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TIME TO BUILD, OPTION VALUE, AND INVESTMENT DECISIONS*

Saman MAJD

University of Pennsylvania, Philadelphia, PA 19104, USA

Robert S. PINDYCK

Massachusetts Institute of Technology, Cambridge, MA 02139, USA

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Investment decisions and outlays are often made sequentially. For example, the rate at which construction proceeds is usually flexible and can be adjusted with the arrival of new information. Traditional discounted cash flow methods which treat the pattern of investment as fixed ignore this flexibility and understate the value of the project. This paper uses contingent claims analysis to derive optimal decision rules and to value such investments. We determine the effects of time to build, opportunity cost and uncertainty on the investment decision. For reasonable parameter values, we show how a simple NPV rule can lead to gross errors.

1. Introduction

Many investment projects have the following characteristics: (i) investment decisions and associated cash outlays occur sequentially over time, (ii) there is a maximum rate at which outlays and construction can proceed – it takes 'time to build' – and (iii) the project yields no cash return until it is completed. The firm's investment problem is to choose a contingent plan for making these sequential – and irreversible – expenditures over time. The arrival of new information might lead the firm to depart from the spending scenario originally planned; the firm might accelerate or decelerate the rate of investment, or simply stop the program in midstream.

Examples of industries for which these characteristics are especially important include aircraft and mining. The production of a new line of aircraft requires engineering, prototype production, testing, and final tooling stages that together can take eight to ten years to complete. The construction of a

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new underground mine, or the development of a large petrochemical plant are projects that usually require at least five or six years, with clear constraints on the pattern of expenditures. In other industries the lead times may be somewhat shorter, but are still important.

Traditional discounted cash flow criteria, which treat the spending pattern as fixed, are inadequate for evaluating such projects. Likewise, neoclassical investment theory,[1] which treats individual units of capital as homogeneous, interchangeable, and individually productive, fails to provide a realistic description of investment behavior under uncertainty. Adapting the neoclassical framework by introducing adjustment costs, whereby the cost of new capital rises with the rate of investment, does not deal with the fundamental problem – most real projects are composed of heterogeneous units of capital that must be installed in sequence, and are unproductive until the project is complete.[2] Indeed, the importance of sequential investment and time to build have been demonstrated by Kydland and Prescott (1982) in the context of a general equilibrium model. They have suggested that such a model yields a much better description of cyclical fluctuations than does the standard adjustment cost framework.

Our paper should be viewed in the context of several recent strands of research, all of which have helped to provide a better microeconomic foundation for investment behavior. First, Roberts and Weitzman (1981) examine projects with sequential outlays using a model that stresses the role of information gathering. In their model, each stage of investment yields information that reduces the uncertainty over the value of the completed project. This is most applicable to R&D projects in which the role of learning is critical.[3] Since the project can be stopped in mid-stream, it might pay to go ahead with the early stages of the project even though *ex ante* the net present value of the entire project is negative. Hence the use of a net present value rule for such projects, particularly one based on a single risk-adjusted discount rate, might reject investments that should be undertaken.

Second, in related papers, Bernanke (1983) and Cukierman (1980) examine investment decisions for which information about project value arrives independently of the cash outlays. They consider incentives to postpone expenditures until more information arrives. In their models the project involves a single expenditure, and there is no time to build. However, the investment expenditure is irreversible, a firm can choose only a subset of the available projects (so that investing in one set of projects excludes all others) and (unlike in Roberts and Weitzman) the firm obtains information before beginning the project. They show that uncertainty over project returns creates an incentive

[1] See Hall and Jorgenson (1967).

[2] For an overview of the adjustment cost literature, see Nickell (1978).

[3] For an application to synthetic fuels, see Weitzman, Newey and Rabin (1981).

(an 'option value') to postpone the investment and wait for more information to arrive, even if the firm is risk neutral.[4] This is just the opposite result from that in Roberts and Weitzman; here a naive net present value rule might *accept* projects that should be rejected or postponed. Both authors use their models to explain the cyclical nature of aggregate investment spending; a recession is associated with greater uncertainty over future cash flows because firms reduce their investment spending until some of that uncertainty is resolved.

The models developed in Roberts and Weitzman, Bernanke, and Cukierman are not explicitly based on valuation in financial markets. Thus, a manager following their investment criteria may not be maximizing the firm's value to stockholders. For example, although the assumption of risk neutrality allows Bernanke and Cukierman to underscore the effects of irreversibility, as distinct from risk aversion, extending their models to a more general setting is not straightforward; the correct risk premium cannot be determined independently of the optimal decision rule.

The third strand of work, and that most closely associated with this paper, is best represented by McDonald and Siegel (1986).[5] They also stress the option value of postponing an irreversible investment, but not as a means of accumulating information. Instead, the payoff from completing the project has a current value consistent with capital market equilibrium. This value fluctuates stochastically over time (independently of any investment expenditures), so that its *future* value is always unknown. Access to the investment opportunity (perhaps purchased or obtained as the result of R&D) is analogous to holding a call option on a dividend-paying common stock, where 'exercising' the option is equivalent to making the investment expenditure. As with such financial options, increased risk increases the incentive to delay the investment expenditure, and for any positive amount of risk, the expenditure is made only when the project's value exceeds costs by a positive amount. These results are similar to those in Bernanke and Cukierman, but for a different reason.

This paper is also concerned with the option value of being able to delay irreversible investment expenditures, but here we focus on a series of expenditures that must be made sequentially, that *cannot exceed some maximum rate*, and that become productive only after the entire sequence is completed. For example, a project requiring a total outlay of $5 million might have a

[4] This notion of an 'option value' is quite different from the one that we develop in this paper. In Bernanke, as in earlier papers such as Arrow and Fisher (1974) and Henry (1974), the option refers to a choice of projects (or irrevocable disposition of a natural resource in Arrow and Fisher) that is foregone once the expenditure has been made.

