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Investments of uncertain cost*

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This paper examines irreversible investment decisions when projects take time to complete and are subject to two types of cost uncertainty. The first is technical uncertainty, i.e., uncertainty over the physical difficulty of completing a project, which is only resolved as the investment proceeds. The second is input cost uncertainty, i.e., uncertainty over the prices of construction inputs or over government regulations affecting construction costs, which is external to the firm. These two types of uncertainty have very different effects on the investment decision. A simple investment rule is derived that maximizes firm value, and is used to analyze the decision to start or continue building a nuclear power plant during the 1980s.

1. Introduction

In most studies of investment under uncertainty, it is the future payoffs from the investment that are uncertain. The emphasis on uncertainty over future payoffs also applies to the growing literature on irreversible investment. Much of that literature [see Dixit (1992), Pindyck (1991), and Dixit and Pindyck (1993) for an overview] studies optimal stopping rules for the timing of sunk costs of known magnitude in exchange for capital whose value fluctuates stochastically.

Sometimes the cost of an investment is more uncertain than the future payoff, particularly for large projects that take considerable time to build. An example is a nuclear power plant, for which total construction costs are hard to predict due to both engineering and regulatory uncertainties. Although the future value of a completed nuclear plant is also uncertain (because electricity demand and

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costs of alternative fuels are uncertain), construction cost uncertainty is much greater than revenue uncertainty, and has deterred utilities from building new plants. There are many other examples, including large petrochemical complexes, the development of a new line of aircraft, and urban construction projects. Moreover, large size is not a requisite. Many R&D projects involve considerable cost uncertainty; the development of a new drug by a pharmaceutical company is an example.

In addition to their uncertain costs, all of the investments mentioned above are irreversible. Expenditures on nuclear power plants, petrochemical complexes, and the development of new drugs are firm- or industry-specific, and hence are sunk costs that cannot be recovered should the investment turn out, *ex post*, to have been a bad one. In each case, the investment could turn out to be bad either because demand for the product is less than anticipated or because the cost of the investment turns out to be greater than anticipated. Whatever the reason, the firm cannot 'disinvest' and recover the money it spent.

This paper studies the implications of cost uncertainty for irreversible investment decisions. With projects that take time to complete, two different kinds of uncertainty arise. The first, which I call *technical* uncertainty, relates to the physical difficulty of completing a project: Assuming prices of construction inputs are known, how much time, effort, and materials will ultimately be required? Technical uncertainty can only be resolved by undertaking the project; actual costs and construction time unfold as the project proceeds.[1] These costs may be greater or less than anticipated if impediments arise or if the work progresses faster than planned, but the total cost of the investment is only known for certain when the project is complete. Also, technical uncertainty is largely diversifiable. It results only from the inability to predict how difficult a project will be, which is likely to be independent of the overall economy.

The second kind of uncertainty relates to *input costs*, and is external to what the firm does. It arises when the prices of labor, land, and materials needed to build a project fluctuate unpredictably, or when unpredictable changes in government regulations change the cost of construction. Prices and regulations change regardless of whether or not the firm is investing, and are more uncertain the farther into the future one looks. Hence input cost uncertainty is particularly important for projects that take time to complete or are subject to voluntary or involuntary delays. Also, this uncertainty may be partly nondiversifiable; changes in construction costs are likely to be correlated with overall economic activity.

[1]This is a simplification, in that for some projects cost uncertainty can be reduced by first undertaking additional engineering studies. The investment problem is then more complicated because one has three choices instead of two: start construction now, undertake an engineering study now, and then begin construction only if the study indicates costs are likely to be low, or abandon the project completely.

This paper derives decision rules for irreversible investments subject to both types of cost uncertainty. For simplicity, I first assume that the value of the completed project is known with certainty, and then show how the model can be extended so that this value is also stochastic. The decision rules I derive allow for the possibility of abandoning the project midstream, and maximize the value of the firm in a competitive capital market. These rules have a simple form: *Invest as long the expected cost to complete the project is below a critical number*. Also, the derivation of the decision rule yields the value of the investment opportunity, i.e., the value of the right to undertake the project. I explore how this value, and the critical expected cost to completion, depend on the type and level of uncertainty.

Both technical and input cost uncertainty increase the value of an investment opportunity. The reason is that the payoff function is $\max[0, V-K]$, where K is the cost and V the value of the completed project. The investment opportunity is like a put option; the holder can sell an asset worth an uncertain amount K for a fixed 'exercise price' V. Like any option, its value is increased by an increase in the variance of the price of the underlying asset. (In my model, the firm actually has a more complicated compound option; it can spend an uncertain amount of money in return for an option to continue the partially completed project.)

However, the two types of uncertainty affect the investment decision differently. Technical uncertainty makes investing more attractive; a project can have an expected cost that makes its conventional NPV negative, but it can still be economical to begin investing. The reason is that investing reveals information about cost, and therefore has a shadow value beyond its direct contribution to the completion of the project; this shadow value lowers the full expected cost of the investment.[2] Also, since information about cost arrives only when investment is taking place, there is no value to waiting.

As an example, a project requires a first phase investment of \$1. Then, with probability 0.5 the project will be finished, and with probability 0.5 a second phase costing \$4 will be required. Completion of the project yields a certain payoff of \$2.8. Since the expected cost of the project is \$3, the conventionally measured NPV is negative. But the conventional NPV ignores the value of the option to abandon the project should the second phase be required. The correct NPV is $-1 + (0.5)(2.8) = \$0.4$, so the firm should proceed with at least the first phase.

Input cost uncertainty makes it less attractive to invest now. A project with a conventional NPV that is positive might still be uneconomical, because costs of construction inputs change whether or not investment is taking place, so there is a value of waiting for new information before committing resources. Also, this

[2] It is analogous to the shadow value of production arising from a learning curve, which lowers the full cost of production; see Majd and Pindyck (1989).

effect is magnified when fluctuations in construction costs are correlated with the economy, or, in the context of the Capital Asset Pricing Model, when the 'beta' of cost is high. The reason is that a higher beta implies that high-cost outcomes are more likely to be associated with high stock market returns, so that the investment opportunity is a hedge against nondiversifiable risk. Put another way, a higher beta raises the discount rate applied to expected future costs, which raises the value of the investment opportunity as well as the benefit from waiting rather than investing now.

For example, suppose an investment can be made now or later. The cost now is \$3, but next period it will either fall to \$2 or rise to \$4, each with probability 0.5, and then remain at that level. Investing yields a certain payoff of \$3.2. Assume the risk-free rate of interest is zero. If we invest now, the project has a conventional NPV of \$0.2. But this NPV ignores the opportunity cost of closing our option to wait for a better outcome (a drop in cost). If we wait, we will only invest if the cost falls to \$2. The NPV if we wait is $(0.5)(3.2 - 2) = \$0.6$, so it is better to wait. Now suppose the beta of cost is high, so that the risk-adjusted discount rate is 25% per period. Because the payoff from completing the project is certain, this discount rate is only applied to cost. Hence the NPV if we wait is now $(0.5)(3.2 - 2/1.25) = \$0.8$. The higher beta raises the present values of net payoffs, and thereby increases both the value of the investment opportunity and the value of waiting.

