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Facets of the generalized permutahedron of a poset

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Abstract

Given a poset P as a precedence relation on a set of jobs with processing time vector p, the generalized permutahedron perm(P, p) of P is defined as the convex hull of all job completion time vectors corresponding to a linear extension of P. Thus, the generalized permutahedron allows for the single machine weighted flowtime scheduling problem to be formulated as a linear programming problem over perm(P, p). Queyranne and Wang [8] as well as von Arnim and Schrader [2] gave a collection of valid inequalities for this polytope. Here we present a description of its geometric structure that depends on the series decomposition of the poset P, prove a dimension formula for perm(P, p), and characterize the facet inducing inequalities under the known classes of valid inequalities.

1. The generalized permutahedron of a poset

The *generalized permutahedron* perm(P, p) is a polytope associated with the following single machine scheduling problem. A set $J = \{1, \dots, n\}$ of n jobs is to be processed on a single machine that can execute at most one job at a time, i.e., the machine is disjunctive. Each job $v \in J$ has a positive processing time p_v and a weight w_v. We impose precedence constraints given by a partially ordered set (poset for short) $P = (J, <_P)$ on the set of jobs, that is we require that a job v can only be started once all jobs u with $u <_P v$ have been finished. Any admissible sequence of jobs corresponds to a linear extension L of the ordering P. The *completion time* C_v^L is the time by which job v is finished when the jobs are processed in the order given by L, i.e., $C_v^L := \sum_{u: u \leqslant_L v} p_u$. Note that we consider only schedules without preemption and without machine idle time.

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We are interested in finding a linear extension L of P that minimizes the weighted mean completion time $(1/n)\sum_{v\in J} w_v C_v^L$ (or equivalently $\sum_{v\in J} w_v C_v^L$). This problem is known to be NP-hard even if all weights w_v are one or all processing times p_v are one (cf., [3] or [4]). Forming the convex hull

$$\mathrm{perm}(P,p) := \mathrm{conv}\{C^L \in \mathbb{R}^J :\ L \text{ is a linear extension of } P\}$$

of all completion time vectors C^L that correspond to a linear extension L of P the single machine scheduling problem can be solved by determining an optimal vertex of perm(P, p) with regard to the linear programming problem

$$\begin{array}{ll} \text{minimize} & \sum_{v\in J} w_v C_v \\ \text{subject to} & C \in \mathrm{perm}(P,p). \end{array}$$

Notice that each feasible completion time vector is in fact a vertex of the polytope perm(P, p). We refer to Pulleyblank [5] for an introduction into the field of polyhedral combinatorics, and to Queyranne and Schulz [7] for an overview on polyhedral approaches to machine scheduling.

If all job processing times p_v are equal to one each completion time vector can be considered as a permutation itself. The resulting polytope is known as the permutahedron of a poset (cf., for instance, [1] and [12], and [7] for further references).

Queyranne and Wang [8] studied a slightly different full dimensional polyhedron $P(J)$ associated with the scheduling problem if machine idle time is allowed,

$$P(J) := \mathrm{conv}(T(J)),$$

where

$$\begin{aligned} T(J) := \{C \in \mathbb{R}^J :\ & C_v \geqslant p_v \text{ for all minimal elements } v \in P, \\ & C_v - C_u \geqslant p_v \text{ for } u <_P v, \\ & C_v - C_u \geqslant p_v \text{ or } C_u - C_v \geqslant p_u \\ & \text{for all incomparable elements } u, v \in P\}. \end{aligned}$$

Since $T(J)$ and therefore $P(J)$ is unbounded from above each valid inequality $\sum_{v\in J} a_v C_v \geqslant \alpha$ for $P(J)$ satisfies $\sum_{v\in J} a_v \geqslant 0$. From this it follows that any completion time vector induced by a schedule with nonzero idle time is contained only in unbounded faces of $P(J)$. Thus perm(P, p) is exactly the unique bounded face of the polyhedron $P(J)$ of maximal dimension. Whereas all valid inequalities for $P(J)$ are also valid for perm(P, p) a facet of $P(J)$ does not necessarily induce a facet of perm(P, p). In this paper we characterize precisely those inequalities among the known classes of valid inequalities for $P(J)$ that induce facets of perm(P, p).

2. Valid inequalities and dimension

The following classes of valid inequalities are known for the generalized permutahedron.

An *ideal* (or *initial set*) of the poset P is a subset $I \subseteq P$ that contains with each $v \in I$ all $u \in P$ with $u <_P v$. For every ideal I we have the *ideal constraint* (or *parallel inequality*)

$$\sum_{v\in I} p_v C_v \geqslant \tfrac{1}{2}p(I)^2 + \tfrac{1}{2}p^2(I). \tag{1}$$

Here $p(I)$ denotes $\sum_{v\in I} p_v$ and $p^2(I)$ stands for $\sum_{v\in I} p_v^2$. The complement $P\backslash I$ of an ideal I is a *filter* (*terminal set*) of the poset P. Each filter F of P induces a *filter constraint*,

$$\sum_{v\in F} p_v C_v \leqslant \tfrac{1}{2}p(F)^2 + \tfrac{1}{2}p^2(F) + p(F)p(P\backslash F).$$

However, the faces of perm(P, p) induced by the filter constraint of F and the ideal constraint of $P\backslash F$ are identical.

