k, v_1^i, \ldots, v_l^i)$ satisfies $\|x_0 - f^i\| \longrightarrow \infty$. Therefore,

$$\|x_0 - f^i\| \leq \|x_0 - \tilde{x}_0^i\| + \|\tilde{x}_0^i - f^i\| \longrightarrow \infty.$$

Because (35) implies $\|x_0 - \tilde{x}_0^i\| \longrightarrow 0$, we must have $\|\tilde{x}_0^i - f^i\| \longrightarrow \infty$. In particular, there exists a number N_1 such that for all $i \geq N_1$,

$$\|\tilde{x}_0^i - f^i\| > \sum_{p=1}^{k} \|u_p\| + 2KRl.$$

Now let $i \geq N_1$ and assume that there are index sets J_1^i and J_2^i such that $J_1^i \cup J_2^i = \{1, \ldots, l\}$ and such that for every index $j \in J_1^i$, we have $0 \leq \mu_j^i < K/q_j^i$, and for every index $j \in J_2^i$, it holds that $0 \leq 1 - \mu_j^i < K/q_j^i$. For the vertex

$$f^i = z_0^i + \sum_{j \in J_2^i} v_j^i$$

of the parallelepiped, it follows with property (vi) from Lemma 5.2 that

$$\begin{aligned}\|\tilde{x}_0^i - f^i\| &= \left\| \sum_{p=1}^{k} \gamma_p^i u_p + \mu_1^i \delta_1^i v_1^i + \cdots + \mu_l^i \delta_l^i v_l^i - \sum_{j \in J_2^i} \delta_j^i v_j^i \right\| \leq \sum_{p=1}^{k} \|u_p\| + \left\| \sum_{j \in J_1^i} \mu_j^i \delta_j^i v_j - \sum_{j \in J_2^i} (1 - \mu_j^i) \delta_j^i v_j^i \right\| \\ &\leq \sum_{p=1}^{k} \|u_p\| + \sum_{j \in J_1^i} \mu_j^i \|v_j^i\| + \sum_{j \in J_2^i} (1 - \mu_j^i) \|v_j^i\| \leq \sum_{p=1}^{k} \|u_p\| + K \sum_{j=1}^{l} \frac{\|v_j^i\|}{q_j^i} \longrightarrow \sum_{p=1}^{k} \|u_p\| + KRl,\end{aligned}$$

which is a contradiction. □

The next technical claim is needed to choose a proper parameter α for the vectors $h_\alpha^i(\delta)$ in Lemma 5.2 that give rise to appropriate GC cuts.

Claim 5.7. *Let $K > 1$, $\mu \in [0,1]$, and $q \in \mathbb{R}$ such that $q \geq 2K$ and $K/q \leq \mu \leq 1 - K/q$. Then there exist integers p_1 and p_2 such that $1 \leq p_1 \leq q/(2K)$ and*

$$p_2 + 1/4 \leq \mu p_1 \leq (p_2+1) - 1/4.$$

Proof of Claim 5.7. We consider three cases. If $1/4 \leq \mu \leq 1 - 1/4$, then $p_1 = 1$ and $p_2 = 0$ satisfy the conditions of the claim. If $\mu < 1/4$, there must exist an integer p such that $1/4 \leq \mu p \leq 1/2 \leq 1 - 1/4$. Then

$$1 \leq \frac{1}{4\mu} \leq p \leq \frac{1}{2\mu} \leq \frac{q}{2K},$$

and we can set $p_1 = p$ and $p_2 = 0$. Finally, if $\mu > 1 - 1/4$, then $1 - \mu < 1/4$ and there must exist an integer p such that $1/4 \leq (1-\mu)p \leq 1/2$. Then

$$1 \leq \frac{1}{4(1-\mu)} \leq p \leq \frac{1}{2(1-\mu)} \leq \frac{q}{2K}.$$

For $p_1 = p$ and $p_2 = p - 1$, we get

$$p_2 + 1/4 < p - 1/2 \leq \mu p \leq p - 1/4 = (p_2+1) - 1/4. \quad \square$$

For the remainder, let us fix a constant K such that $K > 8(2+l)$. For large enough i, the assumptions of Claim 5.7 are satisfied; that is, $q^i_j \geq 2K$ for every $j \in [l]$. Then Claims 5.6 and 5.7 imply that there exists an integer $N(K)$ such that for every $i \geq N(K)$, there exists an index $s \in [l]$ and integer numbers p^i_1 and p^i_2 such that $1 \leq p^i_1 \leq q^i_s/(2K)$ and $p^i_2 + 1/4 \leq \mu^i_s p^i_1 \leq (p^i_2 + 1) - 1/4$. Note that we can write the positive integer p^i_1 as $\lfloor \bar{\alpha}^i_s q^i_s \rfloor$ for some scalar $\bar{\alpha}^i_s$. That is, there exist a number $0 < \bar{\alpha}^i_s < 1/K$ and an integer p^i such that

