Contraction analysis of synchronization in networks of nonlinearly coupled oscillators

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Abstract

Nonlinear contraction theory allows surprisingly simple analysis of synchronisation phenomena in distributed networks of coupled nonlinear elements. The key idea is the construction of a virtual contracting system whose particular solutions include the individual subsystems' states. We also study the role, in both nature and system design, of co-existing "power" leaders, to which the networks synchronize, and "knowledge" leaders, to whose parameters the networks adapt. Also described are applications to large scale computation using neural oscillators, and to time-delayed teleoperation between synchronized groups.

Similarly, contraction theory can be systematically and simply extended to address classical questions in hybrid nonlinear systems. The key idea is to view the formal definition of a virtual displacement, a concept central to the theory, as describing the state transition of a differential system. This yields in turn a compositional contraction analysis of switching and resetting phenomena. Applications to hybrid nonlinear oscillators are also discussed.

Key words: Synchronization, Hybrid Nonlinear Systems, Cooperative Control, Teleoperation, Spike-Based Computation.

1 Introduction

This paper surveys recent applications and extensions of contraction theory [29]. After a brief review of the basic theory in section 2, we consider two broad domains, partial contraction analysis of synchronization phenomena in networks [58,51,48] in section 3, and compositional contraction analysis of hybrid systems [7] in section 4. Applications to cooperative control, large scale computation, and hybrid models of neural oscillators are also discussed. Finally, extensions to time-delayed group teleoperation [60] are studied in section 5.

2 Nonlinear Contraction Analysis

Basically, a nonlinear time-varying dynamic system will be called *contracting* if initial conditions or temporary disturbances are forgotten exponentially fast, i.e., if trajectories of the perturbed system return to their nominal behavior with an exponential convergence rate. It turns out that relatively simple conditions can be given for this stability-like property to be verified, and furthermore that this property is preserved through basic system combinations. Furthermore, the concept of *partial contraction* allows to extend the applications of contraction analysis to include convergence to behaviors or to specific properties (such as equality of state components, or convergence to a manifold) rather than trajectories.

2.1 A Basic Result

In this section, we summarize the basic results of [29], to which the reader is referred for more details.

We consider general time-varying deterministic systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \qquad (1)$$

where $\mathbf{f}$ is an $n \times 1$ nonlinear vector function and $\mathbf{x}$ is the $n \times 1$ state vector. The above equation may also represent the closed-loop dynamics of a controlled system with state feedback $\mathbf{u}(\mathbf{x}, t)$. All quantities are assumed to be real and smooth, by which it is meant that any required derivative or partial derivative exists and is continuous. The basic result of [29] can then be stated as

Theorem 1 *Consider system (1), and assume there exists a uniformly positive definite metric*

$$\mathbf{M}(\mathbf{x}, t) \ = \ \mathbf{\Theta}'(\mathbf{x}, t) \ \mathbf{\Theta}(\mathbf{x}, t)$$

such that the associated generalized Jacobian

$$\mathbf{F} \ = \ \left(\dot{\mathbf{\Theta}} + \mathbf{\Theta}\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)\mathbf{\Theta}^{-1}$$

is uniformly negative definite. Then all system trajectories converge exponentially to a single trajectory, with convergence rate $|\lambda_{max}|$, where λ_{max} is the largest eigenvalue of the symmetric part of $\mathbf{F}$. The system is said to be contracting.

By $\mathbf{\Theta'}$ we mean the Hermitian (conjugate transpose) of $\mathbf{\Theta}$, and by symmetric part of $\mathbf{F}$ we mean $\frac{1}{2}(\mathbf{F}+\mathbf{F}')$. It can be shown conversely that the existence of a uniformly positive definite metric with respect to which the system is contracting is also a necessary condition for global exponential convergence of trajectories. In the linear time-invariant case, a system is globally contracting if and only if it is strictly stable, with $\mathbf{F}$ simply being a normal Jordan form of the system and $\mathbf{\Theta}$ the coordinate transformation to that form. The results immediately extend to the case where the state is in $\mathbb{C}^n$.

An important property is that, under mild conditions, contraction is preserved through system combinations such as parallel, series or hierarchies, translation and scaling in time and state, and certain types of feedback [29,31,49,48].

Example 1 [48]   Consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t) + \mathbf{B}(\mathbf{x},t)\ \mathbf{u}$$

and assume that there exist control primitives $\mathbf{u} = \mathbf{p}_i(\mathbf{x},t)$ which, for any i, make the closed-loop system contracting in some common metric $\mathbf{M}(\mathbf{x})$. Multiplying each equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t) + \mathbf{B}(\mathbf{x},t)\ \mathbf{p}_i(\mathbf{x},t)$$

by a positive coefficient $\alpha_i(t)$, and summing, shows that any convex combination of the control primitives $\mathbf{p}_i(\mathbf{x},t)$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t)+\mathbf{B}(\mathbf{x},t)\ \sum_i\ \alpha_i(t)\ \mathbf{p}_i(\mathbf{x},t) \quad , \quad \sum_i \alpha_i(t) = 1$$

also leads to a contracting dynamics in the same metric. For instance, the time-varying convex combination may correspond to smoothly blending learned primitives in a humanoid robot.

2.2 Partial Contraction

Next we recall the basic principles of partial contraction analysis, which will be a major tool in studying synchronization phenomena. The reader is referred to [58,51,48] for details.

Theorem 2 *Consider a nonlinear system of the form*

$$\dot{\mathbf{x}} \ = \ \mathbf{f}(\mathbf{x},\mathbf{x},t)$$

and assume that the auxiliary system

$$\dot{\mathbf{y}} \ = \ \mathbf{f}(\mathbf{y},\mathbf{x},t)$$

is contracting with respect to $\mathbf{y}$. If a particular solution of the auxiliary $\mathbf{y}$-system verifies a smooth specific property, then all trajectories of the original $\mathbf{x}$-system verify this property exponentially. The original system is said to be partially contracting.

Proof: The virtual, observer-like $\mathbf{y}$-system has two particular solutions, namely $\mathbf{y}(t) = \mathbf{x}(t)$ for all $t \geq 0$ and the solution with the specific property. This implies that $\mathbf{x}(t)$ verifies the specific property exponentially. □

Note that contraction may be trivially regarded as a particular case of partial contraction. Also, consider for instance an original system in the form

$$\dot{\mathbf{x}} \ = \ \mathbf{c}(\mathbf{x},t) \ + \ \mathbf{d}(\mathbf{x},t)$$

where function $\mathbf{c}$ is contracting in a constant metric. The auxiliary contracting system may then be constructed as

$$\dot{\mathbf{y}} \ = \ \mathbf{c}(\mathbf{y},t) \ + \ \mathbf{d}(\mathbf{x},t)$$

and the specific property of interest may consist e.g. of a relationship between state variables.

