

# Investments in flexible production capacity\*

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We examine the technology and capacity choice problem of a multi-output firm facing stochastic demands in a continuous-time framework. The firm can install output-specific capital, or, at greater cost, flexible capital that can be used to produce different outputs. Investment is irreversible. The firm must choose a technology and decide how much capital to install, knowing it can add more later as demand evolves. We formulate the capacity choice problem as a singular stochastic control problem, show that the value of the firm equals the value of its installed capital plus the value of its options to add capacity in the future, and derive an optimal investment rule that maximizes the firm's market value. We also address the analogous problem for a multi-input firm that faces stochastically evolving factor costs, and can install input-specific or flexible capital.

## 1. Introduction

Consider a firm that produces two products, with (possibly interdependent) demands that vary stochastically over time. It can produce these products in one of two ways: by installing and utilizing output-specific capital, or by installing – at greater cost – flexible capital that can be used to produce either product. Investments in all three types of capital are irreversible (the expenditures are sunk costs), but the firm can add more capital later should demand rise. The firm must decide how much of each type of capital to install in order to maximize its market value.

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Problems like this arise with new manufacturing technologies. Automobile companies, for example, produce four- and six-cylinder engines. Demands for the engines are interdependent, and respond to unpredictable changes in gasoline prices, GNP, interest rates, and tastes. In the past, a firm such as GM could invest in capacity specific to four-cylinder engines and/or capacity specific to six-cylinder engines. New technologies allow the same production line to turn out either engine. Given the uncertainty over future demands, the more flexible capacity has an obvious advantage. But it is also more costly. The firm must decide whether the additional cost is justified, and how much capacity to install. Or, consider a firm that produces one product, using either of two alternative factor inputs whose costs vary stochastically over time. The firm can irreversibly invest in input-specific capital, or in a more costly flexible capital that allows the use of either input. Again, the firm must decide which type of capital to use, and how much capacity to install. An example is an electric utility planning new generating capacity. It can build a coal- or oil-burning plant, or a plant designed at the outset to burn either fuel. Which type of plant should be built, and how large should it be?

We develop a framework that addresses these problems, and yields an investment rule that maximizes the firm's market value. As in Bertola (1989) and Pindyck (1988), we focus on irreversible and incremental investment decisions; the firm decides how much capacity to install, given that it can add more later, and given that investment expenditures are sunk costs.<sup>1</sup>

The value of flexibility in plant design was first examined by Fuss and McFadden (1978). More recently, Fine and Freund (1990) studied investments in output-flexible capacity, using a quadratic programming model in which investment occurs in the first period, before demands are known, and production in a second period. Their two-period framework provides insight into the value of flexibility and choice of technology. However, it does not account for the irreversibility of investment, it requires product demands to be independent, and the investment rule it yields does not necessarily maximize the firm's market value.<sup>2</sup> In another recent study, Triantis and Hodder (1990) have shown how one can use contingent claims methods to value a facility that can produce multiple outputs. However, they take the capacity of the facility as fixed, and ignore the investment and technology

<sup>1</sup>Most of the literature on irreversible investment examines the decision to build a discrete project of some fixed size. See, for example, Baldwin (1982), Brennan and Schwartz (1985), Dixit (1989), McDonald and Siegel (1986), Majd and Pindyck (1987), MacKie-Mason (1990), and for an overview, Pindyck (1991).

<sup>2</sup>In a paper related to this one, Kulatilaka (1987) shows how dynamic programming can in principle be used to compare technologies and value of flexibility. As a numerical example, he values a plant that can switch between two inputs (e.g., coal and oil) at some cost, and shows how the value of the plant depends on this switching cost.

choice decisions. Also, their approach requires that the output mix be adjusted only at discrete and pre-specified times.

We also value flexible production capacity, allowing the firm to vary its output mix continuously. We then focus on the firm's decision to irreversibly invest, its choice of technology (flexible or output-specific), and the components of its market value. Pindyck (1988) has shown heuristically how a firm's value can be broken down into the value of its capacity in place and the value of its growth options. Here, we demonstrate this rigorously by showing how the value of the firm solves a singular stochastic control problem. We apply this to the multi-output case, and characterize the free boundary that determines the optimal investment rule. In general, solving for the optimal investment rule when output-specific capital is used can be extremely complicated, but we show that the problem is greatly simplified if the firm's instantaneous profit function is additively separable with respect to the various capital stocks.

Specifically we consider a firm that must decide how much capacity to install to produce one output, the demand for which fluctuates stochastically. Capacity choice is optimal when the present value of the expected cash flow from a marginal unit of capacity equals the total cost of the unit. This total cost includes the purchase and installation cost *plus* the opportunity cost of investing now rather than waiting for new information. An analysis of capacity choice therefore involves two steps. First, the value of an extra unit of capacity must be determined, taking into account that if demand falls the firm need not utilize the unit. Second, the value of the option to invest in this unit must be determined, together with the decision rule for exercising the option. This decision rule is the solution to the optimal capacity problem. It maximizes the net value of the firm: the value of installed capacity net of its cost plus the value of the firm's options to install more capacity.

Now suppose the firm produces two products with interdependent demands that fluctuate stochastically. If it uses product-specific capital, it must decide how much of each type to purchase. This requires the valuation of a marginal unit of each type of capital (which may depend on how much of the other type is installed), the valuation of the options to invest in marginal units of each type, and the rule for exercising the options. That rule again maximizes the firm's net value: the total value of installed capacity of both types net of costs plus the value of the firm's options to add capacity. Or, the firm could install flexible capacity, again choosing an amount to maximize its net value. The optimal choice of technology then boils down to an *ex ante* comparison of net value under each alternative.

This characterization of the investment problem is explained in more detail in the next section. There, we show how the firm's value-maximizing choice of technology and capacity can be found for general demand functions. A

numerical example is presented in section 3. Section 4 discusses the analogous problem of investing in input-flexible capacity. Section 5 concludes and mentions some of the limitations of our approach.

## 2. Optimal investment and technology choice

In this section we show that the value of a firm is equal to the value of its installed capacity plus the value of its options to add capacity in the future. Installed capacity likewise represents a set of options; each unit of capacity gives the firm options to produce at every point over the lifetime of the unit, and can be valued accordingly.<sup>3</sup> Hence, valuing a firm and finding its optimal investment policy can be reduced to a problem of option valuation. This is spelled out below, first for a firm that produces a single output, and then for a firm that produces multiple outputs and must choose between output-specific and flexible capitals.

### 2.1. The single-output firm

Consider a firm facing a demand curve  $P = D(Q; \theta)$ , a cost function  $C(Q)$  [ $C(0) = 0$ ], and a demand shift parameter  $\theta$  that follows a geometric Brownian motion,

$$d\theta = (\mu - \delta)\theta dt + \sigma\theta dz,$$

where  $z$  is a standard Brownian motion, and  $\mu$ ,  $\delta$ , and  $\sigma$  are constants. Assume that there exists a traded asset or a portfolio of traded assets the return of which is perfectly correlated with  $\theta$ , and that this asset or portfolio pays dividends at a rate of  $\delta$  ( $\delta$  may be negative for a portfolio of assets). Thus, markets are dynamically complete in the sense that contingent claims written on  $\theta$  can be priced by taking the expectation of the discounted cash flows under the risk-neutral probability measure. Assume that the riskless rate,  $r$ , is constant.

The firm can install capital one unit at a time, at a sunk cost  $k$  per unit. If  $K$  is the amount of capital in place, then the value of the firm is equal to

$$W(K; \theta) = \max_{\{X_t | X_0 = K\}} E^* \left[ \int_0^\infty e^{-rt} \pi(X_t; \theta_t) dt - \int_0^\infty e^{-rt} k dX_t | \theta_0 = \theta \right], \quad (1)$$

<sup>3</sup>This point is discussed in McDonald and Siegel (1985).

