Solving optimal timing problems in environmental economics

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ABSTRACT

Two papers of Pindyck (2000, 2002) that modeled the control of stock pollutants as optimal stopping problems contained closed form solutions that are incorrect. This paper discusses a subtle error in the derivation and demonstrates how solutions to these and related problems can be obtained numerically. The numerical solutions are contrasted with the ones contained in Pindyck's original papers.

1. Introduction

In two papers Pindyck (2000, 2002) developed option-theoretic models of environmental policy timing when there is a stock pollutant and there is economic and/or environmental uncertainty. The general framework postulated a stochastic process for a stock pollutant so that there is uncertainty over the future level of the pollutant (i.e., environmental uncertainty), as well as a stochastic process that shifts a social cost function, so that there is uncertainty about the future social cost of any particular level of the pollutant stock. These papers addressed the question of when it would be optimal to switch from a high emissions regime to a lower emissions regime.

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Optimal stopping problems in one dimension are relatively easy to solve and, in certain settings, it may even be possible to obtain closed form solutions. In general, the solution involves finding a threshold for the state variable that determines when it is optimal to "stop," i.e., to adopt the policy and reduce emissions. In the continuous time framework, an ordinary differential equation (ODE) has to be solved along with the so-called *value matching* and *smooth pasting* boundary conditions that apply at the threshold. If a general closed form expression is available for the solution of an ODE, the constant parameters and the threshold are simultaneously determined using the conditions at the unknown threshold. In contrast, optimal stopping problems with two or more state variables involve solving a partial differential equation (PDE) instead of an ODE. Closed form solutions may not exist for the relevant PDE, in which case numerical techniques are needed to solve the problems.

In the most general form of Pindyck's models, the state space is two dimensional. Pindyck proposes closed form solutions to his models that involve a set of unknown constants. Although the proposed solution satisfies some of the optimality conditions, the boundary conditions cannot be satisfied unless the "constants" in these conditions are, in fact, functions of the state variables. As a result, the proposed analytical solutions do not provide the correct optimal stopping rules.

In this paper, we discuss how the models developed by Pindyck can be solved using numerical techniques, and we report on the nature of the solutions that we obtained. We describe an efficient numerical solution method based on the collocation approach and the complementarity representation of the optimal stopping problem. The numerical solutions display important qualitative differences from Pindyck's proposed solutions.

To keep the discussion short we briefly review the most general form of the model and the conditions satisfied by the solution. We then discuss a numerical approach to obtain approximations to the optimal value function and decision rule. We close with a presentation of key features of the solution and contrast that with the solutions presented in the original papers.

2. Analytical framework

A stock pollutant M_t is controlled by a flow variable of emissions E_t. It is assumed that the current level of emissions is E_0 but that it can be lowered, at some cost, to $E_1 < E_0$. The stock pollutant M_t evolves according to:

$$dM = (\beta E - \delta M)dt + \sigma_2 dz_2 \tag{1}$$

where z_2 is a standard Brownian motion. The flow of social benefits is given by $B(\theta_t, M_t)$ where θ_t is a social cost variable which evolves stochastically according to:

$$d\theta = \alpha\theta dt + \sigma_1\theta dz_1 \tag{2}$$

It is assumed that the two diffusions are independent, i.e., $\mathcal{E}[dz_1 dz_2] = 0$, and that the cost of reducing the emission flow, $K(E_1)$, is completely sunk.

Pindyck poses the social planner's problem of determining the optimal time to reduce the emissions level and the optimal permanent emissions level. The objective of the policy maker is to maximize the expected discounted social welfare:

$$W = \max_T \quad \mathcal{E}_0 \left[\int_0^\infty e^{-rt} B(\theta_t, M_t)dt - e^{-rT} K(E_1) \right] \tag{3}$$

where T is the unknown optimal time to adopt a once-and-for-all emission reduction policy, $\mathcal{E}_0$ is the conditional expectation operator based on the information at $t = 0$ and r is the discount rate. The social planner's problem is then to determine when to implement the emission reduction policy as a function of the stochastic state variables, M and θ.

The form of the social benefit function $B(\theta_t, M_t)$ plays an important role in the characterization of the optimal policy rule. Pindyck (2000) shows that when $B(\theta, M)$ is linear in the stock pollutant M, the optimal policy becomes independent of M and can be obtained in closed form. The case of quadratic costs is also considered, but in this case the solution does depend on levels of both M and θ. It is this case, in which $B(\theta_t, M_t) = -\theta_t M_t^2$, that concerns us here.