The notion of building a virtual contracting system to prove exponential convergence applies also to control problems. Consider for instance a nonlinear system of the form

$$\dot{\mathbf{x}} \ = \ \mathbf{f}(\mathbf{x},\mathbf{x},\mathbf{u},t)$$

and assume that the control input $\mathbf{u}(\mathbf{x},\mathbf{x}_d,t)$ can be chosen such that

$$\dot{\mathbf{x}}_d \ = \ \mathbf{f}(\mathbf{x}_d,\mathbf{x},\mathbf{u},t)$$

where $\mathbf{x}_d(t)$ is the desired state. Consider now the auxiliary system

$$\dot{\mathbf{y}} \ = \ \mathbf{f}(\mathbf{y},\mathbf{x},\mathbf{u},t)$$

It the auxiliary $\mathbf{y}$-system is contracting, then $\mathbf{x}$ tends $\mathbf{x}_d$ exponentially exponentially, since both are particular solutions of the $\mathbf{y}$-system.

Example 2   Consider a rigid robot model

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \tau$$

and the energy-based controller [50]

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}}_\mathbf{r} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}_\mathbf{r} + \mathbf{g}(\mathbf{q}) - \mathbf{K}(\dot{\mathbf{q}} - \dot{\mathbf{q}}_\mathbf{r}) = \tau$$

with $\mathbf{K}$ a constant s.p.d. matrix. The virtual $\mathbf{y}$-system

$$\mathbf{H}(\mathbf{q})\dot{\mathbf{y}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{y} + \mathbf{g}(\mathbf{q}) - \mathbf{K}(\dot{\mathbf{q}} - \mathbf{y}) = \tau$$

has $\dot{\mathbf{q}}$ and $\dot{\mathbf{q}}_\mathbf{r}$ as particular solutions, and furthermore is contracting, since the skew-symmetry of the matrix $\dot{\mathbf{H}} - \mathbf{2C}$ implies

$$\frac{\mathbf{d}}{\mathbf{dt}}\, \delta\mathbf{y}^\mathbf{T}\mathbf{H}\delta\mathbf{y} = -\mathbf{2}\delta\mathbf{y}^\mathbf{T}(\mathbf{C}+\mathbf{K})\delta\mathbf{y} + \delta\mathbf{y}^\mathbf{T}\dot{\mathbf{H}}\delta\mathbf{y} = -\mathbf{2}\delta\mathbf{y}^\mathbf{T}\mathbf{K}\delta\mathbf{y}$$

Thus $\dot{\mathbf{q}}$ tends to $\dot{\mathbf{q}}_\mathbf{r}$ exponentially. Making then the usual choice $\dot{\mathbf{q}}_\mathbf{r} = \dot{\mathbf{q}}_\mathbf{d} - \lambda(\mathbf{q} - \mathbf{q}_\mathbf{d})$ creates a hierarchy and implies in turn that $\mathbf{q}$ tends to $\mathbf{q}_\mathbf{d}$ exponentially.

Example 3 Consider a convex combination or interpolation between contracting dynamics

$$\dot{\mathbf{x}} = \sum_i \alpha_i(\mathbf{x}, t)\ \mathbf{f}_i(\mathbf{x}, t)$$

where the individual systems $\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x}, t)$ are contracting in a common metric $\mathbf{M}(\mathbf{x})$ and have a common trajectory $\mathbf{x}_o(t)$ (for instance an equilibrium), with all $\alpha_i(\mathbf{x}, t) \geq 0$ and $\sum_i \alpha_i(\mathbf{x}, t) = 1$. Then all trajectories of the system globally exponentially converge to the trajectory $\mathbf{x}_o(t)$. Indeed, the auxiliary system

$$\dot{\mathbf{y}} = \sum_i \alpha_i(\mathbf{x}, t)\ \mathbf{f}_i(\mathbf{y}, t)$$

is contracting (with metric $\mathbf{M}(\mathbf{y})$) and has $\mathbf{x}(t)$ and $\mathbf{x}_o(t)$ as particular solutions.

Other applications of partial contraction are studied in [58,51,59,60] and in [23].

3 Synchronization in Coupled Networks

The results in this section are based on [58,51,59,60].

3.1 Networks with General Connectivity

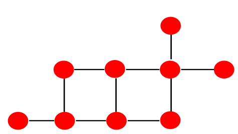

Fig. 1. An example of generally coupled networks.

Consider a coupled network containing n elements

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + \sum_{j \in \mathcal{N}_i} \mathbf{u}_{ji}\ (\mathbf{x}_j, \mathbf{x}_i, \mathbf{x}, t) \qquad i = 1, \ldots, n$$

where $\mathbf{x} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^T$, $\mathbf{u}_{ji}$ is the coupling force from element j to i, and $\mathcal{N}_i$ denotes the set of the active neighbors of element i, which can be very general (Figure 1 illustrates a distributed example). Assume more specifically that the couplings are bidirectional, symmetric, and of the form

$$\mathbf{u}_{ji} \ = \ \mathbf{u}_{ji}\ (\ \mathbf{x}_j - \mathbf{x}_i,\ \mathbf{x},\ t\ )$$

where $\mathbf{u}_{ji}\ (\ \mathbf{0},\ \mathbf{x},\ t\ ) = \mathbf{0}$, and

$$\mathbf{K}_{ji} \ = \ \frac{\partial \mathbf{u}_{ji}\ (\ \mathbf{x}_j - \mathbf{x}_i,\ \mathbf{x},\ t\ )}{\partial(\mathbf{x}_j - \mathbf{x}_i)} \ > \ 0 \qquad \text{uniformly}$$

with $\mathbf{K}_{ji} = \mathbf{K}_{ij}$. For instance, one may have

$$\mathbf{u}_{ji} \ = \ (\ \mathbf{C}_{ji}(t) + \ \mathbf{B}_{ji}(t)\ \|\mathbf{x}_j - \mathbf{x}_i\|\ )\ (\mathbf{x}_j - \mathbf{x}_i)$$

with $\mathbf{C}_{ji} = \mathbf{C}_{ij} > 0$ uniformly and $\mathbf{B}_{ji} = \mathbf{B}_{ij} \geq 0$.

For such a coupled network, all the elements inside will synchronize exponentially if

$$\lambda_{m+1}(\mathbf{L}_\mathbf{K}) > \max_{i=1}^{n} \lambda_{max}(\mathbf{J}_{is}) \qquad \text{uniformly} \qquad (2)$$

where m is the dimension of each single element, $\mathbf{L}_\mathbf{K}$ denotes the weighted Laplacian matrix, and $\mathbf{J}_{is}$ is the symmetric part of $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_i, t)$. The proof [51][58] is based on applying Theorem 2 to the auxiliary system

$$\dot{\mathbf{y}}_i = \mathbf{f}(\mathbf{y}_i, t) + \sum_{j \in \mathcal{N}_i} \mathbf{u}_{ji}\ (\mathbf{y}_j - \mathbf{y}_i,\ \mathbf{x},\ t) - \mathbf{K}_0\ \sum_{j=1}^{n} \mathbf{y}_j + \mathbf{K}_0\ \sum_{j=1}^{n} \mathbf{x}_j(t)$$

where $\mathbf{K}_0$ is a constant symmetric positive definite matrix, whose value can be set arbitrarily according to need. In a general understanding, the condition (2) is equivalent to three requirements:

- the network is connected,
- $\lambda_{max}(\mathbf{J}_{is})$ is upper bounded,
- the coupling strengths are stronger than a threshold.

