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Modularity, Evolution, and the Binding Problem: A View from Stability Theory

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Abstract

Any biological object, and specifically the brain, is the result of evolution. Evolution proceeds by accumulation and combination of stable intermediate states − as is well known, survival of the fittest really means survival of the stable. Simple examples abound: for instance, human emotional response involves both a fast archaic loop bypassing the cortex, and a slower cortical loop; motion control architecture in vertebrates is believed to involve combinations of simple motor primitives. However, in themselves, accumulations and combinations of stable elements have no reason to be stable. Hence the hypothesis that evolution will favor a particular form of stability, which automatically guarantees stability in combination. Such a form of stability, which we refer to as "contraction," can be characterized mathematically. Thus, contraction theory may help guide functional modelling of the central nervous system, and conversely it provides a systematic method to build arbitrarily complex robots out of simpler elements. Furthermore, contraction theory may shed light on the problem of perceptual unity (binding problem) by providing simple models and conditions for the overall convergence of a large number of specialized processing elements connected through networks of feedback loops.

1 Introduction

As our understanding of both brain function and robot design improves, common fundamental questions are starting to emerge, leading us to explore the relations between integrative neuroscience and robotics beyond the most obvious analogies.

While today the evolution and development of cognitive processes is seen as closely linked to the progressive refinement of sensorimotor functions, similarly robotics takes artificial intelligence (AI) beyond its classical conceptual domain by emphasizing the central role of physical interaction with the environment. Of course, the constraints and opportunities of robotics are very different from those of biology. While their physical hardware is far behind nature's, in principle robots can have perfect memory, near-perfect repeatibility, can use mathematics explicitly, and can simulate (imagine) specific actions much faster than humans. The travelling speed of information through an nerve axon is significantly slower than the speed of sound, while that along an electrical wire is closer to the speed of light, i.e., six orders of magnitude faster. Waiting (processing) time at each and every synapse is about 1 ms, probably a major incentive for developing parallel computational architectures. But similar delay problems can also be found in robotics, if one looks not at an autonomous robot, but rather, for instance, at telerobotics over large distances.

While most of robotic theory is founded on physical models and mathematical algorithms, the most fundamental conceptual tool in biology is the theory of evolution. Evolution proceeds by accumulation and combination of stable intermediate states: Darwin's "survival of the fittest" really means survival of the stable (Dawkins, 1976; Simon, 1962). For instance, human emotional response involves both a fast archaic loop bypassing the cortex, and a slower cortical loop (Ledoux, 1996); motion control architecture in vertebrates is believed to involve combinations of simple motor primitives (Bernstein, 1967; Bizzi, et al., 1993).

Conceptually, such accumulations have also been a recurrent theme in cybernetics and AI history (Walter, 1950, 1951; Simon, 1962, 1981; Ashby, 1966; Braitenberg, 1984; Minsky, 1986) under various guises (Brooks, 1986, 1999). They also form the basis of several recent theories of brain function (Tononi, et al., 1998; Dehaene, et al., 1998; Crick and Koch, 1998; Edelman and Tononi, 2000; Grossberg, 2000) and of biological motor control (Bernstein, 1967; Bizzi, et al., 1993; Mussa-Ivaldi, 1997; Wolpert and Kawato, 1998; Tresch, et al., 1999; Jordan and Wolpert, 1999; Thoroughman and Shadmer, 2000; Giszter, et al. 2000).

However, in themselves, accumulations and combinations of stable elements have no reason to be stable. Hence our hypothesis that evolution will favor a particular form of stability, which automatically guarantees stability in combination, since this would considerably reduce (in effect, avoid combinatorial explosion of) trial-and-error as the systems become large and complex. Such a form of stability, which we shall refer to as "contraction," can be characterized mathematically. Thus, contraction theory may help guide functional modelling of the central nervous system, and conversely it provides a systematic method to build arbitrarily complex robots out of simpler elements.

In this paper, we explore some of the possible implications of this hypothesis. In section 2, we define contraction and review its basic properties. In section 3, we discuss some potential applications to physiological modelling. As we shall see, contraction theory may also shed light on the problem of perceptual unity (binding problem) by providing simple models and conditions for the overall convergence of a large number of specialized processing elements connected through networks of feedback loops. Section 4 specifically discusses issues of delays in information transmission. Section 5 offers brief concluding remarks.

2 Modularity and Stability

Basically, a nonlinear dynamic system will be called contracting if initial conditions or temporary disturbances are forgotten exponentially fast, i.e., if trajectories of the perturbed system return to to their nominal behavior with an exponential convergence rate. It turns out that relatively simple conditions can be given for this stability-like property to be verified, and furthermore that this property is preserved through basic system combinations, such as parallel combinations, feedback combinations, and series or hierachies.

