Frustrated Potts: Multiplicity Eliminates Chaos via Reentrance

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The frustrated q-state Potts model is solved exactly on a hierarchical lattice, yielding chaos under rescaling, namely the signature of a spin-glass phase, as previously seen for the Ising ($q = 2$) model. However, the ground-state entropy introduced by the ($q > 2$)-state antiferromagnetic Potts bond induces an escape from chaos as multiplicity q increases. The frustration versus multiplicity phase diagram has a reentrant (as a function of frustration) chaotic phase.

Frustration [1], meaning loops of equal-strength interactions that cannot all be simultaneously satisfied, diminishes ordered phases in the phase diagram and may drastically change the nature of certain ordered phases.[2] For example, the so-called Mattis phase, where spins are ordered in random directions but interactionwise consistently with each other, becomes a spin-glass phase with the introduction of the smallest amount of frustration [3], with residual entropy, unsaturated order at zero temperature, and the chaotic rescaling of the interactions, as measured by a positive Lyapunov exponent. The latter, chaos under scale change, is the signature of the spin-glass phase [4–12], as gauged quantitatively by the Lyapunov exponent. Chaotic interactions under scale change dictate chaotic correlation functions as a function of distance.[13]

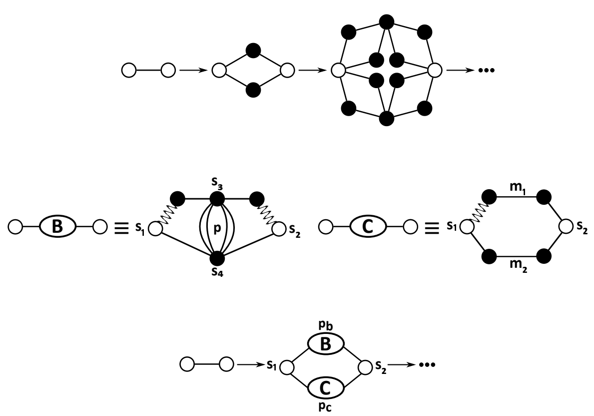

FIG. 1. Top row: Construction of a hierarchical lattice, from Ref.[14]. The renormalization-group solution of a hierarchical lattice proceeds via renormalization group in the opposite direction of its construction. Middle row: The two units used in the construction of the frustrated hierarchical lattice.[4] On the left is the correlation repressing unit and on the right the frustrated unit. The wiggly bonds are infinitely strong antiferromagnetic couplings. Bottom row: The assemblage of the units for the construction of the frustrated hierarchical lattice.

In fact, chaos under rescaling was seen in frustrated systems, with no randomness, with the exact solution of hierarchical lattices.[4–7] Spin-glass chaos was ushered by a sequence of period doublings, which were shown to also convert to chaotic bands under randomness.[5] Spin-glass chaos and its positive Lyapunov exponent was also calculated in the renormalization-group solution of systems where frustration is introduced by quenched randomness.[3]

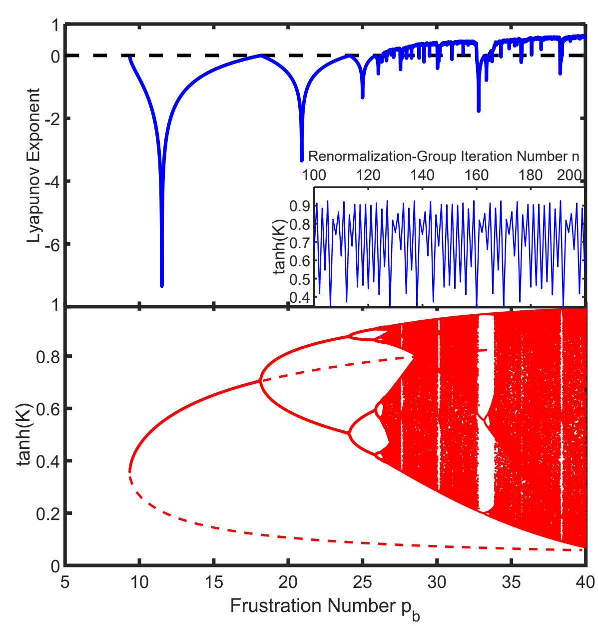

FIG. 2. Lower panel: The onset of chaos, by period doubling, under increased frustration for the $q = 3$-state Potts model. For each p_b, the renormalization-group flows are in the vertical direction. The full lines represent attractive fixed points, limit cycles, and chaotic bands. The dashed lines represent the unstable fixed points, only some of which are shown. The $tanh(K) = 0$ points are the stable fixed points which are the sinks of the disordered phase. The upper panel shows the calculated Lyapunov exponents. The upper inset shows the chaotic renormalization-group trajectory for $p_b = 28$.