It is possible to obtain the expected discounted flow of social benefits given that the emissions level cannot be changed. Using the facts that $\mathcal{E}[\theta_t|\theta_0] = e^{\alpha t}\theta_0$, $\varepsilon[M_t|M_0] = e^{-\delta t}M_0 + (1 - e^{-\delta t})(\beta E/\delta)$ and $Var[M_t|M_0] = (1 - e^{-2\delta t})(\sigma_2^2/2\delta)$, it can be shown that

$$\mathcal{E}_0\left[\int_0^\infty e^{-rt}\theta_t M_t^2\, dt | M_0, \theta_0, E\right] = \frac{\theta_0}{r + 2\delta - \alpha}\left(M_0^2 + \frac{2\beta E}{r + \delta - \alpha}M_0 + \frac{2\beta^2 E^2}{(r - \alpha)(r + \delta - \alpha)} + \frac{\sigma_2^2}{r - \alpha}\right) \quad (4)$$

This expectation, which is a function of the current state variables and the current level of emissions, will be denoted $R(\theta, M, E)$.

The optimal stopping problem can be solved in terms of a value function $W(\theta, M)$ and a stopping curve $\theta^*(M)$ that satisfy the following Hamilton-Jacobi-Bellman equation for values of the state for which it is optimal to continue ($\theta < \theta^*(M)$):

$$rW(\theta, M) = -\theta M^2 + \alpha\theta W_\theta(\theta, M) + (\beta E_0 - \delta M)W_M(\theta, M) + \frac{1}{2}\sigma_1^2\theta^2 W_{\theta\theta}(\theta, M)$$

$$+ \frac{1}{2}\sigma_2^2 W_{MM}(\theta, M) \quad (5)$$

where subscripts denote partial derivatives, e.g., $W_\theta(\theta, M) = \partial W/\partial\theta$. In addition, the following boundary conditions must be satisfied:

$$W(\theta^*(M), M) = R(\theta^*(M), M, E_1) - K(E_1),$$
$$W_\theta(\theta^*(M), M) = R_\theta(\theta^*(M), M, E_1),$$
$$W_M(\theta^*(M), M) = R_M(\theta^*(M), M, E_1), \quad (6)$$
$$W(0, M) = 0,$$
$$\lim_{M \to -\infty} W(\theta, M) - R(\theta, M, E_0) = 0$$

The first boundary condition is the *value matching* condition, the second and third are *smooth pasting* conditions, the fourth condition arises due to the fact that θ has an absorbing barrier at $\theta = 0$ and the fifth boundary condition arises due to the fact that it is never optimal to reset the emissions level if M is infinitely negative and hence, in the limit, the value function should approach $R(\theta, M, E_0)$.

It would be nice if one could find closed form expressions that satisfy these conditions. In his original papers, Pindyck proposes functional forms for W that include unspecified constants. He correctly notes that these functions satisfy the Hamilton-Jacobi-Bellman equation. In order for the proposed solutions to satisfy the boundary conditions, however, the "constants" were solved as functions of θ and M. In doing so, however, the proposed solutions no longer satisfy the Hamilton-Jacobi-Bellman equation, thereby leading to erroneous solutions. This appears to be a relatively common mistake (the second author has reviewed a number of manuscripts with this error) and was also made in an article by Conrad (2000) in this journal.

3. Numerical solution of the models

An alternative formulation of the optimality conditions that leads to a numerical approach for approximating the value function is given by Brekke and Oksendal (1994). In their formulation the continuous d-dimensional state process S is described by a diffusion of the form

$$dS = \mu(S)dt + \sigma(S)dW \quad (7)$$

where μ is a $d \times 1$ drift vector, σ is a $d \times d$ diffusion matrix and W is the d-dimensional vector of independent Wiener processes. An agent can choose between receiving a flow of payments of $f(S)$ per unit time (continuation) and a one-time reward of $U(S)$ (stopping), after which no further rewards are received. The agent's problem is then to maximize the expected discounted flow of payments received and the optimal policy is to choose when to stop:

$$V(S) = \max_T \ \mathcal{E}\left[\int_0^T e^{-rt}f(S_t)dt + e^{-rT}U(S_T)|S_0 = S\right] \quad (8)$$