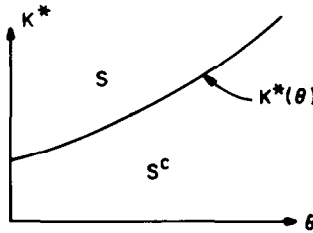


Fig. 1. Free boundary for single-output case.

where  $E^*$  denotes the expectation under the risk-neutral probability measure, and  $\pi(K; \theta) \equiv \max_{0 \leq Q \leq K} D(Q; \theta)Q - C(Q)$  is the instantaneous profit that the firm earns when the installed capacity level is  $K$  and the demand shift parameter is  $\theta$ . The process for the capital stock,  $\{X_t\}$ , is constrained to be an adapted and increasing process, which means that investment is irreversible. We assume that  $\pi(K; \theta)$  is concave in  $K$  and that the maximum in (1) exists and is uniquely attained. Therefore, the value of the firm with  $K$  units of capital in place becomes a solution to a singular stochastic control problem, which can be solved using standard dynamic programming techniques.<sup>4</sup> Subject to certain regularity conditions, the value of the firm must satisfy the Bellman equation [cf. Harrison (1985)],

$$\begin{aligned} \frac{1}{2}\sigma^2\theta^2W_{\theta\theta} + (r - \delta)\theta W_\theta - rW + \pi(K; \theta) &= 0, & (K, \theta) \in S, \\ W_K &= k, & (K, \theta) \in S^c, \end{aligned} \quad (2)$$

where  $S$  is a subset of  $\mathfrak{R}_+^2 = \{(K, \theta): K, \theta \geq 0\}$  such that  $W_K < k$  on  $S$ , and  $S^c$  is the complement of  $S$  relative to  $\mathfrak{R}_+^2$  (see fig. 1). Since the value of a marginal unit of capital is strictly less than its installation cost,  $S$  can be regarded as the no-investment region. Likewise,  $S^c$  can be regarded as the investment region for an immediate addition of capital.

In general, the partial differential eq. (2) is difficult to solve, since it has to be solved differently for the two regions. The curve formed by the intersection of the closure of  $S$  and  $S^c$  defines the free boundary at which it is just optimal to invest. This free boundary must be found as part of the solution. If  $W$  is strictly concave in  $K$ , then  $W_K$  is strictly decreasing in  $K$ . If we assume that  $D(Q; \theta)$  is strictly increasing in  $\theta$  such that  $W_K$  is also strictly increasing in  $\theta$ , then the free boundary can be represented by a smooth curve  $K = K^*(\theta)$ , where  $K^*$  is strictly increasing in  $\theta$ , as shown in fig. 1. However, in general, the shape of the free boundary can be arbitrary.

<sup>4</sup>A stochastic control problem is called singular if the control variable could exhibit both sudden jumps and continuous changes. A control problem in which the control variable is an increasing process is singular [see Harrison (1985)].

If the initial  $(K, \theta)$  lies outside of  $S$ , then the capital stock will be increased immediately up to  $K^*(\theta)$ . If the initial  $(K, \theta)$  lies inside  $S$ , then it is not optimal to add any more capital for the time being. The optimal process for the capital stock,  $\{X_t\}$ , can then be characterized as the maximal adapted increasing process that enforces  $(X_t, \theta_t) \in S$  for all  $t \geq 0$ .

Since the above PDE is difficult to solve directly, we propose an alternative way of finding the value function. Decompose the value of firm:  $W(K; \theta) = V(K; \theta) + F(K; \theta)$ , where

$$V(K; \theta) \equiv E^* \left[ \int_0^\infty e^{-rt} \pi(K; \theta_t) dt \mid \theta_0 = \theta \right] \quad (3)$$

is the value of the capital in place, and  $F(K; \theta) \equiv W(K; \theta) - V(K; \theta)$  is the value of the firm's *growth options*, i.e., the present value of any additional profits, less the cost, from adding more capital later.<sup>5</sup> The optimal capacity decision stated above can now be interpreted as the first-order condition for a value-maximizing firm. That is, if the firm's initial capacity stock is zero, then the firm's optimal capital stock,  $K^*$ , maximizes the firm's net value,  $V(K; \theta) + F(K; \theta) - kK$ . This implies that

$$\frac{\partial V(K^*; \theta)}{\partial K} = k - \frac{\partial F(K^*; \theta)}{\partial K}. \quad (4)$$

Hence, the firm always invests until the value of a marginal unit of capital,  $\partial V(K; \theta)/\partial K$ , equals its total cost, the purchase and installation cost  $k$ , plus the opportunity cost  $-\partial F(K; \theta)/\partial K$ .

To evaluate  $V$  and  $F$ , we define  $\Delta V(K; \theta) \equiv \partial V(K; \theta)/\partial K$  and  $\Delta F(K; \theta) \equiv -\partial F(K; \theta)/\partial K$ . Differentiating (3) with respect to  $K$ ,

$$\Delta V(K; \theta) = E^* \left[ \int_0^\infty e^{-rt} \Delta \pi(K; \theta_t) dt \mid \theta_0 = \theta \right], \quad (5)$$

where  $\Delta \pi(K; \theta) = \partial \pi(K; \theta)/\partial K$ . For given  $K$ , the value of the capital in place  $V(K; \theta)$  and the value of a marginal unit of capital  $\Delta V(K; \theta)$  can be evaluated by direct integration in (3) and (5). The marginal value of the growth options,  $\Delta F(K; \theta)$ , can be treated as a contingent claim on  $\theta$ , and

<sup>5</sup>Note that  $F(K; \theta)$  exceeds the present value of the expected flow of net profits from anticipated future investments, because the firm is not committed to any investment path.

hence satisfies

$$\begin{aligned} \frac{1}{2}\sigma^2\theta^2\Delta F_{\theta\theta} + (r - \delta)\theta\Delta F_{\theta} - r\Delta F &= 0, & \theta \leq \theta^*, \\ \Delta F(K; \theta) &= \Delta V(K; \theta) - k, & \theta \geq \theta^*, \\ \frac{\partial \Delta F(K; \theta^*)}{\partial \theta} &= \frac{\partial \Delta V(K; \theta^*)}{\partial \theta}, \end{aligned} \tag{6}$$

at some critical  $\theta^*$ , which corresponds to the point  $(K, \theta^*)$  on the free boundary where  $K^* = K^*(\theta^*)$ . As we vary  $K$ , we obtain  $\theta^*(K)$  for each  $K$  that satisfies (4). Eq. (6) is an *ordinary* differential equation with a free boundary condition, and is therefore much easier to solve than the partial differential eq. (2).

Once we find the marginal values of a marginal unit of capital and the growth options, the market value of the firm with  $K$  units of capital in place can be obtained as the sum of these marginal values:

$$W(K; \theta) = \int_0^K \Delta V(x; \theta) dx + \int_K^\infty \Delta F(x; \theta) dx, \tag{7}$$

where we have used the fact that  $V(0; \theta) = F(+\infty; \theta) = 0$ .

### 2.2. The multi-output firm

For ease of exposition, and without loss of generality, we consider a firm producing only two outputs, with demand curves  $P_i = D_i(Q_1, Q_2; \theta)$  and cost functions  $C_i(Q_i) [C_i(0) = 0]$ , where  $i = 1, 2$  and  $\theta = (\theta_1, \theta_2)$  is a vector of shift parameters that follow a two-dimensional geometric Brownian motion,

$$d\theta_1 = (\mu_1 - \delta_1)\theta_1 dt + \sigma_1\theta_1 dz_1,$$

$$d\theta_2 = (\mu_2 - \delta_2)\theta_2 dt + \sigma_2\theta_2 dz_2,$$

where  $z_1$  and  $z_2$  are standard Brownian motions with  $E[dz_1 \times dz_2] = \rho dt$ , and  $\mu_i, \delta_i, \sigma_i$ , and  $\rho$  are constants. We assume that there exist two traded assets or two portfolios of traded assets with returns that are perfectly correlated with  $\theta_1$  and  $\theta_2$ , and that these two assets or portfolios pay dividends at rates  $\delta_1$  and  $\delta_2$ , respectively. Thus, markets are dynamically complete in the sense that contingent claims written on  $\theta_1$  and  $\theta_2$  can be priced by taking the expectation of the discounted cash flows under the risk-neutral probability measure.