Incidentally, such a definition fits rather naturally with known data on biological motion perturbation, e.g. perturbation of arm movement (Soechting and Lacquaniti, 1988; Won and Hogan, 1995). Furthermore, it is intrinsic, in the sense that the system's "nominal" behavior needs not be known. Finally, such a form of stability, at least in a local sense, is also a basic prerequisite for any learning, since it guarantees the consistency of the system's behavior in the presence of small disturbances or variations in initial conditions.

2.1 Contraction Analysis

In this section, we summarize the basic results of (Lohmiller and Slotine, 1998), to which the reader is referred for more details. We consider general deterministic systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

where $\mathbf{f}$ is an $n \times 1$ nonlinear vector function and $\mathbf{x}$ is the $n \times 1$ state vector. The above equation may also represent the closed-loop dynamics of a controlled system with state feedback $\mathbf{u}(\mathbf{x}, t)$. All quantities are assumed to be real and smooth, by which it is meant that any required derivative or partial derivative exists and is continuous. The basic result of (Lohmiller and Slotine, 1998) can then be stated as

Theorem 1 *Consider the system (1). If theres exist a uniformly positive definite metric*

$$\mathbf{M}(\mathbf{x}, t) \;=\; \mathbf{\Theta}(\mathbf{x}, t)^T \; \mathbf{\Theta}(\mathbf{x}, t),$$

such that the associated generalized Jacobian

$$\mathbf{F} \;=\; \left(\dot{\mathbf{\Theta}} + \mathbf{\Theta}\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right) \mathbf{\Theta}^{-1}$$

is uniformly negative definite, then all system trajectories then converge exponentially to a single trajectory, with convergence rate $|\lambda_{max}|$*, where* λ_{max} *is the largest eigenvalue of the symmetric part of* $\mathbf{F}$*.*

It can be shown conversely that the existence of a uniformly positive definite metric with respect to which the system is contracting is also a necessary condition for global exponential convergence of trajectories. In the linear time-invariant case, a system is globally contracting if and only if it is strictly stable, with $\mathbf{F}$ simply being a normal Jordan form of the system and $\boldsymbol{\Theta}$ the coordinate transformation to that form.

In this paper, for simplicity we shall concentrate on the global convergence result above. In the case that $\mathbf{F}$ is uniformly negative definite only in a finite region, then the result can be shown to hold for all trajectories starting in the largest ball (with respect to the metric $\mathbf{M}$) contained in that region.

Example 2.1: Consider the gradient descent method for a time-varying cost function $V(\mathbf{x}, t)$

$$\dot{\mathbf{x}} = -\frac{\partial V}{\partial \mathbf{x}}$$

If V is strictly convex, i.e., if $\frac{\partial^2 V}{\partial \mathbf{x}^2} > 0$ uniformly, then this dynamics is contracting, since its Jacobian is uniformly negative definite. □

Example 2.2: Time-invariant contracting systems can be shown to converge to a unique equilibrium point. Indeed, consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, contracting with respect to $\boldsymbol{\Theta}$. One can easily verify that

$$\frac{d}{dt}\,(\boldsymbol{\Theta}\mathbf{f}) \;=\; \mathbf{F}(\boldsymbol{\Theta}\mathbf{f})$$

which implies that $\boldsymbol{\Theta}\mathbf{f}$ and thus $\mathbf{f} = \dot{\mathbf{x}}$ converge exponentially to zero, and therefore that $\mathbf{x}$ converges exponentially to a constant vector.

Similarly, contracting systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}(t))$$

where the input $\mathbf{u}(t)$ is periodic in time, can be shown to converge towards a periodic state of the same period as the input. □

2.2 Combinations of contracting systems

As a form of stability, one of the main features of contraction is that it is automatically preserved through a variety of system combinations. The reader is again referred to (Lohmiller and Slotine, 1998; 2000a) for further details and extensions.

Formally, some of the combination properties of contracting systems are most easily stated used the notion of a virtual displacement, from classical physics. A virtual displacement $\delta\mathbf{x}$ is an infinitesimal displacement *at fixed time*. Coordinate transformations of the form $\delta\mathbf{z} = \boldsymbol{\Theta}(\mathbf{x}, t)\delta\mathbf{x}$ can be performed on virtual displacements – these are much more general than simple coordinate changes, since an explicit $\mathbf{z}$ need not need exist, i.e., the transformation need not be integrable. $\boldsymbol{\Theta}(\mathbf{x}, t)^T\,\boldsymbol{\Theta}(\mathbf{x}, t)$ defines the metric in Theorem 1.