Since technical and input cost uncertainty have different effects on investment, it is important to incorporate both in the analysis. In doing so, the model developed in the next section offers guidance as to the types of projects (e.g., nuclear power plants versus R&D) for which one source of uncertainty or the other will exert the primary influence on investment decisions.

This paper is related to several earlier studies. The value of information gathering has been explored by Roberts and Weitzman (1981), who develop a model of sequential investment similar to mine in that the project can be stopped in midstream, and the process of investing reduces both the expected cost of completing the project as well the variance of that cost. They derive an optimal stopping rule, and show that it may pay to go ahead with the early stages of an investment even though the NPV of the entire project is negative.[3] Grossman and Shapiro (1986) also study investments for which the total effort required to reach a payoff is unknown. They model the payoff as a Poisson arrival, with a hazard rate specified as a function of the cumulative effort

[3]Weitzman, Newey, and Rabin (1981) use this model to evaluate demonstration plants for synthetic fuels, and show that learning about costs could justify these investments. MacKie-Mason (1991) extends the Roberts and Weitzman analysis by allowing for investors (who pay the cost of a project) and managers (who decide whether to continue or abandon the project) to have conflicting interests and asymmetric information. He shows that asymmetric learning about cost leads to inefficient overabandonment of projects. Finally, Zeira (1987) develops a model in which a firm learns about its payoff function as it accumulates capital.

expended. They allow the rate of progress to be a concave function of effort, and focus on the rate of investment, rather than on whether one should proceed or not. The result in this paper complement the work of these authors, but my model is more general in its treatment of cost uncertainty, and yields relatively simple decision rules.

This paper is also related to the basic model of irreversible investment by McDonald and Siegel (1986). They consider the payment of a sunk cost I in return for a project worth V, where V and I evolve as geometric Brownian motions. The optimal investment rule is to wait until V/I reaches a critical value that exceeds one, because of the opportunity cost of committing resources. Also, Majd and Pindyck (1987) study sequential investment when a firm can invest at some maximum rate (so that it takes time to complete a project), the project can be abandoned before completion, and the value of project, received upon completion, evolves as a geometric Brownian motion. In this paper the firm can also invest at a maximum rate, but it is the cost rather than the value of the completed project that is uncertain.

Other related work includes that of Baldwin (1982), who analyzes sequential investment decisions when investment opportunities arrive randomly and the firm has limited resources. She values the sequence of opportunities and shows that a simple NPV rule leads to overinvestment, i.e., there is a value to waiting for better opportunities. Likewise, if cost evolves stochastically, it may pay to wait for cost to fall. Also, Myers and Majd (1984) determine the value of a firm's option to abandon a project in return for a scrap value, S, when the value of the project, V, evolves as a geometric Brownian motion (the firm has a put option to sell a project worth V for a price S), and show how this abandonment value affects the decision to invest.

The basic model is developed in the next section. In section 3, numerical solutions are used to show how the value of the investment opportunity and the optimal investment rule depend on the source and amount of uncertainty, as well as on other parameters. Section 4 analyzes the decision to build a nuclear power plant; it shows how the model can be used in practice and the importance of analyzing technical and input cost uncertainty together. It also illustrates the nature and implications of nuclear plant cost uncertainty during the 1980's. Section 5 discusses some extensions of the basic model, and section 6 concludes.

2. The basic model

Consider an investment in a project whose actual cost of completion is a random variable, $\tilde{K}$, and whose expected cost is $K = \mathrm{E}(\tilde{K})$. The project takes time to complete; the maximum rate at which the firm can (productively) invest is k. Upon completion, the firm receives an asset (e.g., a factory or new drug) whose value, V, is known with certainty.

If there were no uncertainty over the total cost, valuing the investment opportunity and determining the optimal investment rule would be straightforward. The project will take time $T = K/k$ to complete, so the opportunity to invest is worth:

$$F(K) = \max\left[V\mathrm{e}^{-rK/k} - \int_0^{K/k} k\mathrm{e}^{-rt}\mathrm{d}t, 0 \right]$$

$$= \max\left[(V + k/r)\mathrm{e}^{-rK/k} - k/r, 0 \right], \tag{1}$$

where r is the (risk-free) rate of interest. The optimal investment rule is to proceed with the project as long as $F(K) > 0$, i.e., as long as K is less than a critical value, K^*, given by

$$K^* = (k/r)\log(1 + rV/k)\,.$$

If $r = 0$, $F(K) = V - K$ and $K^* = V$. But if $r > 0$, $F(K) < V - K$ and $K^* < V$. The reason is that the payoff V is received only at time T, and must be discounted accordingly, but the cost of the investment is spread out from $t = 0$ to T. Also, note that $F(K)$ is a convex function of K, so uncertainty over cost should increase $F(K)$. Little can be said at this point, however, about the effect of uncertainty on the optimal investment rule.

2.1. Introducing uncertainty

I introduce uncertainty over cost by letting the expected cost to completion, $K(t)$, follow a controlled diffusion process. Suppose for the moment that $K(t)$ is given by

$$\mathrm{d}K = -I\mathrm{d}t + g(I, K)\mathrm{d}z\,, \tag{2}$$

where I is the rate of investment, $z(t)$ is a Wiener process that might or might not be correlated with the economy and the stock market, and $g_I \geqslant 0$, $g_{II} \leqslant 0$, and $g_K \geqslant 0$. Eq. (2) says that the expected cost to completion declines with ongoing investment, but also changes stochastically. Stochastic changes in K might be due to technical uncertainty [in which case $g(0, K) = 0$ and $g_I > 0$], input cost uncertainty [in which case $g(0, K) > 0$], or both. Eq. (2) is a generalization of Roberts and Weitzman (1981), who also model the expected cost to completion as a stochastic process that is controlled by the rate of investment.

I will again assume that there is a maximum rate of investment, k. Let $F(K) = F(K; V, k)$ be the value of the investment opportunity. Then $F(K)$

satisfies

$$F(K) = \max_{I(t)} \mathrm{E}_0\left[V\mathrm{e}^{-\mu\tilde{T}} - \int_0^{\tilde{T}} I(t)\mathrm{e}^{-\mu t}\mathrm{d}t \right], \tag{3}$$

subject to eq. (2), $0 \leqslant I(t) \leqslant k$, and $K(\tilde{T}) = 0$. Here μ is an appropriate risk-adjusted discount rate, and the time of completion, $\tilde{T}$, is stochastic.

For eq. (2) to make economic sense, more structure is needed. In particular, the following conditions should hold: (i) $F(K; V, k)$ is homogeneous of degree one in K, V, and k; (ii) $F_K < 0$, i.e., an increase in the expected cost of an investment should always reduce its value; (iii) the instantaneous variance of $\mathrm{d}K$ is bounded for all finite K and approaches zero as $K \to 0$; and (iv) if the firm invests at the maximum rate k until the project is complete, $\mathrm{E}_0 \int_0^{\tilde{T}} k\mathrm{d}t = K$, so that K is indeed the expected cost to completion. We can meet these conditions and still allow for reasonably general cost structures by letting $g(I, K) = \beta K(I/K)^{\alpha}$, with $0 \leqslant \alpha \leqslant \frac{1}{2}$. This clearly satisfies conditions (i) and (iii). As will become evident later, $0 \leqslant \alpha \leqslant \frac{1}{2}$ rather than $0 \leqslant \alpha < 1$, which also satisfies (i) and (iii), is needed to satisfy (ii). Finally, it is shown in the appendix that (iv) is also satisfied.