Let the infinitesimal generator of state process be given by

$$\mathcal{L} = \sum_{i=1}^{d} \mu_i(S) \frac{\partial}{\partial S_i} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \Sigma_{ij}(S) \frac{\partial^2}{\partial S_i \partial S_j} \tag{9}$$

where $\Sigma_{ij}(S) = \sum_k \sigma_{ik}(S)\sigma_{jk}(S)$. Brekke and Oksendal (1994) shows that the optimal value function $V(S)$ satisfies the set of complementarity conditions

$$rV(S) \geq \mathcal{L}V(S) + f(S) \tag{10}$$

and

$$V(S) \geq U(S) \tag{11}$$

with one of the conditions satisfied with equality at each value of S. An equivalent form of these conditions can be written as

$$0 = \min\left[rV(S) - \mathcal{L}V(S) - f(S), V(S) - U(S)\right] \tag{12}$$

Note that the stopping curve is not explicitly found in this framework. Instead the continuation region is given by the set of points at which $0 = rV(S) - \mathcal{L}V(S) - f(S)$, the stopping region is the set of points at which $0 = V(S) - U(S)$ and the stopping curve is the set of points where both conditions are met.

The solution of this optimal stopping problem necessitates obtaining the unknown value function $V(S)$. An approximation to this unknown function can be obtained using the projection method (Judd, 1998; Miranda and Fackler, 2002). The value function $V(S)$ is approximated by $\phi(S)c$, where ϕ is a set of n basis functions and c is an n-vector of coefficients. The values of c can be obtained by collocation which solves the complementarity condition at a set of n nodal points. The approximate condition then can be written as

$$0 = \min\left[(r\Phi - \mathcal{L}\Phi)c - f, \quad \Phi c - U\right] \tag{13}$$

where Φ is an $n \times n$ matrix of $\phi(S)$ evaluated at the n nodal points, f is an n-vector of $f(S)$ and U is an n-vector of $U(S)$, both evaluated at the nodal points. When piecewise linear functions are used to approximate V, the partial derivatives in $\mathcal{L}\Phi$ are obtained using finite differences. Eq. (13) is a linear complementarity problem which can be solved with a root finding algorithm such as Newton's method. The fact that $\min(a,b)$ has kinks where $a = b$ and is thus non-differentiable can lead to difficulties for Newton's method. It can, however, be replaced by the so-called Fischer–Burmeister function (Fischer, 1992),

$$\lambda(a,b) = a + b - \sqrt{a^2 + b^2}. \tag{14}$$

or by other functions that have the same zero-contours as the min function. Another alternative solution method is the smoothing-Newton method of Qi and Liao (1999).

In general, boundary conditions may need to be imposed to obtain unique correct solutions. In economics applications the appropriate boundary conditions are typically determined by economic considerations. Numerically, the complementarity condition given above is replaced at boundary nodes by the appropriate boundary conditions.

For Pindyck's general model, the optimality condition is

$$0 = \min\left[rW - \alpha\theta W_\theta - (\beta E_0 - \delta M)W_M - \frac{1}{2}\sigma_1^2\theta^2 W_{\theta\theta} - \frac{1}{2}\sigma_2^2 W_{MM} + \theta M^2, W + R(\theta, M, E_1) \right.$$
$$\left. + K(E_1)\right]. \tag{15}$$

Here the stopping value is the negative of the cost of the switch, $K(E_1)$, plus the discounted flow of pollution costs at the new emissions level, $R(\theta, M, E_1)$. This can be solved for a set of nodal points on $[0, \overline{\theta}] \times [0, \overline{M}]$. The state variable M, by definition, can take negative values; however approximating the value function in $[0, \overline{M}]$ would have negligible effects on the optimal policy because the drift term for the process is positive and thus the probability of the state variable taking negative values is very low.

We performed a sensitivity analysis on the approximation region, and determined that a lack of nodal points in the negative region does not alter the optimal policy. The appropriate boundary condition for this problem is $W(0, M) = 0$ because zero is an absorbing barrier for θ; if θ hits zero, it will stay at zero forever and thus the pollutant stock will generate no further costs to society.

