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Uncertainty in the Theory of Renewable Resource Markets

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The natural growth rate of most renewable resource stocks is in part stochastic. This paper examines the implications of such ecological uncertainty for competitive equilibrium in a market with property rights. We show that stochastic fluctuations add a risk premium to the rate of return required to keep a unit of stock *in situ*, and we examine the effects of fluctuations on resource rent. Examples are used to show that extraction can increase, decrease, or be left unchanged as the variance of the fluctuations increases, depending on the extent of market "self-correction". Regulatory implications are also discussed.

1. INTRODUCTION

Renewable resource economics has traditionally been concerned with the study of dynamically optimal harvesting policies given a deterministic function for the natural growth of the resource stock. Issues have included the existence and characteristics of steady-state equilibria for the optimally managed resource, the need for and design of regulatory policies to prevent over-exploitation, and conditions under which (as a social optimum or otherwise) the resource will be exploited to extinction.[1] Much of this work has been based on the assumption of a fixed and exogenous price for the harvested resource (typically resulting in "bang-bang" solutions for the harvesting policy). However some recent papers make price endogenous, and thereby describe how the extraction rate, and the rate of return and asset value of the resource behave in a competitive market with property rights.[2]

For virtually all resources, the natural rate of growth of the stock (or "biomass") is in fact stochastic. This is well appreciated by biologists and ecologists, and a growing body of literature in population ecology has focused on the development of stochastic models of resource growth dynamics, and the characterization of steady-state probability distributions for resource stocks that are either unexploited or else harvested according to some fixed rule.[3]

The presence of "ecological" uncertainty raises interesting questions about the behaviour of renewable resource markets. First, how does such uncertainty affect the value and (expected) rate of return dynamics of the *in situ* resource stock? Second, how does it affect the rate of extraction in a competitive market with property rights? Related to this, what are the implications for the degree of regulation needed in cases where property rights cannot be assigned or maintained? These are the questions that are addressed in this paper.

Other papers have already examined the optimal rate of extraction from a stochastically growing resource stock (typically to maximize the expected flow of utility from net revenue). For example, using a continuous-time geometric random walk for the biomass growth function (i.e. infinite carrying capacity), Gleit (1978) finds the optimal extraction rate that maximizes the expected integral of an isoelastic utility function of net revenue.

He shows that as the variance of the growth rate increases, the optimal extraction rate increases. Smith (1978) solves the same problem but using a reciprocal utility function and the more realistic logistic growth function, and finds the optimal extraction rate reduced by uncertainty. Also, Ludwig (1979) and Ludwig and Varah (1979) use perturbation methods to obtain approximate numerical solutions to the stochastic harvesting problem for a logistic growth function. However, in these and related papers (as in much of the deterministic literature), price is fixed and exogenous.[4]

In this paper the focus is on markets with well-defined property rights, in which prices are determined endogenously—or equivalently, markets in which supply is optimally socially managed. As Levhari, Michener and Mirman (1981) demonstrate clearly, such markets can be analyzed in the standard framework of capital theory. In particular, the dynamics of price and the resource stock are jointly determined by the equilibrium requirement that any unexploited *in situ* unit yield a competitive rate of return, or equivalently, that the net capital gain (properly defined) from holding the unit equal the market rate of interest.

We examine and interpret this equilibrium condition in a stochastic context, i.e. when "asset" growth is in part random. Where possible we point out parallels with exhaustible resource markets. Also, through the use of several examples, we demonstrate the ways in which stock growth uncertainty can influence the rate of resource extraction, and the value of resource "rent". This provides insight into the role of uncertainty in resource markets, and the implications for regulation in cases where property rights cannot be maintained.

In the next section we review competitive market behaviour under certainty, and briefly discuss the rate of return condition that holds in equilibrium. Section 3 sets forth the stochastic model, derives the corresponding rate of return condition, and discusses the ways in which uncertainty affects resource extraction. Section 4 deals with the stochastic analogue of a steady-state equilibrium. Section 5 illustrates the effects of uncertainty on rent and the rate of extraction via three examples, which show how extraction can be increased, decreased, or left unchanged by uncertainty, depending on the characteristics of demand, cost, and stock growth. The concluding section discusses regulatory implications, and suggests directions for further work.

2. MARKET BEHAVIOUR UNDER CERTAINTY

We begin by reviewing the behaviour of renewable resource markets in a deterministic context. As is usually done, we assume the dynamics of the resource stock x is given by an equation of the form

$$\dot{x} = f(x) - q(t) \tag{1}$$

where $f(x)$ is strictly concave, with $f(0) = f(K) = 0$, $K > 0$ is the "carrying capacity", and $q(t)$ is the rate of extraction or "harvest". Total extraction cost is given by $c(x)q$, with marginal cost $c(x)$ decreasing and strictly convex, and $c(0) = \infty$.[5] Finally, price is determined by a downward sloping market demand curve $p(q)$ so that the profit function $\Pi = p(q)q - c(x)q$ is strictly concave.

We assume the market is competitive, but with property rights clearly defined. As Levhari, Michener and Mirman (1981) demonstrate in a discrete-time context, this is equivalent to a centrally planned economy in which $q(t)$ is chosen over time to maximize the flow of discounted net surplus. We therefore refer to this $q(t)$ as the *optimal* extraction rate (as opposed to the rates that would prevail in the open-access or monopoly cases).