[5] Related papers include McDonald and Siegel (1985) and Paddock, Siegel and Smith (1983). Also, Baldwin (1982) analyzes sequential and irreversible investment decisions when investment opportunities arrive randomly. She values the entire sequence of opportunities and shows that, as in McDonald and Siegel (1986), a simple discounted cash flow rule can lead to over-investment.

maximum rate of investment of $1 million per year, so that the minimum time to build is 5 years. Such a project can be viewed as a *compound option*: each unit of investment buys an option on the next unit. Evaluating the project requires a decision rule that determines whether an additional dollar should be spent *given any arbitrary cumulative amount that has already been spent*. That decision rule will depend on the underlying value that the project would have *today* if completed, the remaining expenditure required for completion, as well as parameters describing risk and the opportunity cost of delaying completion.

This paper is in the spirit of recent work on capital budgeting with option-equivalent cash flows.[6] Our approach assumes that the value of a completed project is spanned by a set of traded assets and that the distribution of future values is given. Option pricing techniques are used to derive the relationship between the value of the investment program (what a firm would pay for the right to undertake the program) and the value of the project once completed.

We have several objectives. First, we show how a decision rule, applicable to each stage in the development of a project, can be derived and applied to project evaluation. Second, we show how the value of an investment program and the decision to invest depend on the maximum rate at which expenditures can productively be made (i.e., on the 'time to build'). Finally, we will see how time to build interacts with uncertainty to affect investment spending, and in particular, how the depressive effect of increased uncertainty on investment spending is magnified.

The next section describes the nature of the investment program, and our assumptions regarding the distribution of future values of a completed project. It also outlines our approach to deriving the optimal investment rule. Section 3 presents numerical results for a simple example that shows how risk, opportunity cost, and time to build interact to affect the investment decision. Section 4 uses the model to examine the economic value of construction time flexibility. The concluding section discusses some implications of our results for aggregate investment behavior.

2. A simple model of investment when there is time to build

2.1. The model

Consider a program to build a factory. The program involves a sequence of investment outlays, corresponding to the specific steps involved in construction. The payoff to completing the program is the market value of a *completed* factory. This market value is the present value of the stream of uncertain future cash flows from operating the factory. The owner of the factory,

[6] For related work, see Myers and Majd (1983) and Brennan and Schwartz (1985). For an overview, see Mason and Merton (1985).

receiving these cash flows, earns an equilibrium rate of return as determined by the market.

Note that we are *not* assuming that shares in identical factories are traded in the market and, therefore, have an observable price. We are only assuming that we could calculate the value that would prevail if such shares were traded by applying appropriate capital budgeting methods to the cash flows from the completed factory. This market value will, of course, fluctuate stochastically over time, reflecting new information about future cash flows.

We take the market value of the completed factory, denoted by V, as exogenous, and assume that, during the construction period, it evolves according to the lognormal process:

$$dV = (\mu - \delta)V dt + \sigma V dz, \tag{1}$$

where dz is the increment of a Weiner process. The last term in (1) characterizes the unexpected component of changes in V. The central feature is that future values of V are *always* uncertain, and are distributed lognormally. The degree of uncertainty depends only on how far into the future one looks. Unlike the stylized R&D projects of Roberts and Weitzman where learning takes place at each stage of investment, uncertainty about future values of V is independent of the proportion of the project already completed.[7] Nor is such uncertainty resolved by waiting, as in the models of Bernanke and Cukierman.

The parameter μ is the expected rate of return from owning a completed factory. It is the equilibrium rate established by the capital market, and includes an appropriate risk premium. Eq. (1) says that the expected rate of capital gain on the factory is less than μ; δ represents the opportunity cost of delaying completion of the project.

If the completed factory is infinitely lived, then eq. (1) also represents the evolution of V during the operating period. Specifically, δV will represent the instantaneous rate of cash flow from operating the factory. Because these payouts are not received until construction is completed, δ is the rate of opportunity cost. We assume that δ is constant. In the case of an infinitely lived project, this is consistent with future cash flows being a constant proportion of the market value of the operating factory.[8]

Eq. (1) is an abstraction from most real projects. If variable costs are positive and managers have the option to shut down temporarily when the price of the output is below variable cost, and/or the option to abandon the project

[7] We could introduce learning in our model by making σ a function of the stage of completion. Letting K denote the total amount of investment remaining for completion, we would make $\sigma = \sigma(K)$, $\sigma'(K) > 0$, and $\sigma(0) = 0$. We ignore learning in this paper in order to focus on the implications of time to build.

[8] A constant payout rate, δ, and required return, μ, imply infinite project life:

$$V_0 \equiv \int_0^T \overline{CF}_t e^{-\mu t} dt = \int_0^T \delta V_0 e^{(\mu-\delta)t} e^{-\mu t} dt \quad \Rightarrow \quad T = \infty.$$

Note that this also implies that the expected rate of change of V is $\mu - \delta$.

completely, V will not follow a lognormal process even if the price of the output of the factory does.[9] If variable cost is positive and managers do *not* have the option to shut down (perhaps because of regulatory constraints), V can become negative, again in conflict with the assumption of lognormality. Here we ignore these possibilities (and any other options implicit in operating the completed factory) in order to focus on the options implicit in the construction phase.

There are few real projects that last forever. In principle, future investments (e.g., maintenance) can be made to maintain the productivity of the project indefinitely. To be consistent with eq. (1), we would interpret the cash payout δV, as being net of these future investments. (Another way of handling maintenance investments is discussed below.) But not all projects can be made to last forever by appropriate future maintenance investments. For example, most natural resource projects have finite lives because there is a finite quantity of reserves in the ground. For many oil and gas wells, it is common to assume an exponentially declining extraction rate, but since the price of the natural resource will not be constant, cash flows will not be a constant proportion of project value.