Since we do not allow machine idle time the ideal constraint holds with equality for $I = P$,

$$\sum_{v\in P} p_v C_v = \tfrac{1}{2}p(P)^2 + \tfrac{1}{2}p^2(P). \tag{2}$$

In the absence of precedences between jobs inequalities (1) and Eq. (2) are necessary and sufficient to describe the generalized permutahedron (see Queyranne [6] for sufficiency, Schulz [10] for necessity, or Queyranne and Schulz [7] for both).

A poset P is *series decomposable* if $P = Q \dot\cup R$ with $Q, R \neq \emptyset$ and $q <_P r$ for all $q \in Q$ and all $r \in R$. We write $P = Q * R$ if P admits such a decomposition. A *convex* (or *intermediate*) *set* of P is a subset $C \subseteq P$ such that for $u, x, v \in P$ with $u <_P x <_P v$ and $u, v \in C$ also $x \in C$. For every convex set C that is series decomposable into $C = A * B$ the *convex set constraint* (*series inequality*) is valid for perm(P, p), expressing that all jobs in A have to be scheduled before all jobs in B,

$$\begin{aligned} &p(A)\sum_{v\in B} p_v C_v - p(B)\sum_{v\in A} p_v C_v \\ &\quad\geqslant \tfrac{1}{2}p(A)p(B)(p(A)+p(B)) + \tfrac{1}{2}p(A)p^2(B) - \tfrac{1}{2}p(B)p^2(A). \end{aligned} \tag{3}$$

Let the specific ordering N on $\{u_1, u_2, u_3, u_4\}$ be given by the relations $u_1 <_N u_3$, $u_2 <_N u_4$, and $u_2 <_N u_3$ (cf., Fig. 1). A subset $S \subseteq P$ is called a *spider* if it admits a decomposition $S = N \bowtie R$, defined on $N \dot\cup R$, with the ordering

$$u <_P v \quad \text{if} \quad \begin{cases} u <_N v & \text{and } u, v \in N, \\ u <_R v & \text{and } u, v \in R, \\ u = u_2 & \text{and } v \in R, \\ v = u_3 & \text{and } u \in R, \end{cases} \qquad \text{for } u, v \in S.$$

Here $R \subseteq P\backslash N$ is any subposet and $<_R$ denotes the restriction from P to R.

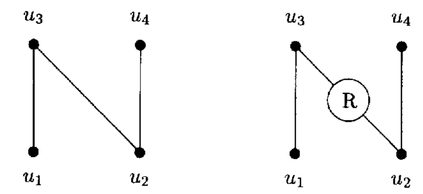

Fig. 1. Hasse diagram of the ordering N and of a spider $N \bowtie R$.

Every spider $S = N \bowtie R$ together with a filter F of the suborder R gives rise to a *spider constraint* (S,F),

$$p(S)\left[(p(F)+p_{u_1})C_{u_3} - \sum_{v\in F\cup\{u_1\}} p_v C_v - \beta C_{u_2}\right] + \beta\sum_{v\in S} p_v C_v$$
$$\geqslant p(S)[\tfrac{1}{2}(p(F)+p_{u_1})(p(F)+p_{u_1}+2p_{u_3}) - \tfrac{1}{2}(p^2(F)+p^2_{u_1}) - \beta p_{u_2}]$$
$$+\tfrac{1}{2}\beta[p(S)^2+p^2(S)], \qquad (4)$$

where $\beta = p(R\backslash F) + p_{u_2}$. Likewise an ideal $I \subseteq R$ induces a *spider constraint* (S,I),

$$p(S)\left[\sum_{v\in I\cup\{u_4\}} p_v C_v - (p(I)+p_{u_4})C_{u_2} + \gamma C_{u_3}\right] - \gamma\sum_{v\in S} p_v C_v$$
$$\geqslant p(S)[\tfrac{1}{2}(p(I)+p_{u_4})^2 + \tfrac{1}{2}(p^2(I)+p^2_{u_4})] + \tfrac{1}{2}\gamma[p(S)^2 - p^2(S)], \qquad (5)$$

where $\gamma = p(R\backslash I) + p_{u_3}$.

A proof of the validity of inequalities (1) through (5) can be found in [2]. Queyranne and Wang [8] prove the validity of the constraints (1) and (3) as well as (4) in case $F = R$, and (5) for $I = R$, for their polyhedron $P(J)$. They also show that for series–parallel orderings $P(J)$ is characterized by (1) and (3). Likewise the polytope perm(P,p) is completely described by (1), (2), and (3) in case of series–parallel orderings (see [1] for the permutahedron and [2] for the general case). Series–parallel orders are orderings in which no four points induce the suborder N. If more generally any five points of P induce at most one N these orderings are called *N-sparse* (or *P_4-sparse*). Their associated generalized permutahedron is completely characterized by (1) through (5), see [2].