I restrict the analysis to $\alpha = 0$ and $\frac{1}{2}$, which corresponds naturally to the two types of cost uncertainty, and which result in simple corner solutions for optimal investment. (As will be discussed in section 5, other values of α result in interior solutions where I is varied in response to changes in the variance of $\mathrm{d}K$.) The case of $\alpha = \frac{1}{2}$ corresponds to technical uncertainty; K can change only if the firm is investing, and the instantaneous variance of $\mathrm{d}K/K$ increases linearly with I/K. When the firm is investing, the expected change in K over an interval Δt is $-I\Delta t$, but the realized change can be greater or less than this, and K can even increase. As the project proceeds, progress will at times be slower and at times faster than expected. The variance of $\tilde{K}$ falls as K falls, but the actual total cost of the project, $\int_0^{\tilde{T}} I\mathrm{d}t$, is only known when the project is completed.

The case of $\alpha = 0$ corresponds to input cost uncertainty; the instantaneous variance of $\mathrm{d}K/K$ is constant and independent of I. Now K will fluctuate even when there is no investment; ongoing changes in the costs of labor and materials will change K irrespective of what the firm does. And since the project takes time to build, the actual total cost of the project is again only known when the project is complete.

To allow for both types of uncertainty, these two cases are combined in a single equation for the evolution of K:

$$\mathrm{d}K = -I\mathrm{d}t + \beta(IK)^{1/2}\mathrm{d}z + \gamma K\mathrm{d}w, \tag{4}$$

where $\mathrm{d}z$ and $\mathrm{d}w$ are the increments of uncorrelated Wiener processes. We will assume that all risk associated with $\mathrm{d}z$ is diversifiable, i.e., $\mathrm{d}z$ is uncorrelated with the economy and the stock market. However, dw may be correlated with the market. Note that eq. (4) combines uncertainty over the amount of effort required to complete a project, uncertainty over the cost of that effort, and uncertainty over the time the project will take.

2.2. *The optimal investment rule*

Given that dw in eq. (4) may be correlated with the market, the risk-free rate of interest cannot be used for the discount rate μ in eq. (3). However, μ can be eliminated from the problem if dw is spanned by existing assets in the economy, i.e., if in principle one could replicate movements in dw with some other asset or dynamic portfolio of assets. The investment problem can then be solved using contingent claims methods. If spanning does not hold, an optimal investment rule could instead be found using dynamic programming, subject to some choice of discount rate μ.

Assuming that spanning holds, let x be the price of an asset or dynamic portfolio of assets perfectly correlated with w, so that dx follows:

$$\mathrm{d}x = \alpha_x x\mathrm{d}t + \sigma_x x\mathrm{d}w\,. \tag{5}$$

By the CAPM, the risk-adjusted expected return on x is $r_x = r + \theta\rho_{xm}\sigma_x$, where θ is the market price of risk[4] and ρ_{xm} is the instantaneous correlation of x with the market portfolio.

The appendix shows that $F(K)$ must satisfy the following differential equation:

$$\tfrac{1}{2}\beta^2 IKF_{KK} + \tfrac{1}{2}\gamma^2 K^2 F_{KK} - IF_K - \phi\gamma KF_K - I = rF\,, \tag{6}$$

where $\phi \equiv (r_x - r)/\sigma_x$. Recall that $r_x = r + \theta\rho_{xm}\sigma_x$. Thus $\phi = \theta\rho_{xm}$. Since θ is an economy-wide parameter, the only project-specific parameter needed to determine ϕ is ρ_{xm}, which is equal to the coefficient of correlation between fluctuations in cost and the stock market.

Note that eq. (6) is the Bellman equation for the stochastic dynamic programming problem given by eq. (3), but with μ replaced by r. Because eq. (6) is linear in I, the rate of investment that maximizes $F(K)$ is always equal to either zero or the maximum rate k:

$$I = \begin{cases} k & \text{for} \quad \tfrac{1}{2}\beta^2 KF_{KK} - F_K - 1 \geqslant 0\,, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

[4]That is, $\theta = (r_m - r)/\sigma_m$, where r_m is the expected return on the market and σ_m is the standard deviation of that return. If we take the New York Stock Exchange Index as the market, over the period 1926–88, $r_m - r \approx 0.08$ and $\sigma_m \approx 0.2$, so $\theta \approx 0.4$.

Eq. (6) therefore has a free boundary at a point K^*, such that $I(t) = k$ when $K \leqslant K^*$ and $I(t) = 0$ otherwise. The value of K^* must be found as part of the solution for $F(K)$. To determine $F(K)$ and K^*, we solve (6) subject to the following boundary conditions:

$$F(0) = V, \tag{8}$$

$$\lim_{K \to \infty} F(K) = 0, \tag{9}$$

$$\tfrac{1}{2}\beta^2 K^* F_{KK}(K^*) - F_K(K^*) - 1 = 0, \tag{10}$$

as well as the 'value matching' condition that $F(K)$ be continuous at K^*. Condition (8) says that at completion, the payoff is V. Condition (9) says that when K is very large, the probability is very small that over some finite time it will drop enough to begin the project. Condition (10) follows from (7), and is equivalent to the 'smooth pasting' condition that $F_K(K)$ be continuous at K^*.

When $I = 0$, eq. (6) has the following simple analytical solution:

$$F = aK^b, \tag{11}$$

where b is the negative root of the quadratic equation $\frac{1}{2}\gamma^2 b(b-1) - \phi\gamma b - r = 0$, i.e.,

$$b = \tfrac{1}{2} + \frac{\phi}{\gamma} - \frac{1}{2\gamma}\sqrt{(\gamma + 2\phi)^2 + 8r}\,. \tag{12}$$

The parameter a is determined from the remaining boundary conditions, together with K^* and the solution for $F(K)$ for $K < K^*$. This must be done numerically, which is relatively easy once eq. (6) has been appropriately transformed.[5] A family of solutions for $K < K^*$ can be found that satisfy condition (8), but a unique solution, together with the value of a, is determined from condition (10) and the continuity of $F(K)$ at K^*.

[5] When $I = k$, eq. (6) has a first-degree singularity at $K = 0$. To eliminate this, make the substitution $F(K) = f(y)$, where $y = \log K$. Then eq. (6) becomes

$$f_{yy}(y) - f_y(y) - \frac{2kf_y(y)}{\beta^2 k + \gamma^2 e^y} = \frac{2k + 2rf(y)}{\beta^2 k e^{-y} + \gamma^2},$$

and boundary conditions (8) to (10) are transformed accordingly.

3. Solution characteristics

The effects of cost uncertainty can be seen by first examining solutions of eq. (6) for the case of pure technical uncertainty, i.e., $\gamma = 0$, and then for the case of pure input cost uncertainty, i.e., $\beta = 0$. Afterwards we will return to the general case.

3.1. Technical uncertainty

When only technical uncertainty is present, eq. (6) reduces to

$$\tfrac{1}{2}\beta^2 IKF_{KK} - IF_K - I = rF\,. \tag{13}$$

In this case, K can change only when investment is taking place, so if $K > K^*$ and the firm is not investing, K will never change and $F(K) = 0$.