Note that if $\mathbf{J}_{is}$ is negative definite, synchronization will occur for any coupling strength even if the network is not connected. On the other hand, if $\mathbf{J}_{is}$ is contracting but based on a non-identity metric, it may be the case that elements are stable by themselves, but they will tend to self-excited oscillations if diffusion interactions are added [58], a phenomenon similar to pioneering work by Turing [56] and Smale [52].

The analysis carries on straightforwardly to other kind of couplings, to unidirectional couplings, and to positive semi-definite couplings. The definition of the "neighbor" sets $\mathcal{N}_i$ is quite flexible. While it may be based simply on position proximity (neighbors within a certain distance of each node), it can be chosen to reflect many other factors, such as similarity, closure, continuity, common region and connectedness.

3.2 Algebraic Connectivity

The three requirements we concluded in the last section actually represents the conditions on both the individual dynamics and the network's geometric structure. For a coupled network, increasing the coupling gain for an existing link or adding an extra link will both improve the convergence process [58].

To see this in more detail, let us assume that all the links within the network are bidirectional (the corresponding graph is called *undirected graph*) with identical coupling gain $\mathbf{K} = \mathbf{K}^T > 0$. Thus,

$$\lambda_{m+1}(\ \mathbf{L_K}\ )\ =\ \lambda_2\ \lambda_{min}(\mathbf{K})$$

where λ_2 is the algebraic connectivity (the second minimum eigenvalue) of the standard graph Laplacian matrix. If both the element dynamics and the coupling gains are fixed, the synchronization condition can be written as

$$\lambda_2\ >\ \hat{\lambda}\ \text{ uniformly}$$

where

$$\hat{\lambda}\ =\ \frac{\max_i \lambda_{max}(\mathbf{J}_{is})}{\lambda_{min}(\mathbf{K})}$$

In fact, we can further transform this condition to the ones based on more explicit properties in geometry [51], such as the graph diameter D (the maximum number of links between two distinct nodes [10])

$$D\ <\ \frac{4}{n\hat{\lambda}}$$

or the mean distance $\bar{\rho}$ (the average number of links between distinct nodes [36])

$$\bar{\rho}\ <\ \frac{2}{\hat{\lambda}(n-1)}\ +\ \frac{n-2}{2(n-1)}$$

These results imply that, different coupling links or nodes may make different contributions to synchronization, because they may play different roles in network structure. Consider a network with open chain structure and another one as a closed ring (see Figure 2 as an illustration). Although the number of coupling links only differ by one in these two cases, with the network size n tending to infinite, the effort to synchronize an open chain network will be four times than that to a closed one [51].

As another example, consider a ring network, a star network and an all-to-all network. With the network size

Fig. 2. Comparison of a network with open chain structure and another one as a closed ring.

$n \to \infty$, the threshold of the coupling strength to synchronize the ring network tends to infinite. It tends to 0 for the all-to-all network, and only needs to have order 1 for the star network. It is much easier to synchronize the star network than to the ring. The reason is that, the central node in the star network performs a global role, which makes the graph diameter keep as constant no matter how big the network size is. Such a star-liked structure is very popular in real world. For instance, the world wide web is composed by many connected subnetworks with star structure.

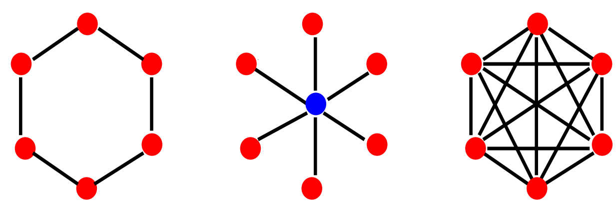

Fig. 3. Comparison of three different kinds of networks.

3.3 Switching Networks

The results above can be extended to analyze the collective behaviors of cooperating moving units with local couplings, where the network structure changes abruptly and asynchronously [21][44][51][58]. The examples include aggregate motions in the natural world, such as bird flocks, fish schools, animal herds, or bee swarms.

Consider such a network. Assume the dynamics of element i is given as the same as that in the last sections, except that $\mathcal{N}_i = \mathcal{N}_i(t)$ denotes the set of the active neighbors at time t. Apply partial contraction analysis to each time interval during which $\mathcal{N}(t) = \bigcup \mathcal{N}_i(t)$ is fixed. If the condition (2) is always true, $\delta\mathbf{z}^T\delta\mathbf{z}$ with $\mathbf{z} = [\mathbf{y}_1, \ldots, \mathbf{y}_n]^T$, the square distance between two neighboring trajectories of the auxiliary system, is continuous in time and upper bounded by a vanishing exponential (though its time-derivative can be discontinuous at discrete instants). Since the particular solution of the auxiliary system in each time interval is $\mathbf{y}_1 = \cdots = \mathbf{y}_n = \mathbf{y}_\infty$, all the elements in the network will reach agreement exponentially as they tend to $\mathbf{y}_1 = \cdots = \mathbf{y}_n$, which is a constant region in the state-space.

As an example, we study a simplified model of schooling or flocking in continuous-time

$$\dot{\mathbf{x}}_i\ =\ \sum_{j \in \mathcal{N}_i(t)} \mathbf{K}_{ji}\ (\mathbf{x}_j - \mathbf{x}_i) \qquad\qquad i = 1, \ldots, n$$

where $\mathbf{x}_i$ denotes the set of the states needed to reach agreements, such as heading, attitude, velocity, etc. $\mathcal{N}_i(t)$ is defined as the set of the nearest neighbors within a certain distance around element i at current time t. Since $\mathbf{J}_{is} = 0$ here, the synchronization condition is satisfied if only the network is connected. Therefore $\forall i$, $\mathbf{x}_i$ converges exponentially to a particular solution, which in this case is a constant value $\bar{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i(0)$. In fact, to reach a group agreement, the network need not be connected for any $t \geq 0$. A generalized condition can be derived which is the same as that obtained in [21] for a discrete-time model. If we separate time into an infinite sequence of bounded intervals starting at $t = 0$, and if the network is connected across each such interval (being connected across a time interval means that the union of the different graphs accounted along the interval is connected), the agreement $\mathbf{x}_1 = \cdots = \mathbf{x}_n$ will be reached asymptotically − a condition akin to persistency of excitation in adaptive systems. The proof is based on the fact that, the closer a subgroup tends to a local agreement, the closer they tend to the global one [51].

