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On the Complexity of Pure-Strategy Nash Equilibria in Congestion and Local-Effect Games

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Rosenthal's congestion games constitute one of the few known classes of noncooperative games possessing pure-strategy Nash equilibria. In the network version, each player wants to route one unit of flow on a single path from her origin to her destination at minimum cost, and the cost of using an arc depends only on the total number of players using that arc. A natural extension is to allow for players controlling different amounts of flow, which results in so-called weighted congestion games. While examples have been exhibited showing that pure-strategy Nash equilibria need not exist anymore, we prove that it is actually strongly NP-hard to determine whether a given weighted network congestion game has a pure-strategy Nash equilibrium. This is true regardless of whether flow is unsplittable or not. In the unsplittable case, the problem remains strongly NP-hard for a fixed number of players. In addition to congestion games, we provide complexity results on the existence and computability of pure-strategy Nash equilibria for the closely related family of bidirectional local-effect games. Therein, the cost of a player taking a particular action depends not only on the number of players choosing the same action, but also on the number of players settling for (locally) related actions.

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1. Introduction. Game theory in general, and the concept of Nash equilibrium in particular, has lately (re)emerged as a "hot topic" in the operations research and computer science literature. The complexity of computing a mixed Nash equilibrium of a finite game given in strategic form is a case in point. Goldberg and Papadimitriou [20] showed that finding a mixed Nash equilibrium in a game with a constant number of players can be reduced to solving a four-player game. Daskalakis et al. [10] showed in turn that the latter problem is PPAD-complete, i.e., it is as difficult as computing a Brouwer fix point of a continuous function from the closed unit ball to itself. Subsequently, Chen and Deng [7] and Daskalakis and Papadimitriou [9] proved that computing mixed Nash equilibria in games with three players is PPAD-complete as well. Eventually, Chen and Deng [8] established the same result for the two-player case.

Although Nash [30] showed that mixed Nash equilibria do exist in any finite noncooperative game, it is well known that pure-strategy Nash equilibria may not, as demonstrated by classical games such as "matching pennies" (e.g., Shor [35]). It is therefore natural to ask which games have pure-strategy Nash equilibria and, if applicable, how difficult it is to find one. In this article, we study these questions for certain classes of weighted congestion games and local-effect games.

Congestion games were introduced by Rosenthal [31], who proved that they are guaranteed to possess pure-strategy Nash equilibria. In fact, Monderer and Shapley [29] showed that every exact potential game is isomorphic to a congestion game. In a congestion game, a player's strategy consists of a subset of resources, and her cost depends only on the number of players choosing the same resources. An important subclass of congestion games can be represented by means of networks.[1] Every player wants to route one unit of flow from her origin to her destination, on a path of minimal cost. The network arcs are the resources, and a player's set of pure strategies consists of the sets of arcs corresponding to paths connecting her origin-destination pair. Fabrikant et al. [12] studied the computational complexity of finding pure-strategy Nash equilibria in congestion games. For symmetric network congestion games, where all players have the same origin-destination pair, they presented a polynomial-time algorithm for computing a pure-strategy Nash equilibrium. On the other hand, they proved that this problem is PLS-complete for symmetric congestion games as well as for asymmetric network

[1] Network congestion games are particularly interesting from a computational point of view because players' strategies can be encoded compactly.

congestion games. A simpler proof of the latter result was given by Ackermann et al. [1], who also showed that this result still holds for affine cost functions.

In weighted congestion games, players control different, integral amounts of flow. Depending on whether players are allowed to split their flows or not, a player's strategy consists of a set of paths with corresponding integer flow values between her origin-destination pair, or a single path. Libman and Orda [25] constructed a simple instance of an unsplittable weighted network congestion game that does not possess a pure-strategy Nash equilibrium. A similar example was presented by Fotakis et al. [16], who also observed that for the special case of affine cost functions, a pure-strategy Nash equilibrium is always guaranteed to exist. Awerbuch et al. [5] derived a tight bound of $(\sqrt{5}+3)/2$ on the pure price of anarchy for this special case. The pure price of anarchy is the ratio of the cost of a worst pure-strategy Nash equilibrium to that of a globally optimal solution. Awerbuch et al. also gave upper bounds for instances with polynomial cost functions of degree greater than one. Tight bounds for this case were later provided by Aland et al. [3]. Goemans et al. [19] showed that a pure-strategy Nash equilibrium need not exist for instances with cost functions that are polynomials of degree of at most two. Milchtaich [27] had earlier shown that weighted congestion games with player-specific cost functions on networks consisting only of parallel arcs do not always have a pure-strategy Nash equilibrium either. This was elaborated on by Gairing et al. [17], who considered different cost functions, slight modifications in the network topology, and both the weighted and the unweighted cases. Milchtaich [28] characterized topological properties of networks that guarantee the existence of pure-strategy Nash equilibria in network congestion games if players control different amounts of flow or cost functions are player specific.

In this article, we prove that the problem of deciding whether a weighted network congestion game with simple, nonlinear cost functions possesses a pure-strategy Nash equilibrium is strongly NP-hard, regardless of whether one considers splittable or unsplittable flows. In the unsplittable case, we are able to show that the problem remains strongly NP-complete even if the number of players is fixed, or if all players have the same origin and destination. We also establish strong NP-completeness for weighted congestion games with affine player-specific cost functions on networks consisting of parallel arcs only.

Leyton-Brown and Tennenholtz [24] introduced local-effect games to model situations in which the use of one resource can affect the cost of using other resources. Local-effect games are, in general, not guaranteed to possess pure-strategy Nash equilibria. However, Leyton-Brown and Tennenholtz showed that so-called bidirectional local-effect games with linear local-effect functions belong to the class of exact potential games, and therefore always have pure-strategy Nash equilibria. The question of whether there exists a polynomial-time algorithm for finding a pure-strategy Nash equilibrium for these games was left open.

We prove that computing a pure-strategy Nash equilibrium is, in fact, PLS-complete. Because the proof uses a tight PLS-reduction, our result implies the existence of instances of bidirectional local-effect games with linear local-effect functions that have exponentially long shortest improvement paths. It also implies that the problem of computing a pure-strategy Nash equilibrium that is reachable from a given strategy state via selfish improvement steps is PSPACE-hard. In addition, we show that, given an initial strategy profile for a bidirectional local-effect game with linear local-effect functions and a positive integer k (unarily encoded), it is strongly NP-complete to decide whether there is a sequence of at most k selfish steps that transforms the initial state into a pure-strategy Nash equilibrium. We also prove that the problem of deciding whether a bidirectional local-effect game with general local-effect functions has a pure-strategy Nash equilibrium is strongly NP-complete.

Before we present the details of our results on weighted congestion games and local-effect games in §§3 and 4, respectively, we conclude this introduction by briefly discussing additional related work on the computational complexity of pure-strategy Nash equilibria. Gottlob et al. [21] considered restrictions of strategic games intended to capture certain aspects of bounded rationality. Among other results, they proved that even in the setting where each player's payoff function depends on the actions of at most three other players and where each player's strategy set consists of at most three actions, the problem of determining whether a strategic game has a pure-strategy Nash equilibrium is NP-complete. This result was strengthened by Fischer et al. [13], who showed that the problem remains NP-hard even if each player has only two actions to choose from and her payoff depends on the actions of at most two other players. Àlvarez et al. [4] studied how various representations of a strategic game influence the computational complexity of deciding the existence of a pure-strategy Nash equilibrium. They showed that this problem is NP-complete when the number of players is large and the number of strategies for each player is constant, whereas the problem is $\sum_2^p$-complete when the number of players is constant and the sizes of the strategy sets are exponential (with respect to the lengths of the strategies). Schoenebeck and Vadhan [34] analyzed the computational complexity of deciding whether pure-strategy Nash equilibria exist in graph games and circuit games. Brandt et al. [6] studied the impact of various notions of symmetry in strategic games on the computational complexity of finding pure-strategy Nash equilibria. Expanding on a line of research started by Ieong et al. [22], who considered singleton congestion games, Ackermann et al. [1] proved that the lengths of all best-response sequences are polynomially bounded in the number of players and resources, in congestion games where the strategy space of each player consists of the bases of a matroid over the set of resources. This especially implies that pure-strategy Nash equilibria for congestion games with this matroid property can be computed in polynomial time, even in the case of player-specific costs (Ackermann et al. [2]). In the latter paper, Ackermann et al. also showed the existence of pure-strategy Nash equilibria in weighted congestion games with the same matroid property.

2. Preliminaries.

Noncooperative games. A *strategic game* is defined by a set N of n players, a finite set of actions S_i for each player $i \in N$, and a payoff or utility function u_i for each player mapping $S := \prod_{i\in N} S_i$ to $\mathbb{Q}$. The set S is called the strategy or action space of the game, and its elements are the pure-strategy states. A *pure-strategy Nash equilibrium* of a strategic game is a state $s = (s_1, s_2, \ldots, s_n) \in S$ such that for each player $i \in N$, $u_i(s) \geq u_i(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n)$, for all $s_i' \in S_i$. Thus, no player can benefit from changing her strategy while the other players retain their current strategies.

Although every game has a Nash equilibrium if players are allowed to randomize over their set of pure actions (Nash [30]), pure-strategy Nash equilibria are, in general, not guaranteed to exist. A fundamental class of strategic games, which always have a pure-strategy Nash equilibrium, are *potential games*. Every game in this class is characterized by the existence of a *potential function* that associates with each strategy profile a real number such that the change in the potential function value of two states differing only in a single player's strategy is positive if and only if the difference in payoff to this particular player is positive. A potential function is *exact* if these two values always coincide.

Congestion games. An *unweighted congestion game* is defined by a set of players $N = \{1, 2, \ldots, n\}$ and a set of resources E. For each player $i \in N$, her set of available strategies is a collection of subsets of the resources; i.e., $S_i \subseteq 2^E$. A nondecreasing cost function $f_e\colon \mathbb{N} \to \mathbb{Q}_{\geq 0}$ is associated with each resource $e \in E$. Given a strategy profile $s = (s_1, s_2, \ldots, s_n) \in S$, the cost of player i is $c_i(s) = -u_i(s) = \sum_{e\in s_i} f_e(n_e(s))$, where $n_e(s)$ denotes the number of players using resource e in s. In other words, in a congestion game each player chooses a subset of resources that are available to her; and the cost to a player is the sum of the costs of the resources used by her, where the cost of a resource depends only on the total number of players sharing this resource.

A *network congestion game* is a congestion game in which the arcs of an underlying directed network represent the resources. Each player $i \in N$ has an origin-destination pair (r_i, t_i), where r_i and t_i are nodes of the network, and the set S_i of pure strategies available to player i is the set of directed (simple) paths from r_i to t_i.