In some resource markets (particularly fisheries), property rights cannot be assigned or enforced; then this extraction rate illustrates an "ideal", and provides a guideline for regulation.

Market equilibrium is given by the solution of

$$\max_q \int_0^\infty [p-c(x)]qe^{-\delta t}dt \tag{2}$$

subject to equation (1), with p taken as exogenous in the maximization, but p and q satisfy the market demand condition. (The discount rate δ is assumed equal to the market rate of interest.) A model of this form has been examined by Berck (1981), and here we only point out its more important features.

First, note that the undiscounted shadow price, or rent, associated with a unit of resource stock is given by:

$$\lambda_u = p - c(x). \tag{3}$$

As for an exhaustible resource, this rent is the scarcity value of the marginal *in situ* unit, i.e. the "user cost" associated with extracting the unit. It is also the price at which the unit would change hands, were there a competitive market for *in situ* stock.[6] Finally, note that competitive producers appear to make positive profits, but those profits are the amortized cost of resource stock ownership (or rental payments to some other owner of the stock).

Solution of this model is straightforward, and yields an equation for the dynamics of price that can be written in terms of resource rent:

$$\frac{d}{dt}(p-c) = \delta(p-c) - f'(x)(p-c) + c'(x)q. \tag{4}$$

This says that the (absolute) rate of capital gain on 1 unit of stock must equal the total cost of holding the unit—i.e. the opportunity cost of foregone interest, less the gain from the increase in the total stock growth rate attributable to that unit (which can be positive or negative), plus the change in total harvesting cost resulting from that unit (which will be negative). If $c'(x)=0$, equation (4) becomes the well known Hotelling (1931) rule for an exhaustible resource, except that now resource rent $(p-c)$ grows at the *net* rate of interest, i.e. the market rate δ less the marginal rate of natural growth $f'(x)$.

Alternatively, we can write equation (4) as a rate of return condition:

$$\frac{(d/dt)(p-c)}{(p-c)} + f'(x) - \frac{c'(x)q}{(p-c)} = \delta \tag{4'}$$

i.e. the total return associated with holding a unit of stock must equal the market rate of interest. Finally, the equation can be written simply in terms of the rate of change of price.

$$\dot{p} = \delta(p-c) - f'(x)(p-c) + c'(x)f(x). \tag{5}$$

Note that by setting $f'(x)=0$, equation (5) describes the price dynamics for a competitively produced exhaustible resource for which extraction cost is a decreasing function of the reserve level, and $f(x)$ the rate of new reserve discoveries.[7]

Equations (4) and (5) hold in competitive market equilibrium at any point in time. A steady-state equilibrium is defined by $\dot{p}=\dot{x}=0$. From (5) we see that in a steady-state equilibrium,

$$p^* = c(x) + \frac{-c'(x)f(x)}{\delta - f'(x)} \tag{6}$$

i.e. price is equal to marginal extraction cost, plus rent. Since $f(x)=q$ if $\dot{x}=0$, in steady-state equilibrium rent is the capitalized value of future increases in cost resulting from a 1-unit reduction in the resource stock. Note that the capitalization is based on a rate that reflects the opportunity cost of the *in situ* unit, namely the market interest rate less the marginal contribution of the unit to the stock growth rate. Again, $\delta - f'(x)$ is a net interest rate, i.e. an internal rate of return for the resource owner; he can invest the proceeds of a harvested unit at the rate δ, but harvesting that unit means reducing the rate of growth of the entire stock by an amount $f'(x)$. Also, observe from (6) that a steady-state equilibrium requires that $f'(x) < \delta$ (or else it would pay to let the resource grow unharvested indefinitely), and that this condition is guaranteed by the concavity of $f(x)$.

The corresponding conditions for alternative market structures are easy to derive. If the resource is extracted by a monopolist, price is replaced by marginal revenue in equations (3)–(6). The monopolist under-extracts, so that the resource stock is always larger, and as for an exhaustible resource, rent is smaller. (Since the monopolist extracts at a lower rate, he attaches a lower value to the marginal *in situ* unit.) If the resource is extracted in a competitive market without property rights (the open-access case), extraction expands until price equals marginal cost, and rent is zero.

The market equilibrium conditions described above are based on a general—but deterministic—model. Let us now re-examine these conditions using an analogous stochastic model.