Parallel combination: Consider two systems of the same dimension, contracting *in the same metric*,

$$\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x}, t) \qquad i = 1, 2$$

Assume further that the metric depends only the state $\mathbf{x}$ and not explicitly on time. Then, any uniformly positive superposition (where $\exists\ \alpha > 0,\ \forall t \geq 0,\ \exists\ i,\ \alpha_i(t) \geq \alpha$)

$$\dot{\mathbf{x}} \ = \ \alpha_1(t)\ \mathbf{f}_1(\mathbf{x}, t) \ + \ \alpha_2(t)\ \mathbf{f}_2(\mathbf{x}, t)$$

is contracting in the same metric. By recursion, this property can be extended to any number of systems.

Hierarchical Combination: Consider two contracting systems, of possibly different dimensions and metrics, and connect them in series, leading to a smooth virtual dynamics of the form

$$\frac{d}{dt}\begin{pmatrix} \delta\mathbf{z}_1 \\ \delta\mathbf{z}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{F}_{11} & \mathbf{0} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{pmatrix} \begin{pmatrix} \delta\mathbf{z}_1 \\ \delta\mathbf{z}_2 \end{pmatrix}$$

Then the overall system is contracting, as long as $\mathbf{F}_{21}$ is bounded. By recursion, the result extends to hierarchies or cascades of contracting systems of arbitrary depths.

Feedback Combination: Consider two contracting systems, of possibly different dimensions and metrics, and connect them in feedback, in such a way that the overall virtual dynamics is of the form

$$\frac{d}{dt}\begin{pmatrix} \delta\mathbf{z}_1 \\ \delta\mathbf{z}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{F}_1 & \mathbf{G} \\ -\ \mathbf{G}^T & \mathbf{F}_2 \end{pmatrix} \begin{pmatrix} \delta\mathbf{z}_1 \\ \delta\mathbf{z}_2 \end{pmatrix}$$

where the matrix $\mathbf{G}(\mathbf{x}_1, \mathbf{x}_2, t)$ is arbitrary (other than being a matrix of partial derivatives). Then the overall system is contracting. The result can of course be extended to any number of systems: with obvious notations, overall contraction is achieved if $\mathbf{G}_{ij} + \mathbf{G}_{ji}^T = \mathbf{0},\ \forall i, j,\ i \neq j$.

Note that, essentially, the convergence rate of parallel combinations is the weighted sum of the individual convergence rates, the convergence rate of feedback combinations is the slowest of the individual convergence rates, and the convergence time-constant of hierarchical combinations is the sum the individual convergence time-constants.

Translation and scaling: It is straightforward to show that if $\mathbf{f}(\mathbf{x}, t)$ defines a contracting dynamics with respect to a *constant* metric, so does any scaled and translated version $\mathbf{f}(a(t)\mathbf{x} - \mathbf{b}(t), t)$, where $a(t)$ and $\mathbf{b}(t)$ are arbitrary differentiable functions and $a(t)$ is uniformly positive definite. This property, combined with the parallel combination property above, can allow contracting dynamics to be used as wavelet-like basis functions in problems of dynamic approximation, estimation, and adaptive control,

$$\dot{\mathbf{x}} = \sum_i \alpha_i(t)\ \mathbf{f}(a_i(t)\mathbf{x} - \mathbf{b}_i(t), t)$$

and thus can provide practical tools for progressive refinement and learning. Note that the translation step is tuned to the scale

$$a_i(t)\ \mathbf{x} \ - \ b_i(t) \ = \ a_i(t)\left(\mathbf{x} \ - \ \frac{\mathbf{b}_i(t)}{a_i(t)}\right)$$

Of course, contraction is also preserved by *any* combination of the above. External inputs can be provided through any subsytem dynamics. Overall contraction also implies that the system will recover exponentially fast from temporary disturbances in any subsystem.

It is straightforward to incorporate adaptive techniques in contraction-based designs if part of the system's uncertainty consists of unknown but constant or slowly-varying parameters (Lohmiller and Slotine, 2000b). Finally, although for simplicity we concentrate on systems described by ordinary differential equations, the discussion extends to large classes of partial differential equations.

3 Some applications

In this section, we discuss some potential applications of the above discussion to physiological modelling and robotic design. Specifically, we discuss composite signals, motor primitives, navigation, prediction, and oscillator synchronisation. We also show how the developement fits naturally with recent theories on the binding problem. In section 4, we will specifically consider some questions linked to information transmission delays in biological and robotic systems.

3.1 Composite variables

Composite signals, i.e., signals representing mixtures of more obvious physical quantities such as position or velocity, are pervasive in the nervous system (Berthoz, 1999). There may be good mathematical reasons for this. Indeed, using the right combination of variables can significantly reduce the complexity of control or estimation problems by enforcing a hierarchy of contracting systems, as we now briefly discuss. Composite variables will also be discussed later in the context of predictor and observer design, and to robustify systems against information transmission delays.