4. Results

We now present the optimal solutions for two of the special cases discussed in Pindyck's original papers. The first includes only economic uncertainty, and the second includes both economic and environmental uncertainty. We also provide some results for a general case not considered in the original papers in which there is both economic and environmental uncertainty and the pollution decay rate is positive. Code for generating the results presented here are available from the second author.
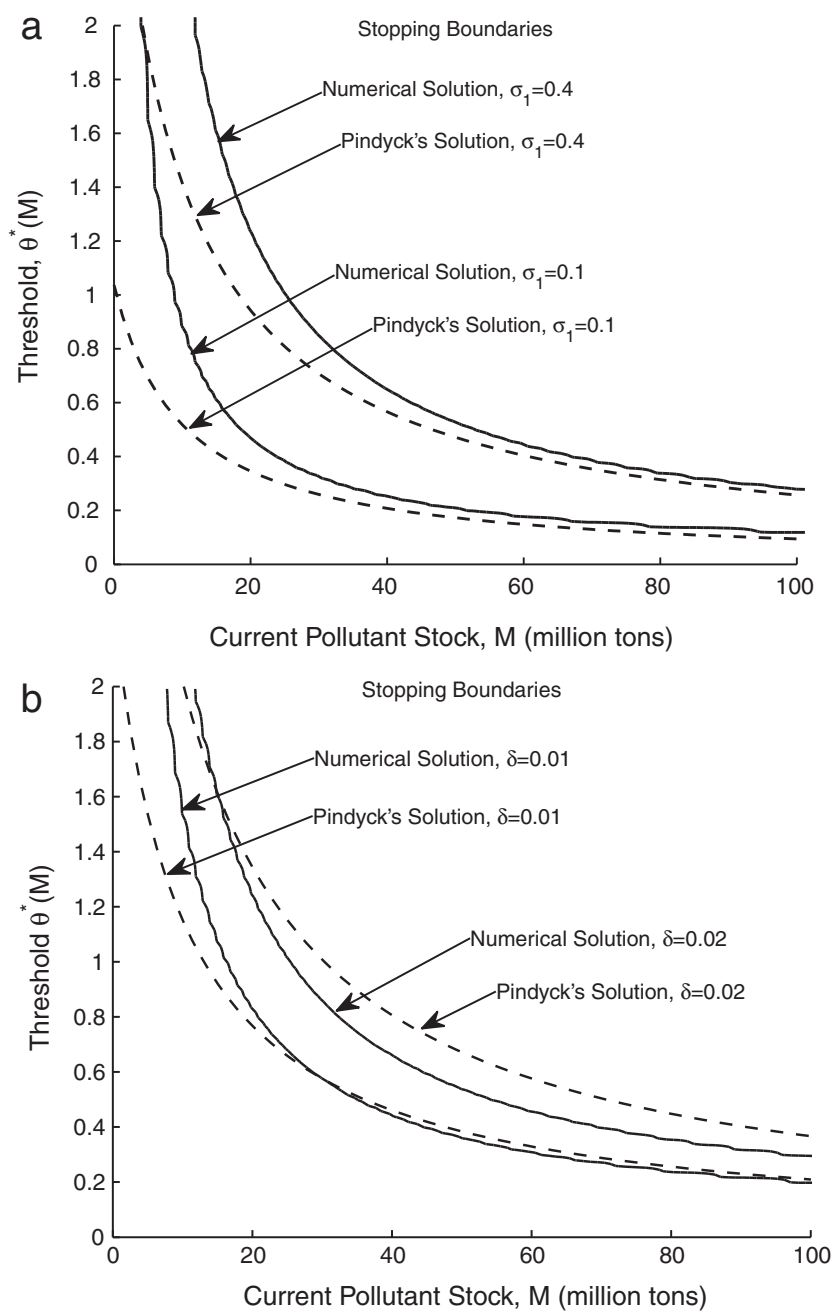