### 2.2.1. Flexible capacity

Suppose the firm uses only flexible capital, which costs  $k_f$  per unit and can be used to produce both outputs. The valuation problem in this case is similar to that of a single-output case. If the current capital stock is  $K_f$ , then the value of the firm is given by

$$W(K_f; \theta) = \max_{\{X_t^f | X_0^f = K_f\}} \mathbb{E}^* \left[ \int_0^\infty e^{-rt} \pi_f(X_t^f; \theta_t) dt - \int_0^\infty e^{-rt} k_f dX_t^f | \theta_0 = \theta \right], \quad (8)$$

where  $\theta = (\theta_1, \theta_2)$  and  $\pi_f(K_f; \theta) \equiv \max_{Q_1 + Q_2 \leq K_f, Q_i \geq 0} D_1(Q_1, Q_2; \theta)Q_1 + D_2(Q_1, Q_2; \theta) - C_1(Q_1) - C_2(Q_2)$  is the instantaneous profit that the firm earns when the flexible capacity level is  $K_f$ , and the stochastic process for the capital stock,  $\{X_t\}$ , is constrained to be an adapted and increasing process. The value of the firm with  $K_f$  units of capital in place must satisfy the Bellman equation

$$\begin{aligned} & \frac{1}{2}\sigma_1^2\theta_1^2W_{\theta_1\theta_1} + \rho\sigma_1\sigma_2\theta_1\theta_2W_{\theta_1\theta_2} + \frac{1}{2}\sigma_2^2\theta_2^2W_{\theta_2\theta_2} + (r - \delta_1)\theta_1W_{\theta_1} \\ & + (r - \delta_2)\theta_2W_{\theta_2} - rW + \pi_f(K_f; \theta) = 0, \quad (K_f, \theta_1, \theta_2) \in S_f, \\ & W_{K_f} = k_f, \quad (K_f, \theta_1, \theta_2) \in S_f^c, \end{aligned} \quad (9)$$

where  $S_f$  is the no-investment region in  $\mathfrak{R}_+^3$  in which  $W_f < k_f$ , and  $S_f^c$  is the complement of  $S_f$  in  $\mathfrak{R}_+^3$ . As before, this equation has a free boundary, corresponding to the intersection of the closure of  $S_f$  and  $S_f^c$ . The free boundary in this case is a surface in three-dimensional space.

Let  $K = K_f^*(\theta_1, \theta_2)$  be the free boundary. If the initial  $(K_f, \theta_1, \theta_2)$  lies outside of  $S$ , then capacity will be increased immediately up to  $K_f^*(\theta_1, \theta_2)$ . If the initial  $(K_f, \theta_1, \theta_2)$  lies inside  $S$ , then it is not optimal to add capacity for the time being. The optimal process for the capital stock can again be characterized as the maximal adapted increasing process that enforces  $(X_t^f, \theta_{1t}, \theta_{2t}) \in S$  for all  $t \geq 0$ .

To determine the market value of the firm, we decompose it as before:  $W(K_f; \theta) = V(K_f; \theta) + F(K_f; \theta)$ , where

$$V(K_f; \theta) \equiv \mathbb{E}^* \left[ \int_0^\infty e^{-rt} \pi_f(K_f; \theta_t) dt | \theta_0 = \theta \right]$$



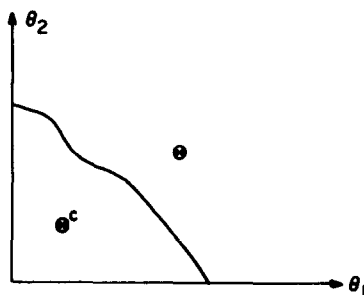


Fig. 2. Critical region – flexible capital.

and

$$F(K_f; \theta) \equiv W(K_f; \theta) - V(K_f; \theta).$$

Also, define the value of an incremental unit of flexible capacity,  $\Delta^f V$ , and the value of the growth options to invest in the future,  $\Delta^f F$ , accordingly. The optimal capacity decision can then be stated as follows:

$$\frac{\partial V(K_f^*; \theta)}{\partial K_f} = k_f - \frac{\partial F(K_f^*; \theta)}{\partial W_f}, \tag{10}$$

i.e., the firm chooses a  $K_f^*$  that maximizes  $W - k_f K_f^*$ , and invests until the value of a marginal unit of capital equals its full cost.

While for given  $K_f$ ,  $V$  and  $\Delta^f V$  can be evaluated by direct integration, the marginal value of the growth options can be determined by the partial differential equation

$$\begin{aligned} & \frac{1}{2} \sigma_1^2 \theta_1^2 \Delta^f F_{\theta_1 \theta_1} + \rho \sigma_1 \sigma_2 \theta_1 \theta_2 \Delta^f F_{\theta_1 \theta_2} + \frac{1}{2} \sigma_2^2 \theta_2^2 \Delta^f F_{\theta_2 \theta_2} + (r - \delta_1) \theta_1 \Delta^f F_{\theta_1} \\ & + (r - \delta_2) \theta_2 \Delta^f F_{\theta_2} - r \Delta F = 0, \end{aligned} \tag{11}$$

for  $\theta \in \Theta$ , and

$$\Delta^f F(K_f; \theta) = \Delta^f V(K_f; \theta) - k, \tag{11'}$$

for  $\theta \in \Theta^c$ , where  $\Theta = \{(\theta_1, \theta_2): (K_f, \theta_1, \theta_2) \in S_f\}$  and  $\Theta^c$  is the complement of  $\Theta$  (see fig. 2). The exact region for  $\Theta$  must be solved along with the solution to the partial differential equation with the additional condition that  $\Delta F$  must be continuously differentiable on the free boundary as well.

Finally, the market value of the firm can be obtained as the sum of the marginal values of capitals and the growth options:

$$\begin{aligned}
 W(K_f; \theta_1, \theta_2) &= \int_0^{K_f} \Delta^f V(x; \theta_1, \theta_2) dx \\
 &+ \int_{K_f}^{\infty} \Delta^f F(x; \theta_1, \theta_2) dx.
 \end{aligned}
 \tag{12}$$

2.2.2. Nonflexible capacity

Suppose instead that each of the firm’s two outputs is produced with a specific type of capital, and that capital of type  $i$  can be installed at a sunk cost of  $k_i$  per unit for  $i = 1, 2$  with  $k_i < k_f$  and  $k_1 + k_2 > k_f$ . If the firm has  $K_i$  units of capital of type  $i$  in place, then its value is

$$\begin{aligned}
 W(K_1, K_2; \theta) &= \max_{(X_1^1, X_1^2, X_0^1=K_1, X_0^2=K_2)} E^* \left[ \int_0^{\infty} e^{-rt} \pi(X_t^1, X_t^2; \theta_t) dt \right. \\
 &\left. - \int_0^{\infty} e^{-rt} (k_1 dX_t^1 + k_2 dX_t^2) \Big| \theta_0 = \theta \right],
 \end{aligned}
 \tag{13}$$

where  $X_t^1$  and  $X_t^2$  are adapted and increasing processes for the two types of capital stocks, and  $\pi(K_1, K_2; \theta) = \max_{0 \leq Q_i \leq K_i} D_1(Q_1, Q_2; \theta)Q_1 + D_2(Q_1, Q_2; \theta)Q_2 - C_1(Q_1) - C_2(Q_2)$ . The value function  $W$  satisfies the Bellman equation

$$\begin{aligned}
 &\frac{1}{2}\sigma_1^2\theta_1^2W_{\theta_1\theta_1} + \rho\sigma_1\sigma_2\theta_1\theta_2W_{\theta_1\theta_2} + \frac{1}{2}\sigma_2^2\theta_2^2W_{\theta_2\theta_2} + (r - \delta_1)\theta_1W_{\theta_1} \\
 &+ (r - \delta_2)\theta_2W_{\theta_2} - rW + \pi(K_1, K_2; \theta) = 0, \quad (K_1, K_2, \theta_1, \theta_2) \in S, \\
 &W_{K_1} = k_1, \quad (K_1, K_2, \theta_1, \theta_2) \in S_1, \\
 &W_{K_2} = k_2, \quad (K_1, K_2, \theta_1, \theta_2) \in S_2, \\
 &W_{K_1} = k_1, \quad W_{K_2} = k_2, \quad (K_1, K_2, \theta_1, \theta_2) \in S_3,
 \end{aligned}
 \tag{14}$$

where  $S$  is the no-investment region in  $\mathfrak{R}_+^4$  in which  $W_{K_1} < k_1$  and  $W_{K_2} < k_2$ ,  $S_1$  ( $S_2$ ) is the investment region for type 1 (type 2) capital in which  $W_{K_1} = k_1$ ,  $W_{K_2} < k_2$  ( $W_{K_2} = k_2$ ,  $W_{K_1} < k_1$ ), and  $S_3$  is the investment region for both types of capital in which  $W_{K_1} = k_1$  and  $W_{K_2} = k_2$ . In general, the free

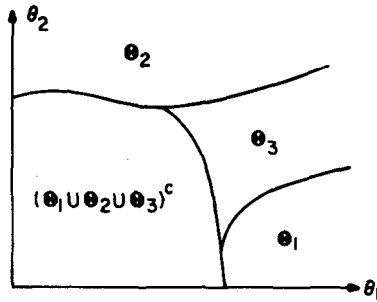


Fig. 3. Critical region – output-specific capital.

boundary for this problem can be very complicated. Fig. 3 shows an example of a free boundary for fixed  $K_1$  and  $K_2$ .