Another important venue for ground-state degeneracy is in the ground-state participation of the multiplicity of spin states, as seen in antiferromagnetic ($q > 2$)-state

Potts models.[15–17] In the current work, we have studied the combination of both effects, chaos from frustration and degeneracy from multiplicity of states. We have exactly solved the q-state Potts models on the frustrated hierarchical model as in Ref.[4] We find that the system escapes chaos through multiplicity and that chaos shows reentrant behavior.

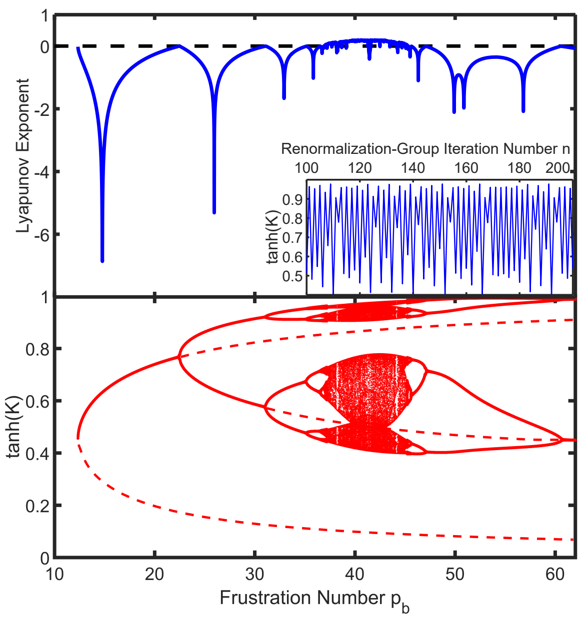

FIG. 3. The $q = 6$-state Potts model, under increased frustration, after the onset of chaos, leaves chaos through a set of reverse period doublings. The upper inset shows the chaotic renormalization-group trajectory for $p_b = 42$.

The model is constructed, by combining two units, embodying the different microscopic effects of competing interactions. The Hamiltonian of the system is

$$-\beta\mathcal{H} = K \sum_{\langle ij \rangle} \delta(s_i, s_j), \qquad (1)$$

where $\beta = 1/k_B T$, at site i the spin $s_i = 1, 2, , ..., q$ can be in q different states, the delta function $\delta(s_i, s_j) = 1(0)$ for $s_i = s_j (s_i \neq s_j)$, and the sum is over all interacting pairs of spins, represented by straight lines in Fig. 1. In unit C, the correlations are repressed but not eliminated along the path of the unit, as $m_2 > m_1$ always and the competing correlation on the longer unit is weaker. Unit B is frustrated: the competing paths are of equal length. By combining in parallel p_c and p_b units C and B, a family of models is created. In this work, we have explored $m_1 = 2, m_2 = 3, p = 4, p_c = 1$ and varying the number p_b of frustrated units. Hierarchical models are solved exactly, by renormalization-group theory, proceeding in the reverse direction of the construction of the hierarchical model and obtaining, by decimating the interior spins

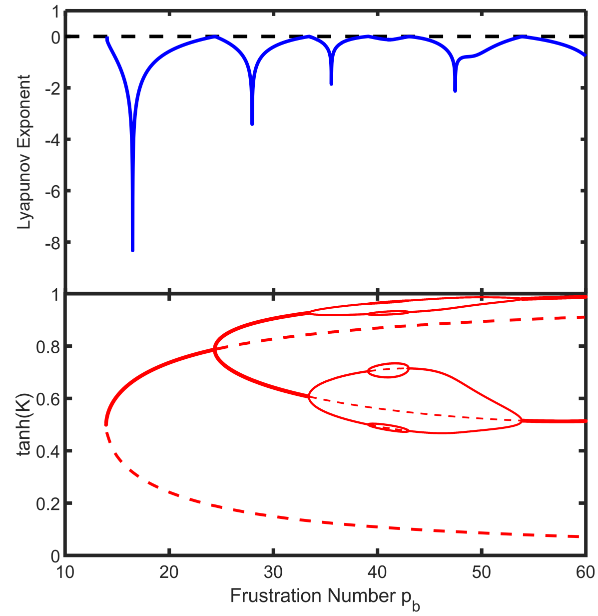

FIG. 4. The $q = 8$-state Potts model, under increased frustration, through a set of period doublings followed by reverse period doublings, bypasses chaos.