Let us assume that P decomposes serially into $P = P_1 * P_2$. Combining the convex set constraint for $P_1 * P_2$ and Eq. (2) gives the ideal inequality for P_1 with reverse inequality sign. Hence we must have equality for P_1,

$$\sum_{v\in P_1} p_v C_v = \tfrac{1}{2}p(P_1)^2 + \tfrac{1}{2}p^2(P_1).$$

Note that every poset P has a unique *series decomposition* $P = P_1 * \cdots * P_k$ where the nonempty suborders $P_1, \ldots, P_k$ are not further series decomposable. We can iterate the above argument to obtain for this decomposition

$$\sum_{v \in P_1 \cup \cdots \cup P_i} p_v C_v = \tfrac{1}{2} p(P_1 \cup \cdots \cup P_i)^2 + \tfrac{1}{2} p^2(P_1 \cup \cdots \cup P_i) \quad \text{for } i = 1, \ldots, k. \tag{6}$$

Theorem 1. *Let P be a poset with series decomposition $P_1 * \cdots * P_k$. Then the system*

$$\sum_{v \in P_i} p_v C_v = \tfrac{1}{2} p(P_i)^2 + \tfrac{1}{2} p^2(P_i) + p(P_i) p(P_1 \cup \cdots \cup P_{i-1}), \quad i = 1, \ldots, k,$$

is a maximal irredundant linear equation system for perm(P, p).

Proof. The validity of the equation system follows by considering the differences of (6) for i and $i-1$. It is obvious that the associated matrix has full row rank. Thus, we only have to show that for any valid equation $\sum_{u \in P} d_u C_u = e$ of perm(P, p) there exist λ_i, $i = 1, \ldots, k$, such that $\lambda_i p_u = d_u$ for all $u \in P_i$.

Let P_i be an arbitrary component of the series decomposition of P with $|P_i| \geqslant 2$. We claim $d_u/p_u = d_v/p_v$ for all $u, v \in P_i$, that is $\lambda_i = d_u/p_u$. Let us distinguish the cases (i) u and v incomparable and (ii) u and v comparable.

(i) If u, v are incomparable, there exists a linear extension L of P such that v follows immediately after u. Let L' be the linear extension obtained from L by interchanging u and v. We write $C = C^L$ and $C' = C^{L'}$ for the associated completion time vectors. By construction we have $C_x = C'_x$ for $x \in P \setminus \{u, v\}$ and $C'_u = C_u + p_v$ as well as $C'_v = C_v - p_u$. Considering the difference $\sum_{x \in P} (d_x C_x - d_x C'_x)$ then gives the claim.

(ii) We may assume $u <_P v$. Let $M(x) := \{y \in P \setminus \{x\} \colon y \text{ is not comparable with } x\}$ denote the set of elements that are incomparable with x. Assume first that v covers u and distinguish two cases. If there exists an element $x \in M(u) \cap M(v)$ we obtain $d_u/p_u = d_x/p_x = d_v/p_v$ using (i). If $M(u) \cap M(v) = \emptyset$ there exist $x \in M(u)$ and $y \in M(v)$ such that x and y are incomparable, since otherwise P_i splits into two series components. Hence we have $d_u/p_u = d_x/p_x = d_y/p_y = d_v/p_v$ using (i) again. If v does not cover u, there is a chain of elements between u and v covering each other. Hence using the same arguments for the covering elements finally gives $d_u/p_u = d_v/p_v$. □

As a corollary we obtain a formula for the dimension of the generalized permutahedron.

Corollary 2. *Let P be a poset with series decomposition $P_1 * \cdots * P_k$. Then*

$$\dim(\text{perm}(P, p)) = |P| - k.$$

In fact, there is the following structural analogy between the decomposition of the poset P and its associated generalized permutahedron perm(P, p). The series composition for posets carries over to the Cartesian product of polytopes. Recall that each face

F of a polytope $Q = Q_1 \times Q_2$ is itself the Cartesian product $F_1 \times F_2$ of faces $F_1 \subseteq Q_1$ and $F_2 \subseteq Q_2$, and $\dim(F) = \dim(F_1) + \dim(F_2)$.

Theorem 3. *Let P be a poset with series decomposition $P_1 * \cdots * P_k$. Then* $\text{perm}(P, p)$ *is the Cartesian product of the polytopes* $\text{perm}'(P_1, p^1), \ldots, \text{perm}'(P_k, p^k)$ *where* $\text{perm}'(P_i, p^i)$ *arises from* $\text{perm}(P_i, p^i)$ *through translation by* $p(P_1 \cup \cdots \cup P_{i-1})\mathbf{1}$,

$$\text{perm}(P, p) = \text{perm}'(P_1, p^1) \times \cdots \times \text{perm}'(P_k, p^k).$$

Here p^i denotes the restriction of the vector p of job processing times to the jobs in P_i.