If initially  $(K_1, K_2, \theta_1, \theta_2)$  lies outside of  $S$ , then the firm should add both two types of capacity up to  $(K_1^*, K_2^*)$  such that

$$\frac{\partial W(K_1^*, K_2^*; \theta)}{\partial K_1} = k_1, \quad \frac{\partial W(K_1^*, K_2^*; \theta)}{\partial K_2} = k_2, \tag{15}$$

or equivalently,  $(K_1^*, K_2^*, \theta_1, \theta_2) \in S_3$ . This amounts to maximizing its net value  $W - k_1 K_1 - k_2 K_2$ . Later, shifts in demand may result in the firm holding excess amounts of some types of capital, although it is still investing in other types. The following condition must hold whenever the firm is adding one of the capital stocks, say  $K_2$ ,

$$\frac{\partial W(K_1, K_2^*; \theta)}{\partial K_2} = k_2,$$

or equivalently,  $(K_1, K_2^*, \theta_1, \theta_2) \in S_2$ . The optimal processes for the capital stocks,  $X^1$  and  $X^2$ , can then be characterized as the maximal adapted and increasing processes that enforce  $(X_t^1, X_t^2, \theta_{1t}, \theta_{2t}) \in S$  for all  $t \geq 0$ .

To solve for the value of the firm, we once again decompose it into two parts:  $W(K_1, K_2; \theta) = V(K_1, K_2; \theta) + F(K_1, K_2; \theta)$ , where

$$V(K_1, K_2; \theta) = E^* \left[ \int_0^\infty e^{-rt} \pi(K_1, K_2; \theta_t) dt \mid \theta_0 = \theta \right].$$

Note that  $V$  is the value of the capital  $K_1$  and  $K_2$  in place, and  $F$  is the value of the firm's options to add more capital.

Let  $\Delta^i V(K_1, K_2; \theta) = \partial V(K_1, K_2; \theta) / \partial K_i$  be the marginal value of the capital in place due to a marginal increase of capacity  $i$ , and let  $\Delta^i F(K_1, K_2; \theta) = -\partial F(K_1, K_2; \theta) / \partial K_i$  be the marginal value of the growth options due to a marginal increase of capacity  $i$ . For given  $K_1$  and  $K_2$ ,  $V$  and  $\Delta^i V$  can be evaluated by direct integration, while  $(\Delta^1 F, \Delta^2 F)$  can be determined as solutions to the partial differential equations

$$\begin{aligned} & \frac{1}{2}\sigma_1^2\theta_1^2\Delta^1 F_{\theta_1\theta_1} + \rho\sigma_1\sigma_2\theta_1\theta_2\Delta^1 F_{\theta_1\theta_2} + \frac{1}{2}\sigma_2^2\theta_2^2\Delta^1 F_{\theta_2\theta_2} + (r - \delta_1)\theta_1\Delta^1 F_{\theta_1} \\ & + (r - \delta_2)\theta_2\Delta^1 F_{\theta_2} - r\Delta^1 F = 0, \end{aligned} \quad (16)$$

for  $\theta \in (\Theta_1 \cup \Theta_3)^c$ , and

$$\begin{aligned} & \frac{1}{2}\sigma_1^2\theta_1^2\Delta^2 F_{\theta_1\theta_1} + \rho\sigma_1\sigma_2\theta_1\theta_2\Delta^2 F_{\theta_1\theta_2} + \frac{1}{2}\sigma_2^2\theta_2^2\Delta^2 F_{\theta_2\theta_2} + (r - \delta_1)\theta_1\Delta^2 F_{\theta_1} \\ & + (r - \delta_2)\theta_2\Delta^2 F_{\theta_2} - r\Delta^2 F = 0, \end{aligned} \quad (17)$$

for  $\theta \in (\Theta_2 \cup \Theta_3)^c$ ,

$$\Delta^1 F(\theta) = \Delta^1 V(\theta) - k_1, \quad \theta \in \Theta_1 \cup \Theta_3, \quad (18)$$

$$\Delta^2 F(\theta) = \Delta^2 V(\theta) - k_2, \quad \theta \in \Theta_2 \cup \Theta_3. \quad (19)$$

(See fig. 3.) Eqs. (18) and (19) must be solved simultaneously with the additional conditions that

$$\frac{\partial \Delta^1 F}{\partial K_2} = \frac{\partial \Delta^2 F}{\partial K_1},$$

and that  $\Delta^1 F$  and  $\Delta^2 F$  is continuously differentiable. The regions  $\Theta_1$ ,  $\Theta_2$ , and  $\Theta_3$  are determined as a part of the solution. In general, solving (18) and (19) simultaneously can be very difficult. However, if the instantaneous profit from the two production processes can be decomposed in such a way that  $\pi(K_1, K_2; \theta) = \pi_1(K_1; \theta) + \pi_2(K_2; \theta)$ , then the value of the capital in place,  $V$ , can also be decomposed into  $V(K_1, K_2; \theta) = V^1(K_1; \theta) + V^2(K_2; \theta)$ . It is then easy to see that the value of the growth options can also be decomposed:  $F(K_1, K_2; \theta) = F^1(K_1; \theta) + F^2(K_2; \theta)$ . Since  $\Delta^i V = \Delta^i V^i$  and  $\Delta^i F = \Delta^i F^i$ , we can interpret  $\Delta^i V^i$  to be the marginal value of the installed capacity of type  $i$ , and  $\Delta^i F^i$  to be the marginal value of the firm's growth options to invest in capacity  $i$  in the future. In this special case,  $\Delta^1 F^1$  satisfies (18) and  $\Delta^2 F^2$  satisfies (19), and they can be solved independently.

Finally, since  $V(0, 0; \theta) = 0$  and  $F^1(\infty; \theta) = F^2(\infty; \theta) = 0$ , we conclude that the market value of the firm with  $K_i$  units of capital  $i$  in place is given by

$$\begin{aligned}
 W(K_1, K_2; \theta) &= \int_0^{K_1} \Delta^1 V^1(x; \theta) dx + \int_0^{K_2} \Delta^2 V^2(y; \theta) dy \\
 &\quad + \int_{K_1}^{\infty} \Delta^1 F^1(x; \theta) dx + \int_{K_2}^{\infty} \Delta^2 F^2(y; \theta) dy. \quad (20)
 \end{aligned}$$

### 2.3. Optimal choice of technology

Now suppose that given the current demand states  $\theta$ , the firm must decide which technology – flexible or output-specific – to adopt, and how much capacity to install.<sup>6</sup> This can be solved as follows. First, calculate the functions  $\Delta^i V(K_1, K_2; \theta)$  and  $\Delta^i F(K_1, K_2; \theta)$ ,  $i = 1, 2$ , and the functions  $\Delta^f V(K_f; \theta)$  and  $\Delta^f F(K_f; \theta)$ . Second, use (11') to obtain the optimal amount of flexible capacity  $K_f^*$ , and (18) and (19) to obtain the optimal amounts of output-specific capitals  $K_1^*$  and  $K_2^*$ . Finally, use (12) and (20) to determine the firm's market value (or the net present value) for each technology as follows:

$$NPV^f = W^f(K_f^*; \theta) - k_f K_f^*,$$

$$NPV^{nf} = W(K_1^*, K_2^*; \theta) - k_1 K_1^* - K_2 K_2^*.$$

We illustrate how this can be done by an example in the next section.