If the completed factory has a finite life, then eq. (1) cannot represent the evolution of V during the operating period. In particular, cash flows from operating the factory will not be a constant proportion of the market value of the factory: the last cash flow is 100 percent of the remaining value. However, under the assumptions discussed above (i.e., no operating costs or operating options during the life of the factory), eq. (1) will still represent the evolution of V during the construction period, but with a different interpretation of the constant, δ.

This can be demonstrated by the following example. Assume that the price of the output of the factory, P, evolves according to the lognormal process $\mathrm{d}P = (\mu - \delta)P\mathrm{d}t + \sigma P \mathrm{d}z$, where μ is the equilibrium rate of return on a security that is perfectly correlated with P. (If the output can be stored, δP is the instantaneous convenience yield from storage, i.e., the flow of convenience benefits from holding inventory.) Let τ be the time at which construction is complete, and T be the operating life of the factory. The value of the factory at completion (i.e., the payoff to the investment program) is given by

$$V(\tau) = \int_0^T \mathrm{E}_\tau\{P(s)\}\mathrm{e}^{-\mu s}\mathrm{d}s = \int_0^T P(\tau)\mathrm{e}^{(\mu-\delta)s}\mathrm{e}^{-\mu s}\mathrm{d}s$$

$$= P(\tau)\left[(1-\mathrm{e}^{-\delta T})/\delta\right] \equiv \phi P(\tau),$$

where ϕ is a constant that depends on δ and T.

[9]For analyses of these options, see McDonald and Siegel (1985), Brennan and Schwartz (1985) and Myers and Majd (1983).

Since the payoff to completing the investment program is proportional to the price of the output at completion, its present value at any earlier time (i.e., before completion) will also be proportional to the contemporaneous price of the output: $V(t) = \phi P(t)$ for $t \leq \tau$. Hence $\mathrm{d}V = (\mu - \delta)V\mathrm{d}t + \sigma V\mathrm{d}z$, during the construction period. Here, δ is still a rate of opportunity cost, but arises from the fact that the expected growth rate of price, and hence of V, is less than the risk-adjusted return on a security that has the same risk as P. Of course the value of an *operating* factory will not evolve according to eq. (1); it will decay faster, and at a time-varying rate, because of the cash flows from operations.[10] But this is irrelevant for the investment decision, which depends only on the dynamics of V up to the time construction is complete.

The example above demonstrates the dynamics for V during the construction period [eq. (1)] do not preclude a completed project with finite life. Of course, eq. (1) will not be appropriate for all projects. More elaborate dynamics, based on different assumptions about the dynamics of output price, operating costs, etc., may be necessary. We proceed with the assumption that the dymanics of V during the construction period are given by eq. (1), in order to focus on the effects of time to build.

If, for some period of time, the payoff to completing the factory is expected to grow at the rate μ (i.e., $\delta = 0$), there will be no opportunity cost from delaying construction, but there will be a savings from delaying the investment expenditure. Hence investment will not occur during such a period. It is because the value of most real projects grows at an expected rate less μ that there is an incentive to invest.

An important assumption in our model is that the factory cannot be built overnight. There is a maximum rate at which construction and investment can proceed – it takes time to build. Because completion of the project requires some minimum amount of time, the payoff from completion is unknown during the construction period. However, we assume that the total required investment is known.

We assume that the *minimum* rate of construction and investment is zero, and that construction can be halted and later resumed without cost. In reality, we would expect some continuing costs associated with maintaining the partially completed factory (e.g., to prevent 'rusting'), and with maintaining the capital and labor resources needed to resume construction. We also assume that investment is completely *irreversible*; capital in place has no alternative use, and therefore zero salvage value. For simplicity, we ignore these added features, although it is straightforward to extend our model to include them.

[10] During the operating period ($t > \tau$) we have

$$V(t) = \int_0^{T+\tau-t} P(s)\mathrm{e}^{(\mu-\delta)s}\mathrm{e}^{-\mu s}\,\mathrm{d}s = P_t\left[(1-\mathrm{e}^{-\delta(T+\tau-t)})/\delta\right] \equiv \phi(t)P(t).$$

From Ito's Lemma, $\mathrm{d}V = (\mu - \delta + \phi_t/\phi)V\mathrm{d}t + \sigma V\mathrm{d}z$, where the proportional rate of change, ϕ_t/ϕ, is strictly negative and varies over time.

For expositional ease, in the example in section 3, we assume that the maximum rate of investment is constant. This is unrealistic for most real projects, where the constraints on the maximum rate of investment generally depend on the stage of construction. Our model allows the maximum rate of investment to be a (known) function of the amount of total investment remaining.

Allowing the maximum rate of investment to vary with the stage of construction also provides a way to account for any future investments required to maintain an infinite life for the completed project. If we assume that the timing and magnitude of these maintenance investments (during the infinite life of the factory) are known with certainty, we can include their present value as a require component of the investment in the final stage of construction. In other words, in the last instant before construction is completed, the investment required to complete the project includes the present value of the future maintenance investments.

To see how the constraint of time to build affects investment decisions, we must determine the *market value of the entire investment program*. This market value is what a value-maximizing firm would pay for the right to undertake the program. It will correspond to an optimal program of investment outlays, which will, of course, be contingent on the evolution of V.

We can characterize this investment decision as an optimal control problem. There are two state variables, the total amount of investment remaining for completion, K, and the current market value of a completed factory, V. The control variable is the rate of investment, I. The problem is to choose the control rule, $I^*(V,K)$, which maximizes the value of the investment program. $I^*(V,K)$ is simply a rule that determines the optimal rate of investment, given the instantaneous values of V and K. It is subject to the constraint $0 \leq I^*(V,K) \leq k$, where k is the maximum rate of investment.