Control problems, for instance, are often greatly simplified by the introduction of intermediate "sliding" variables. A sliding variable is a combination of the instantaneous error and its successive time derivatives. By choosing this combination so that the implicit differential equation it defines is exponentially stable, high-order control problems can be reduced to much easier first-order problems (see e.g., (Slotine and Li, 1991)). For example, in a second-order mechanical system, one may choose

$$s = \dot{\tilde{q}} + \lambda \tilde{q}$$

where $\tilde{q}(t) = q(t) - q_d(t)$ is the tracking error and λ is a strictly positive constant. From the point of view of the previous discussion, such composite variables correspond to creating a hierarchy of contracting systems

$$\dot{s} = \phi(s, t) \qquad \text{contracting by choice of control law}$$

$\dot{\tilde{x}} + \lambda\tilde{x} = s$ contracting by definition of s

The qualitative behavior of the contracting system depends on the actual choice of the composite variable. For instance,

$$\dot{\tilde{x}} + (\lambda_1 + \lambda_2|\tilde{x}|)\tilde{x} = s$$

(where the λ_i are strictly positive constants) reacts faster to larger errors, since its scalar Jacobian is $-(\lambda_1 + 2\lambda_2|\tilde{x}|)$. The dimension of the controlled space can even change in real-time. For instance, the following choice (Slotine and Khatib, 1988)

$$\dot{\mathbf{x}} + \lambda\,\tilde{\mathbf{x}}\ \mathrm{sat}(\frac{V_{max}}{\lambda\|\tilde{\mathbf{x}}\|}) = s$$

corresponds to a ballistic motion at constant velocity V_{max} in the direction opposite to the error vector, followed by a linear position control when the target is in reach:

$$\|\tilde{\mathbf{x}}\| \text{ large} \quad\rightarrow\quad \dot{\mathbf{x}} + V_{max}\,\frac{\tilde{\mathbf{x}}}{\|\tilde{\mathbf{x}}\|} \;=\; s$$

$$\|\tilde{\mathbf{x}}\| \text{ small} \quad\rightarrow\quad \dot{\mathbf{x}} + \lambda\,\tilde{\mathbf{x}} \;=\; s$$

Consider now an arbitrary number of continuously differentiable dynamics

$$\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x}, t)$$

that are all locally contracting in some ball B_i around different operating points $\mathbf{x}_i$ with respect to possibly different $\mathbf{\Theta}_i$. These dynamics can be *sequenced* while preserving overall contraction, as long as switching between dynamics occurs when $\mathbf{x}_i \in B_{i+1}$ (analogously to more classical task sequencing in e.g. (Burridge, et al., 1999)). Composite variables may also be used to control such control switching between different phases of a task, or between stereotyped motions – a preliminary investigation of this hypothesis using psychophysical experiments was performed in (Hanneton, et al., 1998).

Composite variables of the above type will also be used in the next example.

3.2 Motor primitives

Recently, there has been considerable interest in analyzing feedback controllers for biological motor control systems as combinations of simpler elements, or motor primitives (Bizzi, et al., 1993; Mussa-Ivaldi, 1997; d'Avella and Bizzi, 1998; Tresch, et al., 1999). Besides being biological plausible, such a structure is intuitively appealing, as it may yield considerable dimensionality reduction in learning and planning. More recent work aims to confirm this general structure using intrinsic experiments (Giszter and Kargo, 2000; Kargo and Giszter, 2000). Similar goals motivate e.g. (Atkeson et al., 1997; Schaal, 1999; Fod et al., 2000).

(Bizzi, et al., 1993; Mussa-Ivaldi, 1997) stimulate a small number of areas in a frog's spinal cord and measure the resulting torque/angle relations. Force fields appear to add when different areas are stimulated at the same time, so that (Bizzi, et al., 1993; Mussa-Ivaldi, 1997) propose the following biological control inputs

$$\tau = -\sum_1^n k_i(t)\mathbf{f}(\mathbf{q}-\mathbf{q}_i)$$

where each single torque $k_i(t)\mathbf{f}(\mathbf{q}-\mathbf{q}_i(t))$ results from the stimulation of area i in the spinal cord, with $k_i(t) \geq \beta_k > 0$ and $\frac{\partial \mathbf{f}}{\partial \mathbf{q}} \geq \beta \mathbf{I} > \mathbf{0}$. Likely candidates for $k_i(t)$ are sigmoids and pulses (Mussa-Ivaldi, 1997; Berthoz, 1993), and periodic activation patterns from central pattern generators.