### 3. An example

Our model can be simplified considerably, while retaining its basic economic structure, by reducing the number of stochastic state variables to one. We will assume for simplicity that the firm produces two outputs with zero operating costs, but that the demand functions are given by

$$P_1 = -\log \theta - \gamma_{11} Q_1 + \gamma_{12} Q_2,$$

$$P_2 = \log \theta + \gamma_{21} Q_1 - \gamma_{22} Q_2,$$

with  $d\theta = (\mu - \delta)\theta dt + \sigma\theta dz$ , where  $\mu$ ,  $\delta$ , and  $\sigma$  are constants. Therefore, the stochastic component of demand is normally distributed. We will assume

<sup>6</sup>For simplicity, we only allow the firm to invest in a single technology. In general a firm might install a mixture of output-specific and flexible capital. It should be clear that we can still handle this problem by using the singular stochastic control approach, but the solution can become very messy.

that the two outputs are substitutes, so that  $\gamma = \gamma_{12} + \gamma_{21} < 0$ . ( $Q_1$  and  $Q_2$  might be the demands for large and small automobile engines, and  $\theta$  an index of gasoline prices.) For clarity, we also make the demands symmetric:  $\gamma_{11} = \gamma_{22} = b$ . (Then  $\gamma$  drops out of the solution.)

### 3.1. Flexible capital

We first find the optimal investment rule and market value for a firm using flexible capital. Note that the profit generated by an incremental unit of capital, given  $K_f$  already in place, is

$$\Delta\pi_f = \begin{cases} -\log \theta - 2bK_f, & \log \theta \leq -2bK_f, \\ 0, & -2bK_f < \log \theta \leq 2bK_f, \\ \log \theta - 2bK_f, & \log \theta > 2bK_f. \end{cases} \quad (21)$$

Direct integration shows that<sup>7</sup>

$$\Delta^f V(\theta) = \begin{cases} A\theta^{\beta_1} - \frac{\log \theta}{r} - \frac{(r - \delta - \sigma^2/2)}{r^2} - \frac{2bK_f}{r}, & \log \theta \leq -2bK_f, \\ B\theta^{\beta_1} + C\theta^{\beta_2}, & -2bK_f \leq \log \theta \leq 2bK_f, \\ D\theta^{\beta_2} + \frac{\log \theta}{r} + \frac{(r - \delta - \sigma^2/2)}{r^2} - \frac{2bK_f}{r}, & \log \theta \geq 2bK_f, \end{cases} \quad (22)$$

<sup>7</sup>The value of this incremental unit of capital,  $\Delta^f V(K_f; \theta)$ , must satisfy

$$\frac{1}{2}\sigma^2\theta^2\Delta^f V_{\theta\theta} + (r - \delta)\theta\Delta^f V_{\theta} - r\Delta^f V + \Delta\pi_f(\theta) = 0,$$

subject to the boundary conditions

$$\lim_{\theta \rightarrow 0} \Delta^f V(\theta) + (\log \theta)/r + 2bK_f/r + (r - \delta - \sigma^2/2)/r^2 = 0,$$

$$\lim_{\theta \rightarrow +\infty} \Delta^f V(\theta) - (\log \theta)/r + 2bK_f/r - (r - \delta - \sigma^2/2)/r^2 = 0,$$

The first boundary condition says that for  $\theta$  close to zero, the firm can expect to use this unit of capital to produce good 1 and only good 1 for the indefinite future, and  $\Delta^f V(\theta)$  is the corresponding present value of the expected stream of marginal profit. Similarly, the second one says that for  $\theta$  very large, the firm can expect to use the capital to produce good 2 and only good 2 for the indefinite future.

where  $A = \xi_1(e^{2bK_f\beta_1} + e^{-2bK_f\beta_1})$ ,  $B = \xi_1 e^{-2bK_f\beta_1}$ ,  $C = \xi_2 e^{2bK_f\beta_2}$ , and  $D = \xi_2(e^{-2bK_f\beta_2} + e^{2bK_f\beta_2})$ , with

$$\beta_1 = \frac{-(r - \delta - \sigma^2/2)}{\sigma^2} + \frac{1}{\sigma^2} \left[ (r - \delta - \sigma^2/2)^2 + 2r\sigma^2 \right]^{1/2} > 1,$$

$$\beta_2 = \frac{-(r - \delta - \sigma^2/2)}{\sigma^2} - \frac{1}{\sigma^2} \left[ (r - \delta - \sigma^2/2)^2 + 2r\sigma^2 \right]^{1/2} < 0,$$

$$\xi_1 = \frac{1 - \beta_2(r - \delta - \sigma^2/2)/r}{r(\beta_1 - \beta_2)},$$

$$\xi_2 = \frac{1 - \beta_1(r - \delta - \sigma^2/2)/r}{r(\beta_1 - \beta_2)}.$$

To interpret (22), note that the incremental unit of capital is utilized only if  $\log \theta \leq -2bK_f$  or  $\log \theta \geq 2bK_f$ . The first line of (22) says that if  $\log \theta \leq -2bK_f$  so the unit is utilized, its value is the present value of the stream of incremental profits from utilizing the unit indefinitely (the last three terms on the right-hand side), plus the value of the option to stop utilizing the unit should  $i$  rise (the first term). Similarly for the third line of (22), when  $\log \theta \geq 2bK_f$ . The second line applies when the unit is not being utilized. Its value is then the sum of the values of the options to utilize the unit should  $\log \theta$  fall below  $-2bK_f$  (the first term) or rise above  $2bK_f$  (the second term).

Fig. 4 shows  $\Delta^f V$  as a function of  $\log \theta$  for  $b = 1$ ,  $K_f = 0.75$ ,  $r = 0.04$ , and  $\sigma = 0, 0.2, 0.4$ .<sup>8</sup> We let  $\delta = r - \sigma^2/2$ , which makes  $\Delta^f V(\theta)$  symmetric around  $\log \theta = 0$ . When  $\sigma = 0$ ,  $A$ ,  $B$ ,  $C$ , and  $D$  in (22) become zero, so  $\Delta^f V = -\log \theta/r - 2bK_f/r$  if  $\log \theta \leq -2bK_f$ ,  $\Delta^f V = \log \theta/r - 2bK_f/r$  if  $\log \theta \geq 2bK_f$ , and  $\Delta^f V = 0$  otherwise. For our choice of parameter values,  $\Delta^f V > 0$  only if  $\log \theta$  exceeds 1.5 in magnitude. But if  $\sigma > 0$ ,  $\Delta^f V > 0$  for all values of  $\log \theta$ , because of the possibility that  $\theta$  will rise or fall in the future.

<sup>8</sup>The standard deviations of annual changes in the prices of commodities such as oil, gas, copper, and aluminum are 20 to 50 percent. For manufactured goods the numbers are lower (based on producer price indices for 1948-87, they are 11 percent for cereal and bakery goods, 3 percent for electrical machinery, and 5 percent for photographic equipment). But variation in the sales of a product for one company will be much larger than variation in price for the entire industry.

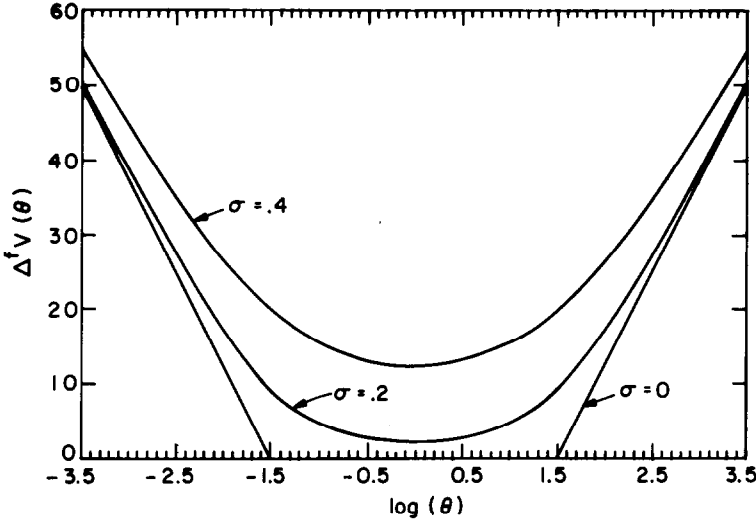


Fig. 4. Value of an incremental unit of flexible capacity ( $K_f = 0.75, \sigma = 0, 0.2, 0.4$ ).