When $r = 0$, eq. (13) has an analytical solution:

$$F(K) = V - K + \beta^2\left(\frac{V}{2}\right)^{-2/\beta^2}\left(\frac{K}{\beta^2+2}\right)^{(\beta^2+2)/\beta^2}, \tag{14}$$

and the critical value of K, K^*, is given by

$$K^* = (1 + \tfrac{1}{2}\beta^2)V\,.$$

Eq. (14) has a simple interpretation. With $r = 0$, $V - K$ would be the value of the investment opportunity where there is no possibility of abandoning the project. The last term is the value of the put option, i.e., the option to abandon the project should cost turn out to be much higher than expected. Note that for $\beta > 0$, $K^* > V$, and K^* is increasing in β. The more uncertainty there is, the greater the value of the investment opportunity, and the larger is the maximum expected cost for which beginning to invest is economical.

When $r > 0$, eq. (13) does not have an analytical solution, but can be solved numerically for different values of β. To choose values for β that are reasonable, we need to relate this parameter to the variance of the project's total cost. The appendix shows that for this case in which $\gamma = 0$, the variance of the cost to completion is given by

$$\mathrm{var}(\tilde{K}) = \left(\frac{\beta^2}{2-\beta^2}\right)K^2\,. \tag{15}$$

Hence, if one standard deviation of a project's cost is 25% of the expected cost, β would be 0.343, and if one standard deviation is 50% of the expected cost, β would be 0.63. Standard deviations of project cost in the range of 25–50% are not unusual, so we will use these values for β in the calculations that follow.

Fig. 1 shows $F(K)$ as a function of K for $V = 10$, $k = 2$, $r = 0.05$, and $\beta = 0, 0.343$, and 0.63. Observe that $F(K)$ looks like the value of a put option, except that $F(K) = 0$ when K exceeds the 'exercise' point K^*. Although $F(K)$ is larger the higher is β, the effect is greatest for larger values of K. Also, the effect of technical uncertainty on the optimal investment rule is moderate; only when $\beta = 0.63$ does K^* substantially exceed its value for the certainty case. In fact, for K^* to increase by 50% (from about 9 to about 13.5), a value of β close to one is required, which in turn implies that the standard deviation of total cost must be about 100% of the expected cost.

Finally, fig. 2 shows how $F(K)$ depends on the maximum rate of investment, k. (Here, $\beta = 0.63$.) As in the certainty case, a larger k implies a larger $F(K)$, because the payoff V is expected to be received earlier, and hence is discounted less. Also, when the investment opportunity is worth more, the critical value K^* is larger.

3.2. Input cost uncertainty

With only input cost uncertainty, eq. (6) becomes

$$\tfrac{1}{2}\gamma^2 K^2 F_{KK} - I F_K - \phi\gamma K F_K - I = rF\,. \tag{16}$$

This is again subject to boundary conditions (8) and (9), but condition (10) is replaced with $F_K(K^*) = -1$. Now K can change whether or not investment is taking place; like a financial put option, $F(K) > 0$ for any finite K.

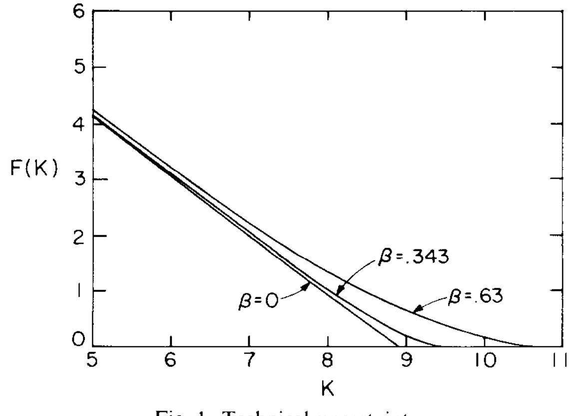

Fig. 1. Technical uncertainty.

Figure shows value of investment opportunity, $F(K)$, as function of expected cost to completion, K, for $\beta = 0, 0.343$, and 0.63, where β describes degree of technical uncertainty. Other parameter values are $V = 10$, $k = 2$, $r = 0.05$, and $\gamma = \phi = 0$. Intersection of $F(K)$ with K axis gives critical expected cost K^*.

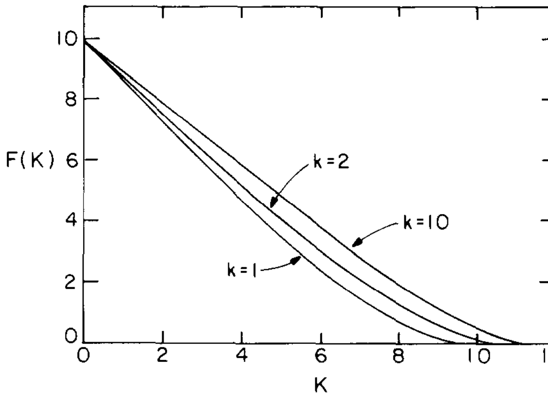

Fig. 2. Changes in maximum rate of investment.

Figure shows value of investment opportunity, $F(K)$, as function of expected cost to completion, K, for three values of maximum rate of investment: $k = 1, 2$, and 10. Only technical uncertainty is present ($\beta = 0.63$, $\gamma = \phi = 0$). Other parameter values are $V = 10$ and $r = 0.05$.

When $\gamma > 0$, eq. (16) has no solution when $r = 0$, because then there would be no reason to ever invest. One would always be better off waiting until K fell close to zero so that the net payoff from investing is larger. It would not matter that substantial time might have to pass for this to happen, because net payoffs would not be discounted.

If $I = 0$, $\tilde{K}$ is lognormally distributed, and γ can be interpreted as the standard deviation of percentage changes per period (in this case, a year) in K. Determining a value for γ that is reasonable depends on the makeup of cost: section 4 shows how this can be done for a specific example. Fig. 3 shows numerical solutions of eq. (16) for $\gamma = 0, 0.2$, and 0.4. (In each case, $V = 10$, $k = 2$, $r = 0.05$, and $\phi = 0$.) Observe that even when γ is 0.2, there is a substantial effect on the value of the investment opportunity (particularly when K is large) and on the critical cutoff K^*. When $\gamma = 0.2$, K^* is about half of what it is when $\gamma = 0$, so that a correct net present value rule would require the payoff from the investment to be about twice as large as the expected cost before the investment is undertaken. This result is similar to the kinds of numerical results obtained by McDonald and Siegel (1986) and Majd and Pindyck (1987) for uncertainty over the payoff to an investment, and shows that the effects of input cost uncertainty can also be quantitatively important.