Because there are no adjustment costs or costs associated with changing the level of investment, the problem has a 'bang-bang' solution: the instantaneous level of investment will be either 0 or k. In turn, the optimal decision rule reduces to a cutoff value for a completed project, $V^*(K)$, such that investment occurs at the maximum rate k for $V > V^*$, and there is no investment otherwise. As we will see, the optimal decision rule $V^*(K)$ is determined simultaneously with the current market value of the investment program.

2.2. Solution

The equilibrium market value of the investment program and the optimal current level of investment, I^*, will depend on the values of the two state variables, V and K. In our model I^* is either k or 0, depending on whether the current value of V is above or below the cutoff value, $V^*(K)$. We will find it convenient to denote the value of the investment program when $V > V^*$ (upper region) by $F(V, K)$, and when $V < V^*$ (lower region) by $f(V, K)$.

Formally, the investment program is a contingent claim. However, it is not a simple contingent claim: at every instant the manger can choose whether or not to invest and continue construction. Hence the project is a compound option, where each expenditure buys an option to make the next expenditure. Although this complicates the solution procedure, the same techniques used to value options in securities markets can be applied to value the investment program.

Using a continuous time framework, Merton (1977) derives the valuation equations for general contingent claims. His approach relies on continuous trading of specified assets to replicate the payoff to the contingent claim. Nevertheless, this approach is also valid when the assets that must be included in the replicating portfolio are not traded in financial markets. What is necessary is a capital market sufficiently complete that the new project does not change the opportunity set available to investors. If this is the case, managers need only calculate the value of the underlying asset, V, that is consistent with the equilibrium valuation model implied by the capital market. For example, if the CAPM holds and the manager can estimate the underlying asset's beta from prices of traded securities, then he can correctly calculate V. Also, given the relationship that must hold between the values of traded options and stocks, he can calculate the value of any contingent claim on V (e.g., this investment program).

Since the market value of the completed factory includes the value of any subsequent operating options, the value of these options must be included in the calculation of V. For example, the manager might have the option of shutting down (temporarily or permanently) the completed factory. Thus the calculation of V might involve more than a simple discounted cash flow analysis. As mentioned above, including the value of such operating options generally will affect the dynamics of V.

The option pricing approach yields a valuation equation relating the value of the contingent claim (the investment program) to the value of the underlying asset (the completed factory). Since the value of the investment program depends on whether the value of the completed factory, V, is above or below V^*, for notational convenience we write a separate valuation equation for each region, i.e., for $F(V, K)$ and $f(V, K)$. It is straightforward to show that F and f must satisfy the following partial differential equations:

$$\left(\tfrac{1}{2}\right)\sigma^2 V^2 F_{VV} + (r-\delta) V F_V - rF - kF_K - k = 0, \tag{2a}$$

$$\left(\tfrac{1}{2}\right)\sigma^2 V^2 f_{VV} + (r-\delta) V f_V - rf = 0, \tag{2b}$$

subject to the boundary conditions

$$F(V,0) = V, \tag{3a}$$

$$\lim_{V\to\infty} F_V(V,K) = e^{-\delta K/k}, \tag{3b}$$

$$f(0,K) = 0, \tag{3c}$$

$$f(V^*,K) = F(V^*,K), \tag{3d}$$

$$f_V(V^*,K) = F_V(V^*,K). \tag{3e}$$

The first boundary condition states that when the project is completed, the value of the investment program is the market value of a completed factory.

As the value of the completed project becomes very large relative to the total investment K, the option 'premium' becomes negligible, and the value of the program approaches the value of the completed project. However, the value of the investment program will increase less rapidly than the value of a completed project. As V becomes large, construction outlays will be made at the maximum rate, k, but there is still a foregone opportunity cost. Hence for very large V, the increase in the value of the investment program for a 1 dollar increase in V is given by

$$1 - \int_0^{K/k} \delta\, e^{(\mu-\delta)t} e^{-\mu t} dt = e^{-\delta K/k}.$$

This condition is shown as (3b) above.

Condition (3c) states that the minimum value of the investment program is zero, and is reached when V is zero. Finally, conditions (3d) and (3e) require that the value of the investment program be continuous and differentiable at the cutoff value V^*.[11]

Eq. (2b) has the analytic solution

$$f(V) = aV^{\alpha}, \tag{4}$$

where

$$\alpha \equiv \left\{ -(r-\delta-\sigma^2/2) + \left[(r-\delta-\sigma^2/2)^2 + 2r\sigma^2\right]^{1/2} \right\} \Big/ \sigma^2.$$

The coefficient a must be determined jointly with the solution for F in the upper region, via the shared boundary conditions (3d) and (3e). This would be

[11] See Merton's (1973) footnote 6 regarding (3e). Intuitively, if a small change in the value of the contingent claim in response to a small change in the value of the underlying asset is greater in one direction than another, moving the free boundary in that direction would result in a net increase in the value of the contingent claim.

straightforward if eq. (2a) also had an analytical solution; since it does not, a numerical approach is required.

First, we eliminate a using eq. (4) and the boundary conditions (3d) and (3e):

$$F(V^*, K) = (V^*/\alpha) F_V(V^*, K). \tag{5}$$

Then we numerically solve eqs. (2a), (2b), and the conditions (3a)–(3c) and (5) using a finite difference method. This procedure transforms the continuous variables V and K into discrete variables, and the partial differential equations into finite difference equations. These equations can be solved algebraically, and the solution proceeds as a backward dynamic program which incorporates the optimal investment decisions at each point. Hence the cutoff value, $V^*(K)$ (the optimal boundary between the two regions), is solved for simultaneously with the value of the investment program.[12] (Details of this procedure are in an appendix, which is available from the authors on request.)