Given  $\Delta^f V(K_f; \theta)$ , we can find  $\Delta^f F(K_f; \theta)$ , the marginal value of the firm's option to invest in this incremental unit of capital.  $\Delta^f F$  must satisfy

$$\begin{aligned} \frac{1}{2}\sigma^2\theta^2\Delta^f F_{\theta\theta} + (r - \delta)\theta\Delta^f F_{\theta} - r\Delta^f F &= 0, & \theta \in [\underline{\theta}^*, \bar{\theta}^*], & (23) \\ \Delta^f F(\theta) &= \Delta^f V(\theta) - k_f, & \theta \geq \bar{\theta}^*, \quad \theta \leq \underline{\theta}^*, \\ \Delta^f F_{\theta}(\bar{\theta}^*) &= \Delta^f V_{\theta}(\bar{\theta}^*), \\ \Delta^f F_{\theta}(\underline{\theta}^*) &= \Delta^f V_{\theta}(\underline{\theta}^*). \end{aligned}$$

Here  $\underline{\theta}^*$  and  $\bar{\theta}^*$  are the lower and upper critical points, i.e., the firm should add capital if  $\theta$  falls below  $\underline{\theta}^*$  or rises above  $\bar{\theta}^*$ .

The solution to (23) is

$$\Delta^f F(\theta) = a_1\theta^{\beta_1} + a_2\theta^{\beta_2}. \tag{24}$$

The critical values  $\underline{\theta}^*$  and  $\bar{\theta}^*$ , as well as  $a_1$  and  $a_2$ , are found by substituting (24) for  $\Delta^f V$  and (24) for  $\Delta^f F$  into the conditions at the critical boundaries and solving numerically. A solution is shown in fig. 5, for  $k_f = 12, \sigma = 0.2$ , and  $K_f, r$ , and  $\delta$  as before. Note that if  $\underline{\theta}^* < \theta < \bar{\theta}^*$ , the total cost of investing in the incremental unit of capital,  $\Delta^f \bar{F}(\theta) + k_f$ , exceeds the value of



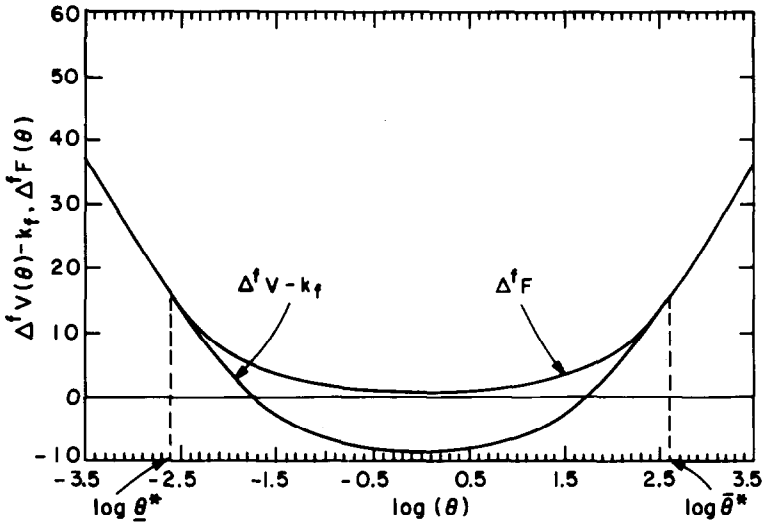


Fig. 5. Optimal investment rule – flexible capacity ( $K_f = 0.75, k_f = 12, \sigma = 0.2$ ).

the unit,  $\Delta^f V(\theta)$ , and so the firm should not invest. Also, recall that  $a_1, a_2, \underline{\theta}^*$ , and  $\bar{\theta}^*$  are functions of  $K_f$ .

As  $K_f$  increases,  $\underline{\theta}^*$  falls and  $\bar{\theta}^*$  rises. Thus, if  $\theta$  is currently less than  $\underline{\theta}^*$  or greater than  $\bar{\theta}^*$ , the firm will add capacity until  $\theta$  just equals one of these critical values. Given this optimal capacity  $K_f^*$ , the net present value of the firm can then be found as in section 2.3.

### 3.2. Output-specific capital

The optimal investment rule for output-specific capital is found in the same way. Direct computation shows that the profits from incremental units of each type of capital, given  $K_1$  and  $K_2$  in place, can be decomposed as  $\pi(K; \theta) = \pi_1(K_1; \theta) + \pi_2(K_2; \theta)$ , which results in

$$\Delta^1 \pi_1 = \begin{cases} -\log \theta - 2bK_1, & \log \theta \leq -2bK_1, \\ 0, & \log \theta \geq -2bK_1, \end{cases} \quad (25)$$

$$\Delta^2 \pi_2 = \begin{cases} 0, & \log \theta \leq 2bK_2, \\ \log \theta - 2bK_2, & \log \theta \geq 2bK_2. \end{cases} \quad (26)$$

The value of an incremental unit of capital of type  $i$  becomes

$$\Delta^1 V^1(\theta) = \begin{cases} A_1 \theta^{\beta_1} - \frac{\log \theta}{r} - \frac{(r - \delta - \sigma^2/2)}{r^2} - \frac{2bK_f}{r}, & \log \theta \leq -2bK_1, \\ B_1 \theta^{\beta_2}, & \log \theta \geq -2bK_1, \end{cases} \quad (27)$$

$$\Delta^1 V^2(\theta) = \begin{cases} A_2 \theta^{\beta_1}, & \log \theta \leq 2bK_2, \\ B_2 \theta^{\beta_2} + \frac{\log \theta}{r} + \frac{(r - \delta - \sigma^2/2)}{r^2} - \frac{2bK_f}{r}, & \log \theta \geq 2bK_2, \end{cases} \quad (28)$$

where  $A_1 = \xi_2 e^{2bK_1\beta_1}$ ,  $B_1 = \xi_1 e^{2bK_1\beta_2}$ ,  $A_2 = \xi_2 e^{-2bK_2\beta_1}$ ,  $B_2 = \xi_1 e^{-2bK_2\beta_2}$ , and  $\beta_1, \beta_2, \xi_1$ , and  $\xi_2$  are defined as above.

The interpretation of (27) and (28) is similar to that of (22). In (27), the incremental unit is utilized only if  $\log \theta \leq -2bK_1$ . Then, its value is the present value of the stream of profits from utilizing it indefinitely, plus the value of the option to stop utilizing it should  $\theta$  rise. The second line of (27) is the value of the option to utilize the unit should  $\log \theta$  fall below  $-2bK_1$ .