3.4 Leader-Followers Networks

The previous results can also be extended to analyze the coupled network with an additional leader [21][26][51][58]. Consider such a system

$$
\begin{aligned}
&\dot{\mathbf{x}}_0 = \mathbf{f}(\mathbf{x}_0, t) \\
&\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + \sum_{j\in\mathcal{N}_i}\mathbf{K}_{ji}\ (\mathbf{x}_j - \mathbf{x}_i) + \gamma_i\ \mathbf{K}_{0i}\ (\mathbf{x}_0 - \mathbf{x}_i) \\
&\quad i = 1, \ldots, n
\end{aligned}
$$

where $\mathbf{x}_0$ is the state of the group leader, $\gamma_i = 0$ or 1, and $\mathcal{N}_i$ does not include the links with $\mathbf{x}_0$. The states of all the followers will converge exponentially to the state of the leader [51][58] if

$$
\lambda_{min}(\mathbf{L_K} + \mathbf{I}^n_{\gamma_i\mathbf{K}_{0i}}) > \max_{i=1}^{n}\lambda_{max}(\mathbf{J}_{is}) \quad \text{uniformly}
$$

where $\mathbf{I}^n_{\gamma_i\mathbf{K}_{0i}}$ is a block diagonal matrix with the i^{th} diagonal entry as $\gamma_i\mathbf{K}_{0i}$. Note that one important condition for leader-following is that, the whole group of $n+1$ elements is connected. Thus the n followers could be either connected together, or there could be isolated subgroups all connected to the leader. Also note that, the network structure here can change from time to time, too.

Note that this result, besides its dubious moral implications, also means that it is easy to detract a group from its nominal behavior by introducing a "covert" element, with possible applications to group control games, ethology, and animal and plant mimicry.

The existence of an additional leader does not always help the followers' network to synchronize. Consider for instance the case when the leader has identical connections to all the followers, i.e., $\forall i,\ \mathbf{K}_{0i} = k\mathbf{I},\ \ k > 0$. Then

$$
\lambda_{min}(\mathbf{L_K} + \mathbf{I}^n_{\gamma_i\mathbf{K}_{0is}}) \ = \ k
$$

This implies that the connections between the leader and the followers do promote the convergence within the followers' network if $\lambda_{m+1}(\mathbf{L_K}) < k$, which is more likely to happen in a network with less connectivity.

Such a leader-following mechanism may cause synchronization propagation [51] in big networks, where the density is not smoothly distributed. Therefore the leader could be a group of elements, which synchronize first due to high connectivity. This leaders group does not have to be independent. They can receive feedback from the followers as well. The inputs from these leaders then facilitate synchronization in low-density regions, where the elements may not be able to synchronize by themselves. The leaders group here is very similar to the concept of core group in infectious disease dynamics [35], which is a group of the most active individuals. A small change in the core group will make a big difference in whether or not an epidemic can occur in the whole population.

Note that synchronization can be made to propagate from the center outward in a more active way, for instance through diffusion of a chemical produced by leaders or high-level elements and having the ability to expand the communication channels it passes through, i.e., to increase the gains through diffusion. Such a mechanism represents a hierarchical combination with gain dynamics. By extending the state, the analysis tools provided here can apply more generally to combinations where the gain dynamics are coupled to each other (with arbitrary connectivity) and to the $\mathbf{x}_i$.

Finally, note that in the spirit of Example 1 (section 2), different leaders $\mathbf{x}_0^j$ of arbitrary dynamics can define different primitives which can be combined. Contraction of the follower dynamics ($i = 1, \ldots, n$)

$$
\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + \sum_{j\in\mathcal{N}_i}\mathbf{K}_{ji}\,(\mathbf{x}_j - \mathbf{x}_i) + \sum_j \alpha_j(t)\,\gamma_i^j\,\mathbf{K}_{0i}^j\,(\mathbf{x}_0^j - \mathbf{x}_i)
$$

is preserved if $\sum_j \alpha_j(t) \geq 1,\ \forall t \geq 0$.

3.5 Knowledge-based Leaders

The leader-following mechanism we introduced in the last section can be considered as power-based, where the leader's dynamics is independent and thus followed by all the others. In fact, there exists another kind of leader role, which is knowledge-based [60,61]. In a knowledge-based network, members' dynamics are initially non-identical. The leader is the one whose dynamics is fixed or changes comparatively slowly. The followers obtain dynamics knowledge from the leader through adaptation. In this sense, if we understand the power leader as the one who tells the others “where to go”, a knowledge leader is the one who indicates “how to go”.

Consider a coupled network containing n elements

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, \mathbf{a}_i, t) + \sum_{j \in \mathcal{N}_i} \mathbf{K}_{ji}(\mathbf{x}_j - \mathbf{x}_i) \quad i = 1, \ldots, n$$

where the connectivity can be general. Assume now that the uncoupled dynamics $\mathbf{f}(\mathbf{x}_i, \mathbf{a}_i, t)$ contains a parameter set $\mathbf{a}_i$ which has a fixed value $\mathbf{a}$ for all the knowledge leaders. Denote Ω as the set of the followers, whose adaptations are based on local interactions

$$\dot{\mathbf{a}}_i \ = \ \mathbf{P}_i \mathbf{W}^T(\mathbf{x}_i, t) \sum_{j \in \mathcal{N}_i} \mathbf{K}_{ji} \ (\mathbf{x}_j - \mathbf{x}_i) \quad \forall \ i \in \Omega$$

where $\mathbf{P}_i > 0$ is constant and symmetric, and $\mathbf{W}(\mathbf{x}_i, t)$ is defined by

$$\mathbf{f}(\mathbf{x}_i, \mathbf{a}_i, t) \ = \ \mathbf{f}(\mathbf{x}_i, \mathbf{a}, t) \ + \ \mathbf{W}(\mathbf{x}_i, t)\tilde{\mathbf{a}}_i$$

with $\tilde{\mathbf{a}}_i = \mathbf{a}_i - \mathbf{a}$.

For such a knowledge-based leader-followers network, we can derive a synchronization condition very similar to (2), which contains no leaders. Consider the Lyapunov-like function

$$V = \frac{1}{2} \ (\ \mathbf{x}^T \mathbf{L}_{\mathbf{K}} \mathbf{x} \ + \ \sum_{i \in \Omega} \tilde{\mathbf{a}}_i^T \mathbf{P}_i^{-1} \tilde{\mathbf{a}}_i \ )$$

which yields

$$\dot{V} = \mathbf{x}^T \ (\ \mathbf{L}_{\mathbf{K}\boldsymbol{\Lambda}} - \mathbf{L}_{\mathbf{K}}^2 \ ) \ \mathbf{x}$$

where

$$\mathbf{L}_{\mathbf{K}\boldsymbol{\Lambda}} \ = \ \mathbf{D} \ \mathbf{I}_{(\mathbf{K}\Lambda)_{ijs}} \ \mathbf{D}^T$$