Proof. Denote by $\text{perm}'(P, p)$ the Cartesian product $\text{perm}'(P_1, p^1) \times \cdots \times \text{perm}'(P_k, p^k)$. Let C^L be the completion time vector of a linear extension L of P. Since L is the series composition of linear extensions L_i of P_i, $i = 1, \ldots, k$, we obtain $C^L = (C^{L_1}, C^{L_2} + \bar{p}_1, \ldots, C^{L_k} + \bar{p}_{k-1})$. Here $\bar{p}_i$ denotes a vector of appropriate dimension with all entries equal to $p(P_1 \cup \cdots \cup P_i)$. Thus $\text{perm}(P, p) \subseteq \text{perm}'(P, p)$. In order to show the reverse inclusion we observe that each vertex of $\text{perm}'(P, p)$ is the Cartesian product of vertices of the polytopes $\text{perm}'(P_1, p^1), \ldots, \text{perm}'(P_k, p^k)$. The rest of the proof is obvious. □

The union of minimal linear descriptions of polytopes Q_1 and Q_2 leads to a minimal linear description of the Cartesian product $Q_1 \times Q_2$. Therefore we obtain the following result.

Corollary 4. *Let P be a poset with series decomposition $P_1 * \cdots * P_k$. Assume, for $i = 1, \ldots, k$, that the system*

$$\sum_{v \in P_i} p_v C_v = \tfrac{1}{2} p(P_i)^2 + \tfrac{1}{2} p^2(P_i),$$

$$\sum_{v \in P_i} a_v^{i,l} C_v \geqslant \alpha^{i,l} \quad \textit{for } l = 1, \ldots, n_i$$

is a complete and minimal description for $\text{perm}(P_i, p^i)$. *Then a complete and minimal description for* $\text{perm}(P, p)$ *is given by the system*

$$\sum_{v \in P_i} p_v C_v = \tfrac{1}{2} p(P_i)^2 + \tfrac{1}{2} p^2(P_i) + p(P_i) p(P_1 \cup \cdots \cup P_{i-1}) \quad \textit{for } i = 1, \ldots, k,$$

$$\sum_{v \in P_i} a_v^{i,l} C_v \geqslant \alpha^{i,l} + a^{i,l}(P_i) p(P_1 \cup \cdots \cup P_{i-1}) \quad \textit{for } i = 1, \ldots, k \textit{ and } l = 1, \ldots, n_i.$$

3. Facets

We turn to characterize which of the constraints (1), (3), (4), and (5) define facets of the generalized permutahedron. In view of Corollary 4 it is sufficient to characterize the facets for components of P that are not series decomposable to obtain a complete and minimal linear description for perm(P, p).

We call C^L *consecutive* for $Q \subseteq P$ if for all $u <_L x <_L v$ with $u, v \in Q$ also $x \in Q$, that is Q is a convex subset of L. The following necessary condition for a constraint to be tight can be proved by revisiting the proofs of validity.

Lemma 5. *Let $Q \subseteq P$ be an ideal, a series decomposable convex set, or a spider. If the corresponding ideal, convex set, or spider inequality is tight for the completion time vector C^L then C^L is consecutive for Q. Furthermore, in case that Q is an ideal, Q is an ideal of L as well.*

First we study ideal constraints.

Theorem 6. *Let P be a non series decomposable poset and I be an ideal of P. If $I_1 * \cdots * I_r$ and $F_1 * \cdots * F_s$ are the series decompositions of I and $P \setminus I$, respectively, then the face of* perm(P, p) *defined by the ideal constraint $\sum_{v \in I} p_v C_v \geqslant \frac{1}{2} p(I)^2 + \frac{1}{2} p^2(I)$ is of dimension $n - (r + s)$.*

Proof. The main observation is that the face induced by an ideal I is itself the generalized permutahedron of an appropriate poset. More precisely, let Q be the poset defined by $Q := I * (P \setminus I)$. Then $\{C \in \text{perm}(P, p)\colon \sum_{v \in I} p_v C_v = \frac{1}{2} p(I)^2 + \frac{1}{2} p^2(I)\} =$ perm(Q, p). The inclusion $\subseteq$ follows from Lemma 5, $\supseteq$ being trivial. The claim now follows from Corollary 2. □

Corollary 7. *Let P be a non series decomposable poset. A nonempty ideal $I \subsetneq P$ induces a facet of* perm(P, p) *if and only if both I and $P \setminus I$ are not series decomposable.*

Next we consider convex set constraints. A convex set $C = A * B$ is called *bipartite* if neither A nor B is series decomposable. First we show that in the case that P is not series decomposable all facet inducing series decomposable convex subsets of P are bipartite. We denote by F_{A*B} the face induced by the convex set $A * B$.