In a *weighted network congestion game*, each player $i \in N$ has a positive integer weight w_i, which constitutes the amount of flow that player i wants to ship from r_i to t_i. In the case of unsplittable flows, the cost of player i adopting strategy s_i in a strategy profile $s = (s_1, s_2, \ldots, s_n) \in S$ is given by $c_i(s) = \sum_{e\in s_i} f_e(\theta_e(s))$, where $\theta_e(s) = \sum_{i\colon e\in s_i} w_i$ denotes the total flow on arc e in s. In integer-splittable network congestion games, a player with weight greater than one can choose a subset of paths on which to route her flow simultaneously; that is, player i's strategy consists of the specification of the r_i-t_i-paths used and the integer amounts of flow routed on them, which sum up to w_i. The corresponding cost is the total cost of the paths that player i uses, weighted by the respective amounts of flow player i routes on them.

An (un)weighted network congestion game is called *symmetric* or a *single-commodity* game if all players have the same origin-destination pair.

In terms of the input size of a weighted network congestion game, we assume that the cost functions are explicitly specified; that is, for each integer value θ with $0 \leq \theta \leq \sum_{i\in N} w_i$ and each arc e, the value $f_e(\theta)$ is given in binary encoding.[2]

Local-effect games. In a *local-effect game* with player set $N = \{1, 2, \ldots, n\}$, all players have the same set of available actions, $\mathcal{A}$. For each action $a \in \mathcal{A}$, there is a nondecreasing cost function $f_a\colon \mathbb{N} \to \mathbb{Q}_{\geq 0}$ that depends merely on the number of players who play this action. Furthermore, for each pair of actions $a, a' \in \mathcal{A}$, $a \neq a'$, a function $f_{a',a}\colon \mathbb{N} \to \mathbb{Q}_{\geq 0}$ expresses the impact of action a' on the cost of action a. Its value depends only on the

[2] Although more compact encodings are often possible, this assumption leads to stronger hardness results, which are the main concern of this paper.

number of players that choose action a'. The functions $f_{a',a}$ are called *local-effect functions*, and it is assumed that $f_{a',a}(0) = 0$. Moreover, local-effect functions are either strictly increasing or identical to zero. For a given strategy state $s = (s_1, s_2, \ldots, s_n) \in \mathcal{A}^n$, $n_a(s)$ denotes the number of players playing action a in s. The cost to a player $i \in N$ for playing action s_i in strategy state s is given by $c_i(s) = f_{s_i}(n_{s_i}(s)) + \sum_{a\in\mathcal{A},\, a\neq s_i} f_{a,s_i}(n_a(s))$. If the local-effect functions $f_{a',a}$ are zero for all $a \neq a'$, the local-effect game is equivalent to a symmetric network congestion game with parallel arcs. A local-effect game is called a *bidirectional* local-effect game if, for all $a, a' \in \mathcal{A}$, $a \neq a'$, and for all $x \in \mathbb{N}$, $f_{a',a}(x) = f_{a,a'}(x)$.

PLS. The complexity class *PLS* was introduced by Johnson et al. [23] to characterize the computational complexity of local search problems. A combinatorial optimization problem Π consists of a collection of instances $(\mathcal{F}, c)$, where $\mathcal{F}$ denotes the set of feasible solutions and $c\colon \mathcal{F} \to \mathbb{Q}$ is the objective function. A combinatorial optimization problem Π together with a neighborhood function $N\colon \mathcal{F} \to 2^{\mathcal{F}}$ belongs to PLS if (a) instances are recognizable in polynomial time and a feasible solution can be computed efficiently, (b) the feasibility of a proposed solution can be checked efficiently and its objective function value can be evaluated in polynomial time, and (c) the neighborhood of a feasible solution can be searched in polynomial time to determine a better feasible solution, if one exists. The computational problem associated with a local search problem is to find, for a given instance $(\mathcal{F}, c)$, a locally optimal solution w.r.t. the neighborhood function N, i.e., an $s \in \mathcal{F}$ such that there is no solution in the neighborhood of s with strictly better cost. A local search problem L_2 in PLS is *PLS-complete* if, for any other problem L_1 in PLS, there are polynomial-time computable functions ϕ and ψ such that ϕ maps instances x of L_1 to instances $\phi(x)$ of L_2, ψ maps solutions of $\phi(x)$ to solutions of x, and if s is a locally optimal solution for the instance $\phi(x)$ of L_2, then $\psi(s, x)$ is a locally optimal solution for x. Such a reduction is called *tight* if for any instance x of L_1 one can identify a subset X of feasible solutions of $\phi(x)$ so that (a) X contains all local optima of $\phi(x)$, (b) for every solution f of x one can construct in polynomial time a solution $s \in X$ such that $\psi(s, x) = f$, and (c) if the transition graph of $\phi(x)$ contains a directed path from $s \in X$ to $s' \in X$ whose internal nodes are not in X, then either $\psi(s, x) = \psi(s', x)$ or the transition graph of x contains an arc from $\psi(s, x)$ to $\psi(s', x)$ (Schäffer and Yannakakis [33]). In particular, the length of a longest path from any solution to a locally optimal solution in the transition graph of L_2 is at least as large as that in the transition graph of L_1.

3. Complexity of weighted congestion games. We begin by giving a high-level description of the common idea that forms the basis of our NP-hardness proofs for the various classes of games. In each case, we take a counterexample, i.e., an instance that does not have a pure-strategy Nash equilibrium, and couple it with an instance of the same class in which the strategy profiles correspond to the feasible solutions in a given instance of an NP-complete problem. We also introduce an additional player who can participate in either game. All other players are limited by cost or structure to participate in "their" part of the game only. The participation of the extra player in the counterexample turns that game into one that has a pure-strategy Nash equilibrium. Therefore, the entire game has a pure-strategy Nash equilibrium if and only if the part of the game corresponding to the NP-complete problem has a pure-strategy Nash equilibrium that prevents the extra player from joining the game, which happens if and only if it corresponds to a Yes-instance. The only deviation from this proof scheme occurs in the case of weighted network congestion games with a fixed number of players. Instead of introducing another player, one of the two players from the counterexample is given access to the other part of the game, which she will be able to take advantage of if and only if the state in this part of the game corresponds to a solution of a Yes-instance.

3.1. Unsplittable flows. Libman and Orda [25] presented an example of a weighted network congestion game with general nondecreasing arc-cost functions that does not have a pure-strategy Nash equilibrium. Fotakis et al. [16] provided a similar instance. We simplify the latter instance and turn it into a gadget to derive the following result.

Theorem 3.1. *The problem of deciding whether a weighted symmetric network congestion game with unsplittable flows possesses a pure-strategy Nash equilibrium is strongly NP-complete.*

Proof. The proof is by reduction from 3-Partition, which is strongly NP-complete (Garey and Johnson [18]). Consider an arbitrary instance of 3-Partition: a finite set $A = \{1, 2, \ldots, 3m\}$ of items ($m \geq 2$), a number $B \in \mathbb{N}$, and a positive integer weight w_i for each item $i \in A$ such that $B/4 < w_i < B/2$ and $\sum_{i\in A} w_i = mB$. We will construct a weighted single-commodity network congestion game that has a pure-strategy Nash equilibrium if and only if A can be partitioned into m disjoint sets $A_1, A_2, \ldots, A_m$ such that $\sum_{i\in A_k} w_i = B$ for $1 \leq k \leq m$.

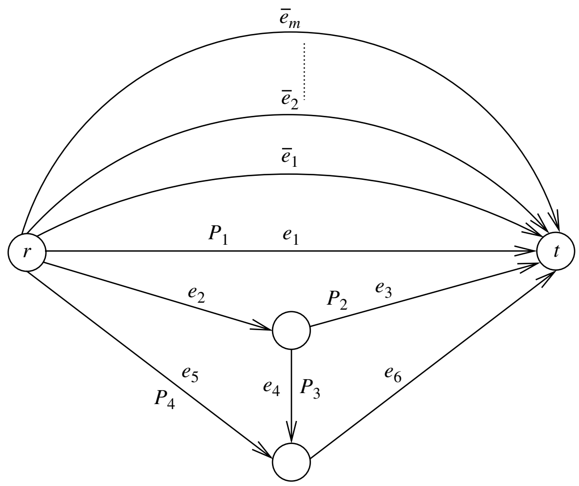

FIGURE 1. Illustration of the weighted single-commodity network congestion game used in the proof of Theorem 3.1.

We introduce a player p_i for each item $i \in A$; the corresponding weight is w_i. In addition, there are three players p_{3m+1}, p_{3m+2}, and p_{3m+3} with weights $w_{3m+1} = \bar{B}$, $w_{3m+2} = 2\bar{B}$, and $w_{3m+3} = \bar{B}/2$, respectively. Here, $\bar{B} := 2mB$. All players have the same origin, r, and destination, t. The network is depicted in Figure 1. It consists of a contracted version of the example by Fotakis et al. [16] and m additional arcs $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_m$, connecting r and t. We denote the r-t-paths in the lower part of the network by $P_1 = (e_1)$, $P_2 = (e_2, e_3)$, $P_3 = (e_2, e_4, e_6)$, and $P_4 = (e_5, e_6)$.