$\mathbf{D}$ is the incidence matrix [10], and $\mathbf{I}_{(\mathbf{K}\Lambda)_{ijs}}$ is a block diagonal matrix whose k^{th} diagonal entry $(\mathbf{K}\Lambda)_{ijs}$ is the symmetric part of $\mathbf{K}_{ij}\Lambda_{ij}$, with

$$\Lambda_{ij} = \int_0^1 \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_j + \chi(\mathbf{x}_i - \mathbf{x}_j), \mathbf{a}, t) \ d\chi$$

corresponding to the k^{th} link. Applying Barbalat’s lemma [50], the states of all the elements will converge together asymptotically if

$$\frac{\lambda_{m+1}^2(\mathbf{L}_{\mathbf{K}})}{\lambda_n(\mathbf{L})} > \max_k \lambda_{max}(\mathbf{K}\boldsymbol{\Lambda})_{ijs}$$

and all the states are bounded. Furthermore [50], $\forall \, i \in \Omega$, $\mathbf{a}_i$ will converge to $\mathbf{a}$ if $\exists \ \alpha > 0, T > 0, \forall t \geq 0$,

$$\int_t^{t+T} \mathbf{W}^T(\mathbf{x}_i, r)\mathbf{W}(\mathbf{x}_i, r)dr \ \geq \ \alpha \mathbf{I}$$

Note that such is the case in a coupled oscillator network as long as oscillations are preserved.

The analysis, detailed in [60],implies that new elements can be added into the network without prior knowledge of the individual dynamics, and that elements in an existing network have the ability to recover dynamic information if temporarily lost. Similar knowledge-based leader-followers mechanism may exist in many natural processes. In evolutionary biology, for instance, knowledge leaders are essential to keep the evolution processes uninvasible or evolutionary stable [43]. In reproduction, the leaders are senior members. The knowledge-based mechanism may also describe evolutionary mutation or disease infection [35], where the leaders are mutants or invaders. Knowledge-based leader-following may also occur in animal aggregate motions or human social activities. In a bird flock, for instance, the knowledge leader can be a junior or injured member whose moving capacity is limited, and which is protected by others through dynamic adaptation.

In fact, knowledge leaders holding different parameters can co-exist in the same network. Assume the dynamics $\mathbf{f}$ contains l parameter sets $\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^l$ with

$$\mathbf{f}(\mathbf{x}_i, \mathbf{a}_i^1, \ldots, \mathbf{a}_i^l, t) \ = \ \mathbf{f}(\mathbf{x}_i, \mathbf{a}^1, \ldots, \mathbf{a}^l, t) \ + \ \sum_{k=1}^{l} \mathbf{W}_k \tilde{\mathbf{a}}_i^k$$

Denoting by $\Omega^1, \Omega^2, \ldots, \Omega^l$ the followers sets corresponding to different parameters, the adaptation laws are, for $k = 1, 2, \ldots, l$,

$$\dot{\mathbf{a}}_i^k \ = \ \mathbf{P}_i^k \mathbf{W}_k^T(\mathbf{x}_i, t) \sum_{j \in \mathcal{N}_i} \mathbf{K}_{ji} \ (\mathbf{x}_j - \mathbf{x}_i) \quad \forall \ i \in \Omega^k$$

Power leaders and knowledge leaders can co-exist in the same network. For instance, a leader guiding the direction may use state measurements from its neighbors to adapt its parameters to the values of the knowledge leaders.

The adaptation law we used corresponds to inserting an integrator in the feedback loop [50]. Such an integrator can be replaced by any operator which preserves the passivity of the mapping from measurement error to parameter error. For instance, the adaptation law could be

$$\hat{\mathbf{a}}_i = \mathbf{a}_i + \mathbf{Q}_i \mathbf{W}^T(\mathbf{x}_i, t) \sum_{j \in \mathcal{N}_i} \mathbf{K}_{ji} \ (\mathbf{x}_j - \mathbf{x}_i)$$

where $\mathbf{Q}_i > 0$ is symmetric and constant, and $\mathbf{a}_i$ is defined as before. Using estimated parameter $\hat{\mathbf{a}}_i$ in the followers' dynamics corresponds to putting an PI block in feedback loop, which improves the convergence rate as compared to having only the I operator.

The adaptive model we described represents a genotype-phenotype mapping, where adaptation occurring in genotypic space is based on the interactions of behavioral phenotypes. The complexity of the genotype-phenotype mapping makes it still a challenge in today's evolutionary biology [43].

Last, both power leaders and knowledge leaders could be virtual, which is common in animal aggregate motions. For instance, a landmark may be used as a virtual power leader. Similarly, when hunting, an escaping prey could specify both where and how.

3.6 Fast Neural Computation

Recent research has explored the notion that artificial spike-based computation, inspired by models of computations in the central nervous system, may present significant advantages for specific types of large scale problems[9][11] [47][55][57]. In this section, we study new models for two common instances of such computation, winner-take-all(WTA)[1][12][22] [33][59] and coincidence detection [17][59]. In both cases, very fast convergence is achieved based on simple networks of FitzHugh-Nagumo neurons.

The FitzHugh-Nagumo(FN) model [8][38] is a well-known simplified version of the classical Hodgkin-Huxley model [16]. Originally derived from the Van der Pol oscillator [50][53], it can be generalized using a linear state transformation to the dimensionless system [37]

$$\begin{cases} \dot{v} = v(\alpha - v)(v-1) - w + I \\ \dot{w} = \beta v - \gamma w \end{cases}$$

where α, β, γ are positive constants, v models membrane potential, w accommodation and refractoriness, and I stimulating current. Simple properties of the FN model can be exploited for neural computations.

The basic network structure computing WTA is illustrated in Figure 4, where each FN neuron receives both an external input and a global inhibition. The global inhibition neuron receives synaptic inputs from the FN neurons. It spikes whenever any FN neuron spikes, after which it slowly converges to a rest steady state. The specific dynamics of the global neuron can be very general.

The linear complexity of the WTA network makes it possible to replace the single global inhibitory neuron with a group of interneurons, each of which only inhibits a set of local FN neurons. In this distributed version, synchronization of the interneurons can be guaranteed by the general nonlinear synchronization mechanisms derived in the last sections. Figure 5 shows a simulation result from [59].

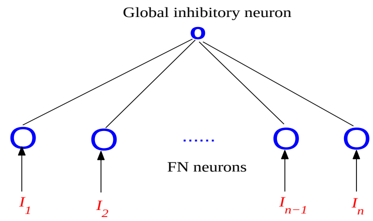

Fig. 4. Neural network computing winner-take-all.

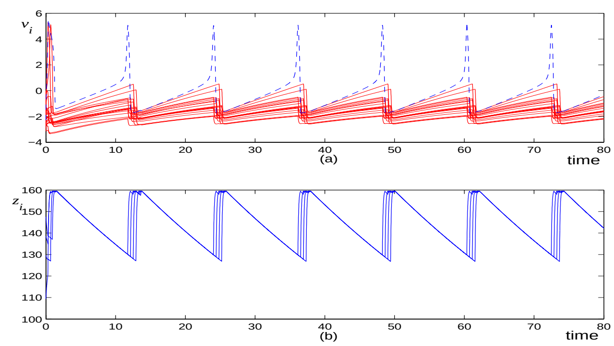

Fig. 5. Simulation result of a distributed WTA computation with four interneurons and twenty inputs. The plots are (a) states v_i versus time and (b) inhibitions z_i versus time.