3. A STOCHASTIC MODEL

Equation (1) says that the change in the resource stock can be predicted exactly from the current stock level and extraction rate. For most resources these changes are in fact partly random. A convenient way of capturing this is to use a stochastic differential equation of the Ito type to describe the stock dynamics. In particular, we replace equation (1) with

$$dx = [f(x) - q(t)]dt + \sigma(x)dz \tag{7}$$

with $\sigma'(x) > 0$, $\sigma(0)=0$, and $dz = \varepsilon(t)\sqrt{dt}$. Here $\varepsilon(t)$ is a serially uncorrelated and normally distributed random variable with unit variance, i.e. $z(t)$ is a Wiener process. Equation (7) implies that the current resource stock is known with certainty, but the instantaneous change in the stock is (in part) random. Also, $\sigma(x)$ is specified so as to ensure that the resource stock x is always non-negative. Stochastic differential equations of this type have found increasing application in economics, but justification for the use of (7) in this paper also has support from the population ecology literature.[8]

We assume that resource firms are risk-neutral. In a competitive market equilibrium (again assuming well-defined property rights) each firm chooses its extraction rate q^i to maximize its expected sum over time of discounted profits, subject to equation (7), and again with price exogenous in the maximization, but price and aggregate extraction q satisfying the market demand function. Before solving this problem it is important to stress that the solution p and q is the same as the solution to the social welfare problem, i.e. p and q maximize the expected sum over time of discounted net surplus.[9]

This problem is best approached using stochastic dynamic programming. The value function for the i-th firm, V^i, is its present value of profits:

$$V^i = V^i(x) = \max_{q^i} E_t \int_t^{\infty} \Pi^i(\tau) e^{-\delta(\tau - t)} d\tau \tag{8}$$

where E_t indicates the expectation at time t, and $\Pi^i = [p - c(x)]q^i$. Then the fundamental equation of optimality is

$$\delta V^i dt = \max_{q^i} \{\Pi^i(t)dt + E_t dV^i\} \tag{9}$$

i.e. for a risk-neutral firm, the required instantaneous return on the resource stock ($\delta V^i dt$) has two components, the instantaneous net cash flow $\Pi^i(t)dt$, and the instantaneous expected capital gain $E_t dV^i$. Also, on the margin any increase in net cash flow will just be offset by a decrease in expected capital gain.

We will assume that there is a large number n of identical firms which own equal shares in a unitized resource stock. (Or, equivalently, firms own their own resource stocks, but with identical growth dynamics, the aggregate of which is given by equation (7), and identical costs.) Then, the fundamental equation of optimality can be written as:[10]

$$\delta V^i = \max_{q^i} \{pq^i - c(x)q^i + [f(x) - nq^i]V_x^i + \tfrac{1}{2}\sigma^2(x) V_{xx}^i\} \tag{10}$$

where subscripts denote partial derivatives, i.e. $V_x^i = \partial V^i / \partial x$. This equation is linear in q^i, so the maximization implies

$$q^i = \begin{cases} q^i_{\max}, & \text{if } p - c(x) > nV_x^i = \tilde{V}_x \\ 0, & \text{if } p - c(x) < nV_x^i = \tilde{V}_x \end{cases} \tag{11}$$

and $\tilde{V} = nV^i$ is the aggregate value of the resource stock to producers. With a downward sloping demand curve, market clearing will ensure total output $q = nq^i$ is such that

$$\tilde{V}_x = p(q) - c(x). \tag{12}$$

Let V be the *social* value function, i.e. the integral of the expected flow of consumer plus producer surplus. By going through the same steps as above to maximize V, it is easily seen that $V_x = \tilde{V}_x$. Thus V_x, i.e. resource rent, is the social and market value of the marginal unit of *in situ* stock. As in the deterministic case, it is just equal to the profit that can be obtained by extracting and selling the unit.

If rent V_x were known, equation (12) could be solved for q:

$$q^*(x) = p^{-1}[V_x + c(x)]. \tag{13}$$

Rent is not known, but the value function $V(x)$ can be found by writing the social optimality condition equivalent to equation (10), and substituting equation (13) for $q(x)$. That yields the following equation that $V(x)$ must satisfy:

$$\delta V = \int_0^{q^*(x)} p(q)dq - c(x)q^*(x) + [f(x) - q^*(x)]V_x + \tfrac{1}{2}\sigma^2(x) V_{xx}. \tag{14}$$

The competitive equilibrium is found by solving this differential equation for $V(x)$, and using (13) to determine $q(x)$. In Section 5, solutions are obtained for three examples to illustrate the effects of uncertainty on resource rent and the rate of extraction. But first, let us examine how uncertainty affects the rate of return on the resource stock.

Differentiate equation (14) with respect to x:

$$\delta V_x = [p - c(x) - V_x]\partial q^*/\partial x - c'(x)q^* + f'(x) V_x + \sigma'(x)\sigma(x) V_{xx} + [f(x) - q^*] V_{xx} + \tfrac{1}{2}\sigma^2(x) V_{xxx}. \tag{15}$$

The terms in the first set of brackets sum to zero, so (15) can be re-written as:

$$\delta V_x = -c'(x)q^* + f'(x) V_x + \sigma'(x)\sigma(x) V_{xx} + (1/dt)E_t d(V_x) \tag{16}$$

Also, equation (12) implies that:

$$(1/dt)E_t d(V_x) = (1/dt)E_t d[p - c(x)]. \tag{17}$$

Now combine equations (12), (16), and (17) and rearrange to obtain a rate of return condition analogous to equation (4′) from the deterministic case:

$$\frac{(1/dt)E_t d(p-c)}{(p-c)} + f'(x) - \frac{c'(x)q^*}{(p-c)} = \delta + \sigma'(x)\sigma(x)A(x) \tag{18}$$

where $A(x) = -V_{xx}/V_x$. $A(x)$ can be thought of as an index of absolute risk aversion; it reflects the premium that resource owners would pay to eliminate stock growth uncertainty.[11]

Because $\sigma'(x) > 0$, the rate of return condition in equation (18) differs from the deterministic case in that the market interest rate δ must be augmented by a "risk premium" equal to the increase in stock growth variance attributable to the marginal *in situ* unit times the index of implicit risk aversion.[12] This "variance effect" increases the expected rate of capital gain needed to hold the marginal *in situ* unit, rather than extract and sell it. Because stochastic fluctuations are costly, it pays to extract more of the resource, thereby reducing the variance of the remaining stock. By itself, this reduces resource rent and increases the rate of extraction. However, rent and the extraction rate are affected in other ways as well.