Fig. 6 shows  $\Delta^1 V^1$  and  $\Delta^2 V^2$  plotted against  $\log \theta$  for  $K_1 = K_2 = 0.75$ , and again,  $b = 1$ ,  $r = 0.04$ ,  $\sigma = 0, 0.2, 0.4$ , and  $\delta = r - \sigma^2/2$ . As with the case of flexible capital, if  $\sigma = 0$  and  $-1.5 < \log \theta < 1.5$ , an extra unit of capital would never be used, and has no value. For  $\sigma > 0$ , an extra unit of capital of either type might be used in the future, and has positive value for all values of  $\log \theta$ . Note that  $\Delta^1 V^1$  and  $\Delta^2 V^2$  have the form of a call option, and increase with  $\sigma$ . Indeed each is the value of an infinite number of (European) call options to produce at every point in the future. Given  $\Delta^1 V^1$  and  $\Delta^2 V^2$ ,  $\Delta^1 F^1$  and  $\Delta^2 F^2$  are found by solving:

$$\frac{1}{2}\sigma^2\theta^2\Delta^1 F_{\theta\theta}^1 + (r - \delta)\theta\Delta^1 F_{\theta}^1 - r\Delta^1 F^1 = 0, \quad \theta \geq \underline{\theta}^*, \quad (29)$$

$$\Delta^1 F^1(\theta) = \Delta^1 V^1(\underline{\theta}^*) - k_1, \quad \theta \leq \underline{\theta}^*,$$

$$\Delta^1 F_{\theta}^1(\underline{\theta}^*) = \Delta^1 V_{\theta}^1(\underline{\theta}^*),$$

$$\lim_{\theta \rightarrow \infty} \Delta^1 F^1(\theta) = 0,$$

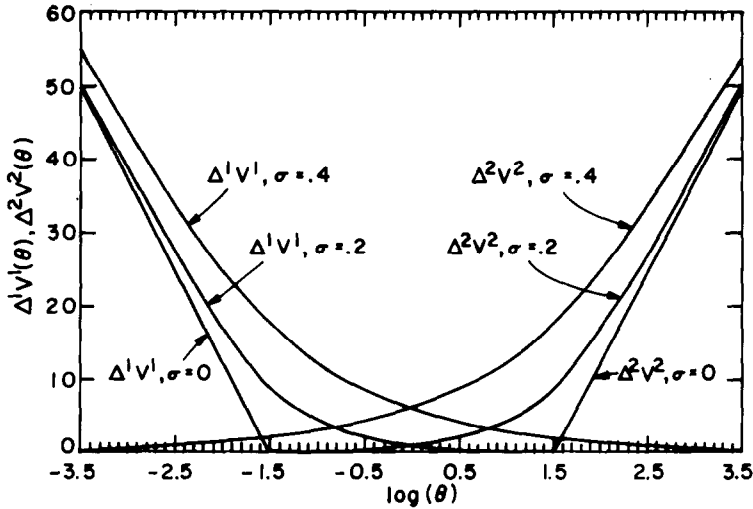


Fig. 6. Value of an incremental unit of output-specific capacity ( $K_1 = K_2 = 0.75$ ,  $\sigma = 0, 0.2, 0.4$ ).

and

$$\begin{aligned} \frac{1}{2}\sigma^2\theta^2\Delta^2F_{\theta\theta}^2 + (r - \delta)\theta\Delta^2F_{\theta}^2 - r\Delta^2F^2 &= 0, & \theta \leq \bar{\theta}^*, & (30) \\ \Delta^2F^2(\theta) = \Delta^2V(\theta) - k_2, & & \theta \geq \bar{\theta}^*, & \\ \Delta^2F_{\theta}^2(\bar{\theta}^*) = \Delta^2V_{\theta}^2(\bar{\theta}^*), & & & \\ \lim_{\theta \rightarrow 0} \Delta^2F^2(\theta) = 0. & & & \end{aligned}$$

The solutions to (29) and (30) are

$$\begin{aligned} \Delta^1F^1(\theta) &= m_1\theta^{\beta_2}, \\ \Delta^1F^2(\theta) &= m_2\theta^{\beta_1}. \end{aligned}$$

Note that  $\underline{\theta}^*$  and  $\bar{\theta}^*$  are again the critical values of  $\theta$ ; the firm adds capital of type 1 if  $\theta$  falls below  $\underline{\theta}^*$ , and adds capital of type 2 if  $\theta$  rises above  $\bar{\theta}^*$ . After substituting in (27) and (28), the boundary conditions can be used to solve for  $\underline{\theta}^*$ ,  $\bar{\theta}^*$ ,  $m_1$ , and  $m_2$ .

A solution is shown in fig. 7 for costs of capital  $k_1 = k_2 = 10$  and  $\sigma = 0.2$ . The critical values of  $\log \theta$  are  $\pm 2.35$ . For  $\log \theta$  inside this range, the value of a unit of either type of capital is less than the total cost of investing in the unit, so the firm does not invest. Again,  $\underline{\theta}^*$ ,  $\bar{\theta}^*$ ,  $m_1$ , and  $m_2$  are all functions of  $K_1$  and  $K_2$ ; as  $K_1(K_2)$  increases,  $m_1$  and  $\theta^*$  fall ( $m_2$  falls and  $\bar{\theta}^*$  rises).

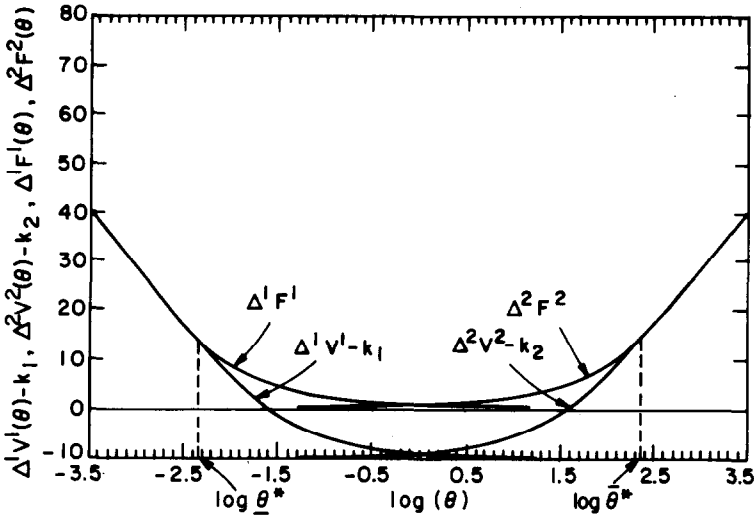


Fig. 7. Optimal investment rule - nonflexible capacity ( $K_1 = K_2 = 0.75$ ,  $k_1 = k_2 = 10$ ,  $\sigma = 0.2$ ).

Thus given the current value of  $\theta$ , the boundary conditions can be used to find the firm's optimal initial capital stocks  $K_1^*$  and  $K_2^*$ . Then, given  $K_1^*$  and  $K_2^*$ , the net present value of the firm can be calculated.

### 3.3. The choice of technology

The ex ante choice of technology requires comparing the net value of the firm using flexible versus output-specific capital. This comparison will depend on the parameters  $\sigma$ ,  $\delta$ , and  $r$ , the capital costs  $k_1$ ,  $k_2$ , and  $k_f$ , as well as the current state of demand, i.e., the value of  $\theta$ .

Table 1 shows the net value of the firm and its components for various values of  $\sigma$  and  $\theta$ , for flexible and nonflexible capital. Note that if  $\sigma = 0$ , the firm observes  $\theta$  and installs as much capital as it will ever need, and the value of its options to grow ( $F^f$  in the flexible case,  $F^1 + F^2$  in the nonflexible) is zero. The total value of the firm is then the same for either technology, so the firm will use the cheaper nonflexible capital. (In the nonflexible case,  $K_1^* = 0$  for all combinations of  $\sigma$  and  $\theta$  shown, but  $F^1$ , the value of the option to install capital of type 1, is positive for  $\sigma > 0$ .) For both technologies, as  $\sigma$  increases, the amount of capital that the firm initially installs falls; although the value of each incremental unit of capital rises with  $\sigma$ , the value of the option to invest in the unit (an opportunity cost) rises even more. For large  $\sigma$ , much of the firm's value comes from its options to grow; for  $\sigma = 0.4$  and

Table 1a  
Value of the firm: Flexible capacity.<sup>a</sup>

$\sigma$	$\log \theta$	$K_f^*$	$V(K_f^*; \theta)$	$F(K_f^*; \theta)$	Total value	Net value
0	1.5	0.50	12.5	0.0	12.5	6.5
	2.5	1.00	37.5	0.0	37.5	25.5
	3.5	1.50	75.0	0.0	75.0	57.0
0.1	1.5	0.35	10.5	1.5	12.0	7.8
	2.5	0.84	36.0	1.5	37.5	27.4
	3.5	1.34	74.6	1.5	76.1	60.0
0.2	1.5	0.27	9.2	5.1	14.3	11.1
	2.5	0.74	34.6	5.4	40.0	31.1
	3.5	1.25	73.3	5.4	78.7	63.7
0.4	1.5	0.22	10.1	18.8	28.9	26.3
	2.5	0.60	33.7	19.9	53.6	46.4
	3.5	1.08	72.5	20.0	92.5	79.5

<sup>a</sup> $k_f = 12$ ,  $k_1 = k_2 = 10$ ,  $r = 0.04$ , and  $\delta = r - \sigma^2/2$ . All of the solutions are symmetric around  $\log \theta = 0$ .