Fig. 1. Stopping boundary, $\theta^*(M)$ (economic uncertainty only). (a) $\delta = 0$, $\sigma_1 = 0.1$ and 0.4. (b) $\sigma_1 = 0.1$, $\delta = 0.01$ and 0.02
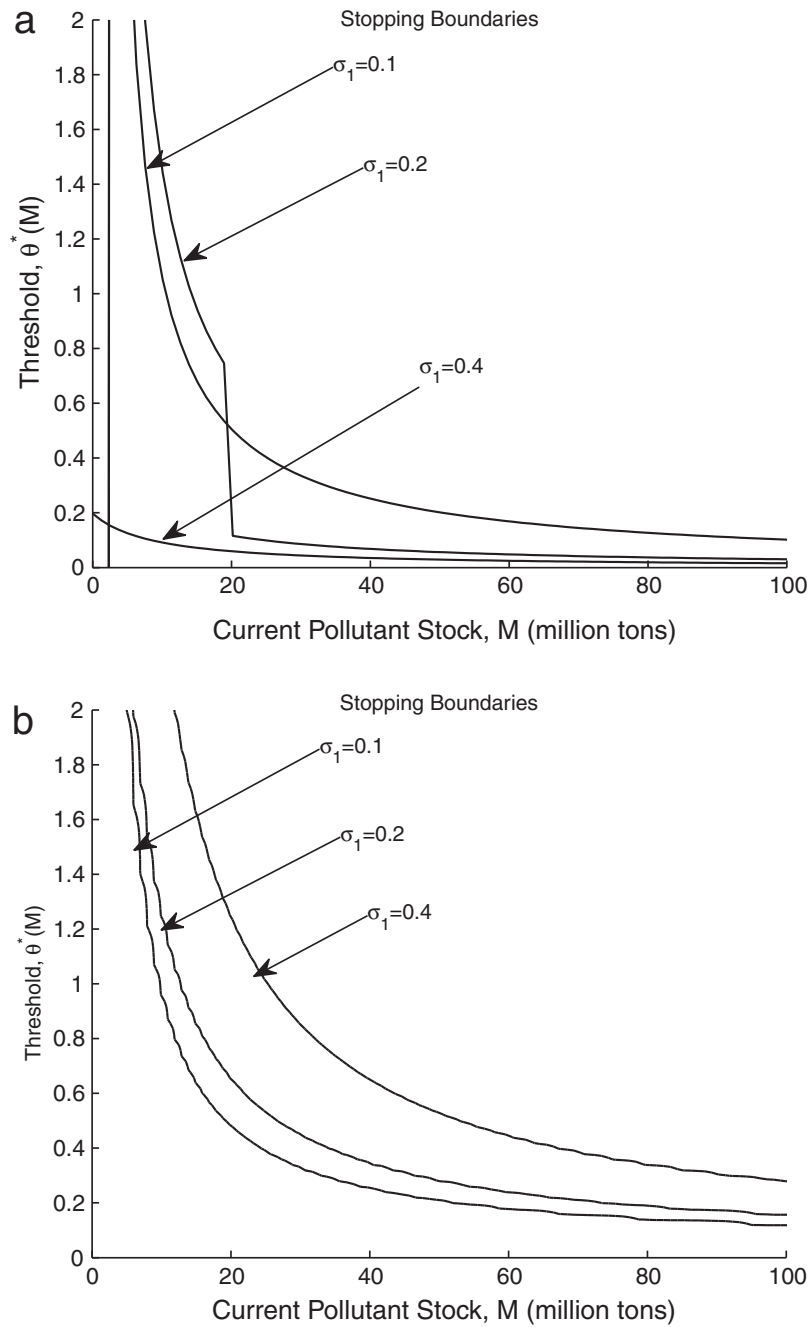
Fig. 2. Stopping boundary, $\theta^*(M)$ (economic and ecological uncertainty with $\delta = 0$). (a) Pindyck's solution, $\sigma_1 = 0.1$, 0.2 and 0.4. (b) Numerical solution, $\sigma_1 = 0.1$, 0.2 and 0.4.

4.1. Economic uncertainty

Pindyck (2000) considered the case in which there was uncertainty only about future pollution costs ($\sigma_2 = 0$). Here Pindyck's proposed solution is compared to the numerical solution using the following parameter values: discount rate $r = 0.04$, emission level $E_0 = 0.3$ million-tons/year, $E_1 = 0$, $\alpha = 0.01$, $\beta = 1$, $K = kE_0 = \$4000$ million. In Fig. 1 the stopping boundaries obtained for the two solutions are displayed, with θ measured in million dollars/(million tons)^2. In Fig. 1a the decay rate of the stock pollutant is set as $\delta = 0$, and the stopping boundaries are drawn for $\sigma_1 = 0.1$ and $\sigma_1 = 0.4$. In Fig. 1b $\sigma_1 = 0.1$ and the stopping boundaries are shown for $\delta = 0.01$ and $\delta = 0.02$.[1]