In contrast to most existing studies, the network's initial state can be set arbitrary in our model, and its convergence is guaranteed in at most two spiking periods, making it particularly suitable to track time-varying inputs. If several neurons receive the same largest input, they all spike as a group. Since FN neurons are independent, they can be added or removed from the network at any time. The model can be extended to compute k-winner-take-all and soft-winner-take-all. In the latter case, a desynchronized spiking sequence is obtained in each stable period.

A similar leader-followers network can be used to compute coincidence detection, which plays a key role in temporal binding [17][28]. By replacing the global inhibition neuron in Figure 4 with an excitory FN neuron, very fast and salient response can be expected for a "many-are-equal" moment [59]. Figure 6 shows a simulation result from [59].

3.7 Fast Inhibition

The dynamics of a large network of synchronized elements can be completely transformed by the addition of

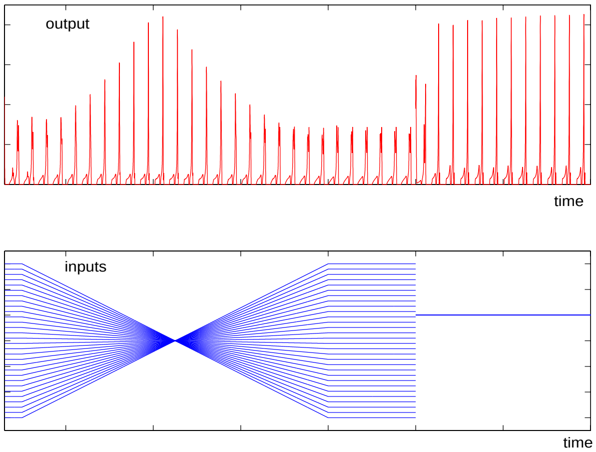

Fig. 6. Simulation result of fast coincidence detection. The output in the upper plot captures the "many-are-equal" moment among the inputs shown in the lower plot.

a single inhibitory coupling link. Start for instance with the synchronized network

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + \sum_{j \in \mathcal{N}_i} \mathbf{K}_{ji} \ (\mathbf{x}_j - \mathbf{x}_i) \qquad i = 1, \ldots, n$$

and add a single inhibitory link between two arbitrary elements a and b

$$\dot{\mathbf{x}}_a = \mathbf{f}(\mathbf{x}_a, t) + \sum_{j \in \mathcal{N}_a} \mathbf{K}_{ja} \ (\mathbf{x}_j - \mathbf{x}_a) + \mathbf{K} \ (-\mathbf{x}_b - \mathbf{x}_a)$$
$$\dot{\mathbf{x}}_b = \mathbf{f}(\mathbf{x}_b, t) + \sum_{j \in \mathcal{N}_b} \mathbf{K}_{jb} \ (\mathbf{x}_j - \mathbf{x}_b) + \mathbf{K} \ (-\mathbf{x}_a - \mathbf{x}_b)$$

The network is contracting for strong enough coupling strengths [51]. Hence, the n elements will be inhibited. If the function $\mathbf{f}$ is autonomous, they will tend to equilibrium points, as we illustrated in Figure 7. If the coupling strengths are not very strong, the inhibitory link will still have the ability to destroy the synchrony, and may then generate a desynchronized spiking sequence as illustrated in Figure 8. Adding more inhibitory couplings preserves the result.

Such inhibition properties may be useful in pattern recognition to achieve rapid desynchronization between different objects. They may also be used as simplified models of minimal mechanisms for turning off unwanted synchronization, as e.g. in epileptic seizures or oscillations in internet traffic. In such applications, small and localized inhibition may also allow one to destroy unwanted synchronization while only introducing a small disturbance to the nominal behavior of the system.

Cascades of inhibition are common in the brain, in a way perhaps reminiscent of NAND-based logic.

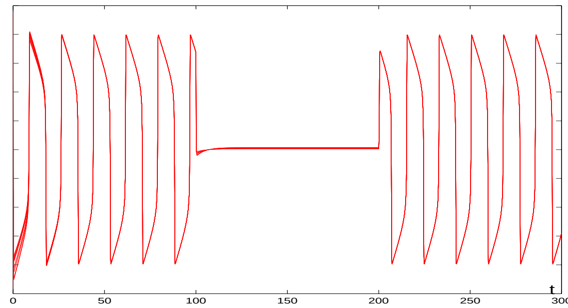

Fig. 7. An example of fast inhibition with a single inhibitory link. The plot shows the states of ten FitzHugh-Nagumo neurons in the time space. The inhibitory link is actived at $t = 100$ and removed at $t = 200$.

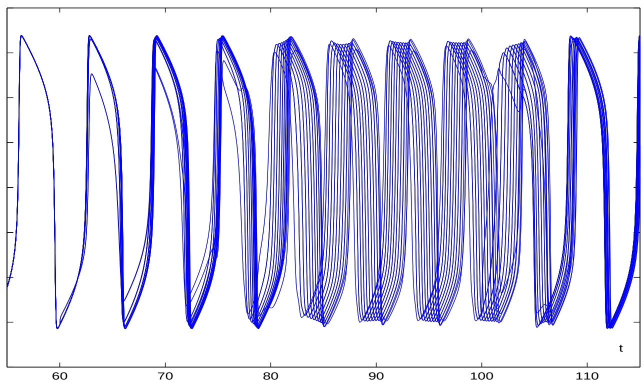

Fig. 8. Fast inhibition with weak single inhibitory link, actived at $t = 60$ and removed at $t = 100$.

4 Hybrid Systems

We now turn to another application and extension of contraction analysis, namely compositional contraction analysis of hybrid systems. The development is based on [7], which systematically extends earlier results in [31].

We let $\overline{\lambda}(.)$ and $\underline{\lambda}(.)$ denote the maximal and minimal eigenvalues of a matrix, and $\overline{\sigma}(.)$ and $\underline{\sigma}(.)$ denote the maximal and minimal singular values of a matrix. Also, $\gamma(.) \equiv \overline{\sigma}(.)/\underline{\sigma}(.)$ denotes the condition number of a matrix, and $\|.\|$ the spectral norm of a matrix, and the subscript s refers to the symmetric part of a matrix. Finally, $\mathbb{R}_+$ refers to the set of non-negative real numbers and $\mathbb{N}_*$ to the set of strictly positive integers.