To see this, note that stochastic fluctuations reduce the value of the resource stock in two ways. First, because the growth function $f(x)$ is concave, stochastic fluctuations in x reduce the expected rate of growth of x (an implication of Jensen's inequality). In effect, stochastic fluctuations increase the physical scarcity of the resource. This increases resource rent, and *reduces* the rate of extraction. Second, because the cost function $c(x)$ is convex, stochastic fluctuations in x increase expected extraction costs over time (again by Jensen's inequality). Because cost-raising fluctuations occur continuously, they create an incentive for resource owners to increase the rate of extraction and thereby reduce the amount of increased cost.[13]

Stochastic fluctuations will thus affect the rate of extraction in three different ways: (i) Because fluctuations reduce the value of the stock, and because their variance is an increasing function of the stock level, there is an incentive to reduce the stock level by harvesting faster. (ii) Fluctuations increase expected extraction costs over time, and this also creates an incentive to extract at a faster rate. (iii) Given a fixed extraction rate, at any stock level x fluctuations reduce the expected growth rate of the stock, and this in turn reduces the extraction rate. Given a particular current stock level x, the net effect of uncertainty on the current rate of extraction is therefore indeterminate.

The problem for a monopolistic resource owner can be solved in the same way, but replacing p by $p(q)$ in equation (10). Equations (12), (14), and (18) will again apply, but with price replaced by marginal revenue.

4. STEADY STATE BEHAVIOUR

In the case of an exhaustible resource, we can ask how uncertainty, monopoly power, etc. affect the rate of extraction as that rate eventually falls to zero. Production from a renewable resource stock need not fall to zero, and we are also interested in the behaviour of the steady-state equilibrium. In a deterministic model the steady-state equilibrium is easily found by setting $\dot{p}=\dot{x}=0$. When the resource stock grows stochastically, that equilibrium can only be described in terms of probability distributions and moments. Depending on $f(x)$, $\sigma(x)$, and $p(q)$, the stock x may eventually fluctuate around some steady-state expected value. Alternatively, the equilibrium $q^*(x)$ might yield a degenerate steady-state distribution for x, i.e. with probability 1 the stock will eventually fall to zero. As we will see, it can be the case that competitive (and optimal) extraction drives the stock to zero with probability 1 in the presence of stochastic fluctuations, but *not* in their absence.

Substituting the equilibrium extraction rate $q^*(x)$ into equation (7) yields a stochastic differential equation that completely describes the evolution of x, i.e. $dx = [f(x)-q^*(x)]dt+\sigma(x)dz$. The steady-state probability distribution for x can then be found from the Kolmogorov forward equation. As Merton (1975) has shown, if that distribution is not degenerate, it is given by:

$$\pi_\infty(x)=\frac{m}{\sigma^2(x)}\exp\left[2\int^x\frac{[f(\nu)-q^*(\nu)]}{\sigma^2(\nu)}\,d\nu\right] \tag{19}$$

with m chosen so that $\pi_\infty(x)$ integrates to unity. From this distribution one can obtain the steady-state expected value for the stock $\bar{x}_\infty$. Similarly, writing

$$dq^*(x)=q^*_x\,dx+\tfrac{1}{2}q^*_{xx}(dx)^2$$

and substituting (7) for dx, one has a stochastic differential equation for q^*, from which one can obtain the steady-state expected extraction rate $\bar{q}^*_\infty$.

Assuming non-degenerate steady-state distributions, the concavity of $f(x)$ ensures that $\bar{q}^*_\infty$ falls as σ^2 increases. However, the effect of increases in σ^2 on the *current* rate of extraction $q^*(x)$ will depend on the particular growth function $f(x)$, and the response of demand to price changes. As we saw earlier, until a steady-state equilibrium is reached, the extraction rate might be higher, lower, or left unchanged. This is best illustrated through some examples.

5. EXAMPLES

As explained above, equations (13) and (14) provide a differential equation that can be solved for the value function $V(x)$ and rent V_x. Then the extraction rate $q^*(x)$ can be determined from equation (13), and the steady-state distribution $\pi_\infty(x)$ from (19). We go through these steps in three examples, in each case determining how V_x, $q^*(x)$, and $\bar{q}^*_\infty$ change as the stock growth variance (and other parameters) are changed.