The reader might note that the investment problem posed above is one of stochastic dynamic programming. Indeed, eqs. (2a) and (2b) are the Bellman equation under risk neutrality. As Cox and Ross (1976, pp. 153–155) have explained, given the current market value of the underlying asset, V, the contingent claims approach is equivalent to dynamic programming with risk neutrality. Any adjustment for risk is embodied in V, so that it is not necessary to know the risk-adjusted rate of return on the contingent claim.

In the next section we apply the solution procedure to a simple and stylized example. This serves to illustrate how the procedure works, and how time to build and uncertainty interact to affect investment decisions.

3. A numerical example

Consider an infinitely-lived project that requires a total investment (K) of \$6 million, which can be spent productively at a rate no faster than \$1 million per year (k). We assume the riskless rate of interest (r) is 2% per year. The value of the underlying asset, V, evolves according to eq. (1); we will vary δ and σ, but as a 'base case', we take $\delta = 0.06$ and $\sigma = 0.20$ (annual rates).

Payout rates on projects can vary enormously from one project to another, so that this value of 6 percent should be viewed as reasonable, but not necessarily representative. The standard deviation of the rate of return on the stock market as a whole has been about 20 percent on average. Although this represents a diversified portfolio of assets, it also includes the effects of

[12]See Hawkins (1982) for a similar model with analytic solutions in both regions. For an overview of numerical methods for solving option problems, see Geske and Shastri (1985). For a useful discussion of finite difference methods, see Brennan and Schwartz (1978).

Table 1

Value of the investment program, f, as a function of the value of the completed project, V, and the total remaining investment, K. (To conserve space, we only show values of the investment program for values of K in multiples of \$1 million and for values of V up to \$42.52 million.) Starred entries indicate the optimal investment rule: for each value of K, investment should be undertaken only if V is above the value corresponding to the starred entry. The value of V corresponding to each starred entry is the cutoff value, $V^*(K)$. The assumed parameters for the problem are: risk-free rate $r = 0.02$, standard deviation $\sigma = 0.20$, rate of opportunity cost $\delta = 0.06$, and maximum rate of investment $k =$ \$1 million per year.

Value of the completed project, V	Total remaining investment, K (millions of dollars)						
	\$6	\$5	\$4	\$3	\$2	\$1	\$0
\$42.52	\$23.70	\$26.47	\$29.37	\$32.42	\$35.62	\$38.98	\$42.52
36.60	19.62	22.12	24.75	27.50	30.39	33.42	36.60
31.50	16.10	18.38	20.76	23.26	25.88	28.62	31.50
27.11	13.07	15.16	17.34	19.62	22.00	24.50	27.11
23.34	10.46	12.38	14.39	16.48	18.67	20.95	23.34
20.09	8.22	10.00	11.85	13.78	15.79	17.89	20.90
17.29	6.23	7.94	9.67	11.46	13.32	15.26	17.29
14.88	4.63	6.18	7.78	9.46	11.19	13.00	14.88
12.81	3.20	4.65	6.17	7.73	9.36	11.05	12.81
11.02	2.02*	3.34	4.77	6.25	7.79	9.37	11.02
9.49	1.22	2.23*	3.57	4.98	6.43	7.93	9.49
8.17	0.74	1.34	2.54	3.88	5.26	6.69	8.17
7.03	0.44	0.81	1.65*	2.93	4.26	5.62	7.03
6.05	0.27	0.49	1.00	2.12	3.39	4.70	6.05
5.21	0.18	0.29	0.60	1.42*	2.65	3.91	5.21
4.48	0.10	0.18	0.36	0.86	2.00	3.23	4.48
3.86	0.06	0.11	0.22	0.52	1.45	2.64	3.86
3.32	0.04	0.06	0.13	0.31	0.98*	2.13	3.32
2.86	0.02	0.04	0.08	0.19	0.59	1.70	2.86
2.46	0.01	0.02	0.05	0.11	0.36	1.32	2.46
2.12	0.01	0.01	0.03	0.07	0.21	1.00	2.12
1.82	0.00	0.01	0.02	0.04	0.13	0.73*	1.82
1.57	0.00	0.01	0.01	0.02	0.08	0.44	1.57
1.35	0.00	0.00	0.00	0.02	0.05	0.27	1.35
1.16	0.00	0.00	0.00	0.01	0.03	0.16	1.16
1.00	0.00	0.00	0.00	0.01	0.02	0.10	1.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00*

leverage on equity returns, and, therefore, might be a reasonable number for an average asset.

As discussed in the appendix, the solution procedure requires a discretization of the variables V and K; for this example, we assume investment outlays are made quarterly; i.e., K is measured in discrete units of \$0.25 million.

The base case solution is shown in table 1. Each entry is the value of the investment program for different levels of V and K. Entries with an asterisk

Table 2

Cutoff value of completed project, V^*, above which investment occurs, for different values of the rate of opportunity cost, δ, and standard deviation, σ. Also shown is V^{**}, the cutoff value adjusted for foregone cash flows due to the opportunity cost, δV, when construction proceeds at the maximum rate (risk-free rate $r = 0.02$, total remaining investment $K = \$6$ million, maximum rate of investment $k = \$1$ million per year). *Note*: Present value of investment outflow at maximum rate of investment $= K^* = \$5.65$ million.