Lemma 8. *Let $A * B$ be a convex set of P, and suppose A or B are in turn series decomposable into $A = A_1 * A_2$ or $B = B_1 * B_2$, respectively. Then*

$$F_{A*B} = F_{A_1*A_2} \cap F_{A_2*B} \quad \textit{or} \quad F_{A*B} = F_{A*B_1} \cap F_{B_1*B_2}.$$

Proof. We show only $F_{A*B} \subseteq F_{A_1*A_2} \cap F_{A_2*B}$. The other inclusions are proved similarly. Let C be a feasible completion time vector that satisfies the convex set constraint induced by $A * B$ with equality. If we multiply this equation with $p(A_2)$ and subtract $p(B)$ times the convex set constraint induced by $A_1 * A_2$ we obtain $C \in F_{A_2*B}$. On the other hand subtracting $p(A)$ times the convex set constraint of $A_2 * B$ gives $C \in F_{A_1*A_2}$. □

To give a complete characterization of facet defining series decomposable convex sets and facet defining spiders we need the concept of a contracted ordering. In the *contracted ordering* P/Q a convex subset $Q \subsetneq P$ is replaced by a single element $q \notin P$, i.e., on the set $(P\backslash Q) \cup \{q\}$ we define an ordering by distinguishing three cases: $u <_{P/Q} v$ if $u, v \in P\backslash Q$ with $u <_P v$; $q <_{P/Q} v$ if v is greater than some u in Q; and likewise $u <_{P/Q} q$ if u is less than some v in Q. The comparabilities implied by transitivity have to be added, of course.

The faces induced by ideals could be viewed as generalized permutahedra induced by an extended ordering of P. For convex sets and spiders we need two generalized permutahedra.

If $Q \subseteq P$ then the situation inside Q is governed by the generalized permutahedron $\text{perm}(Q, p^Q)$ associated with Q, where p^Q denotes the restriction of the vector p of job processing times to the jobs in Q. Every completion time vector $C \in \text{perm}(P, p)$ defines an element $C^Q \in \text{perm}(Q, p^Q)$, as follows. Let L be the linear extension of P with $C = C^L$ then $C^Q_v := \sum_{u \in Q, u \leqslant_L v} p_u$ for $v \in Q$. Equivalently we could associate with L the restricted linear extension L_Q of L to Q and set $C^Q := C^{L_Q}$. We call C^Q the *induced completion time vector* (*w.r.t.* C).

If C is consecutive for Q then the induced completion time vector C^Q equals the restriction of C to Q plus an offset, i.e., $C_v = C^Q_v +$ const. for all $v \in Q$. The jobs of $P\backslash Q$ are divided into two sequences separated by the jobs of Q. Hence a consecutive C can be viewed as an element of $\text{perm}(P/Q, p^{P/Q})$ and of $\text{perm}(Q, p^Q)$, where P/Q is the contracted order, and $p^{P/Q}$ is given by $p^{P/Q}_v = p_v$ for $v \in P\backslash Q$ and $p^{P/Q}_q = p(Q)$. This idea yields a dimension formula stated in the following theorem.

Theorem 9. *Let P be a non series decomposable poset, $Q \subseteq P$ a convex set, and F be a face of* $\text{perm}(Q, p^Q)$. *Let T be the set of completion time vectors $C \in \text{perm}(P, p)$ that are consecutive for Q and such that $C^Q \in F$. Then there exists a bijection φ between T and the set of ordered pairs $(\overline{C}, \widetilde{C})$, where $\overline{C}$ is a vertex of* $\text{perm}(P/Q, p^{P/Q})$ *and $\widetilde{C}$ is a vertex of F. Furthermore* $\dim(T) = \dim(\text{perm}(P/Q, p^{P/Q})) + \dim(F)$.

Proof. First we establish the claimed bijection φ. Let C be a consecutive vertex of $\text{perm}(P, p)$ with $C^Q \in F$ and let L be the linear extension with $C^L = C$, then $\varphi(C) := (\overline{C}, C^Q)$ is defined by $\overline{C}_v = C_v$ for $v \in P\backslash Q$ and $\overline{C}_q = \sum_{v \in P\backslash Q, v <_L u} p_v + p(Q)$ with $u \in Q$. Observe that since C is consecutive $\overline{C}_q$ is well defined, and $\overline{C} \in \text{perm}(P/Q, p^{P/Q})$. The inverse mapping φ^{-1} is given by $\varphi^{-1}(\overline{C}, \widetilde{C}) = C$ with $C_v = \overline{C}_v$ for $v \in P\backslash Q$ and $C_v = \widetilde{C}_v + \overline{C}_q - p(Q)$ for $v \in Q$. We leave it to the reader to verify the formula for φ^{-1}.

To prove the dimension formula we will now show that there exist $k + l - 1$ affinely independent vectors in T if and only if there are k affinely independent completion time vectors in $\text{perm}(P/Q, p^{P/Q})$ and l affinely independent completion time vectors in the face F of $\text{perm}(Q, p^Q)$.