Fig. 4 shows the dependence of $F(K)$ and K^* on ϕ, i.e., on the extent to which fluctuations in K are correlated with the economy and the stock market. Recall that $\phi = \theta\rho_{xm} = \theta\rho_{Km}$. A reasonable value for θ, the market price of risk, is 0.4 (see footnote 4), so we would expect ϕ to be less than this, perhaps on the order of 0.1 to 0.3. Fig. 4 shows $F(K)$ for $\phi = 0, 0.3$, and for illustrative purposes, 0.6. As is clear from this figure, a value of ϕ on the order of 0.1 will have only

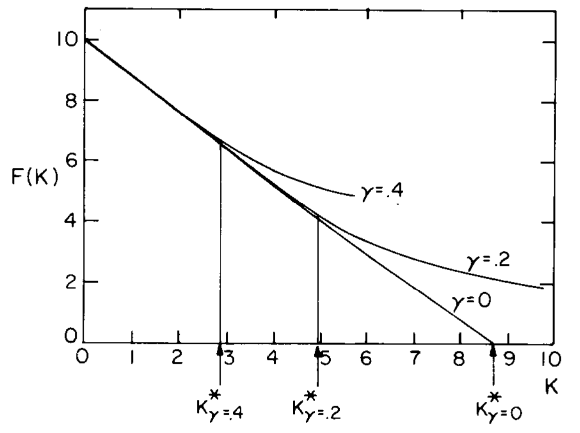

Fig. 3. Input cost uncertainty.

Figure shows value of investment opportunity, $F(K)$, as function of expected cost to completion, K, and critical expected cost, K^*, for $\gamma = 0$, 0.02, and 0.4, where γ is annual standard deviation of percentage changes in cost due to input cost fluctuations. Other parameter values are $V = 10$, $k = 2$, $r = 0.05$, $\beta = 0$, and $\phi = 0$.

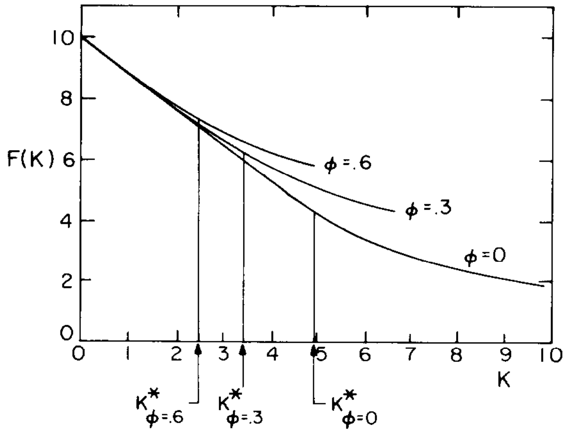

Fig. 4. Input cost uncertainty with systematic risk.

Figure shows value of investment opportunity, $F(K)$, as function of expected cost to completion, K, and critical expected cost, K^*, for $\phi = 0$, 0.3, and 0.6. Only input cost uncertainty is present ($\gamma = 0.2$, $\beta = 0$). Other parameter values are $V = 10$, $k = 2$, and $r = 0.05$.

a negligible effect on $F(K)$ and K^*. For a value of 0.3, however, the effect is large, and reduces K^* by around 25% compared to $\phi = 0$. Thus, input cost uncertainty with a large systematic componet can have a substantial impact on the decision to invest.

4.3. The general case

The value of the investment opportunity and the critical expected cost K^* can be found for any combination of β, γ, and ϕ by numerically solving eq. (6) and its associated boundary conditions. Since increases in β and γ (or ϕ) have opposite effects on K^*, it is useful to determine the net effect for combinations of these parameters.

Table 1 shows K^* as a function of both β and γ, for $\phi = 0$, $V = 10$, $k = 2$, and $r = 0.05$. Note that K^* decreases with γ and increases with β, but is much more sensitive to changes in γ. Whatever the value of β, a γ of 0.5 reduces K^* to about a fifth of its value when $\gamma = 0$. Also, this drop in K^* would be even larger if there were a systematic component to the input cost uncertainty. Thus for many investments, and particularly for large industrial projects for which input costs fluctuate, increasing uncertainty is likely to depress investment. The opposite will be the case only for investments like R&D programs, for which technical uncertainty is far more important and β could easily exceed 1.

Table 2 shows $F(K; \beta, \gamma)$ as a function of β and γ for $K = 8.92$, which is the value of K^* when $\beta = \gamma = 0$. This is the 'premium' in the value of the investment opportunity that results from the two sources of cost uncertainty. Note that this premium is increasing in both β and γ, but is again more sensitive to γ. Also, if γ is large (say, 0.5), this premium changes very little when β is increased.

The use of this model for investment decisions requires estimates of β and γ, and secondarily, an estimate of ϕ or ρ_{Km}. This requires estimating confidence intervals around projected cost for each source of uncertainty. To break cost

Table 1

Critical cost to completion, K^*, as a function of β and γ.[a]

Degree of technical uncertainty, β	Degree of input cost uncertainty, γ					
	0	0.1	0.2	0.3	0.4	0.5
0	8.9257	6.6113	4.9463	3.7524	2.8857	2.2559
0.1	8.9844	6.6504	4.9756	3.7720	2.9016	2.2681
0.2	9.1309	6.7578	5.0537	3.8330	2.9468	2.3032
0.3	9.3750	6.9385	5.1855	3.9307	3.0225	2.3608
0.4	9.7168	7.1875	5.3711	4.0674	3.1274	2.4438
0.5	10.156	7.5098	5.6104	4.2480	3.2617	2.5488
0.6	10.693	7.9053	5.8984	4.4629	3.4277	2.6758
0.7	11.328	8.3691	6.2402	4.7168	3.6230	2.8271
0.8	12.051	8.8965	6.6309	5.0146	3.8477	3.0005
0.9	12.861	9.5020	7.0801	5.3467	4.1016	3.1982
1.0	13.770	10.166	7.5732	5.7178	4.3848	3.4180

[a]Critical cost K^* is found by solving eq. (6) with its associated boundary conditions for $F(K)$. The parameters β and γ measure the degrees of technical and input cost uncertainty, respectively. Other parameter values are $V = 10$, $k = 2$, $r = 0.05$, and $\phi = 0$.

Table 2
Value of investment opportunity, $F(K)$, as a function of β and γ.[a]

Degree of technical uncertainty, β	Degree of input cost uncertainty, γ					
	0	0.1	0.2	0.3	0.4	0.5
0	0	1.0877	2.1553	3.1588	4.0535	4.8345
0.1	0.1384	1.0915	2.1596	3.1599	4.0565	4.8371
0.2	0.2026	1.0983	2.1642	3.1670	4.0606	4.8409
0.3	0.2428	1.1149	2.1753	3.1747	4.0692	4.8456
0.4	0.3924	1.1434	2.1956	3.1878	4.0810	4.8595
0.5	0.5199	1.1918	2.2277	3.2146	4.0974	4.8746
0.6	0.7499	1.2650	2.2697	3.2440	4.1240	4.8920
0.7	0.9067	1.3652	2.3280	3.2837	4.1572	4.9184
0.8	1.1664	1.4942	2.3998	3.3401	4.1978	4.9487
0.9	1.3606	1.6848	2.4939	3.4024	4.2460	4.9884
1.0	1.6034	1.8724	2.5996	3.4764	4.3021	5.0323

[a]Evaluated at K equal to the critical cost K^* corresponding to $\beta = \gamma = 0$.

uncertainty into technical and input cost components, note that the first is independent of time, whereas the variance of cost due to the second component grows linearly with the time horizon. Thus, a value for γ is found by estimating the standard deviation of cost T years into the future, assuming no investment takes place prior to that time. This estimate, $\hat{\sigma}_T$, could come from experience with construction costs, or from an accounting model of cost combined with variance estimates for individual inputs. Then, $\hat{\gamma} = \hat{\sigma}_T/\sqrt{T}$. Likewise, using eq. (15) and an initial estimate of expected cost, $K(0)$, a value for β can be based on an estimate of the time-independent standard deviation of $\tilde{K}$. The next section illustrates this in the context of a specific example – the decision to build a nuclear power plant.