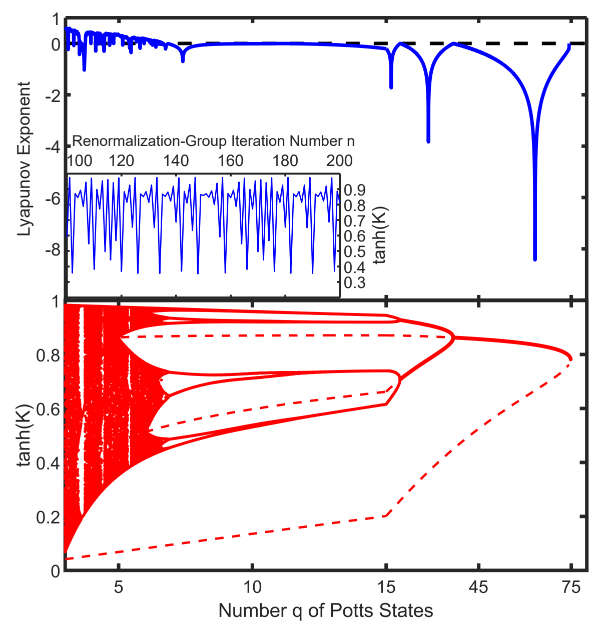

FIG. 5. As the number q of Potts states increases, the system leaves chaos through reverse period doublings. Note the horizontal scale change in the figure at $q = 15$. For this calculation, $p_b = 40$. The upper inset shows the chaotic renormalization-group trajectory for $q = 5$.

of each level (black circles in Fig. 1), obtaining recursion relations $K' = K'(K)$ exactly.[14, 18, 19]

The onset of the chaotic bands, as the number p_b of frustrated units is increased, is shown in Fig. 2, for the number of Potts states $q = 3$. The calculated Lyapunov exponents [20, 21],

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left| \frac{dK_{k+1}}{dK_k} \right| , \tag{2}$$

where K_k is the interaction at step k of the renormalization-group trajectory. The sum in Eq.(2) is to be taken within the asymptotic trajectory, so that we throw out the first 100 renormalization-group iterations to eliminate the transient points and subsequently use 600 iterations in the sum in Eq.(2), which assures convergence in the chaotic bands. It is seen that the Lyapunov exponent is non-positive outside chaos, barely touching zero at each period doubling.

However, for $q = 6$ in Fig. 3, as frustration is increased, the system leaves chaos through a series of reverse period doublings (foldings). We thus have non-chaos reentrance [22] around the chaotic phase. For $q = 8$, shown in Fig. 4, doublings and foldings succeed, but chaos has disappeared. We can therefore look for reverse bifurcations as the number of Potts states q is increased for fixed fixed frustration. This is seen in Fig. 5, where it is indeed seen that chaos disappears as the antiferromagnetic degeneracy of the Potts model is increased by increasing q.

The complete frustration versus multiplicity phase diagram has been calculated and is given in Fig. 6, clearly showing reentrance. Note the stability of the chaotic phase around $q = 3$.

The ousting of frustrated chaos by the multiplicity of local states could have a relevance to the mathematical Ising-type modeling of societal collective behavior.[23]

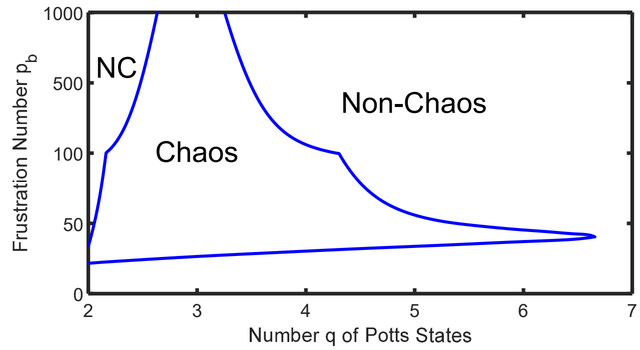

FIG. 6. The frustration-multiplicity chaos phase diagram of the system, shows the qp_b combinations for which chaos occurs. NC stands for non-chaos. The vertical scale changes at $p_b = 100$. Note the reentrance, as a function of frustration, of non-chaos.

ACKNOWLEDGMENTS

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