Assume we have $x^1, \ldots, x^k$ affinely independent completion time vectors of $\text{perm}(P/Q, p^{P/Q})$ and $y^1, \ldots, y^l$ affinely independent completion time vectors of F.

Using φ^{-1} we obtain $k \cdot l$ completion time vectors Z^{ij} for $i = 1, \ldots, k$ and $j = 1, \ldots, l$ of perm(P, p) with

$$Z_u^{ij} = \begin{cases} x_u^i, & \text{if } u \in P \setminus Q, \\ y_u^j + r^i, & \text{if } u \in Q, \end{cases}$$

where r^i is given by $x_q^i - p(Q)$. Observe that $Z^{ij} = Z^{i1} + Z^{kj} - Z^{k1}$. Hence all vectors Z^{ij} are in the affine span of the $k + l - 1$ vectors Z^{i1}, $i = 1, \ldots, k$, and Z^{kj}, $j = 2, \ldots, l$. We now prove that these vectors are linearly independent.

To do this we think of these vectors as row vectors of a $(k + l - 1, |P|)$-matrix and multiply each column u with the processing time $p_u \neq 0$. The rank of the matrix remains unchanged. We label the resulting row vectors by z^i, that is

$$z_u^i = \begin{cases} p_u x_u^i, & \text{if } u \in P \setminus Q \\ p_u(y_u^1 + r^i), & \text{if } u \in Q \end{cases} \qquad \text{for } i = 1, \ldots, k,$$

$$z_u^i = \begin{cases} p_u x_u^k, & \text{if } u \in P \setminus Q \\ p_u(y_u^{i-k+1} + r^k), & \text{if } u \in Q \end{cases} \qquad \text{for } i = k + 1, \ldots, k + l - 1.$$

The sum $\sum_{u \in P} z_u^i$ of the components of each z^i gives Eq. (2) of perm(P, p) and is therefore a constant independent of i.

Assume the vectors $z^1, \ldots, z^{k+l-1}$ satisfy the equations

$$\sum_{i=1}^{k+l-1} \lambda^i z_u^i = 0 \quad \text{for } u \in P \tag{7}$$

with $\lambda^i \in \mathbb{R}$. We take the sum over all components to obtain

$$\sum_{i=1}^{k+l-1} \lambda^i = 0. \tag{8}$$

On the other hand we sum (7) over the components in Q,

$$\sum_{u \in Q} \sum_{i=1}^{k+l-1} \lambda^i z_u^i = 0.$$

Using the definition of z_u^i with $u \in Q$ and exchanging summation gives

$$0 = \sum_{i=1}^{k} \lambda^i \sum_{u \in Q} p_u(y_u^1 + r^i) + \sum_{i=k+1}^{k+l-1} \lambda^i \sum_{u \in Q} p_u(y_u^{i-k+1} + r^k).$$

Since $y^i \in \text{perm}(Q, p^Q)$ Eq. (2) for Q yields

$$0 = \sum_{i=1}^{k} \lambda^i (\tfrac{1}{2}(p^2(Q) + p(Q)^2) + p(Q) r^i)$$

$$+\sum_{i=k+1}^{k+l-1}\lambda^i(\tfrac{1}{2}(p^2(Q)+p(Q)^2)+p(Q)r^k).$$

With the help of (8) we conclude

$$\sum_{i=1}^{k}\lambda^i r^i + r^k\sum_{i=k+1}^{k+l-1}\lambda^i = 0.$$

Hence we can rewrite those equations of (7) with $u\in Q$ as

$$\begin{aligned}0 &= \sum_{i=1}^{k+l-1}\lambda^i z_u^i\\ &= \sum_{i=1}^{k}\lambda^i p_u(y_u^1+r^i)+\sum_{i=k+1}^{k+l-1}\lambda^i p_u(y_u^{i-k+1}+r^k)\\ &= p_u\left(\left(\sum_{i=1}^{k}\lambda^i\right)y_u^1+\sum_{i=k+1}^{k+l-1}\lambda^i y_u^{i-k+1}\right).\end{aligned}$$

Since the vectors y^i are affinely independent we can conclude $\lambda^i = 0$ for $i = k+1,\dots,k+l-1$. Revisiting (7) for the jobs $u\in P\backslash Q$ as well as the single job $\bar{u}\in Q$ with $y_{\bar{u}}^1 = p(Q)$, we obtain from the affine independence of the vectors $x^1,\dots,x^k$ that also $\lambda^i = 0$ for $i = 1,\dots,k$. Hence the vectors z^i are linearly independent and therefore $\dim(T)\geqslant\dim(\mathrm{perm}(P/Q, p^{P/Q}))+\dim(F)$. It is a not too hard exercise to establish the reverse inequality by constructing hyperplanes containing T from those that describe the affine hull of $\mathrm{perm}(P/Q, p^{P/Q})$ and F, respectively. □

If the constraint induced by a set Q is tight for a completion time vector C then we know from Lemma 5 that C is consecutive for Q. This implies $C_v = C_v^Q +$ const. for all $v\in Q$. Let $ax\geqslant b$ be a convex set or a spider constraint. Observe that for those constraints the component sum vanishes, $a(P)=0$, and we have for all C with $C = C^Q + \text{const.}\mathbf{1}$ the equality $\sum a_v C_v = \sum a_v C_v^Q$. Hence the convex set (spider) constraint of $\mathrm{perm}(P,p)$ induced by Q is tight for C if and only if the convex set (spider) constraint induced by Q of $\mathrm{perm}(Q,p^Q)$ is tight for C^Q. We are now in a position to characterize facets induced by convex sets and spiders, respectively.