Standard deviation, σ	Cutoff value	Annual rate of opportunity cost, δ (millions of dollars)		
		0.02	0.06	0.12
0.10	V^*	11.02	9.03	12.43
	V^{**}	9.77	6.30	6.05
0.20	V^*	20.09	11.02	12.81
	V^{**}	17.82	7.69	7.03
0.40	V^*	121.51	24.53	20.09
	V^{**}	107.77	17.11	9.78

denote the cutoff value, $V^*(K)$. For example, a project with \$5 million of investment outlays remaining has a cutoff value $V^*(K)$ of \$9.49 million: if V is currently \$9.49 million or more it pays to invest this quarter, otherwise it does not (although one would resume investing should V later rise above \$9.49 million). At this critical level the value of the contingent claim is \$2.23 million; this is the equilibrium market value of the right to the investment program.[13]

Observe that table 1 can be used to make optimal investment decisions as construction of this project proceeds (i.e., as K falls from \$6 million to zero). It can also be used to evaluate any project requiring a total outlay of \$1, \$2,..., \$6 million, but with the same values for the risk-free rate, r, the rate of opportunity cost, δ, the standard deviation, σ, and the maximum rate of investment, k.

We are interested in the sensitivity of the investment decision to the parameters σ, δ, and k. This decision is summarized by the cutoff value $V^*(K)$. Table 2 shows, for the initial investment decision (i.e., when $K = 6$), how the cutoff value changes in response to changes in σ and δ. (The middle entry in table 2 corresponds to the base case shown in table 1.)

Observe that V^* increases when σ is increased: i.e., with greater risk, the value of a completed project today would have to be higher to induce investment. Like most financial options, the value of the investment program, f, is a convex function of the value of the underlying asset, V, and therefore increases as the standard deviation of V increases. Recall that the only reason to invest at *any* value of V is the opportunity cost δ, which in our example of

[13] To conserve space, the table shows only values of the investment program for values of K in multiples of \$1 million and for values of V up to \$42.52 million.

an infinitely-lived project represents the foregone cash flows. Because one is not obliged to exercise the option to invest, greater uncertainty over the future payoffs can only increase the value of the contingent claim, and increase the incentive to hold it rather than exercise it. This is an important point made by McDonald and Siegel (1986).

The dependence of V^* and δ is less obvious. One might expect that a higher opportunity cost of delaying the project, would reduce the cutoff value, V^*, and increase the incentive to invest. This would indeed be the case if the project could be built instantly, as in the model of McDonald and Siegel (1986). But the fact that it takes time to build the project creates a countervailing effect. The payoff from the project, V, is only obtained at completion and must be adjusted for the foregone cash flows during construction [the expected rate of growth of V is only $(\mu - \delta)$]. Time to build therefore reduces the value of the payoff at completion, and as δ increases, it reduces it by a larger amount. This in turn reduces the incentive to invest, i.e., increases the *current* critical cutoff value V^*. As table 2 shows, for high rates of opportunity cost this second effect predominates; for $\sigma = 0.10$ and 0.20, V^* rises when δ is increased from 0.06 to 0.12.

It is useful to calculate the critical cutoff value *net* of the present value of the expected flow of opportunity cost (δV), assuming that investment expenditures are made at the maximum rate. This value, V^{**}, is simply

$$V^{**} = V^* - \int_0^{K/k} \delta V^* \mathrm{e}^{(\mu-\delta)t} \mathrm{e}^{-\mu t} \mathrm{d}t = V^* \mathrm{e}^{-\delta K/k}, \tag{6}$$

where the second term on the right is the present value of the expected flow of opportunity cost (e.g., foregone rent) during the construction period.

Values for V^{**} are shown for each case in table 2. Increasing δ increases the opportunity cost of delaying the project (leading to a lower critical cutoff value), and also increases the opportunity cost *necessarily* incurred because of time to build (leading to a higher cutoff value). V^{**} corrects for the latter, and, as shown in the table, for any value of σ, it declines as δ increases.

Table 2 also shows the importance of the contingent nature of the investment program. A 'naive' discounted cash flow criterion would ignore flexibility during the construction period, and assume a fixed scenario for the investment outlays. Under this naive criterion, one would invest if the present value of investment outlays under the assumed scenario is less than the present value of the payoff at completion. Assuming investment occurs at the maximum rate, the present value of the payoff at completion is the current value of a completed project, V, less the foregone cash flows during construction, which is given by

$$\hat{V}(t) = V(t) - \int_0^{K/k} \delta V(t) \mathrm{e}^{-\delta\tau} \mathrm{d}\tau = V(t) \mathrm{e}^{-\delta K/k}, \tag{7}$$

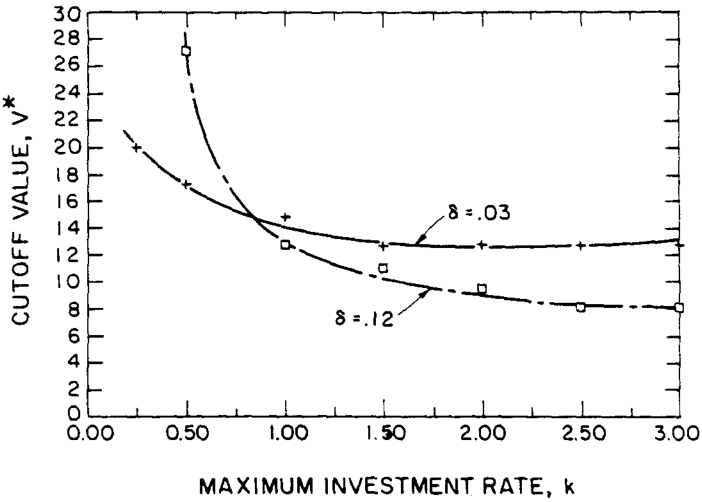

Fig. 1. Cutoff value for investment, V^*, as a function of the maximum rate of investment, k, corresponding to two different levels of opportunity cost, δ. V^* is the value of a completed project above which it is optimal to proceed with the next stage of investment.

and the present value of investment outlays is given by

$$K^* = \int_0^{K/k} k\mathrm{e}^{-r\tau}\mathrm{d}\tau = (1 - \mathrm{e}^{-rK/k})k/r. \tag{8}$$

For our example, the present value of investment outlays made at the maximum rate is $K^* = \$5.65$ million. Even making a rough correction for time to build by subtracting off the foregone cash flows as in eq. (6), the critical cutoff value (which would then be V^{**}) is still significantly higher than K^* for any reasonable value of σ and δ, and *much* higher if σ is large and/or δ is small. The discretionary nature of the investment program increases the threshold still further; V^* is significantly larger than V^{**}, particularly for large values of δ. For our base case of $\sigma = 0.20$ and $\delta = 0.06$, V^* is \$11.02 million, about *double* the present value of the investment outlays K^*.