The nondecreasing arc-cost functions are defined as follows:

$$f_{\bar{e}_k}(x) := \begin{cases} x, & \text{if } x < (m+1)B, \\ 240\bar{B}, & \text{otherwise,} \end{cases} \quad \text{for } k = 1, \ldots, m,$$

$$f_{e_1}(x) := \begin{cases} 12\bar{B}, & \text{if } x \le \bar{B}, \\ 120\bar{B}, & \text{if } \bar{B} < x \le 2\bar{B}, \\ 228\bar{B}, & \text{otherwise,} \end{cases}$$

$$f_{e_2}(x) := \begin{cases} 1\bar{B}, & \text{if } x \le \bar{B}, \\ 2\bar{B}, & \text{if } \bar{B} < x \le 2\bar{B}, \\ 8\bar{B}, & \text{otherwise,} \end{cases}$$

$$f_{e_3}(x) := \begin{cases} 16\bar{B}, & \text{if } x \le \bar{B}, \\ 18\bar{B}, & \text{if } \bar{B} < x \le 2\bar{B}, \\ 20\bar{B}, & \text{otherwise,} \end{cases}$$

$$f_{e_4}(x) := \begin{cases} 1\bar{B}, & \text{if } x \le \bar{B}, \\ 40\bar{B}, & \text{if } \bar{B} < x \le 2\bar{B}, \\ 79\bar{B}, & \text{otherwise,} \end{cases}$$

$$f_{e_5}(x) := \begin{cases} 10\bar{B}, & \text{if } x \le \bar{B}, \\ 12\bar{B}, & \text{if } \bar{B} < x \le 2\bar{B}, \\ 14\bar{B}, & \text{otherwise,} \end{cases}$$

$$f_{e_6}(x) := \begin{cases} 2\bar{B}, & \text{if } x \le \bar{B}, \\ 10\bar{B}, & \text{if } \bar{B} < x \le 2\bar{B}, \\ 12\bar{B}, & \text{otherwise.} \end{cases}$$

TABLE 1. Possible defections in the subgame defined by the subnetwork of the network displayed in Figure 1 that consists of arcs $e_1, e_2, \ldots, e_6$ and players p_{3m+1} and p_{3m+2} only.

s_{3m+1}	s_{3m+2}	Deviator → New strategy	Current cost/$\bar{B}$	Improved cost/$\bar{B}$
P_1	P_1	$p_{3m+1} \to P_3$	$c_{3m+1} = 228$	$c'_{3m+1} = 4$
P_1	P_2	$p_{3m+1} \to P_3$	$c_{3m+1} = 12$	$c'_{3m+1} = 11$
P_1	P_3	$p_{3m+2} \to P_2$	$c_{3m+2} = 52$	$c'_{3m+2} = 20$
P_1	P_4	$p_{3m+2} \to P_2$	$c_{3m+2} = 22$	$c'_{3m+2} = 20$
P_2	P_1	$p_{3m+2} \to P_2$	$c_{3m+2} = 120$	$c'_{3m+2} = 28$
P_2	P_2	$p_{3m+1} \to P_1$	$c_{3m+1} = 28$	$c'_{3m+1} = 12$
P_2	P_3	$p_{3m+1} \to P_1$	$c_{3m+1} = 24$	$c'_{3m+1} = 12$
P_2	P_4	$p_{3m+1} \to P_1$	$c_{3m+1} = 17$	$c'_{3m+1} = 12$
P_3	P_1	$p_{3m+2} \to P_2$	$c_{3m+2} = 120$	$c'_{3m+2} = 26$
P_3	P_2	$p_{3m+2} \to P_4$	$c_{3m+2} = 26$	$c'_{3m+2} = 24$
P_3	P_3	$p_{3m+1} \to P_1$	$c_{3m+1} = 99$	$c'_{3m+1} = 12$
P_3	P_4	$p_{3m+1} \to P_1$	$c_{3m+1} = 14$	$c'_{3m+1} = 12$
P_4	P_1	$p_{3m+2} \to P_2$	$c_{3m+2} = 120$	$c'_{3m+2} = 20$
P_4	P_2	$p_{3m+1} \to P_3$	$c_{3m+1} = 12$	$c'_{3m+1} = 11$
P_4	P_3	$p_{3m+1} \to P_1$	$c_{3m+1} = 22$	$c'_{3m+1} = 12$
P_4	P_4	$p_{3m+1} \to P_1$	$c_{3m+1} = 26$	$c'_{3m+1} = 12$

Suppose there is a partition $A_1, A_2, \ldots, A_m$ of A such that $\sum_{i \in A_k} w_i = B$ for $1 \le k \le m$. Consider the strategy state s in which player p_i chooses the arc $\bar{e}_k$ such that item $i \in A_k$, and $s_{3m+1} = P_3$, $s_{3m+2} = P_4$, and $s_{3m+3} = P_1$. The cost to player p_i, for $i \in A$, is B. If such a player would route her flow on one of the other arcs $\bar{e}_\ell$, $\ell \neq k$, her cost would increase to $B + w_i$. By choosing a path P_k, $k \in \{1, 2, 3, 4\}$, this player's cost would increase to at least $4\bar{B}$. Player p_{3m+1} experiences a cost of $14\bar{B}$. A change to path P_1, P_2, or P_4 would result in a cost of $120\bar{B}$, $17\bar{B}$, or $26\bar{B}$. For player p_{3m+2}, the cost in state s is $24\bar{B}$. Switching to path P_1, P_2, or P_3 increases her cost to $228\bar{B}$, $26\bar{B}$, or $99\bar{B}$. Player p_{3m+3} has a cost of $12\bar{B}$. Routing her flow on path P_2, P_3, or P_4 would result in an increased cost of $18\bar{B}$, $54\bar{B}$, or $26\bar{B}$. Finally, every player p_{3m+1}, p_{3m+2}, and p_{3m+3} would increase her cost to $240\bar{B}$ by switching to some single-arc path $\bar{e}_k$, $k \in \{1, 2, \ldots, m\}$. Thus, no player can decrease her cost by routing flow over another path: s is a pure-strategy Nash equilibrium.

For the other direction, we claim that every pure-strategy Nash equilibrium s of this game has the following properties:

(a) Players p_{3m+1}, p_{3m+2}, and p_{3m+3} play a strategy in $\{P_1, P_2, P_3, P_4\}$.

(b) Every player p_i, $i \in \{1, 2, \ldots, 3m\}$, chooses an arc $\bar{e}_k$, for some $1 \le k \le m$.

Property (a) clearly holds for players p_{3m+1} and p_{3m+2}, for if one of these players chose an arc $\bar{e}_k$ for some $k \in \{1, 2, \ldots, m\}$, she would experience a cost of $240\bar{B}$. She could decrease her cost to at most $228\bar{B}$ by switching to some path P_k, $k \in \{1, 2, 3, 4\}$. For property (b), suppose there is a player p_i, $i \in \{1, 2, \ldots, 3m\}$, routing her flow on a path P_k, $k \in \{1, 2, 3, 4\}$. In this case, the cost of player p_i is at least $4\bar{B}$. Given that we have already established property (a) for players p_{3m+1} and p_{3m+2}, the total weight of players using an arc $\bar{e}_k$ in s is at most $2mB$. Therefore, there must exist some $k \in \{1, 2, \ldots, m\}$ such that $\sum_{i:\, s_i = \bar{e}_k} w_i \le 2B$. By switching to arc $\bar{e}_k$, player p_i can decrease her cost to, at most, $f_{\bar{e}_k}(2B + w_i) < 3B$. To show (a) for player p_{3m+3}, suppose that she uses an arc $\bar{e}_k$, $k \in \{1, 2, \ldots, m\}$. Then, only players p_{3m+1} and p_{3m+2} use one of the paths P_1, P_2, P_3, and P_4. However, the congestion game restricted to players p_{3m+1} and p_{3m+2} and strategies P_1, P_2, P_3, and P_4 does not have a pure-strategy Nash equilibrium—a contradiction.[3] Table 1 lists the 16 possible combinations that we need to consider to show that in each case at least one of the two players can decrease her cost by routing her flow on a different path. Hence, the only way that the entire game can have a pure-strategy Nash equilibrium is for player p_{3m+3} to play a strategy in $\{P_1, P_2, P_3, P_4\}$.

Given a pure-strategy Nash equilibrium s, we can now define a partition of A by setting $A_k := \{i \in A\colon s_i = \bar{e}_k\}$, $k = 1, 2, \ldots, m$. We claim that these sets define a solution to the 3-PARTITION problem. Suppose this is not the case. Then, because of (a), there exists an index $k \in \{1, 2, \ldots, m\}$ such that $\sum_{i:\, s_i = \bar{e}_k} w_i < B$. The current cost of player p_{3m+3}, using a path P_k, $k \in \{1, 2, 3, 4\}$, is at least $4\bar{B}$. By switching to arc $\bar{e}_k$, this player can decrease her cost to, at most, $f_{\bar{e}_k}(B - 1 + mB) = (m+1)B - 1$. This contradicts the assumption of s being a Nash equilibrium.

To complete the proof, we note that the problem of deciding whether a weighted network congestion game with unsplittable flows has a pure-strategy Nash equilibrium is in NP. Indeed, one can verify in polynomial time that a given strategy state is a Nash equilibrium by conducting a shortest-path computation for each player. □

[3] This subgame coincides with the contracted version of the instance by Fotakis et al. [16], to which we referred earlier.

While the NP-hardness of the corresponding decision problem for weighted network congestion games with player-specific payoff functions follows immediately, we can actually strengthen this result.

THEOREM 3.2. *The problem of deciding whether a weighted network congestion game with parallel arcs and affine player-specific cost functions possesses a pure-strategy Nash equilibrium is strongly NP-complete.*

PROOF. The problem is obviously in NP. To show NP-completeness, we reduce, as before, from 3-PARTITION. We are given a set $A = \{1, 2, \ldots, 3m\}$ of items, a number $B \in \mathbb{N}$, and a positive integer weight w_i for each item $i \in A$ such that $B/4 < w_i < B/2$ and $\sum_{i\in A} w_i = mB$. We will construct a weighted network congestion game with parallel arcs only and player-specific cost functions such that it has a pure-strategy Nash equilibrium if and only if A can be partitioned into m disjoint sets $A_1, A_2, \ldots, A_m$ such that $\sum_{i\in A_k} w_i = B$, for $1 \le k \le m$.

We introduce a player p_i for each item $i \in A$; the corresponding weight is w_i. There are four additional players p_{3m+i}, for $i = 1, 2, 3, 4$, with $w_{3m+1} = 1$, $w_{3m+2} = 2$, $w_{3m+3} = 3$, and $w_{3m+4} = 1$. All players want to ship flow from r to t in a network of parallel arcs $e_1, e_2, \ldots, e_{m+3}$ connecting r and t.