In each example, $\sigma(x)=\sigma x$, i.e. percentage changes in the stock dx/x have a random component that is normally distributed. This specification is intuitively appealing, and is used widely in the population ecology literature. In addition, all of the examples utilize an isoelastic demand function, $q(p)=bp^{-\eta}$, and isoelastic marginal cost function, $c(x)=cx^{-\gamma}$. In each succeeding example we (a) increase the elasticity of demand η, (b) change the growth function so that it is increasingly skewed to the left, and (c) reduce the elasticity of marginal cost γ. (The resulting demand, growth, and cost functions are all quite plausible.)

In each of the examples the extraction rule $q^*(x)$ is linear, and since $f(x)$ is concave, the steady-state expected extraction rate $\bar{q}^*_\infty$ falls as σ^2 increases (in Example 3, $\bar{q}^*_\infty = 0$ for any $\sigma^2 > 0$). By changing $f(x)$, η, and γ, we observe very different effects of uncertainty on rent and the rate of extraction across the examples. This in turn has interesting implications for the regularion of renewable resource stocks in the presence of uncertainty.

Example 1

In this first example we use the logistic function for $f(x)$, i.e.

$$f(x) = rx(1 - x/K) \tag{20}$$

where K is the carrying capacity. This function is symmetric around a maximum at $K/2$. We set the elasticity of demand, η, equal to $\frac{1}{2}$, and the elasticity of marginal cost, γ, equal to 2. Equation (14) then becomes:

$$\delta V = -2b(V_x + c/x^2)^{1/2} + [rx - (r/K)x^2]V_x + \tfrac{1}{2}\sigma^2 x^2 V_{xx}. \tag{21}$$

The solution to this equation is:[14]

$$V(x) = -\phi_1/x - \phi_1 r/\delta K \tag{22}$$

where

$$\phi_1 = \frac{2b^2 + 2b[b^2 + c(r + \delta - \sigma^2)^2]^{1/2}}{(r + \delta - \sigma^2)^2}.$$

Rent is therefore

$$V_x = \phi_1/x^2 \tag{23}$$

and the extraction rate is

$$q^*(x) = b(\phi_1 + c)^{-1/2}x. \tag{24}$$

Observe that $\partial\phi_1/\partial\sigma^2 > 0$. Stochastic fluctuations reduce V, the total social value of the resource, which must always be the case since V is concave (so that $A(x) > 0$). In this example, fluctuations also increase resource rent and therefore reduce the extraction rate at every stock level. To see why, let us first examine the steady-state equilibrium for this example.

It is easily shown that a non-degenerate steady-state distribution for x exists if

$$\sigma^2 < 2r - 2b(\phi_1 + c)^{-1/2}.$$

(Otherwise $x \to 0$ with probability 1.)[15] If this condition is met, x has the following gamma distribution in equilibrium:

$$\pi_\infty(x) = \frac{(2r/\sigma^2 K)^{(2\theta_1/\sigma^2 - 1)} x^{2\theta_1/\sigma^2 - 2} e^{-2rx/\sigma^2 K}}{\Gamma(2\theta_1/\sigma^2 - 1)} \tag{25}$$

where

$$\theta_1 = r - b(\phi_1 + c)^{-1/2},$$

and $\Gamma(\ )$ is the gamma function. From (25) we determine that

$$\bar{x}_\infty = K[1 - \sigma^2/2r - b/r(\phi_1 + c)^{1/2}] \tag{26}$$

and

$$\bar{q}^*_\infty = K[(1-\sigma^2/2r)b/(\phi_1+c)^{1/2} - b^2/r(\phi_1+c)]. \tag{27}$$

It is straightforward (but tedious) to show that $\partial\bar{x}_\infty/\partial\sigma^2 < 0$ and $\partial\bar{q}^*_\infty/\partial\sigma^2 < 0$.[16] Also, letting $c \to \infty$, we see that the expected steady-state unexploited stock is

$$\bar{x}^u_\infty = K(1-\sigma^2/2r),$$

which falls as σ^2 increases.

Because the growth function $f(x)$ is concave, an increase in σ^2 increases the physical scarcity of the resource stock by reducing its expected growth rate, and this increases rent. In this example, this increase in rent outweighs the decreases in rent associated with the convexity of $c(x)$ and the fact that $\sigma'(x) = \sigma > 0$. As a result, the extraction rate is reduced at every stock level x. As the next two examples show, increasing the elasticity of demand and skewing the growth function $f(x)$ to the left reduces the effect of changes in σ^2 on the physical scarcity of the resource, so that the overall effect on rent and the extraction rate can be reversed.

Example 2

In this example we use the Gompertz function for $f(x)$:

$$f(x) = rx \log(K/x) \tag{28}$$

where K is again the carrying capacity. Note that this function is skewed to the left relative to the logistic function of Example 1, and has a maximum at K/e. We set both the elasticity of demand η and the elasticity of marginal cost γ to 1. Equation (14) becomes:

$$\delta V = -b + b \log b - b \log(V_x + c/x) + rx \log(K/x) V_x + \tfrac{1}{2}\sigma^2 x^2 V_{xx}, \tag{29}$$

the solution to which is:

$$V(x) = \frac{b}{(r+\delta)} \log x + \phi_2 \tag{30}$$

where

$$\phi_2 = \frac{1}{\delta}\left\{ b \log\left[\frac{b(r+\delta)}{1+(r+\delta)c}\right] - b + (r \log K - \sigma/2)/(r+\delta)\right\}.$$

Resource rent V_x and the extraction rate $q^*(x)$ are given by:

$$V_x = b/(r+\delta)x \tag{31}$$

and

$$q^*(x) = \frac{b(r+\delta)}{b+(r+\delta)c} x. \tag{32}$$

Now rent and the extraction rate are unaffected by stochastic fluctuations (although the total value of the resource stock is again reduced, since $\partial\phi_2/\partial\sigma^2 < 0$). Compared to the logistic function, the Gompertz function rises more rapidly for small x, and then falls more slowly. This and the more elastic demand function increase the extent of market self-correction when the resource stock falls randomly. This can also be seen from the steady-state properties.