We can obtain further insight into the ways in which uncertainty and time to build interact in affecting the investment decision by calculating V^* for different values of k, the maximum rate of investment. Fig. 1 shows V^* as a function of k for $\delta = 0.03$ and 0.12, and $K = 6$.[14] Observe that if the rate of opportunity cost is small ($\delta = 0.03$), changes in k have very little effect on V^*.

[14] Our calculations are subject to numerical error because of the finite difference approximation. Absent such errors, the points plotted in figs. 1, 2, and 3 would lie on smooth curves.

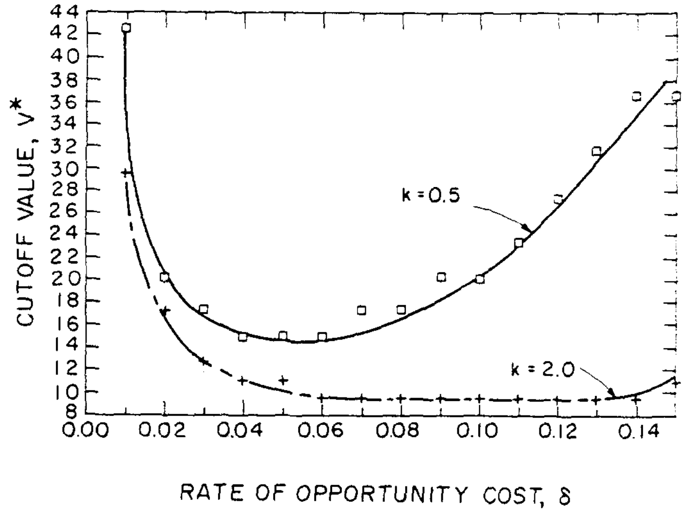

Fig. 2. Cutoff value for investment, V^*, as a function of the opportunity cost, δ, corresponding to two different values for the maximum rate of investment, k. V^* is the value of a completed project above which it is optimal to proceed with the next stage of investment.

(V then has an expected rate of growth close to μ, the equilibrium market rate.) Hence the ability to speed up construction has little effect on the value of the investment program, or on the investment decision. However, if the rate of opportunity cost is large ($\delta = 0.12$), V^* is fairly sensitive to k. Small values of k correspond to long minimum construction times. Hence the minimum present value of the opportunity cost during the construction period is large, reducing the value of the investment program, and increasing the *current* critical value V^*. [If V^* is adjusted for the flow of opportunity cost during the construction period, the resulting cutoff value (V^{**}) will not be very sensitive to k.] Thus time to build is more important for investment decisions where most of the return on the underlying asset is in the form of a payout stream rather than price appreciation.

Fig. 2 shows V^* as a function of δ, for $k = 0.5$ (a 12-year minimum construction period) and 2.0 (a 3-year minimum construction period). In both cases, V^* falls as δ is increased from 0.01 to 0.04. (Remember that as the rate of opportunity cost becomes small, the critical value for investment becomes large; in the limit of zero opportunity cost one would never exercise the right to invest.) However, as δ increases, the effect on V^* depends on the maximum rate of investment. If that maximum rate is high, V^* remains low over a wide range of δ (but is still 30–50 percent greater than the present value of the investment outlays). But if k is small, V^* can depend critically on δ. Thus for projects where the minimum time to build is long, knowledge of the rate of opportunity cost δ is particularly critical input to the investment decision.

We have used numerical examples to illustrate how investment decisions are affected by the sequential and contingent nature of construction outlays. The difference between the results of our calculations and those based on a 'naive' application of DCF rules will depend on the parameters of the problem, but as table 2 shows, for very reasonable parameter values, these differences can be large. Indeed, the range of values for σ in table 2 (0.1 to 0.4) is quite conservative. For many projects σ will exceed 0.4, so that the naive DCF rule will be grossly misleading.

4. The value of construction time flexibility

Many projects can be built with alternative construction technologies. An important way in which these technologies can differ is in terms of flexibility over the rate of construction. Generally, technologies offering greater flexibility are more costly, so that increased cost must be balanced against the value of increased construction time flexibility. Our model provides a straightforward way to determine the value of that increased flexibility.

In our model, construction time flexibility is measured by the maximum rate of construction, k. Higher k corresponds to greater flexibility, i.e., a shorter minimum construction time, K/k. The value of the investment program $f(V,K)$ increases as k increases, and the change in f corresponding to a change in k measures the value of the extra flexibility. This value of extra flexibility will depend on V and K, as well as other parameters of the problem, such as δ and σ.

We can determine the incremental value of construction time flexibility by examining the way in which the value of the investment program f changes as k changes. We calculate and compare the values of the investment program f for different values of k, holding all other variables constant. In particular, since alternative construction technologies are assumed to lead to the same completed project, its current value V must also be held constant. The incremental value of flexibility is then given by the slope of $f(k)$.[15]

Fig. 3 shows the results of such a calculation for the base case parameters from the preceding section. We show $f(k)$ for two different values of the completed project, $V_1 = 10$ and $V_2 = 15$. As fig. 3 shows, for each value of the

[15]Another measure of the incremental value of flexibility is the change in $f(V, K; k)/K$ (the value of the investment program per dollar of total required investment) corresponding to changes in the minimum construction times K/k. Note, however, that $f(V, k; k)$ is not linear homogeneous in K, so that the resulting measure will still depend on K.