Let $f_{i,k}$ denote the cost function of player p_i for arc e_k, and let $K := 3(mB + 7) + 1$. For $i = 1, 2, \ldots, 3m$ and $k = 1, 2, \ldots, m + 3$, we define

$$f_{i,k}(x) := \begin{cases} x, & \text{if } k \in \{1, 2, \ldots, m\}, \\ K, & \text{otherwise.} \end{cases}$$

For the remaining players p_{3m+i}, $i \in \{1, 2, 3, 4\}$, we set

$$f_{3m+1,k}(x) := \begin{cases} K, & \text{if } k \in \{1, 2, \ldots, m+1\}, \\ 7, & \text{if } k = m+2, \\ 2x, & \text{if } k = m+3, \end{cases}$$

$$f_{3m+2,k}(x) := \begin{cases} K, & \text{if } k \in \{1, 2, \ldots, m+1\}, \\ 2x, & \text{if } k = m+2, \\ 5, & \text{if } k = m+3, \end{cases}$$

$$f_{3m+3,k}(x) := \begin{cases} K, & \text{if } k \in \{1, 2, \ldots, m\} \cup \{m+2\}, \\ 3x, & \text{if } k = m+1, \\ 2x, & \text{if } k = m+3, \end{cases}$$

$$f_{3m+4,k}(x) := \begin{cases} x, & \text{if } k \in \{1, 2, \ldots, m\}, \\ K, & \text{if } k \in \{m+1, m+2\}, \\ B+1, & \text{if } k = m+3. \end{cases}$$

Assume that we are given a YES-instance of the partition problem. Then there is a partition $A_1, A_2, \ldots, A_m$ of A such that $\sum_{i\in A_k} w_i = B$ for $1 \le k \le m$. Consider the strategy state s in which player p_i chooses the arc e_k with $i \in A_k$, $i = 1, 2, \ldots, 3m$, and $s_{3m+1} = e_{m+3}$, $s_{3m+2} = e_{m+2}$, $s_{3m+3} = e_{m+1}$, and $s_{3m+4} = e_{m+3}$. In s, each player p_i corresponding to an item $i \in A$ has a cost of B. Switching to a different arc in $\{e_1, e_2, \ldots, e_m\}$ increases her cost to $B + w_i$, and routing her flow on an arc in the set $\{e_{m+1}, e_{m+2}, e_{m+3}\}$ results in a cost of K, which is no improvement either. Player p_{3m+1} has a cost of four. The only other arc yielding a cost less than K is e_{m+2}. However, switching to this arc results in a higher cost of seven. Player p_{3m+2} has a cost of four in state s. Changing her strategy to e_{m+3} gives a new cost of five; all other arcs have cost K for this player. Player p_{3m+3} with current cost nine can decrease her cost neither by using arc e_{m+3}, which would yield a cost of 10, nor by taking one of the other arcs, which would result in a cost of K. Player p_{3m+4}'s cost is $B + 1$ in s. Switching to an arc e_k, for some $1 \le k \le m$, results in the same cost; all other arcs would increase her cost. Hence, s is a pure-strategy Nash equilibrium.

For the other direction of the proof, we first observe that any pure-strategy Nash equilibrium of the constructed game has the following properties:

(a) Each player p_i, $i \in \{1, 2, \ldots, 3m\}$, uses an arc in $\{e_1, e_2, \ldots, e_m\}$.
(b) None of the players p_{3m+1}, p_{3m+2}, p_{3m+3} plays a strategy in $\{e_1, e_2, \ldots, e_m\}$.
(c) Player p_{3m+4} plays strategy e_{m+3}.

TABLE 2. Possible defections in the subgame restricted to players p_{3m+1}, p_{3m+2}, p_{3m+3} and strategies e_{m+1}, e_{m+2}, e_{m+3}, as discussed in the proof of Theorem 3.2. This game is similar to instances described by Milchtaich [27].

s_{3m+1}	s_{3m+2}	s_{3m+3}	Defector → New strategy	Current cost	Improved cost
e_{m+2}	e_{m+2}	e_{m+1}	$p_{3m+1} \to e_{m+3}$	$c_{3m+1} = 7$	$c'_{3m+1} = 2$
e_{m+2}	e_{m+2}	e_{m+3}	$p_{3m+2} \to e_{m+3}$	$c_{3m+2} = 6$	$c'_{3m+2} = 5$
e_{m+2}	e_{m+3}	e_{m+1}	$p_{3m+1} \to e_{m+3}$	$c_{3m+1} = 7$	$c'_{3m+1} = 6$
e_{m+2}	e_{m+3}	e_{m+3}	$p_{3m+3} \to e_{m+1}$	$c_{3m+3} = 10$	$c'_{3m+3} = 9$
e_{m+3}	e_{m+2}	e_{m+1}	$p_{3m+3} \to e_{m+3}$	$c_{3m+3} = 9$	$c'_{3m+3} = 8$
e_{m+3}	e_{m+2}	e_{m+3}	$p_{3m+1} \to e_{m+2}$	$c_{3m+1} = 8$	$c'_{3m+1} = 7$
e_{m+3}	e_{m+3}	e_{m+1}	$p_{3m+2} \to e_{m+2}$	$c_{3m+2} = 5$	$c'_{3m+2} = 4$
e_{m+3}	e_{m+3}	e_{m+3}	$p_{3m+3} \to e_{m+1}$	$c_{3m+3} = 12$	$c'_{3m+3} = 9$

Properties (a) and (b) follow immediately from the fact that for any player there exists a strategy with cost strictly smaller than K, i.e., in any Nash equilibrium each player pays less than K. For property (c), we first observe that in any Nash equilibrium the only possible strategies for player p_{3m+4} are e_k, $1 \le k \le m$, and e_{m+3}. Suppose that p_{3m+4} does not use arc e_{m+3}. Then, by properties (a) and (b), only p_{3m+1}, p_{3m+2}, and p_{3m+3} play strategies in $\{e_{m+1}, e_{m+2}, e_{m+3}\}$. However, the congestion game restricted to these three players and strategies does not have a pure-strategy Nash equilibrium, yielding a contradiction. In fact, we only need to consider all possibilities for the three players to choose their strategies from $\{e_{m+1}, e_{m+2}, e_{m+3}\}$. We can exclude from the start all possibilities that imply a cost of K for one of the players. Eight possible combinations, which are listed in Table 2, remain to be considered. In consequence, the only way for the whole game to have a pure-strategy Nash equilibrium is if player p_{3m+4} uses arc e_{m+3}.

Given a pure-strategy Nash equilibrium s of the constructed game, we can now associate a partition of the item set with the m groups of players who route their flows on arcs e_k with $1 \le k \le m$. We claim that the so defined sets, i.e., $A_k := \{i \in A\colon s_i = e_k\}$, $1 \le k \le m$, form a solution to the 3-PARTITION problem. Suppose this is not true. Then there exists an index $k \in \{1, 2, \ldots, m\}$ such that the total weight of players routing their flows on e_k is at most $B - 1$. Consider player p_{3m+4}, who, by property (c), plays strategy e_{m+3} in any pure-strategy Nash equilibrium. Her current cost is $B + 1$. By switching to arc e_k, she can decrease her cost to at most B. However, this contradicts the assumption of s being a Nash equilibrium. Therefore, $\sum_{i \in A_k} w_i = B$ for $1 \le k \le m$. □

The following result shows that deciding the existence of pure-strategy Nash equilibria in asymmetric weighted network congestion games with unsplittable flows remains strongly NP-complete, even if the number of players is fixed. Note that this result does not render Theorem 3.1 obsolete, because that theorem dealt with symmetric games.

THEOREM 3.3. *The problem of deciding whether a weighted network congestion game with a fixed number of players has a pure-strategy Nash equilibrium is strongly NP-complete.*

PROOF. We reduce from ARC-DISJOINT PATHS: Given a directed graph $G = (N, A)$ and a set of node pairs $(r_1, t_1), (r_2, t_2), \ldots, (r_k, t_k)$, does there exist a collection of arc-disjoint paths $P_1, P_2, \ldots, P_k$, where P_i is an r_i-t_i-path? This problem is NP-complete, even in the case of only two terminal pairs (Fortune et al. [15]).

Let $G = (N, A)$, (r_1, t_1), and (r_2, t_2) be an instance of ARC-DISJOINT PATHS with two terminal pairs. We will construct a congestion game whose underlying network will consist of two building blocks. The first component is obtained from the original network G by replacing each arc $a \in A$ by a path consisting of three arcs $\bar{e}^1$, $\bar{e}^2$, and $\bar{e}^3$. We denote the resulting graph by G'. The second building block is the network that we already used as a gadget in the proof of Theorem 3.1. It consists of four different r-t-paths, which we will call Q_k here, for $k = 1, 2, 3, 4$. We will refer to it as the "lower" part of the new network. See Figure 2 for an illustration.

Both building blocks are connected as follows. Assume that $A = \{a_1, a_2, \ldots, a_m\}$. We introduce an arc $\bar{e}_0$ between r and the starting node of the second arc $\bar{e}_1^2$, which was one of three arcs replacing the original arc $a_1 \in A$. Furthermore, for $1 \le i \le m - 1$, we create an arc $\bar{e}_i$ connecting the end node of $\bar{e}_i^2$ with the start node of $\bar{e}_{i+1}^2$. Finally, there is an arc $\bar{e}_m$ from the end node of $\bar{e}_m^2$ to the terminal node t. We denote the r-t-path formed by the arcs $\bar{e}_k$, $k = 0, 1, \ldots, m$, and $\bar{e}_i^2$, $i = 1, 2, \ldots, m$, by Q_5.

The game has four players. Players p_1 and p_2 wish to route one unit of flow from r_1 to t_1 and from r_2 to t_2, respectively. Players p_3 and p_4 have weights $w_3 = m$ and $w_4 = 2m$, to be sent from r to t.

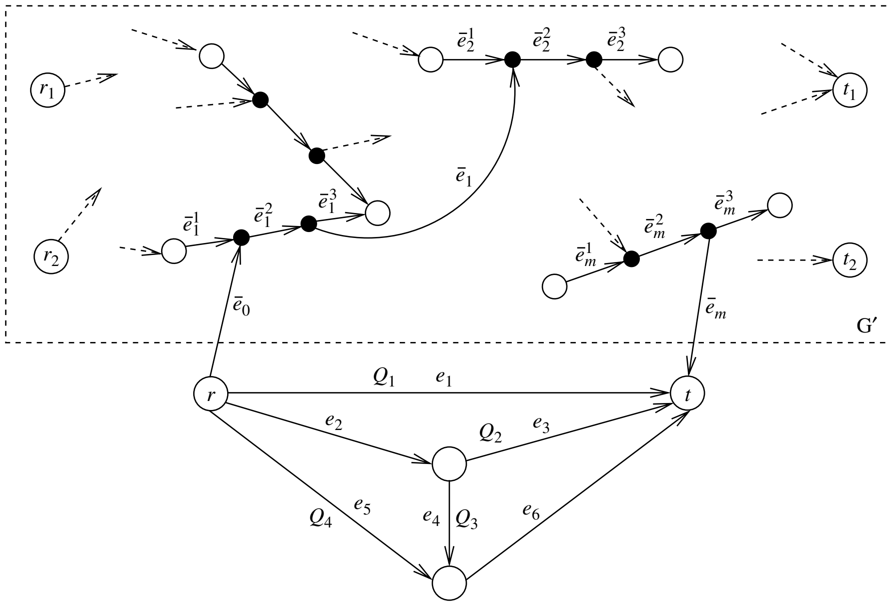

FIGURE 2. Illustration of the weighted network congestion game created in the proof of Theorem 3.3.