$$p^i + 1/4 \leq \mu^i_s \lfloor \bar{\alpha}^i_s q^i_s \rfloor \leq (p^i+1) - 1/4. \tag{37}$$

Define $\alpha^i \in \mathbb{R}^l_+$ by

$$\alpha^i_j = \begin{cases} \bar{\alpha}^i_s, & \text{if } j = s \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\|\alpha^i\|_\infty \leq 1$. Now let $\bar{\delta} = (1, \ldots, 1)$ and take $h^i := h^i_{\alpha^i}(\bar{\delta})$ according to Lemma 5.2 from property (x). For some sufficiently large number N_2, we can assume that for every $i \geq N_2$,

$$h^i \perp u_p \quad \text{for } p \in [k] \tag{38}$$

$$\alpha^i_j - 1/K \leq w^i_j h^i \leq \alpha^i_j + 1/K \quad \text{for } j \in [l] \tag{39}$$

$$h^i v^i_j = \lfloor \alpha^i_j q^i_j \rfloor \quad \text{for } j \in [l] \tag{40}$$

$$0 \geq r_s h^i \quad \text{for } s \in [m] \tag{41}$$

$$|h^i a^i| \leq C \|a^i\|^2, \tag{42}$$

where $C > 0$ is a constant. By (33) and arguing as in part (b), R has been chosen large enough so that every point $x \in F$ can be written as

$$x = x_0 + u + \sum_{j=1}^{l} \lambda_j \delta_j w^i_j,$$

for some $u \in U$ and $\lambda_j \in [0,1]$ and $\delta_j \in \{-1, 1\}$ for $j \in [l]$. With (34), (36), (38), and (40), we get for every $x \in F$,

$$\begin{aligned} h^i x &= h^i x_0 + h^i u + \sum_{j=1}^{l} \lambda_j h^i \delta_j w^i_j = h^i x_0 + \sum_{j=1}^{l} \lambda_j h^i \delta_j w^i_j \\ &= h^i z^i_0 + \mu^i_1 h^i v^i_1 + \cdots + \mu^i_l h^i v^i_l - \frac{a^i_0 - a^i x_0}{\|a^i\|^2} h^i a^i + \sum_{j=1}^{l} \lambda_j h^i \delta_j w^i_j \\ &= h^i z^i_0 + \mu^i_s \lfloor \bar{\alpha}^i_s q^i_s \rfloor - \frac{a^i_0 - a^i x_0}{\|a^i\|^2} h^i a^i + \sum_{j=1}^{l} \lambda_j h^i \delta_j w^i_j. \end{aligned}$$

For large enough i we get with (35) and (42),

$$\left|\frac{a_0^i - a^i x_0}{\|a^i\|^2} h^i a^i\right| \le 1/K.$$

Consequently, with (39) and $0 \le \bar{\alpha}_s^i < 1/K$, we obtain

$$\left|-\frac{a_0^i - a^i x_0}{\|a^i\|^2} h^i a^i + \sum_{j=1}^{l} \lambda_j h^i \delta_j w_j\right| \le \left|\frac{a_0^i - a^i x_0}{\|a^i\|^2} h^i a^i\right| + \sum_{j=1}^{l} |h^i w_j| \le \frac{1}{K} + \bar{\alpha}_s^i + \frac{l}{K} \le \frac{2+l}{K} < \frac{1}{8}.$$