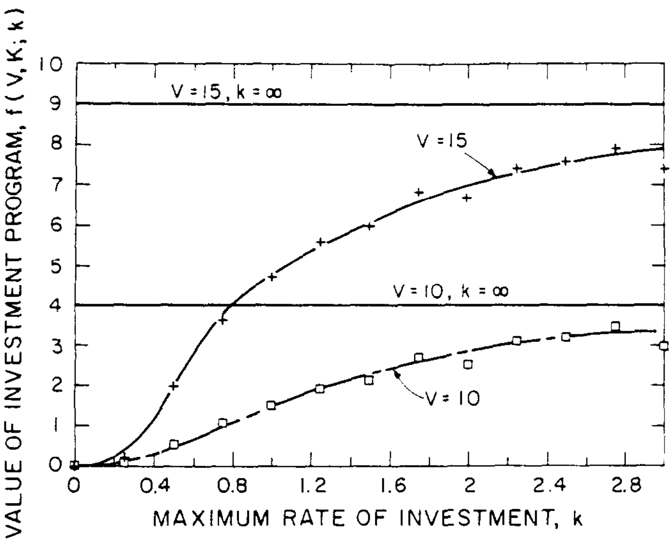

Fig. 3. Value of the investment program, f, as a function of the maximum rate of investment, k, corresponding to two different values for a completed project, V. The horizontal lines represent the asymptotes corresponding to an infinite rate of investment (i.e., no time to build).

completed project, the value of the investment program increases as k increases. Note also that the incremental value of flexibility falls as k increases. For $V=10$, the value of the investment program with *maximum* flexibility (corresponding to $k=\infty$) is 4.0, and for $V=15$ it is 9.0, and these values are shown as horizontal lines in the figure.[16]

Consider two different construction technologies with the same total construction cost $K=6$, but with different maximum rates of investment ($k=0.5$ for the first, and $k=1.0$ for the second). At $V=10$, the incremental value of the more flexible technology ($k=1$) is $\Delta f/\Delta k = 0.977/0.5 = 1.954$. This incremental value will be higher if the value of the completed project is higher; at $V=15$, the incremental value is 5.520.

In general, greater flexibility might be accompanied by a different total investment K. Because the value of the investment program, $f(V, K; k)$, is not linear homogeneous in K (see footnote 15), we cannot isolate the value of the

[16] The case of $k=\infty$ means there is no time to build. This corresponds to a perpetual call option on a stock paying a constant proportional dividend, with exercise price K. The analytical solution is $f(V) = aV^{\alpha}$ for $V \leq V^*$ and $f(V) = V - K$ for $V > V^*$. Here α is given in eq. (4), $a = (V^* - K)/V^*)^{\alpha}$, and $V^* = \alpha K/(\alpha - 1)$ is the cutoff value above which the option is exercised (i.e., the factory is built). In our example, $V^* = 8.6$. Since $V = 10$ and 15 exceeds this critical value, $f(V, K; \infty) = V - K$. See Merton (1973) for a derivation. Note that this is also the model used in McDonald and Siegel (1986).

greater flexibility in such cases simply by comparing $f(V, K; k)/K$ for each technology. However, we can still rank the technologies by comparing $f(V, K; k)$.

5. Concluding remarks

We have shown how optimal investment rules can be determined for projects with sequential investment outlays and maximum construction rates. An important feature of such projects is that the pattern of expenditures can be adjusted as new information arrives. For such projects, we have shown that traditional discounted cash flow criteria based on a fixed pattern of expenditures can lead to grossly incorrect investment decisions. As our calculations for different values of σ, δ, and k have illustrated (table 2, and figs. 1, 2 and 3), the effects of time to build are greatest when uncertainty is greatest, when the opportunity cost of delay is greatest, and when the maximum rate of construction is lowest.

There are some important caveats. First, our simplifying assumption that V is lognormally distributed and that the payout rate, δ, is constant will be exact for very few projects, and for some projects may be a poor approximation. Second, our optimal investment rule critically depends on the current value of a completed project, V, as well as the parameters σ, and δ. We have assumed that these numbers are known, but in fact it may be difficult or impossible to estimate them accurately. Third, in some cases the value of the completed project, and the rate of opportunity cost δ are endogenous to the problem. This would be the case, for example, if the value of the completed project and its cash flows are affected by potential entry by competitors. Then the values of δ and V, as well as the optimal decision rule, must be determined simultaneously (e.g., as a Nash equilibrium for the resulting non-cooperative game).

Although there are many situations where the specific assumptions of our model will not be satisfied, we believe that the qualitative results will continue to hold. In particular, uncertainty is likely to have a depressive effect on the level of investment, an effect which is likely to be magnified when there is time to build.

Our primary focus in this paper has been investment decisions from the point of view of a single firm. However, our results also have implications for the behavior of aggregate investment spending, and in particular the role of risk in the economy. As in the models of Bernanke, Cukierman and McDonald and Siegel (1983), we find that investment decisions can be extremely sensitive to the level of risk (which we measure by the parameter σ). Indeed, this sensitivity is greater than that suggested by traditional investment models. In our model, this greater sensitivity is due to the flexibility that the firm has in making sequential investment outlays; in the models of Bernanke and

Cukierman, it is due to the reduction of uncertainty that results from learning. For different reasons, therefore, our results reinforce the view that aggregate investment spending is likely to be highly sensitive to changes in perceived risk.[17]

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[17]Fischer and Merton (1984) have documented the close empirical connection between aggregate investment and the level of the stock market. Our results (like those of the authors mentioned above) suggest that aggregate investment spending might also be sensitive to the *volatility* of the stock market.

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