We define the cost of the arcs that replaced $a_i \in A$ as follows:

$$f_{\bar{e}_i^1}(x) := f_{\bar{e}_i^3}(x) := \begin{cases} 0, & \text{if } 0 \le x < m, \\ K, & \text{if } x \ge m, \end{cases}$$

$$f_{\bar{e}_i^2}(x) := \begin{cases} 0, & \text{if } 0 \le x < m+2, \\ K, & \text{if } x \ge m+2, \end{cases}$$

where $K := 237m + 1$. Moreover, for $0 \le i \le m$,

$$f_{\bar{e}_i}(x) := \begin{cases} 0, & \text{if } 0 \le x \le m, \\ K, & \text{if } x > m. \end{cases}$$

The cost functions f_{e_i}, for $i = 1, 2, \ldots, 6$, in the lower part of the network, are defined in the same way as the corresponding functions in the proof of Theorem 3.1; however, $\bar{B}$ is replaced by m.

Let us first assume that the given instance of ARC-DISJOINT PATHS is a YES-instance, i.e., there exist arc-disjoint paths P_1 and P_2 in G connecting r_1 and t_1, and r_2 and t_2, respectively. We will abuse notation and denote the corresponding paths in G' also by P_1 and P_2. Consider the strategy state s in which players p_1 and p_2 choose P_1 and P_2, respectively, whereas player p_3 uses Q_5, and p_4 routes her flow on Q_2. We claim that s is a Nash equilibrium. Because at most one of the players p_1 and p_2 uses a particular arc in G', the cost of any arc $\bar{e}_i^1, \bar{e}_i^2, \bar{e}_i^3$, $1 \le i \le m$, in s is zero. Therefore, p_1 and p_2 have zero cost and play optimal strategies. Similarly, the cost of player p_3 is zero, because in addition to the last observation, the total weight on any arc $\bar{e}_i$, $0 \le i \le m$, is m, i.e., the cost for using each of these arcs is zero. Finally, player p_4 uses the cheapest path in the lower part of the network. By routing her flow on any path sharing arcs with G', her cost would increase to at least K. Thus, no player can decrease her cost by routing flow over another path; s is indeed a pure-strategy Nash equilibrium.

Let us now assume that we are given a pure-strategy Nash equilibrium s of the constructed game. Because any arc in G', if used by player p_4, induces a cost of $K > 237m$, this player is always better off using a path in $\{Q_1, Q_2, Q_3, Q_4\}$. We further observe that there is no r_1-t_1-path or r_2-t_2-path that shares any arc with the lower part of the network, i.e., neither player p_1 nor player p_2 will use such an arc. However, if we restrict the game to players p_3 and p_4 and the lower part of the network, it follows from the same reasoning used in the proof of Theorem 3.1 that this subgame does not have a pure-strategy Nash equilibrium. Therefore, player p_3 has to choose a path intersecting G'. Arcs $\bar{e}_i^1$ and $\bar{e}_i^3$, $1 \le i \le m$, are very expensive if the load is greater than or equal to $w_3 = m$; hence, the only way for p_3 to route her flow in a Nash equilibrium is to use path Q_5. Because she cannot decrease her cost by switching to a path in the lower part of the network, the cost for using

Q_5 must be smaller than K. This implies that at most one of the players p_1 and p_2 uses an arc $\bar{e}_i^2$, $1 \le i \le m$ (otherwise, the total weight on such an arc would be $m+2$). Similarly, neither p_1 nor p_2 can use any arc $\bar{e}_k$ for $k \in \{0, 1, \ldots, m\}$. This, in turn, implies that both p_1 and p_2 only use G' to route their flows. Furthermore, the corresponding paths of these two players in G have to be disjoint, i.e., the Arc-Disjoint Paths instance is a Yes-instance. □

3.2. Integer-splittable flows. Rosenthal [32] gave an example of an asymmetric weighted network congestion game that does not have a pure-strategy Nash equilibrium if players are allowed to split their flows (see Figure 5). Interestingly, the same game possesses a pure-strategy Nash equilibrium if each player has to route her flow on a single path. The following result shows that one cannot efficiently decide the existence of pure-strategy Nash equilibria in network congestion games with integer-splittable flows, unless $\mathrm{P} = \mathrm{NP}$.

Theorem 3.4. *The problem of deciding whether a weighted network congestion game with integer-splittable flows possesses a pure-strategy Nash equilibrium is strongly NP-hard, even if there is only one player with weight two, and all other players have unit weights.*

Proof. The reduction is from Monotone 3Sat, which is known to be NP-complete (Garey and Johnson [18]). Consider an instance of Monotone 3Sat with set of variables $X = \{x_1, x_2, \ldots, x_n\}$ and set of three-variable clauses $C = \{c_1, c_2, \ldots, c_m\}$. Each clause contains either only negated variables or only unnegated variables.

We will create a game that has one player p_x for every variable $x \in X$ with weight $w_x = 1$, origin x, and destination $\bar{x}$. Moreover, each clause $c \in C$ gives rise to a player p_c with weight $w_c = 1$, origin c, and destination $\bar{c}$. There are three more players p_1, p_2, and p_3 with weights $w_1 = 1$, $w_2 = 2$, and $w_3 = 1$ and origin-destination pairs (r, t_1), (r, t_2), and (r, t_3), respectively. For every variable $x \in X$, the network contains two disjoint paths P_x^1 and P_x^0 from x to $\bar{x}$. Path P_x^0 consists of $2|\{c \in C \mid x \in c\}| + 1$ arcs, and P_x^1 has $2|\{c \in C \mid \bar{x} \in c\}| + 1$ arcs with cost functions as shown in Figure 3. For each origin-destination pair $(c, \bar{c})$, we introduce two disjoint paths P_c^1 and P_c^0 from c to $\bar{c}$. Path P_c^1 consists of only two arcs. The paths P_c^0 have seven arcs each and are constructed for $c = c_j$ in the order $j = 1, 2, \ldots, m$ as follows. For a positive clause $c = c_j = (x_{j_1} \vee x_{j_2} \vee x_{j_3})$ with $j_1 < j_2 < j_3$, path P_c^0 starts with the arc connecting c to the first inner node v_1 on path $P_{x_{j_1}}^1$ that has only two incident arcs so far. The second arc is the unique arc (v_1, v_2) of path $P_{x_{j_1}}^1$ that has v_1 as its start vertex. The third arc connects v_2 to the first inner node v_3 on path $P_{x_{j_2}}^1$ that has only two incident arcs so far. The fourth arc is the only arc (v_3, v_4) on $P_{x_{j_2}}^1$ with start vertex v_3. From v_4, there is an arc to the first inner node v_5 on $P_{x_{j_3}}^1$ that has only two incident arcs so far, followed by (v_5, v_6) of $P_{x_{j_3}}^1$. The last arc of P_c^0 connects v_6 to $\bar{c}$. Figure 3 illustrates this construction. For a negative clause $c = c_j = (\bar{x}_{j_1} \vee \bar{x}_{j_2} \vee \bar{x}_{j_3})$, we proceed in the same way, except that we choose the inner nodes v_i from the upper variable paths $P_{x_{j_1}}^0$, $P_{x_{j_2}}^0$, and $P_{x_{j_3}}^0$.

The strategy set of player p_x is $\{P_x^1, P_x^0\}$. We will interpret it as setting the variable x to *true* (*false*) if p_x sends her unit of flow over P_x^1 (P_x^0). Note that player p_c can only choose between the paths P_c^1 and P_c^0. This is due to the order in which the paths P_c^0 are constructed. Depending on whether player p_c sends her unit of flow over path P_c^1 or P_c^0, the clause c will be satisfied or not.

The second part of the network consists of all origin-destination pairs and paths for players p_1, p_2, and p_3 (see Figure 4). Player p_1 can choose between paths $U_1 = \{(r, t_2), (t_2, t_1)\}$ and $L_1 = \{(r, t_1)\}$. Player p_2 is the only

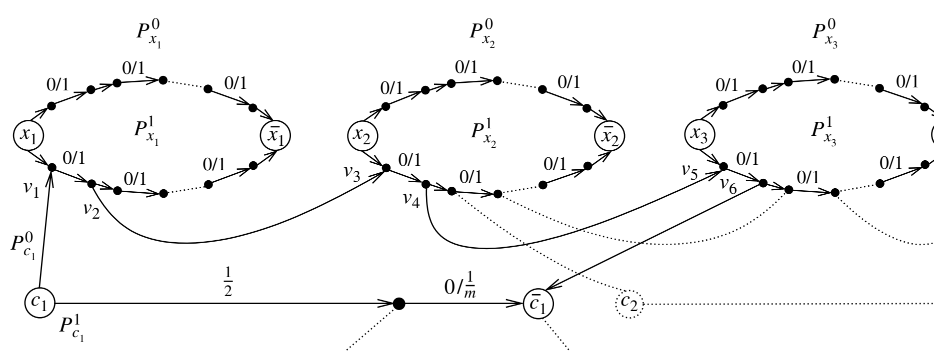

Figure 3. Part of the constructed network corresponding to a positive clause $c_1 = (x_1 \vee x_2 \vee x_3)$. The notation a/b defines a cost per unit flow of value a for load 1 and b for load 2. For any other arc, the cost does not depend on the amount of flow on that arc. Arcs without specified values have zero cost.

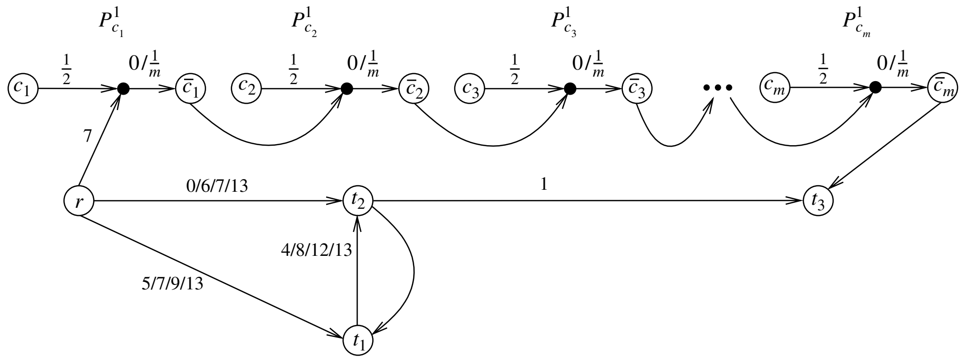

FIGURE 4. Part of the network used by players p_1, p_2, and p_3. A single number a on an arc defines a constant cost of a per unit flow for this arc. Unlabeled arcs have cost zero.

player who can split her flow; that is, she can route her two units either both over path $U_2 = \{(r, t_2)\}$, both over path $L_2 = \{(r, t_1), (t_1, t_2)\}$, or one unit on the upper and the other unit on the lower path; i.e., her strategy set is $S_2 = \{L_2, U_2, L_2U_2\}$. Finally, player p_3 has three possible paths from which to choose. The upper path U_3 shares an arc with each clause path P^1_c and has some additional arcs to connect these. The paths $M_3 = \{(r, t_2), (t_2, t_3)\}$ and $L_3 = \{(r, t_1), (t_1, t_2), (t_2, t_3)\}$ have only arcs with the paths of p_1 and p_2 in common. The cost functions are defined in Figure 4.