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[1] The parameters used here are similar to those used by Pindyck except that the scales have been changed to make the values closer to unity. Specifically M is denominated in million tons rather than tons and monetary units are in million dollars rather than dollars. Such rescaling is important for obtaining numerical results.
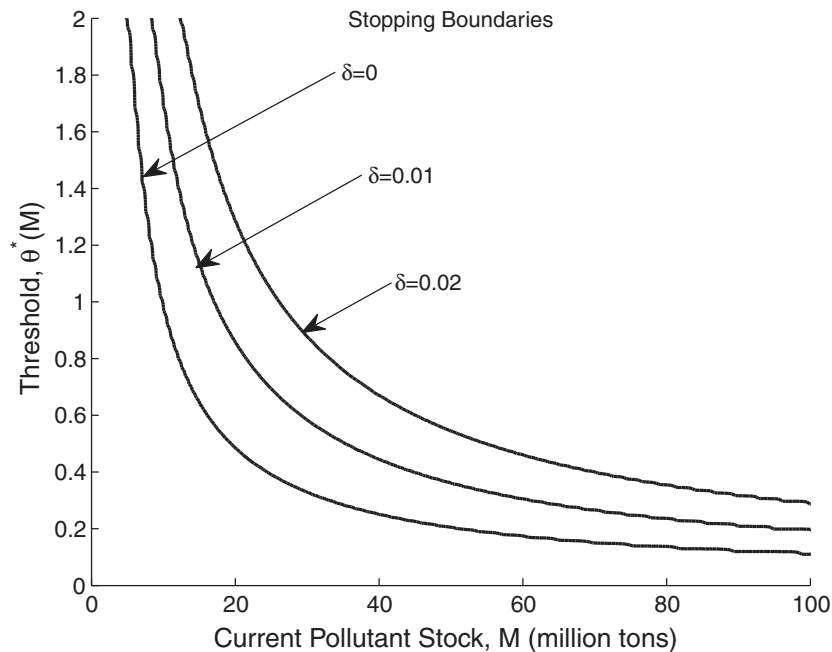
Fig. 3. Stopping boundary, $\theta^*(M)$ (general model).

Uncertainty and the decay rate of the stock pollutant both play important roles in the policy adoption. An increase in either uncertainty and the depreciation rate raises the policy adoption threshold as expected. There are, however, quantitative differences in the proposed and numerical solutions. In addition, a significant qualitative difference lies in the fact that for Pindyck's proposed solution $\theta^*(M)$ crosses the $M=0$ axis. This leads to the counterintuitive (and incorrect) conclusion that it can be optimal to incur the sunk cost of emissions reduction even when the pollution level is zero.

4.2. Economic and ecological uncertainty

Pindyck (2002) considered the case in which there was uncertainty about both future pollution costs and the stock of pollution. He proposed a solution when the decay rate of the stock pollutant is $\delta=0$. In this section, we first provide numerical solutions when $\delta=0$ and then explore the impact of alternative values of δ on the optimal stopping curves.

The optimal stopping boundaries for Pindyck's proposed solution and for the numerical solution are illustrated in Fig. 2 for the case of $\delta=0$ and the parameter values $r=0.04$, $E_0=0.3$ million-tons/year, $\alpha=0.01$, $\sigma_2=1$ million tons, $\beta=1$, and $K=kE_0=\$4000$ million. Fig. 2a shows the optimal stopping boundaries of Pindyck's solution and Fig. 2b shows the optimal stopping boundaries obtained from the numerical solution. In both cases, the boundaries are shown for σ_1 equal to 0.1, 0.2 and 0.4. Although not evident in the original paper, for some parameter values there can be values of M at which the proposed stopping boundaries exhibit discontinuities and, as a result, are not necessarily increasing as the social cost uncertainty increases.[2] These discontinuities are evident in Fig. 2a. On the other hand, the results of the numerical solution indicate that an increase in uncertainty always results in a higher optimal stopping threshold, and no discontinuities occur, as is seen in Fig. 2b.

Numerical solutions can be easily obtained for positive decay rates. To illustrate, we solved the general model numerically for δ equal to 0, 0.01 and 0.02 (with $\sigma_1=0.1$). The resulting optimal stopping boundaries are shown in Fig. 3. As expected an increase in decay rate of the stock pollutant leads to a higher value of the stopping threshold θ^*. It should be noted that the often-used geometric Brownian motion assumption is not needed with numerical solutions. Other, possibly more realistic, processes could therefore be used to describe the dynamics of the cost variable θ.

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[2] The discontinuities arise at point where the γ parameter, defined in Eq. (44) of Pindyck (2002), is equal to either 1 or 0.

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