This implies that for every $x \in F$,

$$h^i z_0^i + \mu_s^i \lfloor \bar{\alpha}_s^i q_s^i \rfloor - 1/8 \le h^i x \le h^i z_0^i + \mu_s^i \lfloor \bar{\alpha}_s^i q_s^i \rfloor + 1/8,$$

and with (37), it follows that for every $x \in F$,

$$(h^i z_0^i + p^i) + 1/8 \le h^i x \le (h^i z_0^i + p^i + 1) - 1/8. \tag{43}$$

Now observe that (41) implies that $h^i x$ is maximized over P by a point in F. Therefore, using (43) and that $z_0^i \in \mathbb{Z}^n$, we have that $h^i x \le h^i z_0^i + p^i$ is a GC cut for P. Moreover, (43) implies that this cut is violated by every point in F; that is,

$$(h^i x \le h^i z_0^i + p^i) \cap F = \varnothing.$$

Arguing as at the beginning of part (c), we can find a rational polyhedron $Q \supseteq P$ such that $(h^i x \le h^i z_0^i + p^i)$ is also a GC cut for Q and such that

$$Q \cap (h^i x \le h^i z_0^i + p^i) \subseteq (ax \le a_P).$$

The facet normals of Q together with h^i imply the desired set S of the lemma. □

As the proof of the above lemma shows, for every nonrational face-defining inequality $ax \le a_P$ of P, the GC procedure will separate every point in $P \cap (ax = a_P)$ that is not contained in the maximal rational affine subspace of $(ax = a_P)$.

Corollary 5.2. *Let P be a polytope and let $F = P \cap (ax = a_P)$ be a face of P. If V_R denotes the maximal rational affine subspace of $(ax = a_P)$, then $P' \cap F \subseteq V_R$.*

Lemma 5.3 gives us the tools to complete the first step of the main proof.

Corollary 5.3. *Let P be a polytope in $\mathbb{R}^n$. Then there exists a finite set $S \subseteq \mathbb{Z}^n$ such that $C_S(P) \subseteq P$.*

Proof. Let $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$ for some matrix A and some vector b. Let A^1 denote the set of vectors corresponding to rows of A that define rational facet-defining inequalities of P and let A^2 denote the set of vectors associated with the nonrational facet-defining inequalities of P. By means of Lemma 5.3, for every nonrational facet-defining inequality $ax \le a_P$ of P, there exists a finite set $S_a \subseteq \mathbb{Z}^n$ such that $C_{S_a}(P) \subseteq (ax \le a_P)$. Therefore, the finite set

$$S := \left(\bigcup_{a \in A^2} S_a\right) \cup A^1$$

satisfies $C_S(P) \subseteq P$. □

5.2. Step 2. In this section, we show a property of the GC closure that is sometimes referred to as *homogeneity*: the GC closure of a face of a polytope is equal to the intersection of the GC closure of the polytope with the face. This property is well known for rational polytopes (see, e.g., Schrijver [12]) but to our knowledge has not yet been shown for nonrational polytopes. We first prove a kind of rotation lemma.

Lemma 5.4. *Let P be a polytope and let $F = P \cap (ax = a_P)$ be a face of P. Let V_R denote the maximal rational affine subspace of $(ax = a_P)$ and assume that $V_R \neq \varnothing$. If $cx \le \lfloor c_F \rfloor$ is a GC cut for F and facet-defining for F', then there exists a GC cut $\bar{c}x \le \lfloor \bar{c}_P \rfloor$ for P such that*

$$(ax = a_P) \cap V_R \cap (\bar{c}x \le \lfloor \bar{c}_P \rfloor) = (ax = a_P) \cap V_R \cap (cx \le \lfloor c_F \rfloor).$$

Proof. Let $V_R = x_0 + \operatorname{span}(u_1, \ldots, u_k)$, where $x_0 \in (ax = a_P) \cap \mathbb{Q}^n$ and $u_1, \ldots, u_k \in \mathbb{Z}^n$, $k \le n-1$. Note that $V_R = \{x_0\}$ is possible. Furthermore, assume that $P \subseteq (ax \le a_P)$. Now consider a GC cut $cx \le \lfloor c_F \rfloor$ for F that is facet-defining for F'. Moreover, assume that $\hat{x}$ is a vertex of F that maximizes c over F. Let $r_1, \ldots, r_m$ denote all edge directions of P that emanate from vertices in F to vertices of P that are not in F. Note that for $s \in [m]$,

$$r_s a < 0. \tag{44}$$

According to Corollary 4.1, there exists a sequence $\{a^i\} \subseteq \mathbb{Z}^n$ such that $a^i \perp u_j$ for $j \in [k]$ and such that