Given a satisfying truth assignment, we define a strategy state s of the game by setting the strategy of player p_x to be P^1_x for a true variable x, and P^0_x otherwise. Each player p_c plays P^1_c. Furthermore, $s_1 = L_1$, $s_2 = U_2$, and $s_3 = M_3$. It is easy to show that no player can decrease her cost by unilaterally switching to another strategy; i.e., the defined strategy configuration is a pure-strategy Nash equilibrium.

For the other direction, we first observe that any pure-strategy Nash equilibrium s has the following properties: (a) player p_3 does not use path U_3, (b) the cost of player p_3 is at least eight, and (c) each player p_c routes her unit flow over path P^1_c. Property (a) follows from the fact that the subgame shown in Figure 5 with players p_1 and p_2 only does not have a pure-strategy Nash equilibrium.[4] Thus, p_3 will use either the middle or the lower path. No matter how many other players use an edge of the lower path, the cost of p_3 using L_3 is at least 10. The only possibility for p_3 to face a cost strictly less than eight is if she uses the middle path and at most one additional unit of p_1 or p_2 is routed over arc (r, t_2). Let us consider the case $s_1 = L_1$, $s_2 = L_2$, and $s_3 = M_3$ first. Then p_2 has a cost of 34, and she can decrease her cost by switching to strategy U_2 with a new cost of 14. If $s_1 = L_1$, $s_2 = L_2U_2$, and $s_3 = M_3$, the cost of player p_2 is 17. Choosing strategy U_2 instead yields a lower cost of 14. The last case to consider is $s_1 = U_1$, $s_2 = L_2$, and $s_3 = M_3$. Then, player p_2 has a cost of 30, which can be decreased by switching to strategy L_2U_2, leading to a cost of 16. Consequently, property (b) holds for any pure-strategy Nash equilibrium of the game. For property (c), suppose there is a player p_c, $c \in C$, routing her unit flow on P^0_c. By (a) and (b), we know that p_3 uses either the lower or the middle path with a cost of at least eight. Consider a change of p_3 to the upper path U_3. Her new cost would be at most $7 + (m-1)(1/m) < 8$, which would contradict that s is a Nash equilibrium.

We claim that the truth assignment that sets a variable x to *true* if the corresponding player uses P^1_x, and x to *false* otherwise, satisfies all clauses. Suppose that all variables of a positive clause $c = (x_1 \vee x_2 \vee x_3)$ are *false*; i.e., $s_{x_i} = P^0_{x_i}$ for $i = 1, 2, 3$. By property (c), player p_c uses P^1_c. Because of (a), her current cost is 1/2. Choosing path P^0_c instead would decrease the cost to zero, which contradicts the assumption of s being a Nash equilibrium. A similar argument holds for a negative clause. □

Note that we have not claimed that the problem of deciding whether a weighted network congestion game with integer-splittable flows possesses a pure-strategy Nash equilibrium is in NP. Although this can be easily shown if all cost functions are convex, it follows from a result by Meyers and Schulz [26] that, in general, the problem of deciding whether a given strategy profile is a pure-strategy Nash equilibrium is in itself a co-NP-complete problem.

4. Bidirectional local-effect games. Leyton-Brown and Tennenholtz [24] presented a characterization of local-effect games that have an exact potential function and that are therefore guaranteed to possess pure-strategy Nash equilibria. One of these subclasses is that of bidirectional local-effect games with linear local-effect

[4] This subgame coincides with the instance originally conceived by Rosenthal [32], to which we alluded earlier.

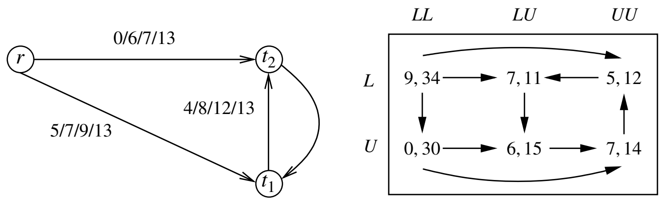

FIGURE 5. On the left: Subgame with two players without pure-strategy Nash equilibrium. On the right: Digraph of all strategy states and improving moves.
Note. The players' weights are one and two, respectively.

functions. However, without linear local-effect functions, deciding the existence of pure-strategy Nash equilibria is difficult.

THEOREM 4.1. *The problem of deciding whether a bidirectional local-effect game possesses a pure-strategy Nash equilibrium is strongly NP-complete.*

PROOF. The proof uses a reduction from 3-PARTITION. Consider an arbitrary instance of 3-PARTITION with finite set $A = \{1, 2, \ldots, 3m\}$ of items, a number $B \in \mathbb{N}$, and a positive integer weight w_i for each item $i \in A$ such that $B/4 < w_i < B/2$ and $\sum_{i\in A} w_i = mB$. We may assume that $B \geq 12$. We will construct a bidirectional local-effect game such that it has a pure-strategy Nash equilibrium if and only if A can be partitioned into m disjoint sets $A_1, A_2, \ldots, A_m$ such that $\sum_{i\in A_k} w_i = B$ for $1 \leq k \leq m$.

The action set $\mathcal{A}$ consists of $3m^2 + 2m + 3$ actions that are available to $4m + 3$ players. For each item $i \in A$, there is a set of m corresponding actions $a_i^1, a_i^2, \ldots, a_i^m$; we will make sure that in any Nash equilibrium of the game exactly one player will pick one of these actions, for each item. The remaining actions are denoted by d_j and $\bar{d}_j$, $j = 1, 2, \ldots, m$, and h_1, h_2, and h_3. We define the cost functions for actions $a \in \{a_i^j\colon i = 1, 2, \ldots, 3m,\ j = 1, 2, \ldots, m\} \cup \{d_j\colon j = 1, 2, \ldots, m\}$ as

$$f_a(x) := \begin{cases} 0, & \text{if } x \leq 1,\\ K, & \text{otherwise,}\end{cases}$$

where $K := 3(4m + 3)B + 18$. For $j = 1, 2, \ldots, m$,

$$f_{\bar{d}_j}(x) := \begin{cases} B, & \text{if } x \leq 1,\\ K, & \text{otherwise.}\end{cases}$$

Furthermore, we have

$$f_{h_1}(x) := f_{h_2}(x) := \begin{cases} 0, & \text{if } x = 0,\\ B+1, & \text{if } x = 1,\\ 2B+6, & \text{if } x = 2,\\ 3B+12, & \text{if } x = 3,\\ Bx, & \text{otherwise,}\end{cases}$$

$$f_{h_3}(x) := \begin{cases} 0, & \text{if } x = 0,\\ B+1, & \text{if } x = 1,\\ 2B+2, & \text{if } x = 2,\\ 3B+12, & \text{if } x = 3,\\ Bx, & \text{otherwise.}\end{cases}$$

The local-effect functions between actions h_1, h_2, and h_3 are given by

$$f_{h_1,h_2}(x) := \begin{cases} 0, & \text{if } x = 0,\\ B+1, & \text{if } x = 1,\\ 2B+4, & \text{if } x = 2,\\ Bx, & \text{otherwise,}\end{cases}$$

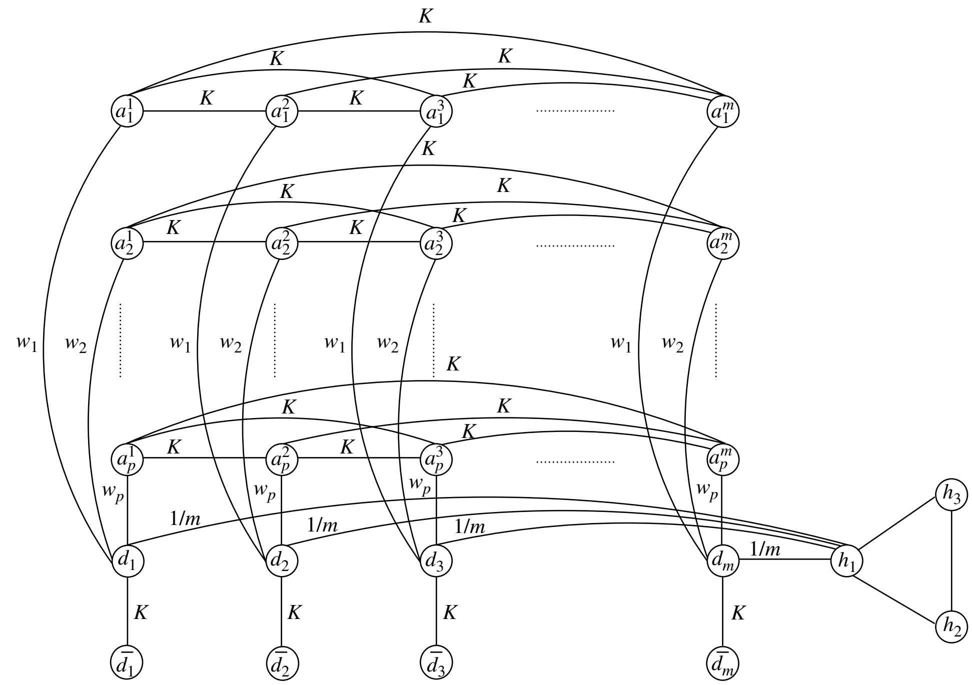

FIGURE 6. Illustration of the "local-effect graph" of the game constructed in the proof of Theorem 4.1.
Note. An arc label α corresponds to the coefficient of a linear local-effect function $f_{a,a'}(x) = \alpha x$. Moreover, $p := 3m$.

$$f_{h_1,h_3}(x) := \begin{cases} 0, & \text{if } x = 0, \\ B+3, & \text{if } x = 1, \\ 2B+4, & \text{if } x = 2, \\ Bx, & \text{otherwise,} \end{cases}$$

$$f_{h_2,h_3}(x) := \begin{cases} 0, & \text{if } x = 0, \\ B+1, & \text{if } x = 1, \\ 2B+5, & \text{if } x = 2, \\ Bx, & \text{otherwise.} \end{cases}$$

All other local-effect functions are defined in Figure 6.