However time-varying spring gains $k_i(t)$ do not generally guarantee stable mechanical motions. Imagine a spring which is very weak when it is being elongated and very strong when it puts back energy in the system. This spring represents an energy source and the system can become unstable. Instead, assume that an intermediate composite variable is used, for instance $\mathbf{s} = \mathbf{q} + T\dot{\mathbf{q}}$, with T a strictly positive constant, and introduce corresponding damping, so as to get the dynamics in first-order form

$$\mathbf{H}\dot{\mathbf{s}} = -\sum_1^n k_i(t)\mathbf{f}(\mathbf{s}-\mathbf{s}_i(t))$$

Note that the sliding variable, which replaces pure position as the argument of the primitives, can be interpreted simply as a first-order *prediction* of position, with prediction time T. As discussed in section 3.1, however, many structures for s are actually possible, as long as they define contracting dynamics. Biological plausibility of specific choices may be studied as physiological data in motion (rather than isometric data) becomes available.

The "open-loop" terms $k_i(t)$ (and, if non-zero, $\mathbf{s}_i(t)$) describing the desired trajectory, may themselves be the outputs of contracting dynamics in the brain, time-advanced because of the significant nerve transmission delays. The primitives in $(\mathbf{s}-\mathbf{s}_i)$ may then be generated at the spinal chord level through high-bandwidth few-synapse connections (combined with the natural viscoelastic properties of the muscles). Note that different metrics may be used at each level of this hierarchy. This is rather natural, since e.g. typically motion control is done in intrinsic coordinates, while planning is done in extrinsic coordinates − indeed, it has even been suggested that some aspects of motion planning in the brain may use an affine metric, perhaps exploiting commonalities with visual processing (Pollick and Sapiro, 1997; Flash, et al., 1997).

Finally, note some that not all motion primitives need to be contracting: some can be repulsive (as in obstacle avoidance, for instance when arm motion should avoid hitting the torso), as long as the overall sum is contracting.

3.3 Navigation

In the human vestibular system, otolithic organs measure linear acceleration, while semi-circular canals estimate angular velocity (through heavily damped angular acceleration signals). As is well known, this configuration is essentially the same as in the so-called strapdown problem in inertial navigation, where angular position (Euler angles) $\mathbf{x} = (\psi, \theta, \phi)^T$ and inertial position $\mathbf{r}$ are computed from angular velocity ω and acceleration γ, measured in intrinsic (body-fixed) coordinates,

$$\dot{\mathbf{x}} = \mathbf{H}^{-1}\omega \tag{2}$$

$$\ddot{\mathbf{r}} = \dot{\mathbf{v}} = \mathbf{A}\gamma \tag{3}$$

with

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\psi & \cos\theta\sin\psi \\ 0 & -\sin\psi & \cos\theta\cos\psi \end{pmatrix}$$

Using $\mathbf{\Theta} = \mathbf{AH}$, where $\mathbf{A}(\mathbf{x})$ is the orthonormal transformation matrix from intrinsic to inertial coordinates, leads for (2) to the generalized Jacobian (Lohmiller and Slotine, 2000a)

$$\mathbf{F} = \mathbf{0}$$

Thus, (2) and (3) represent a hierarchy of three indifferent systems. Noticing that $\partial\mathbf{A}/\partial\mathbf{x}$ is bounded, this shows that the basic strapdown algorithm is marginally contracting.

Strict contraction can then be ensured either by addition of dissipation, or by appropriate combination with other sensors (such as vision), and thus the above stucture can form the basis of a navigation system.

3.4 Prediction and Observers

Prediction is one of the main activities of the brain (Berthoz, 1999). It can be found in many aspects, such as anticipating the trajectory of a ball to be caught, avoiding moving obstacles, using the last hours of sleep to prepare the body to awake, and even perhaps in the unreasonable effectiveness of the placebo effect. Prediction also plays a key role in active sensing, perception (as opposed to just sensing). In the nervous system, sensory information is selected, filtered, or simplified at every sensory relay.

Which brings us to the general question of observers, which are mathematical algorithms used to compute or predict the internal state of a dynamic system given partial measurements. An observer is typically composed of a system simulation (using a perhaps coarse *internal model*), guided and corrected by actual *measurements* on the system. Furthermore, in the case of active sensing and under certain conditions, the observer can *select* the measurement or set of measurements to best improve the estimate – at any instant, the most relevant measurement to be made can be specified a priori. Kalman filters and their extensions are probably the best known observers for linear systems, and are designed to be optimal in some specific sense (see (Dickmanns, 1998) for a recent discussion of active sensing applications), but similar ideas can apply to nonlinear systems as well. Also note that observers may be viewed as generalizing the notion of a content-addressable memory – as in the case of Marcel Proust's madeleine leading to the eight volumes of "Remembrance of Things Past."