Theorem 10. *Let P be a non series decomposable poset and $A*B\subset P$ be a convex set. Then $A*B$ defines a facet of* $\mathrm{perm}(P,p)$ *if and only if $A*B$ is bipartite and the contraction $P/(A*B)$ is not series decomposable.*

Proof. From Lemma 8 follows that $A*B$ has to be bipartite.

For the face F in Theorem 9 we choose the generalized permutahedron of the convex set $A*B$ itself. Then the set of all vertices $C\in\mathrm{perm}(P,p)$ for which the convex set constraint of $A*B$ is tight equals the set T defined in Theorem 9. So $A*B$ induces a facet if and only if $\dim(T) = \dim(\text{perm}(P, p)) - 1 = |P| - 2$. By Theorem 9 and Corollary 2 we obtain

$$\begin{aligned}\dim(T) &= \dim(\text{perm}(P/(A*B), p^{P/(A*B)})) + \dim(\text{perm}(A*B, p^{A*B})) \\ &= |P| - |A*B| + 1 \\ &\quad -\#(\text{components of the series decomposition of } P/(A*B)) + |A*B| - 2.\end{aligned}$$

Hence the bipartite convex set $A * B$ induces a facet if and only if the number of components in the series decomposition of $P/(A * B)$ is one. □

We call a spider S *convex* if S is a convex set of P. By Lemma 5 only convex spiders induce nontrivial faces. The following condition is necessary and sufficient for spiders to induce facets.

Theorem 11. *Let P be a poset that is not series decomposable, and let $S \subseteq P$ be a convex spider. A spider constraint (S,F) or (S,I) defines a facet of the generalized permutahedron* perm(P, p) *if and only if the contracted poset P/S is not series decomposable.*

Proof. If the ordering P is a spider itself it is quite easy to construct $|P| - 1$ affinely independent completion time vectors satisfying a particular spider inequality with equality. Hence every spider constraint defines a facet.

Therefore in the general case we choose for the face F in Theorem 9 the facet of perm(S, p^S) induced by the spider constraint. Now we continue as in the proof of Theorem 10. □

We conclude this section with some remarks on the case that P has series decomposition $P = P_1 * \cdots * P_k$. Any ideal $I \subseteq P$ has the form $I = P_1 * \cdots * P_i * \hat{I}$ for some $i \in \{0, \ldots, k-1\}$ where $\hat{I}$ is an ideal of the suborder P_{i+1}. It defines a facet of perm(P, p) if and only if $\hat{I}$ defines a facet of perm(P_{i+1}, p^{i+1}). The conditions obtained on $\hat{I}$ are exactly those stated in Corollary 7.

A bipartite convex set $A * B$ defines a facet if and only if

(a) $A * B \subsetneq P_i$ and $P_i/(A * B)$ is not series decomposable, or

(b) $A = P_i$, $B \subsetneq P_{i+1}$ for some $i \in \{1, \ldots, k-1\}$ and $P_{i+1} \backslash B$ is not series decomposable, or

(c) $A \subsetneq P_i$, $B = P_{i+1}$, for some $i \in \{1, \ldots, k-1\}$ and $P_i \backslash A$ is not series decomposable.

This is seen as follows.

Since A is not series decomposable, $A \cap P_i \neq \emptyset$ for some $i \in \{1, \ldots, k\}$ implies $A \subseteq P_i$ and the same holds for B. Hence for a bipartite convex set $A * B$ we have either (i) $A * B \subsetneq P_i$ for some $1 \leqslant i \leqslant k$ or (ii) $A \subseteq P_i$ and $B \subseteq P_{i+1}$ for some $1 \leqslant i \leqslant k-1$. If $A * B$ is of type (ii) then it follows from Lemma 5 that the set of those C in perm(P, p) for which the convex set constraint induced by $A * B$ is tight equals the permutahedron perm(Q, p), where Q is defined as $Q := P_1 * \cdots * (P_i \backslash A) * A * B * (P_{i+1} \backslash B) * \cdots * P_k$. Hence $A * B$ defines a facet if and only if $\dim(Q) = |P| - k - 1$. By Corollary 2 we obtain that this is true if and only if $A * B$ has form (b) or (c). The proof for case (i) follows from Theorem 10 and Corollary 4.