$$\|a^i\| \|\bar{a}^i - \bar{a}\| \longrightarrow 0, \tag{45}$$

where $\bar{a}^i = a^i / \|a^i\|$ and $\bar{a} = a/\|a\|$. Because $r_s \bar{a} < 0$ by (44), it follows with (45) that there exists a constant $\beta > 0$ such that $r_s \bar{a}^i \le -\beta$ for large enough i. Hence, noting that $\|a^i\| \longrightarrow \infty$ because of $a \in \mathbb{R}^n \setminus \mathbb{Q}^n$, there exists a constant $\beta > 0$ and an $N_1 \in \mathbb{N}$ such that $r_s a^i \le -\beta$ for all $s \in [m]$ and $i \ge N_1$. Let $M := \max_{s \in [m]} \{c r_s\}$. If $M \le 0$, then $\hat{x}$ also maximizes c over P and, hence, $cx \le \lfloor c_F \rfloor$ is a GC cut for P. Therefore, assume that $M > 0$. Let $p \in \mathbb{Z}^n$ and $q \in \mathbb{N}$ with $q \ge 1$ such that $x_0 = p/q$. We define the constant $K := q\lceil (1/\beta) M \rceil$ and vectors $\bar{c}^i := c + K a^i$ for every $i \ge N_1$. Note that $K \in \mathbb{Z}$ and therefore $\bar{c}^i \in \mathbb{Z}^n$. We have for $s \in [m]$,

$$r_s \bar{c}^i = r_s (c + K a^i) \le r_s c - K\beta \le 0,$$

which implies that for $i \ge N_1$, the vector $\bar{c}^i$ is maximized over P by a point in F. Now let $\hat{x}^i \in \arg\max\{a^i x \mid x \in F\}$. We obtain for every $i \ge N_1$,

$$\begin{aligned} \max\{\bar{c}^i x \mid x \in P\} &= \max\{\bar{c}^i x \mid x \in F\} \le \max\{cx \mid x \in F\} + K \max\{a^i x \mid x \in F\} \\ &= c\hat{x} + K a^i \hat{x}^i = c_F + K a^i x_0 + K a^i (\hat{x}^i - x_0). \end{aligned}$$

With (45), the boundedness of F, and $a\hat{x}^i = a x_0 = a_P$, we get

$$|a^i(\hat{x}^i - x_0)| = \|a^i\| |(\bar{a}^i - \bar{a})(\hat{x}^i - x_0)| \le \|a^i\| \|\bar{a}^i - \bar{a}\| \|\hat{x}^i - x_0\| \longrightarrow 0.$$

Therefore, for any $\varepsilon > 0$, there exists an $N_\varepsilon \in \mathbb{N}$ such that $|K a^i (\hat{x}^i - x_0)| \le \varepsilon$ for all $i \ge N_\varepsilon$. In particular, we can choose i large enough so that

$$\bar{c}_P^i = \max\{\bar{c}^i x \mid x \in P\} < \lfloor c_F \rfloor + K a^i x_0 + 1.$$

Observe that $K a^i x_0 \in \mathbb{Z}$. Consequently,

$$\bar{c}^i x \le \lfloor \bar{c}_P^i \rfloor \le \lfloor c_F \rfloor + K a^i x_0 \tag{46}$$

is a GC cut for P. Furthermore, it has to hold that $\lfloor \bar{c}_P^i \rfloor = \lfloor c_F \rfloor + K a^i x_0$: First, observe that Corollary 5.3 implies that $F' \subseteq F$ and, therefore, $F' \subseteq P' \cap F$. Because $cx \le \lfloor c_F \rfloor$ is by assumption facet-defining for F', there must exist a point $\tilde{x} \in F'$ such that $c\tilde{x} = \lfloor c_F \rfloor$. Note that $F' \subseteq V_R$, according to Corollary 5.2, implies that $\tilde{x} \in V_R$ and, thus, $a^i \tilde{x} = a^i x_0$. Furthermore, we have $\tilde{x} \in P' \cap F$ because of $F' \subseteq P' \cap F$. In particular, $\tilde{x}$ satisfies the GC cut $\bar{c}^i x \le \lfloor \bar{c}_P^i \rfloor$. Consequently,

$$\bar{c}^i \tilde{x} = c\tilde{x} + K a^i \tilde{x} = \lfloor c_F \rfloor + K a^i x_0 \le \lfloor \bar{c}_P^i \rfloor.$$