Specifically, consider a dynamic system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

If the system is contracting, an observer can be constructed just by copying the dynamics

$$\dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}}, t)$$

Indeed, the estimate $\hat{\mathbf{x}}$ will then converge towards $\mathbf{x}$ exponentially. In the more general case, if a *partial* or indirect measurement of the state is available ($\dim \mathbf{y} \leq \dim \mathbf{x}$)

$$\mathbf{y} \ = \ \mathbf{y}(\mathbf{x})$$

under certain conditions one can choose a function $\mathbf{g}$ such that the dynamics

$$\dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}}, t) + \mathbf{g}(\hat{\mathbf{y}} - \mathbf{y})$$

converges towards $\mathbf{x}$ exponentially, where $\hat{\mathbf{y}} \ = \ \mathbf{y}(\hat{\mathbf{x}})$. Note the prediction/correction (or equivalently, simulation/feedback) structure of the above equation. The choice of $\mathbf{g}$ can be simplified by using an appropriate composite variable.

Example 3.1: Consider the Van der Pol oscillator

$$\ddot{x} + (x^2 - 1)\dot{x} + f(x, t) = 0$$

where x is measured and $v = \dot{x}$ to be estimated. Defining the composite variable

$$\hat{v} = \bar{v} + kx \qquad k > 1$$

the observer

$$\dot{\bar{v}} + (x^2 + k - 1)\hat{v} + f(x, t) = 0$$

leads to exponential convergence of the estimate $\hat{v}$ to v. Indeed, one has

$$\dot{\hat{v}} + (x^2 - 1)\hat{v} + k(\hat{v} - v) + f(x, t) = 0$$

where $\hat{v} = v$ is a particular solution and the Jacobian $-\ (x^2 + k - 1)$ is uniformly negative definite.

The role of the internal model can also be well understood on this example. Consider instead the problem of estimating velocity given position measurements, based on the simpler system model

$$\ddot{x} = 0$$

Proceeding as before leads to the observer

$$\dot{\bar{v}} + \alpha\hat{v} = 0$$

$$\hat{v} = \bar{v} + \alpha x$$

But this is nothing but filtered differentation

$$\hat{v} \ = \ \alpha \left(1 - \frac{\alpha}{p + \alpha} \right) x \ = \ \frac{\alpha \, p}{p + \alpha} \, x$$

(with p the Laplace variable), which indeed converges exponentially to the correct value for the system model. □

It can be shown that bounds on the quality of the internal model translate into bounds on the quality of the estimation. Furthermore, one can trade off modelling quality versus measurement precision, as we discuss next.

Assume now that the measurements occur instead at *discrete* time instants. The prediction/correction parts of the observer are then performed in sequence, with prediction occuring between measurements, and correction when the new measurement becomes available

$$\dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}}, t) \qquad \text{simulation between measurements}$$

$$\hat{\mathbf{x}}^{new} = \hat{\mathbf{x}}^{old} + \mathbf{g}(\hat{\mathbf{y}}^{old} - \mathbf{y}, t) \qquad \text{discrete update based on measurement}$$

In this form, it becomes clear that if several possible measurements or sets of measurements can be performed, one may try at each step to select the most *relevant* (i.e., the measurement or set of measurements i which will best contribute to improving the estimate $\hat{\mathbf{x}}$)

$$\hat{\mathbf{x}}^{+} = \hat{\mathbf{x}}^{-} + \mathbf{g}_i(\hat{\mathbf{y}}_i^{-} - \mathbf{y}_i, t)$$

Similarly to the Kalman filters used in (Dickmanns, 1998) for linear systems, this can be achieved by computing, along with the state estimate itself, the corresponding quadratic bounds R on the estimation error (Lohmiller and Slotine, 2000b), and then selecting accordingly the measurement i which minimizes

$$R^{\text{new}} \;=\; \bar{\lambda}_i \; e^{\lambda_i T_i} \;\; R^{\text{old}} \;+\; \|\mathbf{d}_{\text{model}} \; T_i \;+\; \mathbf{d}_i\|_{\max}$$

Note the trade-off between internal model precision, represented by $\mathbf{d}_{\text{model}}$, measurement quality, represented by the error bound $\mathbf{d}_i$, and measurement frequency $1/T_i$ (with λ_i the expansion rate between measurements, and $\bar{\lambda}_i$ the discrete expansion rate at measurement time).

Finally, by analogy with sensory relays, note that hierarchies of contracting observers are themselves contracting.

3.5 Oscillator Synchronisation

Rythmic phenomena are pervasive in physiology. These include, for instance, the rhythmic motor behaviors used in locomotion and driven by central pattern generators, as in walking, swimming, or flying, automatic mechanisms such as breathing and heart cycles, and intrinsic pacemakers in the brain (Kandel, et al., 2000; Dowling, 1992). As we mentioned earlier, a time-invariant contracting system driven by a periodic input converges exponentially to a periodic state of the same period. The results can be made more specific in the practically important case of oscillator synchronisation, using tools quite similar to those studied for observers.