The facets induced by convex sets of the form (b) and (c) are also induced by certain ideals. To see this notice that $C = P_i * B$ is convex and bipartite with $P_{i+1} \backslash B$ not series decomposable if and only if $I = P_1 * \cdots * P_i * B$ is an ideal with B and $P_{i+1} \backslash B$ not series decomposable. Analogously, $C' = A * P_{i+1}$ is convex and bipartite with $P_i \backslash A$ not series decomposable if and only if $I' = P_1 * \cdots * P_{i-1} * (P_i \backslash A)$ is an ideal and both A and $P_i \backslash A$ are not series decomposable. It is an easy computation to show that the facets induced by C and I as well as those induced by C' and I', respectively, are identical.

4. N-sparse posets

If the poset P is N-sparse it is proved in [2] that perm(P, p) is completely described by the linear system (1)–(5). In the case of a non series decomposable poset the facet inducing ideal, convex set, and spider constraints define mutually distinct facets. Hence the next theorem follows from Corollary 4, Corollary 7, Theorem 10, and Theorem 11.

Theorem 12. *Let P be an N-sparse poset with series decomposition $P_1 * \cdots * P_k$. Then* perm(P, p) *is completely and minimally described by the following linear system (in each case i ranges from 1 to k):*

(i) *The equations*

$$\sum_{v \in P_i} p_v C_v = \tfrac{1}{2} p(P_i)^2 + \tfrac{1}{2} p^2(P_i) + p(P_i) p(P_1 \cup \cdots \cup P_{i-1}).$$

(ii) *The ideal inequalities*

$$\sum_{v \in I} p_v C_v \geqslant \tfrac{1}{2} p(I)^2 + \tfrac{1}{2} p^2(I)$$

for all ideals $I = P_1 * \cdots * P_{i-1} * \hat{I}$, where the nonempty sets $\hat{I}$ and $P_i \backslash \hat{I}$ are not series decomposable.

(iii) *The convex set inequalities*

$$p(A) \sum_{v \in B} p_v C_v - p(B) \sum_{v \in A} p_v C_v$$
$$\geqslant \tfrac{1}{2} p(A) p(B)(p(A) + p(B)) + \tfrac{1}{2} p(A) p^2(B) - \tfrac{1}{2} p(B) p^2(A)$$

for $A * B \subsetneq P_i$ convex and bipartite with $P_i/(A * B)$ not series decomposable.

(iv) *The spider inequalities* (S,F),

$$p(S)\left[(p(F)+p_{u_1})C_{u_3}-\sum_{v\in F\cup\{u_1\}}p_vC_v-\beta C_{u_2}\right]+\beta\sum_{v\in S}p_vC_v$$

$$\geqslant p(S)[\tfrac{1}{2}(p(F)+p_{u_1})(p(F)+p_{u_1}+2p_{u_3})-\tfrac{1}{2}(p^2(F)+p_{u_1}^2)-\beta p_{u_2}]$$

$$+\tfrac{1}{2}\beta[p(S)^2+p^2(S)],$$

with $\beta=p(R\backslash F)+p_{u_2}$. *Here* $S=N\bowtie R\subseteq P_i$ *can be any convex spider such that* P_i/S *is not series decomposable, and* F *is any filter of* R.

(v) *The spider inequalities* (S,I),

$$p(S)\left[\sum_{v\in I\cup\{u_4\}}p_vC_v-(p(I)+p_{u_4})C_{u_2}+\gamma C_{u_3}\right]-\gamma\sum_{v\in S}p_vC_v$$

$$\geqslant p(S)[\tfrac{1}{2}(p(I)+p_{u_4})^2+\tfrac{1}{2}(p^2(I)+p_{u_4}^2)]+\tfrac{1}{2}\gamma[p(S)^2-p^2(S)],$$

with $\gamma=p(R\backslash I)+p_{u_3}$. *Again,* $S=N\bowtie R\subseteq P_i$ *can be any convex spider such that* P_i/S *is not series decomposable, and* I *is any ideal of* R.

We note that (i)–(iii) completely and minimally describe the generalized permutahedra of series–parallel orderings. This gives a correction to the statement of Theorem 4.6 in [1] (see also [12]).

5. Concluding remarks

The availability of an explicit complete description of the generalized permutahedron of an N-sparse poset by means of linear equations and inequalities suggests the related single machine sequencing problem be solvable in polynomial time. Indeed, this has been shown by Schulz [11] (see also [10]) who developed an $\mathcal{O}(n^2)$ combinatorial algorithm. This in particular implies that the separation problem associated with the generalized permutahedron of an N-sparse poset can be solved efficiently. But whereas we are aware of combinatorial algorithms solving the separation problems restricted to the family of ideal and spider inequalities, respectively, in polynomial time, even for arbitrary posets, there is no direct algorithm known for the whole class of convex set constraints (there is an implicit one, however, for a broader class, see [7] for details).

Here we have shown that several of the known inequalities for the generalized permutahedron of a poset are facet defining. This motivates and justifies in particular their use in algorithms of cutting plane type to solve scheduling problems. Their usefulness is confirmed by first computational results (see [9]).

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