Using differential displacements $\delta\mathbf{x}$ at fixed time (which will be further discussed later), the associated differential dynamics of system (1) is represented by the equation:

$$\frac{d}{dt} \, \delta\mathbf{z}(t) \;=\; \mathbf{F} \, \delta\mathbf{z}(t) \tag{3}$$

where $\delta\mathbf{z} = \mathbf{\Theta}(\mathbf{x}, t) \, \delta\mathbf{x}$. System (1) is contracting if (and only if [29])

$$\exists \; \alpha < 0, \;\; \overline{\lambda}(\mathbf{F}_s) \leq \alpha \tag{4}$$

4.1 Discrete-time Systems

Analogous results to section 2 hold for discrete-time systems given by

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), k) \qquad (5)$$

where $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$. Defining similarly the differential displacement $\delta\mathbf{x}(k)$, the corresponding differential dynamics is represented by

$$\delta\mathbf{z}(k+1) = \mathbf{F}\,\delta\mathbf{z}(k) \qquad (6)$$

where the differential displacement transformation $\delta\mathbf{z}(k) = \mathbf{\Theta}(\mathbf{x}, k)\delta\mathbf{x}(k)$ is used, with $\mathbf{\Theta}(\mathbf{x}, k)$ an $n \times n$ matrix of a uniformly bounded inverse, and $\mathbf{M}(\mathbf{x}, k) = \mathbf{\Theta}(\mathbf{x}, k)'\mathbf{\Theta}(\mathbf{x}, k)$ the metric. Let $\mathbf{J}(\mathbf{x}, k) = \partial\mathbf{f}(\mathbf{x}, k)/\partial\mathbf{x}(k)$ be the Jacobian of the vector field of interest, then the generalized Jacobian is given as $\mathbf{F}(\mathbf{x}, k) = \mathbf{\Theta}(\mathbf{x}(k+1), k+1)\,\mathbf{J}\,\mathbf{\Theta}(\mathbf{x}(k), k)^{-1}$.

The analogue to Theorem 1 for discrete-time systems [29] is given next.

Theorem 3 *A discrete-time system given by Equation (5) is contracting if and only if* $\exists$ *a metric* $\mathbf{M}(\mathbf{x}, k)$ *such that the generalized Jacobian* $\mathbf{F}$ *is uniformly negative definite. This is equivalent to* $\exists\ \beta < 1$ *such that* $\overline{\sigma}(F) \leq \beta$.

4.2 Compositional Contraction Analysis

The compositional description of contraction of nonlinear systems is presented using a generalized differential state transition matrix in a metric. This approach, along with the equivalent variational conditions based on a vector field's Jacobian [29], will become a basis for a unified simple characterization of the transition and stability of hybrid nonlinear systems. The result not only yields generalizations and relaxations of the existing conditions for stability of switched systems and resetting systems, including earlier work based on contraction [31], but also spans systems combining switching and resetting, which will be referred to as switched-resetting systems.

The approach is unified across both continuous-time and discrete-time systems based on the description of contraction through the differential state transition matrix.

4.2.1 Results

Let the initial differential displacement be defined as $\delta\mathbf{x}_o \equiv d\mathbf{x}(t_o)$, where the initial condition vector $\mathbf{x}_o = \mathbf{x}(t_o)$. Then the differential displacement is given by:

$$\delta\mathbf{x}(t) \ = \ \frac{\partial\mathbf{x}(\mathbf{x}_o, t)}{\partial\mathbf{x}_o}\ \delta\mathbf{x}_o \qquad (7)$$

where $\Phi_J(t, t_o) \equiv \partial\mathbf{x}(\mathbf{x}_o, t)/\partial\mathbf{x}_o$ will be referred to as the differential state transition matrix. The generalized differential state transition matrix satisfies $\delta\mathbf{z}(t) = \Phi_F(t, t_o)\delta\mathbf{z}_o$ and is given by:

$$\Phi_F(t, t_o) \ = \ \mathbf{\Theta}(t)\ \Phi_J(t, t_o)\ \mathbf{\Theta}_o^{-1} \qquad (8)$$

where $\mathbf{\Theta}_o = \mathbf{\Theta}(\mathbf{x}(t_o), t_o)$ and $\mathbf{\Theta}(t) = \mathbf{\Theta}(\mathbf{x}(t), t)$. Identically in the discrete-time case, $\Phi_J(k, k_o) = \partial\mathbf{x}(\mathbf{x}_o, k)/\partial\mathbf{x}_o$ will be referred to as the differential state transition matrix, with the generalized differential state transition matrix is given by $\Phi_F(k, k_o) = \mathbf{\Theta}(k)\Phi_J(k, k_o)\mathbf{\Theta}_o^{-1}$, where $\mathbf{\Theta}_o = \mathbf{\Theta}(\mathbf{x}(k_o), k_o)$ and $\mathbf{\Theta}(k) = \mathbf{\Theta}(\mathbf{x}(k), k)$. Note that explicit dependence on $\mathbf{x}$ has been omitted in the notation for simplicity.

Theorem 4 *A system given by Equation (1)(or (5)) is contracting if and only if* $\exists$ *a metric* $\mathbf{M}(\mathbf{x}, t)$ *(or* $\mathbf{M}(\mathbf{x}, k)$*) and* $\eta < 0$ *such that the generalized differential state transition matrix* Φ_F *satisfies* $\|\Phi_F(t, t_o)\| \leq e^{\eta(t-t_o)}$ *(or* $\|\Phi_F(k, k_o)\| \leq e^{\eta(k-k_o)}$*).*

Proof: To prove sufficiency assume that $\|\Phi_F(t, t_o)\| \leq e^{\eta(t-t_o)}$. Then

$$\begin{aligned}\|\delta\mathbf{z}(t)\| &= \|\Phi_F(t, t_o)\delta\mathbf{z}_o\| \leq \|\Phi_F(t, t_o)\|\,\|\delta\mathbf{z}_o\| \\ &\leq \|\delta\mathbf{z}_o\|\, e^{\eta(t-t_o)}\end{aligned}$$

To prove necessity take $\|\delta\mathbf{z}(t)\| \leq \|\delta\mathbf{z}_o\|\, e^{\eta(t-t_o)}$. Given $\|\delta\mathbf{z}_o\| \neq 0$ define a normalized state vector $\mathbf{v} = \frac{\delta\mathbf{z}_o}{\|\delta\mathbf{z}_o\|}$ and thus

$$\|\delta\mathbf{z}(t)\| = \|\Phi_F(t, t_o)\ (\|\delta\mathbf{z}_o\|\mathbf{v})\ \| = \|\delta\mathbf{z}_o\|\|\Phi_F(t, t_o)\mathbf{v}\|$$

Hence

$$\begin{aligned}\sup\|\delta\mathbf{z}(t)\| &= \|\delta\mathbf{z}_o\| \sup_{\|\mathbf{v}\|=1} \|\Phi_F(t, t_o)\mathbf{v}\| = \|\delta\mathbf{z}_o\|\|\Phi_F(t, t_o)\| \\ &\leq \|\delta\mathbf{z}_o\|\, e^{\eta(t-t_o)}\end{aligned}$$

which yields

$$\|\Phi_F(t, t_o)\| \leq e^{\eta(t-t_o)}$$

Since the normalized vector $\mathbf{v}$ can be chosen arbitrarily this completes the proof. The proof for the discrete-time case is identical with replacing t by k and t_o by k_o. $\square$

4.2.2 Remarks

- The differential state transition matrix is the same as the state transition matrix for a linear system since $\Phi = \mathbf{f}(t, t_o)$ only. This obviously does not generally hold for nonlinear systems because $\Phi = \mathbf{f}(t, t_o, \mathbf{x}_o)$. The statement also holds for the discrete-time case.