Together with (46), we obtain $\lfloor \bar{c}_P^i \rfloor = \lfloor c_F \rfloor + K a^i x_0$. It follows that

$$\begin{aligned} (\bar{c}^i x \le \lfloor \bar{c}_P^i \rfloor) \cap (a^i x = a^i x_0) &= (cx + K a^i x \le \lfloor c_F \rfloor + K a^i x_0) \cap (a^i x = a^i x_0) \\ &= (cx \le \lfloor c_F \rfloor) \cap (a^i x = a^i x_0). \end{aligned}$$

As $V_R \subseteq (a^i x = a^i x_0)$, this implies for $\bar{c} := \bar{c}^i$ for some large enough i,

$$(\bar{c}x \le \lfloor \bar{c}_P \rfloor) \cap V_R = (cx \le \lfloor c_F \rfloor) \cap V_R.$$

The lemma follows. □

With this observation, we can prove the homogeneity property for arbitrary polytopes.

Corollary 5.4. *Let P be a polytope and let F be a face of P. Then $F' = P' \cap F$.*

Proof. For the first direction $F' \subseteq P' \cap F$, observe that $F \subseteq P$ implies $F' \subseteq P'$. Furthermore, $F' \subseteq F$ because of Corollary 5.3. Hence, $F' \subseteq P' \cap F$.

For the second direction, let $F = P \cap (ax = a_P)$ be a face of P and let $cx \leq \lfloor c_F \rfloor$ be a GC cut for F that is facet defining for F'. If $(ax = a_P) \cap \mathbb{Q}^n = \varnothing$ Corollary 5.2 implies $P' \cap F = \varnothing \subseteq F'$. Therefore, assume that $(ax = a_P) \cap \mathbb{Q}^n \neq \varnothing$; that is, the maximal rational affine subspace V_R of $(ax = a_P)$ is nonempty. By Lemma 5.4, there exists a GC cut for P that satisfies

$$(ax = a_P) \cap V_R \cap (\bar{c}x \leq \lfloor \bar{c}_P \rfloor) = (ax = a_P) \cap V_R \cap (cx \leq \lfloor c_F \rfloor).$$

Together with Corollary 5.2, that is, $P' \cap F \subseteq V_R$, we obtain $P' \cap F \subseteq (cx \leq \lfloor c_F \rfloor)$. □

5.3. Step 3. In this subsection, we show that if for some finite set $S \subseteq \mathbb{Z}^n$ of vectors $C_S(P) \subseteq P$ and $C_S(P) \cap \operatorname{rbd}(P) \subseteq P'$, no more than a finite number of GC cuts have to be added to $C_S(P)$ to obtain the closure P'. In fact, we prove that this is true for arbitrary bounded convex sets.

Dadush et al. [4] proved this property for full-dimensional convex sets K. The key observation in this case was that one can find an ε-ball around every interior point of K such that the ball is fully contained in K. Because any additional undominated cut for K' must separate a vertex of $C_S(K)$ in the strict interior of K, it must be derived from an inequality for which the boundary of the associated half-space is shifted by at least ε. However, for valid inequalities $ax \leq a_P$ with $\|a\| > 1/\varepsilon$ this is not possible. As a consequence, only cuts that are associated with normal vectors of a certain bounded norm need to be considered and their number is finite.

For a lower-dimensional convex set K the situation is a somewhat different. Any additional undominated cut would have to separate a point v in the *relative* interior of K and no ε-ball around v is fully contained in K. All we can guarantee is that there exists an ε-ball whose intersection with the affine hull of K and, hence, also with the affine hull of $C_S(K)$, is contained in K. Therefore, any cut that separates v has to correspond to a half-space $(cx \leq c_K)$ for which the intersection of its boundary $(cx = c_K)$ with the affine hull $\operatorname{aff}(C_S(K))$ is shifted by at least ε within $\operatorname{aff}(C_S(K))$. This intersection of $(cx = c_K)$ with $\operatorname{aff}(C_S(K))$ is a lower-dimensional rational affine subspace, say H^*. Similar to the full-dimensional case, there is only a finite number of these affine subspaces H^*, which are shifted within $\operatorname{aff}(C_S(K))$ by at least a distance of ε by the rounding operation in the GC procedure. However, an infinite number of hyperplanes in $\mathbb{R}^n$ has the same intersection H^* with $\operatorname{aff}(C_S(K))$. Consequently, there is an infinite number of GC cuts that could separate a point in the relative interior of $C_S(K)$. Yet we show that among all rational valid inequalities $cx \leq c_K$ with the same intersection $(cx = c_K) \cap \operatorname{aff}(C_S(K))$, there will be one that corresponds to a GC cut that dominates every other GC cut associated with a valid inequality in this equivalence class. For this reason, no more than one GC cut for each of the finitely many H^*'s has to be added to the description of $C_S(K)$.