Example 3.2: Consider two identical Van der Pol oscillators in series

$$\ddot{x}_1 + \alpha(x_1^2 - 1)\dot{x}_1 + \omega^2 x_1 \;=\; \alpha k(\dot{x}_2 - \dot{x}_1)$$

$$\ddot{x}_2 + \alpha(x_2^2 - 1)\dot{x}_2 + \omega^2 x_2 = 0$$

The first equation can be written, with $\dot{x}_2$ viewed as an input,

$$\ddot{x}_1 + \alpha(x_1^2 + k - 1)\dot{x}_1 + \omega^2 x_1 = \alpha k \dot{x}_2$$

which can easily be shown to be contracting for $k > 1$ (Combescot and Slotine, 2000). Furthermore, since $x_1 = x_2$ is an obvious particular solution, $x_1 \rightarrow x_2$ exponentially.

The result immediately extends to cascade of oscillators of the form

$$\begin{aligned}
\ddot{x}_1 + \alpha(x_1^2 - 1)\dot{x}_1 + \omega^2 x_1 &= \alpha k(\dot{x}_2 - \dot{x}_1) \\
\ddot{x}_2 + \alpha(x_2^2 - 1)\dot{x}_2 + \omega^2 x_2 &= \alpha k(\dot{x}_3 - \dot{x}_2) \\
\cdots\cdots \\
\ddot{x}_n + \alpha(x_n^2 - 1)\dot{x}_n + \omega^2 x_n &= \alpha k(\dot{x}_{n+1} - \dot{x}_n) \\
\ddot{x}_{n+1} + \alpha(x_{n+1}^2 - 1)\dot{x}_{n+1} + \omega^2 x_{n+1} &= 0
\end{aligned}$$

which all converge exponentially to x_{n+1}. Indeed, this represents a hierarchy of n contracting systems driven by x_{n+1}, with $x_1 = x_2 = \ \ldots\ = x_n = x_{n+1}$ as a particular solution.

The result may also be combined with the observer of Example 3.1 when only position measurerents are available. □

A recent discussion of oscillators in the context of robotics can be found in (Williamson, 1999).

3.6 On the binding problem

The binding problem has long been one of the central themes of neuroscience. It is the question of how, based on external inputs from multiple sensory modalities, hundreds of specialized processing elements distributed in the brain give rise to a single unified perception of the world. In vision alone, for instance, some cortical areas process edges, others shape, motion, depth, color, and so on. While there is extensive literature on the subject (see (Kandel, et al., 2000)), recently it has been emphasized that the problem may be approached as one involving *overall convergence* of large clusters of specialized dynamic processing elements connected *through dense networks of feedback loops* (Tononi, et al., 1998; Dehaene, et al., 1998; Edelman and Tononi, 2000), primarily in the fast, thalamo-cortical system.

The previous discussion, of course, provides simple sufficient conditions for such systems to actually converge, and further for their states to vary smoothly and consistently as their inputs change: one only needs each of the processing dynamics to be contracting, and the feedback connection to enforce the required structure in some appropriate metric. While precisely determining to what extent this happens in the brain would be a formidable task, the principle could at least be used to achieve similar "decentralized" integration in artificial systems.

As mentioned in section 2.2, and *regardless of the network's size*, the overall convergence rate will be merely the slowest of the individual convergence rates. This means in particular that any new input or data supplied to a subsystem will quickly propagate through and be processed by the entire system. Similarly, if driven by a signal periodic in time, the entire system would quickly converge to an overall behavior at the same period. Note that this is in sharp contrast with a hierarchical or series structure, where convergence time rapidly increases with system size, and information flow is unidirectional.

For instance, if the individual building blocks are time-invariant, then, given any constant inputs, the overall system will tend towards a unique equilibrium, with an exponential convergence rate equal to that of the slowest subsystem. It may be viewed as solving, in a distributed fashion, the set of coupled algebraic equations defined by $\dot{\mathbf{x}}_i = \mathbf{0} \ , \forall i$. Of course, some of the building blocks may themselves consist of system combinations, such as sums or hierarchies of contracting sub-blocks.

In the next section, we consider similar questions of asynchronous distributed computation when the time-delay of information transmission *between* subsystems is significant. In particular, we show that, in principle, binding can still be achieved in those cases, although of course convergence rate is then limited by the transmission delays.

4 Transmission delays

Delays are central in shaping the organisation and the performance of the central nervous system. Besides the fast cortico-thalamo-cortical feedback loops mentioned above, the brain uses also unidirectional, "long" loops or rings, for instance cortico-cerebello-thalamo-cortical loops. Similarly, information takes about 1/10 second to travel from the brain to the hand, including muscle delays. When playing a very fast passage on the piano, there is, literally, no time for conscious feedback control. Many aircraft cockpit displays are analog rather than digital so as to minimize reaction time. In robotics, similar questions of delays occur in the context of force-reflecting teleoperation for instance, or in handling computational delays, as in visual processing.