A non-degenerate steady-state distribution exists if

$$\sigma^2 < 2r \log K - 2b(r+\delta)/[b+(r+\delta)c],$$

and since $\sigma^2 \geqq 0$, this also requires $K > 1$ and

$$b < r \log K[b+(r+\delta)c]/(r+\delta).$$[17]

(Again, $x \to 0$ with probability 1 otherwise.) The steady-state distribution is:

$$\pi_\infty(x) = (m/\sigma^2)x^{-(r/\sigma^2)\log x + 2\theta_2/\sigma^2 - 2} \tag{33}$$

where

$$\theta_2 = r \log K - b(r+\delta)/[b+(r+\delta)c],$$

and

$$m = \sigma^3(\pi/r)^{1/2} \exp(-\theta_2^2/r\sigma^2 + \theta_2/r - \sigma^2/4r).$$

The steady-state expected stock and extraction rate are therefore given by:

$$\bar{x}_\infty = Ke^{-b(r+\delta)/r[b+(r+\delta)c] - \sigma^2/4r} \tag{34}$$

$$\bar{q}_\infty^* = b(r+\delta)\bar{x}_\infty/[b+(r+\delta)c]. \tag{35}$$

Again, $\partial \bar{q}_\infty^*/\partial \sigma^2 < 0$ because of the concavity of $f(x)$. However, if the conditions needed for a non-degenerate steady-state distribution are satisfied, $\partial \bar{q}_\infty^*/\partial \sigma^2$ is smaller in magnitude than in Example 1.[18] Here the scarcity-related increase in rent resulting from an increase in σ^2 is smaller, and is just offset by the reduction in rent due to convex extraction costs and the risk premium. The net result is that at any current stock level x, rent and the extraction rate are unchanged as σ^2 increases, even though the steady-state expected stock and extraction rate fall.

Also observe that the parameters can be such that x and q fall to zero with probability 1 if the stock is exploited, but x has a non-degenerate distribution in the absence of exploitation. This is the case if

$$2r \log K - 2b(r+\delta)/[b+(r+\delta)c] < \sigma^2 < 2r \log K. \tag{36}$$

Thus it may be socially optimal to exploit a resource stock to extinction when there are stochastic fluctuations, although this would not be the case in a world of certainty.

Example 3

This example shows how stochastic fluctuations can increase the rate of extraction. The growth function is now:

$$f(x) = rx^{1/2} - rx/K^{1/2}. \tag{37}$$

K is again the carrying capacity, and the function has its maximum at $K/4$, so it is even more skewed to the left. For small x, the expected (absolute) change in the stock is of order $x^{1/2}$ but the standard deviation of that change is of order x. As a result, the unexploited resource stock does not have a non-degenerate steady-state distribution, i.e. for any $\sigma^2 > 0$, x will eventually fall to zero. The exploited stock will likewise fall to zero, but we can ask how the rate of extraction (along the path to $q = 0$) changes as σ^2 rises.

To complete the example, we set the elasticity of demand $\eta=2$, and the elasticity of marginal cost $\gamma=1/2$. Equation (14) becomes

$$\delta V = b(V_x + c/x^{1/2})^{-1} + r(x^{1/2} - x/K^{1/2})V_x + \tfrac{1}{2}\sigma^2 x^2 V_{xx}, \tag{38}$$

which has the solution:

$$V(x) = 2\phi_3 x^{1/2} + r\phi_3/\delta \tag{39}$$

where

$$\phi_3 = -c/2 + \tfrac{1}{2}[c^2 + 4b/(2\delta + r/K^{1/2} + \sigma^2/4)]^{1/2}.$$

Rent and the extraction rate are therefore:

$$V_x = \phi_3 x^{-1/2} \tag{40}$$

and

$$q^*(x) = \frac{bx}{(\phi_3 + c)^2}. \tag{41}$$

Since $\partial\phi_3/\partial\sigma^2 < 0$, stochastic fluctuations reduce rent and *increase* the extraction rate at any stock level x. This is a consequence of the larger elasticity of demand and more skewed growth function. Here an increase in σ^2 again increases rent through its effect on resource growth,[19] but this is outweighed by the reduction in rent occurring because $c''(x) > 0$ and $A(x) > 0$.