In the following lemma, we formalize this observation and prove the finite augmentation property for arbitrary bounded convex sets.

Lemma 5.5. *Let K be a convex and compact set in $\mathbb{R}^n$. If there exists a finite set $S \subseteq \mathbb{Z}^n$ such that*
(i) *$C_S(K) \subseteq K$ and*
(ii) *$C_S(K) \cap \operatorname{rbd}(K) \subseteq K'$,*
then K' is a rational polytope.

Proof. Because $C_S(K)$ is a rational polytope, we can assume that $\operatorname{aff}(C_S(K)) = w_0 + W$, where $w_0 \in \mathbb{Q}^n$ and where W is a rational linear vector space. Let $\mathscr{V}$ denote the finite set of vertices of $C_S(K)$. Assumption (i) implies $\mathscr{V} \subseteq K$. Because of assumption (ii), any GC cut for K that separates a point in $C_S(K)\backslash K'$ must also separate a vertex in $\mathscr{V}\backslash \operatorname{rbd}(K) \subseteq \operatorname{ri}(K)$. We will show that for each of the finitely many vertices of $C_S(K)$ in the relative interior of K, one only has to consider a finite set of GC cuts.

First, observe that because of $\mathscr{V}\backslash \operatorname{rbd}(K) \subseteq \operatorname{ri}(K)$ and because the number of vertices of $C_S(K)$ is finite, there exists an $\varepsilon > 0$ such that for every $v \in \mathscr{V}\backslash \operatorname{rbd}(K)$,

$$(v + B(0, \varepsilon)) \cap \operatorname{aff}(K) \subseteq K. \tag{47}$$

Consequently,

$$(v + B(0, \varepsilon)) \cap \operatorname{aff}(C_S(K)) \subseteq K. \tag{48}$$

Now let us fix a vertex v of $C_S(K)$ in the relative interior of K; that is, $v \in \mathscr{V}\backslash \operatorname{rbd}(K)$. Furthermore, let $c \in \mathbb{Z}^n$. We will consider two cases, depending on whether K is full-dimensional or not. If $\dim(K) = n$, then $\operatorname{aff}(K) = \mathbb{R}^n$ and with (47), $(v + B(0, \varepsilon)) \subseteq K$. We get

$$\lfloor c_K \rfloor = \lfloor \max\{cx \mid x \in K\} \rfloor \geq cv + \max\{cx \mid x \in B(0, \varepsilon)\} - 1 = cv + c\left(\varepsilon \frac{c}{\|c\|}\right) - 1 = cv + \varepsilon\|c\| - 1.$$

If $\|c\| \geq 1/\varepsilon$, then the GC cut associated with the normal vector c does not separate the vertex v. Hence, we only need to consider GC cuts with normal vectors c satisfying $\|c\| < 1/\varepsilon$, and their number is finite.

In the remainder of the proof, let us assume that $\dim(K) < n$ and, therefore, $\dim(\text{aff}(C_S(K)) =: k < n$. Since $\dim(W) = k$, we can rename the indices such that there exist integers p_{ij} and $q_{ij} \geq 1$, for $i = 1, \ldots, n-k$ and $j = 1, \ldots, k$, such that for every $w \in W$,

$$w_{k+i} = \sum_{j=1}^{k} \frac{p_{ij}}{q_{ij}} w_j.$$

In words, any point in W is uniquely determined by its first k components. Moreover, we can find an upper bound for the norm of each point $w \in W$ that is a function of the norm of the vector $(w_1, \ldots, w_k)$, that is, the restriction of w to its first k components: Since

$$\|w\|^2 = w_1^2 + \cdots + w_k^2 + \left(\sum_{j=1}^{k} \frac{p_{1j}}{q_{1j}} w_j\right)^2 + \cdots + \left(\sum_{j=1}^{k} \frac{p_{n-k,\,j}}{q_{n-k,\,j}} w_j\right)^2,$$

there exist rational constants $\alpha_i > 0$, for $i \in [k]$, and α_{ij}, for $1 \leq i < j \leq k$, such that

$$\|w\|^2 = \alpha_1 w_1^2 + \cdots + \alpha_k w_k^2 + \sum_{1 \leq i < j \leq k} \alpha_{ij} w_i w_j.$$