Table 1b  
Value of the firm: Nonflexible capacity.<sup>a</sup>

$\sigma$	$\log \theta$	$K_2^*$	$V^2(K_2^*; \theta)$	$F^2(K_2^*; \theta)$	$F^1(K_1^*; \theta)$	Total value	Net value
0	1.5	0.55	13.1	0.0	0.0	13.1	7.6
	2.5	1.05	38.1	0.0	0.0	38.1	27.6
	3.5	1.55	75.6	0.0	0.0	75.6	60.1
0.1	1.5	0.40	11.3	1.5	0.0	12.8	8.8
	2.5	0.90	37.0	1.5	0.0	38.5	29.5
	3.5	1.40	75.6	1.5	0.0	77.1	63.1
0.2	1.5	0.29	9.4	5.5	0.2	15.1	12.2
	2.5	0.79	35.0	5.5	0.0	40.5	32.6
	3.5	1.29	73.5	5.4	0.0	78.9	66.0
0.4	1.5	0.15	5.3	19.5	2.9	27.8	26.2
	2.5	0.65	31.0	19.3	1.4	51.7	45.2
	3.5	1.65	69.6	19.3	0.7	89.6	73.1

<sup>a</sup>In all cases shown,  $K_1^* = 0$ , so  $V^1(K_1^*; \theta) = 0$ .

$\log \theta = 1.5$ , these options account for more than half of total value, with either technology.

In the example in table 1, flexible capital makes the net value of the firm higher only when  $\sigma$  is 0.4. (It is misleading to compare total values. With equal amounts of installed capacity, a firm using the flexible technology will always have a higher total value. But flexible capital is more expensive, and, as table 1 shows, the amounts of installed capacity differ in the two cases.)

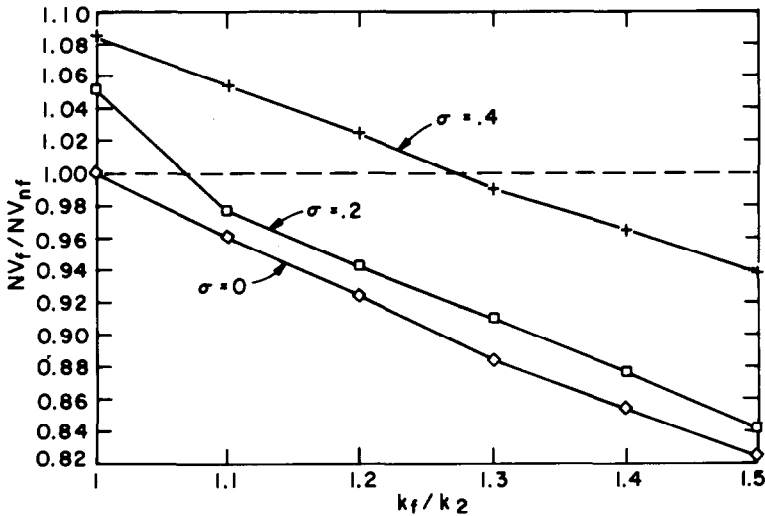


Fig. 8. Ratio of net values vs. ratio of capital costs ( $\log \theta = 2.5$ ).

Fig. 8 shows how the choice of technology depends on relative capital costs for  $\sigma = 0, 0.2, 0.4$  and  $\log \theta = 2.5$ . (There,  $k_1$  and  $k_2$  are fixed at 10, and the optimal amount of flexible capacity and corresponding net value of the firm are calculated as  $k_f$  is varied between 10 and 15.) When  $\sigma = 0.2$ , the ratio of net values exceeds 1 only when  $k_f/k_2$  is less than about 1.07.

These results illustrate how a value-maximizing choice of technology and capacity can be calculated, and how they depend on various parameters. One should not infer that the net benefit of flexible capital is low; our example is based on a specific production technology and specific demand functions, and our solutions apply to a limited range of parameter values.

#### 4. Investments in input-flexible capacity

The analogous investment problem that arises with input-flexible capacity can be treated in the same way. To see this, consider a firm facing the following nonstochastic demand curve for its single output:  $P = a - bQ$ . Suppose the firm must use, in addition to capital, one of two variable inputs whose costs,  $c_1$  and  $c_2$ , vary stochastically:

$$dc_i = \alpha_i c_i dt + \sigma_i c_i dz_i, \quad i = 1, 2,$$

with  $E(dz_1 dz_2) = \rho dt$ . The firm can (irreversibly) purchase and install input-flexible capacity at a cost  $k_f$  per unit, or input-specific capacity at a

(lower) cost  $k_1$  or  $k_2$ . Each unit of capacity allows the firm to produce one unit of output using one unit of the corresponding input.

This technology and capacity choice problem can be solved using the approach of section 2. The profit generated by an incremental unit of flexible capacity at time  $t$  is given by

$$\Delta\pi(K_f; c) = \max[0, a - 2bK_f - \min(c_1, c_2)].$$

For an incremental unit of input-specific capacity of type 1, the profit is

$$\begin{aligned} \Delta^1\pi(K_1, K_2; c) = \min\{ & \max[0, a - 2bK_1 - c_1], \\ & \max[0, c_2 - c_1, a - 2b(K_1 + K_2) - c_1]\}. \end{aligned}$$

(Similarly for  $\Delta^2\pi$ .)  $\Delta^iV$ ,  $i = 1, 2$ , and  $f$  again can be calculated by integration,  $\Delta^iF$  can be determined by a set of partial differential equations with certain boundary conditions. The solutions of these equations give the optimal capacity levels, and the approach of section 2.3 can be used to find the market value of the firm for each technology.

In general, a solution requires numerical methods. However, the problem is much simpler if only one input cost is stochastic, and the other is constant. (This would apply, say, to an electric utility choosing among a coal-fired plant, an oil-fired plant, or a plant that can burn either fuel – coal prices fluctuate little compared to oil prices.) An analytical solution can then be found similar to the one presented in section 3.

## 5. Conclusions

We have shown how the value-maximizing choice of technology and capacity can be found in a way that is consistent with the irreversibility of investment, the fact that capacity in place need not always be utilized, and the existence of a competitive capital market. First, the value of an incremental unit of capacity of each type is determined. Second, the value of the firm's option to invest in this unit is determined, together with the optimal exercise rule. The latter yields the firm's optimal initial capacity, and the corresponding net value of the firm can be calculated. The choice of technology can then be made by comparing ex ante net values.

Our example suggests that irreversibility and uncertainty can have a substantial effect on the capacity the firm initially installs; note from table 1 that  $K^*$  falls rapidly as  $\sigma$  is increased, for both technologies. This is consistent with recent studies of irreversible investment, but some restrictive assumptions may have exaggerated this effect. For example, by assuming the firm can incrementally invest, we ignored the lumpiness of investment. We

also ignored depreciation (if capital becomes obsolete rapidly, the opportunity cost of investing will be small). And, as mentioned earlier, our numerical results apply to a simplified model and a limited range of parameter values. This also limits the generality of our finding that flexible capital is preferred only if its cost premium is low.

In addition, we made the simplifying assumption that the firm can invest in only a single technology, whereas in general it may be optimal to install a mixture of output-specific and flexible capital. (The problem can be solved for the more general case, but the algebra is much messier.)

Other caveats deserve mention. We ignored scale economies, which could make cost increase with the number of products the firm produces, creating an incentive to produce only one output (and use nonflexible capital). Except for capital costs (and constant average variable costs), only demands affect the output mix in our model. [For a model that shows implications of scale economies, see de Groote (1987).] And we ignore strategic aspects of flexibility. As Vives (1986) and others have shown, flexibility can have a negative value in a small numbers environment because with it the firm is less able to commit itself to a particular output level or product mix.<sup>9</sup>

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<sup>9</sup>But flexibility with respect to factor inputs can provide a strategic advantage in bargaining. See Kulatilaka and Marks (1988).



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