Assume that the given 3-PARTITION instance is a YES-instance; that is, there is a partition $A_1, A_2, \ldots, A_m$ of the item set A such that $\sum_{i \in A_k} w_i = B$ for $1 \le k \le m$. Consider the following strategy state s of the local-effect game. For $i = 1, 2, \ldots, 3m$ and $j = 1, 2, \ldots, m$, let

$$n_{d_j}(s) = 1, \quad n_{\bar{d}_j}(s) = 0, \quad n_{h_1}(s) = 0, \quad n_{h_2}(s) = 1, \quad n_{h_3}(s) = 2, \qquad \text{and} \qquad n_{a_i^j}(s) = \begin{cases} 1, & \text{if } i \in A_j, \\ 0, & \text{otherwise.} \end{cases}$$

We show that no player can decrease her cost by switching to another action. First, a player choosing some action a_i^j pays a cost of w_i due to a local effect from action d_j. Switching to action a_i^k for some $k \neq j$ does not change the player's cost. Any other action apart from h_1, h_2, or h_3 would imply a cost of at least K. Switching to action h_1, h_2, or h_3 leads to a cost of at least $B + 1 > w_i$. Thus, none of these players can decrease her cost. Now consider a player with action d_j. Her cost is $\sum_{i \in A_j} w_i = B$. Switching to some action a_i^j, d_k, or $\bar{d}_k$ with $k \neq j$ increases her cost to at least K. Action $\bar{d}_j$ implies an equal cost of B. Switching to h_1, h_2, or h_3 results in a cost of at least $B + 1$. Hence, such a player has no incentive to change her strategy either. The player playing action h_2 has a cost of $3B + 6$. Switching to some action in $\mathcal{A} \setminus \{h_1, h_2, h_3\}$ increases her cost to at least K. A change to action h_1 or h_3 results in a new cost of $3B + 6$ or $3B + 12$, respectively. Finally, a player with action h_3 has cost $3B + 3$ in state s. She would also strictly increase her cost by switching to some action in $\mathcal{A} \setminus \{h_1, h_2, h_3\}$. A change to action h_1 would result in a cost of $3B + 6$, whereas switching to h_2 increases her cost by four cost units. Consequently, the defined state is a pure-strategy Nash equilibrium.

TABLE 3. Possible defections in the subgame restricted to actions h_1, h_2, and h_3 and three players "…, under the assumption that $\sum_{j=1}^{m} n_{d_j}(s) < m$"

n_{h_1}	n_{h_2}	n_{h_3}	Defector → New strategy	Current cost	Improved cost
3	0	0	$h_1 \to h_2$	$\geq 3B+12$	$3B+5$
0	3	0	$h_2 \to h_3$	$3B+12$	$3B+6$
0	0	3	$h_3 \to h_2$	$3B+12$	$3B+6$
2	1	0	$h_1 \to h_3$	$\geq 3B+7$	$3B+5$
2	0	1	$h_1 \to h_2$	$\geq 3B+9$	$3B+3$
1	2	0	$h_2 \to h_3$	$3B+7$	$3B+5$
0	2	1	$h_2 \to h_3$	$3B+7$	$3B+3$
1	0	2	$h_3 \to h_2$	$3B+5$	$3B+3$
0	1	2	$h_2 \to h_1$	$3B+6$	$\leq 3B+5+(m-1)/m$
1	1	1	$h_1 \to h_3$	$\geq 3B+5$	$3B+3$

For the other direction of the proof, we make the following observations. Let s be a pure-strategy Nash equilibrium. Then, for each $i \in \{1, 2, \ldots, 3m\}$, at most one player chooses a strategy from the set $\{a_i^j\colon j \in \{1, 2, \ldots, m\}\}$. Otherwise, at least one of the players could decrease her current cost of at least K to less than $3(4m+3)B < K$ by switching to action h_3. By the same token, for each $j \in \{1, 2, \ldots, m\}$, at most one player chooses an action from $\{d_j, \bar{d}_j\}$ in s. It follows that at least three players play an action from $\{h_1, h_2, h_3\}$. Suppose there were more than three. Then, there existed either an $i \in \{1, 2, \ldots, 3m\}$ such that $\sum_{j=1}^{m} n_{a_i^j}(s) = 0$ or some $j \in \{1, 2, \ldots, m\}$ with $n_{d_j}(s) + n_{\bar{d}_j}(s) = 0$. In the first case, one of the players currently playing h_1, h_2, or h_3 could decrease her cost from at least $B+1$ to at most $w_i < B$ if she made the switch to any of the actions corresponding to item a_i. In the second case, switching to $\bar{d}_j$ would decrease the cost of some player to B. Consequently,

$$n_{h_1}(s) + n_{h_2}(s) + n_{h_3}(s) = 3, \tag{1}$$

$$\sum_{j=1}^{m} n_{a_i^j}(s) = 1 \quad \text{for } i = 1, 2, \ldots, 3m, \qquad \text{and} \tag{2}$$

$$n_{d_j}(s) + n_{\bar{d}_j}(s) = 1 \quad \text{for } j = 1, 2, \ldots, m. \tag{3}$$

The key insight left to show is that $n_{d_j}(s) = 1$ for $j = 1, 2, \ldots, m$. Assume that $\sum_{j=1}^{m} n_{d_j}(s) < m$. Consider the subgame defined by actions h_1, h_2, and h_3 and three players. Table 3 shows that this subgame does not have a pure-strategy Nash equilibrium. Therefore, the only way s can be a pure-strategy Nash equilibrium is that the local effects on h_1 due to actions d_j sum up to one (see the row next to last in Table 3). Consequently, $n_{d_j}(s) = 1$ for $j = 1, 2, \ldots, m$.

Now consider a player with action d_j in state s. Because s is a Nash equilibrium, she cannot decrease her cost by switching to action $\bar{d}_j$. Therefore, the current cost of this player has to satisfy $\sum_{i=1}^{3m} n_{a_i^j}(s) w_i \leq B$ (note that by (3), $n_{\bar{d}_j}(s) = 0$). Because of (2), we have $\sum_{i=1}^{3m} \sum_{j=1}^{m} n_{a_i^j}(s) w_i = mB$, implying

$$\sum_{i=1}^{3m} n_{a_i^j}(s)\, w_i = B \quad \text{for } j = 1, 2, \ldots, m.$$

Hence, a solution to the partitioning problem is given by $A_j := \{i \in A\colon n_{a_i^j}(s) = 1\}$, $j = 1, 2, \ldots, m$. $\square$

If the number of players or the number of actions in a local-effect game is fixed, then the number of possible strategy combinations of all players is polynomially bounded in the input size. Hence, deciding the existence of pure-strategy Nash equilibria is solvable in polynomial time in these cases.

As mentioned before, bidirectional local-effect games with linear local-effect functions belong to the class of exact potential games (Leyton-Brown and Tennenholtz [24]); in particular, pure-strategy Nash equilibria are guaranteed to exist. However, it turns out that computing one is at least as hard as finding a local optimum for several combinatorial optimization problems with efficiently searchable neighborhoods.

THEOREM 4.2. *The problem of computing pure-strategy Nash equilibria for bidirectional local-effect games with linear local-effect functions is PLS-complete.*

PROOF. We reduce from MAX-CUT with the flip-neighborhood, which is PLS-complete (Schäffer and Yannakakis [33]):[5] Given a complete graph on n nodes with nonnegative integral edge weights w_{ij}, for $i \neq j$, find a partition of the nodes into two sets L and R such that the sum of the weights of edges between nodes in L and R cannot be increased by moving a single node from one set to the other.

We construct a bidirectional local-effect game with linear local-effect functions as follows. There are n players with common action set $\mathcal{A}$ that contains two actions a_i^L and a_i^R for each node $i = 1, 2, \ldots, n$. Let $K := n \max_{i \neq j} w_{ij}$. For each action $a \in \mathcal{A}$, $f_a(x) := 0$ if $x \leq 1$, and $f_a(x) := K$, otherwise. The local-effect functions are given for $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, by $f_{a_i^L, a_j^L}(x) := f_{a_i^R, a_j^R}(x) := w_{ij}x$. Furthermore, $f_{a_i^L, a_i^R}(x) := Kx$ for $i \in \{1, 2, \ldots, n\}$. All local-effect functions not defined so far are identical to zero. This definition of local effects ensures that in any pure-strategy Nash equilibrium s of the game,

$$n_{a_i^L}(s) + n_{a_i^R}(s) = 1 \quad \text{for } i = 1, 2, \ldots, n. \tag{4}$$

For if not, at least one player would be able to decrease her present cost of at least K by switching from her current action to an action a_i^L for which $n_{a_i^L}(s) + n_{a_i^R}(s) = 0$. Ergo, we can associate with any pure-strategy Nash equilibrium s a unique cut in the graph by defining $L(s) := \{i\colon n_{a_i^L}(s) = 1\}$ and $R(s) := \{i\colon n_{a_i^R}(s) = 1\}$.

We show next that for any pure-strategy Nash equilibrium s of the game, the corresponding cut is indeed a local optimum of the MAX-CUT instance. Because s is a Nash equilibrium, no player can decrease her cost by switching to another action. In particular, a player with action a_i^L, $i \in \{1, 2, \ldots, n\}$, cannot improve by switching to action a_i^R. Using (4), this implies

$$\sum_{\substack{j=1\\ j\neq i}}^{n} w_{ij} n_{a_j^L}(s) \leq \sum_{\substack{j=1\\ j\neq i}}^{n} w_{ij} n_{a_j^R}(s). \tag{5}$$

With the definition of $L(s)$ and $R(s)$, it follows that $\sum_{j\in R(s)} w_{ij} - \sum_{j \in L(s)} w_{ij} \geq 0$. Thus, moving node i from $L(s)$ to $R(s)$ does not increase the weight of the associated cut. Similarly, one can show that shifting a node from $R(s)$ to $L(s)$ cannot improve the cut either. We may conclude that the described transformation is indeed a PLS-reduction. ☐

Because the reduction is actually a *tight* PLS-reduction, we obtain the following two results for free (see, e.g., Yannakakis [36]).

COROLLARY 4.1. *There are instances of bidirectional local-effect games with linear local-effect functions that have exponentially long shortest improvement paths.*

COROLLARY 4.2. *For a bidirectional local-effect game with linear local-effect functions, the problem of finding a pure-strategy Nash equilibrium that is reachable from a given strategy state via selfish improvement steps is PSPACE-complete.*

The following observation underlines that finding a pure-strategy Nash equilibrium for bidirectional local-effect games with linear local-effect functions is indeed hard. It was inspired by similar results of Fischer [14] for some local search problems.