Using $\alpha_{ij} w_i w_j \leq (1/2)|\alpha_{ij}| w_i^2 + (1/2)|\alpha_{ij}| w_j^2$ and defining $\alpha_{ji} := \alpha_{ij}$ for every $1 \leq i < j \leq k$, we obtain

$$\|w\|^2 \leq \left(\alpha_1 + \frac{1}{2}\sum_{j=2}^{k} |\alpha_{1j}|\right) w_1^2 + \cdots + \left(\alpha_k + \frac{1}{2}\sum_{j=1}^{k-1} |\alpha_{kj}|\right) w_k^2.$$

Let us define $\alpha := \max_{i \in [k]} \{\alpha_i + (1/2) \sum_{j=1,\, j \neq i}^{k} |\alpha_{ij}|\}$ and observe that α is a positive constant that only depends on W. For any $w \in W$, we have

$$\|w\| \leq \sqrt{\alpha} \|(w_1, \ldots, w_k)\|. \tag{49}$$

Moreover,

$$cw = c_1 w_1 + \cdots + c_k w_k + \sum_{i=1}^{n-k} c_{k+i} \left(\sum_{j=1}^{k} \frac{p_{ij}}{q_{ij}} w_j\right) = \sum_{j=1}^{k} \left(c_j + \sum_{i=1}^{n-k} \frac{p_{ij}}{q_{ij}} c_{k+i}\right) w_j.$$

Let $L\colon \mathbb{R}^n \to \mathbb{R}^k$ denote the affine map that is defined for $j \in [k]$ by

$$L_j(x) := x_j + \sum_{i=1}^{n-k} \frac{p_{ij}}{q_{ij}} x_{k+i}. \tag{50}$$

Then for every $w \in W$,

$$cw = \sum_{j=1}^{k} L_j(c) w_j = L(c)(w_1, \ldots, w_k). \tag{51}$$

Let $w^c = (w_1^c, \ldots, w_n^c) \in W$ such that $(w_1^c, \ldots, w_k^c) = L(c)$. Then (49) implies $\|w^c\| \leq \sqrt{\alpha}\|L(c)\|$ and, therefore,

$$\frac{1}{\sqrt{\alpha}\|L(c)\|} w^c \in B(0, 1) \cap W.$$

Using $\text{aff}(C_S(K)) = v + W$, we get

$$v + \frac{\varepsilon}{\sqrt{\alpha}\|L(c)\|} w^c \in ((v + B(0, \varepsilon)) \cap \text{aff}(C_S(K))),$$

and by (48),

$$v + \frac{\varepsilon}{\sqrt{\alpha}\|L(c)\|} w^c \in K.$$

Therefore, (51) implies

$$\lfloor c_K \rfloor \geq \max\{cx \mid x \in K\} - 1 \geq cv + \frac{\varepsilon}{\sqrt{\alpha}\|L(c)\|} cw^c - 1 = cv + \frac{\varepsilon}{\sqrt{\alpha}} \|L(c)\| - 1.$$

Note that for $\|L(c)\| \geq \sqrt{\alpha}/\varepsilon$, the GC cut associated with c does not separate v. Because of (50), there exists for each $j \in [k]$ an integer $q_j \geq 1$ such that $L_j(c)$ is an integral multiple of $1/q_j$. Therefore, the number of vectors $L(c) \in \mathbb{R}^k$ with $\|L(c)\| < \sqrt{\alpha}/\varepsilon$ is finite. However, there is an infinite number of integral vectors c in $\mathbb{R}^n$ that are mapped to the same rational vector $L(c)$ in $\mathbb{R}^k$. Let $\mathscr{A}$ denote the set of rational vectors $a \in \mathbb{R}^k$ such that for each $j \in [k]$, a_j is an integral multiple of $1/q_j$ and such that $\|a\| < \sqrt{\alpha}/\varepsilon$. For every $a \in \mathscr{A}$, we define $N(a) := \{c \in \mathbb{Z}^n \mid L(c) = a\}$. Let