Consider n such subsystems, of possibly different dimensions

$$\dot{\mathbf{z}}_i \ = \ \mathbf{f}_i(\mathbf{z}_i, t) \ + \ \sum_j \mathbf{G}_{ij} \tau_{ij} \qquad\qquad i = 1, ..., n$$

where the sum is performed over each loop j connecting subsystem i with other subsystems. The loops are assumed to be separate, i.e., not to share common links, although each subsystem may be part of many loops. The $\mathbf{G}_{ij}$ are constant, and for each loop j the τ_{ij} have the same dimension. Examples of this architecture include meshes or webs of arbitrary size, with bidirectional coupling along each link, as well as parallel unidirectional rings of arbitrary length. Inputs to the overall system can be provided through any of the subsystem dynamics.

While delays are inherent to the system physics, the overall system's stability properties depend on which variables are actually transmitted. Define a composite input variable at the subsystem i along the loop j

$$\mathbf{u}_{ij} \;=\; \mathbf{G}_{ij}^T \mathbf{z}_i \;+\; \mathbf{K}_j \; \tau_{ij}$$

and a corresponding composite output variable

$$\mathbf{y}_{ij} \;=\; \mathbf{G}_{ij}^T \mathbf{z}_i \;-\; \mathbf{K}_j \; \tau_{ij}$$

where $\mathbf{K}_j$ is a constant symmetric positive definite matrix which can be selected arbitrarily for each loop j. Next, use these variables to transmit information along the loop j from subsystem i_1 to subsystem i_2 , with time delay $T_{i_1 j}$

$$\mathbf{u}_{i_2 j}(t) \;=\; \mathbf{y}_{i_1 j}(t - T_{i_1 j})$$

Extending the derivation in (Lohmiller and Slotine, 2000a), the rate of change of differential length can then be computed as

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{n}\left(\delta\mathbf{z}_i^T\delta\mathbf{z}_i + \frac{1}{2}\sum_{j}\int_{t-T_{ij}}^{t} \delta\mathbf{y}_{ij}^T\mathbf{K}_j^{-1}\delta\mathbf{y}_{ij}\; d\tau\right) = \sum_{i=1}^{n}\delta\mathbf{z}_i\frac{\partial\mathbf{f}_i}{\partial\mathbf{z}_i}\delta\mathbf{z}_i$$

Assuming that the individual subsystems are all contracting, this in turn shows asymptotic contraction of the overall system.

The role of the composite variables above may be best understood from the physical analogy which inspired it (Niemeyer and Slotine, 1991). If we restrict the above discussion to mechanical systems, essentially the above transformations make the transmission channels mimic flexible mechanical beams – while such beams transmit waves bidirectionally with pure delays, they contain no source of energy and therefore are inherently stable (Anderson and Spong, 1989). Thus, the composite variables are referred to as "wave" (or scattering) variables. Some further flexibility in their choice can be obtained by noticing that contraction is preserved through any orthonormal cooordinate change on the $\mathbf{z}_i$, and that the inputs τ_i can be redefined through any constant invertible transformation. A preliminary investigation of the role of wave variables in the control of biological movement, and in particular of their relevance to models of the intermediate cerebellum, is proposed in (Massaquoi and Slotine, 1996).

In the case that the individual subsystems are time-invariant, the system tends towards a unique equilibrium, which is therefore independent of the delays. Furthermore, if constant external inputs are introduced, the system may be viewed as performing, in a distributed fashion, the associated algebraic computations $\dot{\mathbf{z}}_i = \mathbf{0}$, $\forall i$, with constraints of the form $\mathbf{u}_{i_1 j} = \mathbf{y}_{i_2 j}$ between adjacent subsystems i_1 and i_2 along the loop j, leading to $\mathbf{G}_{i_1 j}^T \mathbf{z}_{i_1 j} = \mathbf{G}_{i_2 j}^T \mathbf{z}_{i_2 j}$ and $\sum_i \tau_{ij} = \mathbf{0}$.

5 Concluding remarks

The above program is of course rather ambitious, and the results presented preliminary. However, we believe that it may lead to a better understanding of complex systems built in stages, such as biological systems, and thus may be viewed as suggesting a dynamic version of the pioneering work of (Simon, 1962). A specific discussion of the oculo-motor system in this context is presented in (Berthoz and Slotine, 2000). Finally, note that, in principle, the discussion can also extend to effective interaction between multiple robots, be they electromechanical or biochemical.

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