4. Example – The construction of nuclear power plants

This section examines the decision to start or continue building a nuclear power plant in the context of market conditions in late 1982 or 1983. This was about three years after Three Mile Island and a time of considerable uncertainty over nuclear plant construction costs, which had begun rising sharply. Many utilities faced difficult decisions about whether to go ahead with planned or ongoing construction, and some utilities canceled plants that were well on their way towards completion.[6] Examining this investment problem will show how

[6]For example, Virginia Electric Power canceled its Northanna III and IV units, which were 10% completed, Public Service of Indiana canceled Marble Hill (35% completed), Washington Public Power Supply Systems canceled four of its five plants (5% to 50% completed), and Cleveland Electric Illuminating canceled its Zimmer plant, which was more than 90% completed.

the model can be used and provide insight into the evolution of nuclear power in the U.S.

To apply the model, we need estimates of the expectation and variance of the cost of building a kilowatt of nuclear generating capacity, a decomposition of that variance into technical and factor cost components, the maximum rate of investment, and the value of the unit of capacity. The last two numbers are relatively straightforward. Perl (1987, 1988) has shown that given the prices of alternative fuels during the early- and mid-1980s, the value of a unit of capacity was about $2,000, with fluctuations in real terms within only a $\pm$ 10% range. (Unless otherwise noted, all numbers are in 1985 constant dollars.) The *actual* construction time for nuclear plants varied through time and across plants during the late 1970s and 1980s, from six to as long as sixteen years, but tended to move proportionally with realized costs, and increased over the years as (real) costs increased. During the early 1980s, however, estimates of *expected* construction time were clustered around ten years, so a good estimate of the maximum rate of investment is 10% of expected cost.

To estimate the expectation, variance, and variance decomposition of cost, I use survey data on individual nuclear plant costs published by the Tennessee Valley Authority (TVA) and a cross-section regression analysis by Perl (1987, 1988) that explains differences in these costs across plants. The TVA obtained quarterly estimates of expected cost for nuclear plants planned or under construction in the U.S. These numbers, published in the TVA's 'Costs per Kilowatt Report for U.S. Nuclear Plants', provide data on the expected cost of a kilowatt of generating capacity on a plant-by-plant basis. The variance of cost and its decomposition can be estimated from the time-series and cross-sectional variation of these numbers, using the fact that the variance of cost due to technical uncertainty is independent of time, but the variance due to input cost fluctuations grows with the time horizon.

In any year, expected costs per kilowatt will vary across the 50 to 60 plants in the TVA survey, but part of this variation can be explained by differences in the type of plant, the experience of the contractor, region of the country, etc. Consider the cross-section regression:

$$COST_{it} = a_0 + a_1 X_{1it} + a_2 X_{2it} + \cdots + \varepsilon_i , \tag{17}$$

where $COST_{it}$ is expected cost for plant i in year t, and the X_{it}'s are a set of explanatory variables. This regression filters out the predictable part of the cross-sectional variation. Then, for plant i in year t, an estimator of the variance of cost due to technical uncertainty is the variance of the cross-sectional forecast error for $COST_{it}$ from the regression equation (17), given the values of X_{1it}, X_{2it}, etc. that apply to plant i.

A lower bound on this variance is the (squared) standard error of the regression, which would be the variance of the forecast error if, for plant i, X_{kit}

for each k were equal to its cross-sectional mean. In general, the X_{kit}'s for any plant will differ from the means, so the variance of the forecast error will exceed the squared standard error of the regression. (The reason is that the true coefficients a_1, a_2, etc. are unknown, and only estimated.) An upper bound on the variance of the forecast error is the cross-sectional sample variance of $COST_{it}$. Hence, I consider values of β in eq. (6) that correspond to forecast error variances ranging from the squared standard error of the regression to the sample variance.

Perl ran such regressions in logarithmic form for 1977–1985, using the TVA data on $COST$ for the last quarter of each year, with a set of up to ten explanatory variables that included: the log of the real wage, the log of the net design electric rating (reflecting the scale of the plant), the log of the experience of the architect/engineer (measured in number of plants designed), and dummy variables for the region of the country, the type of rock foundation, whether the plant was the first or subsequent built by the utility, whether it was a boiling water reactor, whether the utility served as its own construction manager, and whether the plant had a complex cooling tower. (Only variables that were statistically significant were retained, so regressions for some years included only a subset of the above.) I infer values of β from his results, using the 1982 data and regression. Converting to levels, the mean expected cost for that year was \$1,435 per kilowatt, with a standard error of regression of 17%. This is a lower bound on the standard deviation of the cross-sectional forecast error, and using eq. (15), implies $\beta = 0.24$.[7] The upper bound is the sample standard deviation, which for 1982 was 46% of expected cost and corresponds to $\beta = 0.59$.

Next, I estimate the variance due to input cost uncertainty by fitting the annual time series for mean expected cost to a geometric random walk. The drift and standard deviation of percentage changes in mean expected cost are 0.12 and 0.06, respectively, for 1977–1985, and 0.11 and 0.07 for 1977–1982. Since I consider decisions at the end of 1982, I use the latter numbers. However, an estimate of the drift based on six years of data (1977–1982) is very imprecise, and an expected real rate of increase of mean cost of 5% per year would have been reasonable at the time. This rate of increase would yield an estimated standard deviation of 0.20, so 0.07 to 0.20 is used as a reasonable range for γ in eq. (6). Also, most input cost uncertainty was due to continual and unpredictable regulatory change, rather than factor price fluctuations. Since regulatory change is largely uncorrelated with the economy, I set $\phi = 0$.

Table 3 shows solutions for $\beta = 0, 0.24$, and 0.59 and $\gamma = 0, 0.07$, and 0.20. In each case, $V =$ \$2,000 per kilowatt, $k =$ \$144 per year (10% of the \$1,435 mean

[7] Note that this accounts for construction experience and movement down the learning curve. For a discussion of the impact of experience on nuclear plant *operating* costs, see McCabe (1991). McCabe also examines technology adoption with uncertain operating cost, and argues that utilities buy a mix of technologies in order to reduce the variance of operating cost.

Table 3

Critical cost of kilowatt of capacity at end of 1982.[a]

Degree of technical uncertainty, β	Degree of input cost uncertainty, γ		
	0	0.07	0.20
0	$K^* = 1550$ $F(\bar{K}) = 121$	$K^* = 1251$ $F(\bar{K}) = 194$	$K^* = 867$ $F(\bar{K}) = 465$
0.24	$K^* = 1609$ $F(\bar{K}) = 131$	$K^* = 1260$ $F(\bar{K}) = 201$	$K^* = 871$ $F(\bar{K}) = 469$
0.59	$K^* = 1881$ $F(\bar{K}) = 215$	$K^* = 1293$ $F(\bar{K}) = 228$	$K^* = 887$ $F(\bar{K}) = 487$

[a]Based on $V = \$2{,}000$ per kilowatt, $r = 0.045$, $k = \$144$ per year, and $\phi = 0$. Mean expected cost was $\bar{K} = \$1{,}435$ per kilowatt.

expected cost in 1982), $\phi = 0$, and $r = 0.045$. (The average yield on three-year and ten-year Treasury bonds in 1982 was 13%. Using the 1979–1982 average rate of inflation of 7% in the PPI and 10% in the CPI as estimates of expected inflation puts the real risk-free rate, r, at about 3–6%.) The table shows the critical expected cost to completion, K^*, and the value of the utility's investment option (per kilowatt) for an actual expected cost equal to the mean of \$1,435.