It is also interesting to compare these results with those for a monopolist resource owner. We can do that in this example because the elasticity of demand exceeds unity. For the monopoly case, price $p(q)$ in equations (13) and (14) is replaced by the marginal revenue function $MR(q)$, and then (14) is solved as usual for $V(x)$. The result is:

$$V^m(x) = 2\phi_{3m} x^{1/2} + r\phi_{3m}/\delta \tag{42}$$

where

$$\phi_{3m} = -c/2 + \tfrac{1}{2}[c^2 + 3b/(2\delta + r/K^{1/2} + \sigma^2/4)]^{1/2}.$$

Rent and the extraction rate are:

$$V_x^m = \phi_{3m} x^{-1/2} \tag{43}$$

and

$$q_m^*(x) = \frac{bx}{4(\phi_{3m} + c)^2}. \tag{44}$$

Production is lower than in the competitive case, and rent is smaller. (The monopolist views the marginal *in situ* unit as less scarce because he will extract less anyway.) The total value of the resource stock, V^m, is also lower, as we would expect. Uncertainty, however, has the same effect as in the competitive case, reducing rent and increasing the extraction rate for any stock level.

5. CONCLUSIONS

1. As for an exhaustible resource, an *in situ* unit of renewable resource stock is an asset which, in a competitive market, must yield a total return that is competitive. Stochastic

fluctuations in the stock add a risk premium to that required return, to the extent that the marginal unit contributes to the variance of the fluctuations. Also, because marginal extraction cost is a convex function of the stock, stochastic fluctuations will on average increase that cost over time. At any stock level, these effects reduce rent and increase the rate of extraction.

2. Whether the resource stock eventually fluctuates around a non-zero steady-state expected value or falls to zero, an increase in the variance of stock fluctuations reduces the expected growth rate of the stock. This increases rent and reduces the extraction rate. As a result, the overall effect of an increase in variance may be to decrease the extraction rate $q^*(x)$, increase it, or leave it unchanged. As the examples demonstrate, when the growth function is more skewed to the left (i.e. higher rates of growth at lower stock levels) and demand is more elastic, the result is more likely to be an increase in the extraction rate. This is intuitively appealing in that a skewed growth function and more elastic demand function provide stronger homeostatic mechanisms for the correction of random decreases in the stock.

3. This has implications for the degree of regulation needed when property rights cannot be assigned and maintained—the case for many resource stocks. The conventional wisdom has been that ecological uncertainty increases the need for regulation. However, we have seen in Section 5 that more or less regulation may be in order depending on the extent and speed of self-correction in the system, i.e. depending on the biology of the resource and the characteristics of market demand. If the natural recovery of the resource stock after a sharp drop is rapid, and if demand is sufficiently elastic so that the extraction rate falls when the stock is low, then *less* regulation may be needed. Furthermore, there may be instances when it is optimal to allow a competitive market to eventually drive the resource stock to zero, even though that might not be the case in a world of certainty. Uncertainty may have important implications for regulation, but those implications will have to be carefully worked out for the particular resource market in question.

Now some caveats are needed, particularly with respect to our conclusions regarding regulation. The model developed here contains a number of simplifications, some of which may overstate the degree of homeostasis that actually exists in renewable resource markets.

Perhaps most important, in this model stochastic fluctuations in x are continuous in time, and the current resource stock x can be observed without error. The optimal extraction rate $q^*(x)$ is likewise assumed to adapt continously over time. This kind of continuous observation and adaptation may be too much to expect, however, in actual markets. If stock observations occur with error and significant lags, if demand and extraction respond only slowly to price changes, and if firms make optimization errors, the "second-best" extraction policy might be much more conservationist. This is especially likely if the resource growth function is such that a small amount of "over-harvesting" (e.g. following a stochastic drop in stock size) can result in a catastrophic collapse of the stock.

Further work is needed to deal with such problems as errors and lags in stock observations, dynamically sub-optimal behaviour on the part of resource firms, and other aspects of market structure that could exacerbate the effect of uncertainty, or otherwise increase the chances for catastrophic over-exploitation of an unregulated open-access stock. One approach that may provide insight into these problems is to introduce investment in quasi-fixed capital stocks (e.g. fishing fleets) into models such as this one. Such capital stocks impose rigidities on extraction levels, rigidities which have been shown in deterministic models to (temporarily) cause severe over-exploitation.[20] Introducing quasi-fixed capital may greatly complicate a stochastic model, but results might be obtained via numerical approximation techniques, as in Ludwig (1979).

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NOTES

1. The literature in this area is a large one. For a clear and thorough introduction and overview, see Clark (1976). Detailed discussions of these issues can also be found in Beddington, Watts and Wright (1975), Berck (1979), Clark (1973), Clark and Munro (1975), and Lewis and Schmalensee (1977).

2. See Berck (1981) and Levhari, Michener and Mirman (1981). These papers are more in the spirit of the exhaustible resource literature.

3. See, for example, Beddington and May (1977), Goel and Richter-Dyn (1974), May (1974), May, Beddington, Horwood and Shepherd (1978), Reed (1978), and Tuckwell (1974).