THEOREM 4.3. *Given an instance of a bidirectional local-effect game with linear local-effect functions, a pure-strategy profile s_0, and an integer $k > 0$ (unarily encoded), it is strongly NP-complete to decide whether there exists a sequence of at most k selfish steps that transforms s_0 into a pure-strategy Nash equilibrium.*

PROOF. Given a sequence $(s_0, s_1, \ldots, s_n)$ of pure-strategy profiles, we can check in polynomial time whether it is a sequence of at most k self-improving steps such that s_n is a pure-strategy Nash equilibrium, i.e., the problem is in NP. The proof of strong NP-completeness is by reduction from 3-PARTITION.

Consider an arbitrary instance of 3-PARTITION with item set $A = \{1, 2, \ldots, 3m\}$, positive integer weights w_i with $B/4 < w_i < B/2$ for $i \in A$, where $B \in \mathbb{N}$ and $\sum_{i\in A} w_i = mB$. Without loss of generality, we may assume that $m > 2$. We construct an instance of a bidirectional local-effect game with linear local-effect functions and

[5] Our original proof of this result (Dunkel and Schulz [11]) used a reduction from POSNAE3FLIP (Schäffer and Yannakakis [33]). The new, shorter proof presented here was inspired by a proof of Ackermann et al. [1] for the PLS-completeness of finding a pure-strategy Nash equilibrium in congestion games with linear cost functions. In fact, it was pointed out by an anonymous referee that there is a close relationship between congestion games with linear cost functions and local-effect games with linear local-effect functions. In particular, the following reduction shows that the result by Ackermann et al. [1] is stronger than Theorem 4.2. Given a congestion game with strategy sets $S_1, S_2, \ldots, S_n \subseteq 2^E$, let m_e be the slope of the linear cost function associated with resource $e \in E$. Define a bidirectional local-effect game as follows: The action set is $\mathcal{A} := \bigcup_{i=1}^n S_i$. Moreover, for $a \in \mathcal{A}$, $f_a(x) := \sum_{e\in a} m_e x + M(x-1)$. In addition, $f_{a', a}(x) := \sum_{e \in a \cap a'} m_e x$ if $a \in S_i$ and $a' \in S_j$ with $i \neq j$. If $a, a' \in S_i$, then $f_{a', a}(x) := Mx$. Here, M is a sufficiently large constant.

a pure-strategy profile s_0 such that there exists a sequence of at most $k = 4(m-1)$ selfish steps that transforms s_0 into a pure-strategy Nash equilibrium if and only if the 3-Partition instance is a Yes-instance, i.e., A can be partitioned into m disjoint subsets $A_1, A_2, \ldots, A_m$ such that $\sum_{i\in A_j} w_i = B$ for $j = 1, 2, \ldots, m$.

The action set $\mathscr{A}$ of the local-effect game contains for each item $i \in A$ actions a_i^j for $j = 1, 2, \ldots, m$. In addition, there are actions a^j for $j = 1, 2, \ldots, m$, and actions b_l for $l = 1, 2, \ldots, k+1$. To simplify notation, we introduce index sets $J := \{1, 2, \ldots, m\}$, $L_1 := \{1, 2, \ldots, m-1\}$, $L_2 := \{m, m+1, \ldots, k+1\}$, and $L := L_1 \cup L_2$. The game has $3m + k + 2$ players. We also let $D := kB + 1$ and $K := D^2 + 1$. Then, the cost functions are defined as follows:

$$f_{a^j}(x) := D + \frac{x-1}{k+2} \qquad \text{for all } j \in J,$$

$$f_{a_i^j}(x) := \begin{cases} x, & \text{if } x \le 1, \\ Kx, & \text{otherwise,} \end{cases} \qquad \text{for all } i \in A \text{ and } j \in J,$$

$$f_{b_l}(x) := \begin{cases} (D+B+1)x, & \text{if } x \le 1, \\ Kx, & \text{otherwise,} \end{cases} \qquad \text{for all } l \in L_1,$$

$$f_{b_l}(x) := \begin{cases} (D+B)x, & \text{if } x \le 1, \\ Kx, & \text{otherwise.} \end{cases} \qquad \text{for all } l \in L_2.$$

For the slopes of the linear local-effect functions, the reader is referred to Figure 7. Finally, we define the pure-strategy profile s_0 by

$$\begin{aligned} n_{a^1}(s_0) &:= 1, \\ n_{a^j}(s_0) &:= 0 \quad \text{for all } j \in J\backslash\{1\}, \\ n_{a_i^1}(s_0) &:= 1 \quad \text{for all } i \in A, \\ n_{a_i^j}(s_0) &:= 0 \quad \text{for all } i \in A \text{ and } j \in J\backslash\{1\}, \\ n_{b_l}(s_0) &:= 1 \quad \text{for all } l \in L. \end{aligned}$$

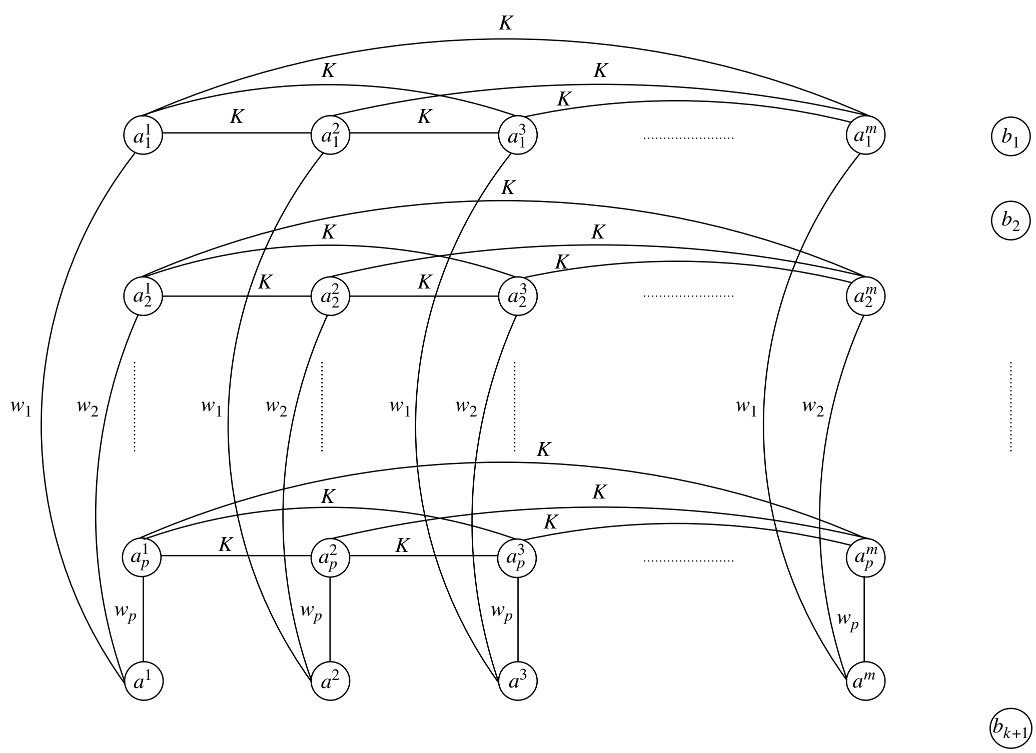

Figure 7. Illustration of the local-effect graph of the game constructed in the proof of Theorem 4.3.
Note. An arc label α corresponds to the coefficient of a linear local-effect function $f_{a,a'}(x) = \alpha x$. Moreover, $p := 3m$.

Assume that the 3-Partition instance is a Yes-instance. We consider the following sequence of players' moves. First, the $3(m-1)$ players with an action a_i^1 in s_0 such that $i \in A_j$ for some $j \neq 1$ switch to action a_i^j, in arbitrary order. They are followed by the $m-1$ players with actions b_l, $l \in L_1$, who move, in arbitrary order, to actions a^{l+1}. It is not difficult to see that this sequence has the desired properties. Furthermore, the resulting state s, i.e.,

$$\begin{aligned} n_{a^j}(s) &= 1 && \text{for all } j \in J, \\ n_{a_i^j}(s) &= \begin{cases} 1, & \text{if } i \in A_j, \\ 0, & \text{otherwise,} \end{cases} && \text{for all } i \in A \text{ and } j \in J, \\ n_{b_l}(s) &= 0 && \text{for all } l \in L_1, \\ n_{b_l}(s) &= 1 && \text{for all } l \in L_2. \end{aligned}$$

defines a pure-strategy Nash equilibrium of the game.

For the other direction, assume that there exists a sequence of at most k self-improving steps that transforms s_0 into a pure-strategy Nash equilibrium s of the local-effect game. We claim that s satisfies

$$\sum_{j \in J} n_{a_i^j}(s) = 1 \quad \text{for all } i \in A. \tag{6}$$

It cannot happen that $\sum_{j\in J} n_{a_i^j}(s) > 1$ for some $i \in A$ because then there would be a player with cost at least K, who could certainly improve. Therefore, suppose there is an $i \in A$ such that $\sum_{j\in J} n_{a_i^j}(s) = 0$. Then, at least $k+3$ players play an action in $\{a^j \mid j \in J\} \cup \{b_l \mid l \in L\}$, each having a cost of at least D. Because there is some $j \in J$ such that $n_{a^j}(s) \leq (3m+k+2)/m \leq 7$, any one of these players can decrease her cost by switching to action a_i^j with new cost at most $1 + 7w_i < D$. This is a contradiction, and hence (6) must hold. Consequently,

$$\sum_{i \in A} w_i \left(\sum_{j \in J} n_{a_i^j}(s) \right) = mB.$$

In the last step of the proof, we show that

$$\sum_{i \in A} w_i n_{a_i^j}(s) = B \quad \text{for all } j \in J; \tag{7}$$

i.e., $A_j := \{i \in A \mid n_{a_i^j}(s) = 1\}$ defines a solution to the partition problem. Suppose (7) does not hold. Then there is some $j \in J$ with

$$\sum_{i \in A} w_i n_{a_i^j}(s) \leq B - 1.$$

We claim that this implies $n_{b_l}(s) = 0$ for all $l \in L$. If there was an l with $n_{b_l}(s) = 1$, then the cost of the corresponding player would be at least $D + B$. Because s was obtained from s_0 by a sequence of at most k steps, and because $n_{a^j}(s_0) \leq 1$ for all $j \in J$, we have $n_{a^j}(s) \leq k+1$. The player with action b_l can decrease her cost by changing to action a^j, because $f_{a^j}(k+2) + \sum_{i\in A} w_i n_{a_i^j}(s) < D + B$. Therefore, $n_{b_l}(s) = 0$ for all $l \in L$. However, because of $n_{b_l}(s_0) = 1$ for all $l \in L$, at least $k+1$ players have to change their action in order to reach strategy state s. This is a contradiction; i.e., (7) must hold, and the sets A_j as defined above are a feasible solution to the 3-Partition instance. ☐

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