Observe that absent input cost uncertainty ($\gamma = 0$), K^* ranges from \$1,609 to \$1,881, so that these investments would have been largely economical. (Technical uncertainty increases K^* by 4–21% compared to its value of \$1,550 when $\beta = \gamma = 0$.) But input cost uncertainty lowers K^* considerably, making the average plant uneconomical. Even for $\gamma = 0.07$, in most cases it would have been preferable to wait to see how regulations (and the expected costs they implied) evolved. And for $\gamma = 0.20$, it would have been economical to stop construction on plants that were 40% complete. This would seem to justify the decisions that some utilities made at the time to cancel planned or ongoing construction. Also, the TVA surveys were available to all U.S. utilities, so presumably they could have performed the same analysis.

The results are not very sensitive to the maximum rate of investment, k. Taking $\beta = 0.24$ and $\gamma = 0.07$, if $k = 288$ (so expected construction time is five years instead of ten), K^* rises to \$1,397. If $k = 96$ (so construction is expected to take fifteen years), K^* falls to \$1,154. Thus for a reasonable range of expected construction times, K^* varies by $\pm$ 10%.

These results show that for nuclear plants, the investment decision is most affected by input cost uncertainty, even though there is substantial technical uncertainty. The results also show the importance of incorporating both types of uncertainty in the analysis, rather than treating them separately. Note from the table that the dependence of K^* on β is much less when γ is 0.07 or 0.20 than it is when γ is 0. If one first calculated the change in K^* due to, say, a β of 0.59

(holding $\gamma = 0$) and then the percentage change due to a γ of 0.07, the result would be a K^* of about \$1,518 rather than the correct value of \$1,293.

5. Extensions of the model

This section show how the model can be extended to account for uncertainty over the future value of the completed project and to allow for more general processes for $K(t)$.

5.1. Uncertainty over the value of the completed project

Suppose the evolution of K is again given by eq. (4), but V also evolves stochastically:

$$dV = \alpha_v V dt + \sigma_v V dz_v \,, \tag{18}$$

where dz_v is assumed to be uncorrelated with dz or dw. Future values of V are thus log-normally distributed, and since the project takes time to complete, the payoff is always uncertain. For simplicity, we will assume that all risk is diversifiable. Then we can use dynamic programming, discounting with the risk-free rate of interest.

The value of the investment opportunity is again given by eq. (3), but with V now stochastic and hence replaced by $V(\tilde{T})$. The Bellman equation is

$$rF = \max_{I(t)} \{ -I(t) - IF_K + \tfrac{1}{2}\beta^2 IKF_{KK} + \tfrac{1}{2}\gamma^2 K^2 F_{KK} + \alpha_v VF_V$$

$$+ \tfrac{1}{2}\sigma_v^2 V^2 F_{VV} \} \,. \tag{19}$$

This equation is linear in I, and eq. (7) again applies. The optimal rule is to invest whenever $K \leqslant K^*(V)$. Eq. (19) is an elliptic partial differential equation with a free boundary along the line $K^*(V)$. The solution must satisfy the following boundary conditions: (i) $F(0, V) = V$, (ii) $\lim_{V \to 0} F(K, V) = 0$, (iii) $\lim_{K \to \infty} F(K, V) = 0$, (iv) $\frac{1}{2}\beta^2 K^* F_{KK}(K^*, V) - F_K(K^*, V) - 1 = 0$, and $F(K, V)$ and $F_K(K, V)$ continuous at $K^*(V)$. Condition (ii) reflects the fact that zero is an absorbing barrier for V; the other conditions are interpreted as before.

When $K > K^*(V)$, so that $I = 0$, eq. (19) has the following analytical solution:

$$F(K, V) = m(K/V)^{\omega} \,, \tag{20}$$

where

$$\omega = \left(\tfrac{1}{2} + \frac{\alpha_v - \sigma_v^2}{\gamma^2 + \sigma_v^2}\right)\left(1 - \sqrt{1 + \frac{2r(\gamma^2 + \sigma_v^2)}{(\gamma^2 + 2\alpha_v - \sigma_v^2)^2}}\right). \tag{21}$$

When $K < K^*(V)$, we can use the continuity of $F(K, V)$ and $F_K(K, V)$ at K^* to eliminate m:

$$F(K^*, V) = (K^*/\omega)F_K(K^*, V). \tag{22}$$

Eq. (19) together with conditions (i) and (22) can be solved numerically using a finite difference method. The boundary, $K^*(V)$, is found simultaneously with $F(K, V)$.

5.2. Generalizing the process for $K(t)$

We imposed restrictions on $K(t)$ that resulted in a simple investment rule and let us clearly differentiate between two types of cost uncertainty. We let $K(t)$ follow:

$$\mathrm{d}K = -I\mathrm{d}t + \beta K(I/K)^{\alpha}\mathrm{d}z, \tag{23}$$

with $\alpha = 0$ or $\frac{1}{2}$. Now suppose $0 < \alpha < \frac{1}{2}$. We will again assume that dz is diversifiable and that V is fixed and certain. Then the Bellman equation is

$$rF = \max_{I(t)} \{ -I(t) - IF_K + \tfrac{1}{2}\beta^2 I^{2\alpha} K^{2(1-\alpha)} F_{KK} \}. \tag{24}$$

Maximizing with respect to I gives the optimal investment rule in terms of $F(K)$:

$$I^*(K) = \left[\frac{\alpha\beta^2 K^{2(1-\alpha)} F_{KK}}{1 + F_K}\right]^{1/(1-2\alpha)}. \tag{25}$$

Substituting $I^*(K)$ into eq. (24) yields the following nonlinear differential equation for $F(K)$:

$$rF = 1 + F_K - (\alpha\beta^2 K^{2-2\alpha} F_{KK})^{1/(1-2\alpha)}(1 + F_K)^{-2\alpha/(1-2\alpha)}. \tag{26}$$

To find $F(K)$, eq. (26) must be solved (numerically) subject to conditions (8) and (9).

Eq. (26) has solutions for which $-1 < F_K \leqslant 0$ and $F_{KK} > 0$. (At $K = 0$, F_K must be greater than -1 as long as construction takes finite time and the discount rate is positive. Likewise, F_{KK} must remain finite as $K \to 0$.) Note from eq. (25) that $I \to 0$ as $K \to 0$, so for small K, I falls as the net payoff $V - K$ rises. This is the opposite of Grossman and Shapiro's (1986) finding that I rises as the net payoff rises when there are decreasing returns to effort. In my model there are constant returns to effort; I falls because the variance of $\tilde{K}$ falls as K falls, so that the shadow value of learning falls.