4. In a sense, Smith's (1978) work is an exception, since he maximizes the expected flow of discounted utility from consumption, and ignores harvesting costs. His extraction rate is therefore the same as that for a competitive market (with property rights) and an endogenous price equal to marginal utility. In related work, Hutchinson and Fischer (1979) obtain approximate numerical solutions for the maximization of the expected flow of discounted net revenues (taking price fixed), for alternative stochastic versions of the logistic model. Reed (1979), using a discrete-time model and again taking price as fixed, shows that under broad cost assumptions the optimal harvesting policy is given by a feedback rule that forces the resource stock back to an optimal escapement level as rapidly as possible. Also, Smith (1980) derives the steady-state distribution for a logistic growth function with fixed harvesting effort, and then estimates the parameters (including the effect of uncertainty) using data for a particular fishery. Andersen (1981) uses Smith's steady-state distribution results to examine the effects of uncertainty, but in a static (steady-state equilibrium) context.

5. Some authors make marginal extraction cost constant (or zero), and impose the condition $x \geqq 0$. But this simply means that marginal extraction cost is *highly* convex, e.g. is of the form $c(x) = c_0 + c_1 x^{-n}$, with $n \to \infty$. It is more realistic to make $n > 0$, but finite.

6. Of course this requires that property rights could be assigned and enforced. In either case, rent represents the social value of the unit, and best measures *in situ* resource scarcity, as it does for an exhaustible resource. For discussions, see Brown and Field (1978) and Fisher (1979).

8. See, for example, Goel and Richter-Dyn (1974), May (1974), Turelli (1977), and Tuckwell (1974). Also, Merton (1975) used an Ito process of this type to describe the labour force dynamics in a stochastic model of economic growth. In most applications $\sigma(x) = \sigma x$, i.e. future values of x are lognormally distributed. Of course there are resource markets for which this description of uncertainty would be a poor one; for example, a seasonal fish stock like salmon.

9. That is, uncertainty does not change the fact that the competitive equilibrium is optimal (if firms are risk-neutral). This is shown formally and in general by Lucas and Prescott (1971). Of course for many resource markets this model of perfect competition with property rights will not apply, and we must then view the solution as an "ideal", and a guideline for regulatory policy.

10. Using Ito's Lemma, write the stochastic differential dV^i as:

$$dV^i = V^i_x dx + \tfrac{1}{2} V^i_{xx} (dx)^2.$$

Substituting (7) for dx, and noting that $q = nq^i$ and $E_t(dz) = 0$,

$$E_t dV^i = [f(x) - nq^i] V^i_x dt + \tfrac{1}{2}\sigma^2(x) V^i_{xx} dt.$$

Equation (10) is obtained by substituting this expression in eqn. (9). For an introduction to the techniques used in this paper, see Chow (1979) and Merton (1971, 1975).

11. The value function V is a derived utility function, and $A(x)$ measures its curvature. Firms are risk-neutral, but since Π is concave, V is concave, and $A(x) > 0$.

12. Recall that we restrict $\sigma'(x)$ to be positive so that the resource stock is always non-negative. If $\sigma'(x) = 0$ the expected rate of return dynamics is the same as in the deterministic case, i.e. for any x, the expected percentage rate of growth of rent is unaffected by uncertainty—although the *level* of rent may differ.

13. This can be seen by writing dc as:

$$dc = c'(x) dx + \tfrac{1}{2} c''(x)(dx)^2 \tag{i}$$

so that

$$(1/dt) E_t dc = c'(x)[f(x) - q^*] + \tfrac{1}{2}\sigma^2(x) c''(x). \tag{ii}$$

Substituting (ii) into equation (18) and rearranging yields an equation for the expected dynamics of price analogous to equation (5) for the deterministic case:

$$(1/dt)E_t dp = \delta(p-c) - f'(x)(p-c) + c'(x)f(x) + \sigma'(x)\sigma(x)A(x)(p-c) + \tfrac{1}{2}\sigma^2(x)c''(x). \quad \text{(iii)}$$

Since $c(x)$ is convex by assumption, the last term in (iii) is positive, and for any value of x has the effect of increasing the expected rate of growth of price. This result also applies to an exhaustible resource when there are stochastic reserve fluctuations and marginal extraction cost is a convext function of reserves. Note that setting $f(x) = f'(x) = \sigma'(x) = 0$ in equation (iii) gives eqn. (15) on page 1209 of my earlier paper (1980).

14. Smith (1978) solves a problem identical to this, except with zero extraction cost. Also note that in this example, $\int p(q)dq = -b^2/q + A$. We set the arbitrary constant $A = 0$, which is why $V(x)$ is negative (but increasing in x).

15. A sufficient condition for a degenerate distribution is that $\int_0^\infty \pi_\infty(x)dx$ is unbounded.

16. If $c = 0$, $\partial \bar{x}_\infty / \partial \sigma^2 = 0$ and $\partial \bar{q}_\infty^* / \partial \sigma^2 < 0$, i.e. the reduction in the extraction rate is just sufficient to leave the expected stock unchanged.

17. The latter requirements result from the fact that as $x \to 0$, the natural growth rate of the stock approaches $r \log K$.

18. This is easiest to see by setting $c = 0$ in both examples (for algebraic simplicity), and applying the necessary conditions for a non-degenerate steady-state distribution.

19. The stock will eventually fall to zero even if there is no extraction. However, rent, the relevant measure of scarcity, is the sacrifice (i.e. reduced PDV of future profits) that results from extraction of the marginal *in situ* unit.

20. See, for example, Clark, Clarke and Munro (1979).

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