$$c^a \in \arg\min_{c \in N(a)} \{\lfloor c_K \rfloor - cv\}.$$

Observe, that c^a is well defined: Since $v \in K$, we have for any $c \in N(a)$,

$$\lfloor c_K \rfloor - cv \geq \max\{cx \mid x \in K\} - 1 - cv \geq -1.$$

Furthermore, because v is a vertex of the rational polytope $C_S(K)$, it holds that $v \in \mathbb{Q}^n$. Hence, there exist an integer vector $\bar{v} \in \mathbb{Z}^n$ and an integer $q_v \geq 1$ such that $v = \bar{v}/q_v$. Consequently, the set $\{\lfloor c_K \rfloor - cv \mid c \in N(a)\}$ contains only multiples of $1/q_v$ and is bounded from below.

Finally, observe that the GC cut $c^a x \leq \lfloor c^a_K \rfloor$ dominates every other GC cut associated with a vector in $N(a)$ in $\operatorname{aff}(C_S(K))$: For this, consider an arbitrary point $x \in \operatorname{aff}(C_S(K))$ that satisfies $c^a x \leq \lfloor c^a_K \rfloor$. We can write $x = v + w$, for some $w = (w_1, \ldots, w_n) \in W$. Using (51), we get

$$c^a x = c^a v + c^a w = c^a v + L(c^a)(w_1, \ldots, w_k) = c^a v + a(w_1, \ldots, w_k) \leq \lfloor c^a_K \rfloor;$$

that is,

$$a(w_1, \ldots, w_k) \leq \lfloor c^a_K \rfloor - c^a v.$$

By the definition of c^a it follows that for every $c \in N(a)$,

$$cx = cv + cw = cv + a(w_1, \ldots, w_k) \leq cv + \lfloor c_K \rfloor - cv = \lfloor c_K \rfloor.$$

That is, if x satisfies the GC cut $c^a x \leq \lfloor c^a_K \rfloor$, it also satisfies every other GC cut $cx \leq \lfloor c_K \rfloor$ such that $c \in N(a)$. Consequently, for each vector $a \in \mathscr{A}$, we only need to consider a single GC cut. Since $|\mathscr{A}|$ is finite, this completes the proof. ☐

5.4. Step 4. We are finally prepared to prove the main result of this paper. By drawing from the insights of the previous three subsections and using an inductive argument, we prove that the GC closure of any polytope is a rational polytope.

Theorem 5.1. *The GC closure P' of a nonrational polytope P is a rational polytope.*

Proof. The proof is by induction on the dimension $d \leq n$ of $P \subseteq \mathbb{R}^n$. Let $n \geq 1$ be arbitrary. The base case, $d = 0$, is trivially true. Therefore, assume that $d \geq 1$. By Corollary 5.3, we know that there exists a finite set $S_1 \subseteq \mathbb{Z}^n$ such that

$$C_{S_1}(P) \subseteq P.$$

Let $\{F_i\}_{i \in I}$ denote the set of facets of P and assume that $F^i = P \cap (a^i x = a^i_P)$. By the induction assumption for $d - 1$, we know that F_i' is a rational polytope for every $i \in I$. That is, there exists a finite set $S_i \subseteq \mathbb{Z}^n$ such that $C_{S_i}(F_i) = F_i'$. According to Lemma 5.4, we can find for every GC cut for F_i that is facet defining for F_i' a GC cut for P that has the same impact on the maximal rational affine subspace of $(a^i x = a^i_P)$. Furthermore, by Corollary 5.2, F_i' is contained in this rational affine subspace. Hence, for every $i \in I$, there exists a finite set $\bar{S}_i \subseteq \mathbb{Z}^n$ such that $C_{\bar{S}_i}(P) \cap F_i = F_i'$. Because $\operatorname{rbd}(P) = \bigcup_{i \in I} F_i$, the set $S = S_1 \cup (\bigcup_{i \in I} \bar{S}_i)$ satisfies $C_S(P) \subseteq P$ and $C_S(P) \cap \operatorname{rbd}(P) \subseteq P'$. By Lemma 5.5, P' is a rational polytope. ☐

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