6. Conclusions

The model developed in this paper, as well as such predecessors as Roberts and Weitzman (1981) and Grossman and Shapiro (1986), belongs to a broad class of optimal search problems analyzed by Weitzman (1979). In what he characterized as a 'Pandora's box' problem, one must decide how many investment opportunities with uncertain outcomes should be undertaken, and in what order. In this paper, each dollar spent towards completion of a project is a single investment opportunity, and the uncertain outcome is the amount of progress that results. The model developed here is more general in that expected outcomes can evolve stochastically even when no investment is taking place (input cost uncertainty), but is more restrictive in that the order in which dollars are spent is predetermined.

One advantage of this model is that it leads to a simple investment rule that is relatively easy to apply in practice. Also, the restrictions that have been imposed on the process for the expected cost to completion, $K(t)$, allow us to clearly differentiate between two types of cost uncertainty. As we have seen in the previous section, some of the restrictive assumptions in the model can be relaxed (e.g., that V is nonstochastic), but at the cost of added computational complexity. Other restrictions can be relaxed as well. For example, we can relax the restriction that technical uncertainty is the same for each phase of the project (i.e., the uncertainty over the first third of a project's anticipated cost is the same as for the last third) by making β in eq. (13) a function of K. As long as $\beta(K)$ is a smooth monotonic function, it is reasonably straightforward to obtain numerical solutions for $F(K)$.

The sources and amounts of cost uncertainty will vary greatly across different projects. However, based on the ranges of parameter values that would apply to the bulk of large capital investments, factor cost uncertainty is likely to be more important than technical uncertainty in terms of its effect on the investment rule and the value of the investment opportunity. We saw that this is clearly the case for investments in nuclear power plants. The opposite may be the case for some R&D projects. And although we found that the critical cost to completion, K^*, is not very sensitive to the degree of technical uncertainty, β, this finding was based on the assumption, discussed above, that the uncertainty is the same across all phases of the project. Increases in the critical cost may be much larger if a project's uncertainty is largely resolved during its early phases.

Appendix

A.1. Mean and variance of $\tilde{K}$

Here I show that if $K(t)$ follows a controlled diffusion of the form

$$\mathrm{d}K = -k\mathrm{d}t + \beta K(k/K)^{\alpha}\mathrm{d}z\,, \tag{A.1}$$

then $K(t)$ is indeed the expected cost to completion. Let

$$M(K) = \mathrm{E}_t\left[\int_t^{\tilde{T}} k\mathrm{d}\tau | K(t)\right], \tag{A.2}$$

where $\tilde{T}$ is the first passage time for $K = 0$. We will show that $M(K) = K$.

We make use of the fact that the functional $M(K)$ must satisfy the Kolmogorov backward equation corresponding to (A.1):

$$\tfrac{1}{2}\beta^2 k^{2\alpha} K^{2-2\alpha} M_{KK} - kM_K + k = 0\,, \tag{A.3}$$

subject to the boundary conditions (i) $M(0) = 0$ and (ii) $M(\infty) = \infty$. [See Karlin and Taylor (1981, ch. 15).] Clearly $M(K) = K$ is a solution of (A.3) and the associated boundary conditions. Now consider a more general solution of the form $M(K) = K + h(K)$, where $h(K)$ is an arbitrary function of K. By direct integration,

$$h_K(K) = C\exp\left[\frac{2K^{2\alpha-1}}{(2\alpha-1)\beta^2 k^{2\alpha}}\right]. \tag{A.4}$$

But since $\lim_{K\to\infty} h_K(K) = C$, the constant C must equal zero to satisfy boundary condition (ii). Hence $M(K) = K$.

For the case of $\alpha = \frac{1}{2}$ (technical uncertainty), we can also find the variance of the cost to completion, i.e.,

$$\mathrm{var}(K) = \mathrm{E}_t\left[\int_t^{\tilde{T}} k\mathrm{d}\tau | K\right]^2 - K^2(t)\,. \tag{A.5}$$

Let $G(K) = \mathrm{E}_t[\int_t^{\tilde{T}} k\mathrm{d}\tau | K]^2$. Then $G(K)$ must satisfy the following Kolmogorov equation:

$$\tfrac{1}{2}\beta^2 kKG_{KK} - kG_K + 2kK = 0\,, \tag{A.6}$$

subject to the boundary conditions $G(0) = 0$ and $G(\infty) = \infty$. [See Karlin and Taylor (1981, p. 203).] The solution to (A.6) is $G(K) = 2K^2/(2-\beta^2)$, so the variance is

$$\operatorname{var}(K) = \left(\frac{\beta^2}{2-\beta^2}\right)K^2 . \tag{A.7}$$

A.2. Derivation of eq. (6)

Given a replicating asset or portfolio whose price x follows eq. (5), we can value the firm's investment opportunity as a contingent claim. First, denote $\delta \equiv r_x - \alpha_x$. Now, consider the following portfolio: hold the investment opportunity, worth $F(K)$, and sell short n units of the asset with price x. The value of this portfolio is then $\Phi = F(K) - nx$, and the instantaneous change in this value is $\mathrm{d}\Phi = \mathrm{d}F - n\mathrm{d}x$. Since the expected rate of growth of x is $\alpha_x < r_x$, the short position will require a payment stream over time at the rate $n(r_x - \alpha_x)x = n\delta x$. Also, insofar as investment is taking place, holding the investment opportunity implies a payment stream $I(t)$. Thus over an interval $\mathrm{d}t$, the total return on the portfolio is $\mathrm{d}F - n\mathrm{d}x - n\delta x\mathrm{d}t - I(t)\mathrm{d}t$.

Next, using Ito's Lemma, write $\mathrm{d}F$ as

$$\begin{aligned}\mathrm{d}F &= F_K\mathrm{d}K + \tfrac{1}{2}F_{KK}(\mathrm{d}K)^2\\ &= -IF_K\mathrm{d}t + \beta(IK)^{1/2}F_K\mathrm{d}z + \gamma KF_K\mathrm{d}w + \tfrac{1}{2}\beta^2 IKF_{KK}\mathrm{d}t\\ &\quad + \tfrac{1}{2}\gamma^2K^2F_{KK}\mathrm{d}t .\end{aligned}$$

Substituting (5) for $\mathrm{d}x$, the total return on the portfolio over an interval $\mathrm{d}t$ is therefore:

$$\begin{aligned}&-IF_K\mathrm{d}t + \beta(IK)^{1/2}F_K\mathrm{d}z + \gamma KF_K\mathrm{d}w + \tfrac{1}{2}\beta^2IKF_{KK}\mathrm{d}t + \tfrac{1}{2}\gamma^2K^2F_{KK}\mathrm{d}t\\ &- n\alpha_x x\mathrm{d}t - n\sigma_x x\mathrm{d}w - n\delta x\mathrm{d}t - I\mathrm{d}t.\end{aligned}$$

By setting $n = \gamma KF_K/\sigma_x x$, we can eliminate the terms in $\mathrm{d}w$, and thereby remove nondiversifiable risk from the portfolio. With n chosen this way, the only risk the portfolio carries is diversifiable, and hence the expected rate of return on the portfolio must be the risk-free rate, r. Using this value of n and equating the expected portfolio return to $r(F - nx)\mathrm{d}t$ yields eq. (6) for $F(K)$.

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