- The unified result in Theorem 4 implies that $\eta \equiv \alpha$ in equation (4) and $\eta \equiv \ln \beta$ in Theorem 3. In this regard, $|\eta|$ will be referred to as the *contraction rate*.
- The composition property of the state transition matrix holds for the differential state transition matrix by uniqueness of solutions of the linear differential equation (2), in the same metric: $\Phi_F(t_3, t_1) = \Phi_F(t_3, t_2)\Phi_F(t_2, t_1), \quad \forall t_1, t_2, t_3$.
- The composition property holds for the discrete-time case but in forward-time only (unless Φ is invertible): $\Phi_F(j, k) = \Phi_F(j, i)\Phi_F(i, k), \quad \forall j \geq i \geq k$

Establishing the equivalent condition for contraction based on the differential state transition matrix, Theorem 4, along with the compositional property will become the basis for hybrid systems contraction analysis. This is the case since there is no difference between the required condition for contraction for continuous-time and discrete-time systems when viewed via the generalized transition matrix. In fact, the composition property of this operator will allow for the description of the complicated evolution and transition of hybrid systems to be captured by simple means. Upon producing required conditions for stability of a hybrid system, the verification of the required conditions on individual systems can then be performed using the equivalent variational conditions, equation (4) and Theorem 3. This is the case since a closed form explicit expression for the differential transition matrix is not always possible and it is only used as an intermediate analysis tool. Note that although the state transition matrix is only useful for (possibly time-varying) linear systems, the differential version applies to the nonlinear case and is all what is needed to characterize contraction.

Contraction of a hybrid system, though it retains the same formal definition, has a slightly different interpretation. A hybrid system can be seen as a family of *particular systems* which can be activated depending on how the system's solutions trigger the logic-based transitions. Each of these systems is characterized by a unique transition operator described by a composition of individual dynamics active during certain time intervals. Then all solutions of a particular system will converge exponentially to a single *particular trajectory* if the hybrid system is contracting, and similarly for all particular systems. In some cases, all particular trajectories of these particular systems are the same, e.g. a common fixed point for all switching sequences of a switched system. In other important cases, the solutions of the overall hybrid system are steered to different pathways and different particular trajectories, by design. In such cases, the overall hybrid system is still said to be contracting if it is contracting for each particular system, as solutions that do not converge to the same particular trajectory are not solutions of the same particular system.

4.3 Switched Hybrid Systems

Switched hybrid systems are defined as systems combining continuous state variables, governed by differential equations or difference equations, and discrete states, governed by discrete (symbolic) dynamics. These discrete states govern the switching between different vector fields. In this paper as in most of the existing hybrid literature, the case of infinitely fast switching is not considered, and thus the set of switches is countable.

Definition 1
(i) A continuous-time switched system is defined by

$$\dot{\mathbf{x}}(t) = f_q(\mathbf{x}(t), t) \qquad\qquad q(t)^+ = g(\mathbf{x}(t), t, q(t))$$

(ii) A discrete-time switched system is defined by

$$\mathbf{x}(k+1) = f_q(\mathbf{x}(k), k) \qquad\qquad q(k)^+ = g(\mathbf{x}(k), k, q(k))$$

Here $x \in \mathbb{R}^n$ is the continuous state and $q \in \mathbb{N}_*$ is the discrete state, which is the switching index. The switching can be triggered by a state event, a time event or discrete state history, i.e., memory. Also continuous and discrete times $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$, respectively.

Two theorems providing sufficient conditions for contraction of switched hybrid systems are presented. The first is a generalization of the well-known condition of a common quadratic Lyapunov function [27,46] for fixed point stability of systems of the form $\dot{\mathbf{x}} = f_q(\mathbf{x})$.

Theorem 5 *A continuous-time (or discrete-time) switched system is contracting if the individual systems are contracting with respect to metrics $\mathbf{M}_i = \mathbf{\Theta}_i^{'}\mathbf{\Theta}_i$ which are equal at the switching times. This means that $\mathbf{\Theta}_i(t_i) = \mathbf{\Theta}_{i+1}(t_i)$ (or $\mathbf{\Theta}_i(k_i) = \mathbf{\Theta}_{i+1}(k_i)$) $\forall\, i$ where t_i (or k_i) is the i^{th} switching time.*

Proof: Let $\Phi_{J_i}(t_i, t_{i-1})$ be the transition matrix for the i^{th} system associated with Jacobian J_i and active during the period $\Delta t_{si} = t_i - t_{i-1}$. The generalized transition matrix is Φ_{Fi} associated with a generalized Jacobian $\mathbf{F}_i$. Then the overall transition matrix $\Phi_J(t, t_o)$ can be written as a composition of individual systems' transition matrices:

$$\begin{aligned}\|\Phi_J(t, t_o)\| &= \|\Phi_{J_n}(t, t_n)\Phi_{J_{n-1}}(t_n, t_{n-1}) \ldots \\ &\quad \ldots \Phi_{J_3}(t_3, t_2)\Phi_{J_2}(t_2, t_1)\Phi_{J_1}(t_1, t_o)\| \\ &= \|\mathbf{\Theta}_n(t)^{-1}\Phi_{\mathbf{F}_n}(t, t_n)\mathbf{\Theta}_n(t_n)\mathbf{\Theta}_{n-1}(t_n)^{-1} \ldots \\ &\quad \ldots \mathbf{\Theta}_2(t_1)\mathbf{\Theta}_1(t_1)^{-1}\Phi_{\mathbf{F}_1}(t_1, t_o)\mathbf{\Theta}_1(t_o)\|\end{aligned}$$

Using $\mathbf{\Theta}_i(t_i) = \mathbf{\Theta}_{i+1}(t_i)$, this yields

$$\begin{aligned}\|\Phi_J(t,t_o)\| &= \|\mathbf{\Theta}_n(t)^{-1}\Phi_{\mathbf{F}_n}(t,t_n)\Phi_{\mathbf{F}_{n-1}}(t_n,t_{n-1})\ldots \\ &\quad \ldots\Phi_{\mathbf{F}_3}(t_3,t_2)\Phi_{\mathbf{F}_2}(t_2,t_1)\Phi_{\mathbf{F}_1}(t_1,t_o)\mathbf{\Theta}_1(t_o)\| \\ &\leq \|\mathbf{\Theta}_n(t)^{-1}\|\|\Phi_{\mathbf{F}_n}(t,t_n)\|\|\Phi_{\mathbf{F}_{n-1}}(t_n,t_{n-1})\|\ldots \\ &\quad \ldots\|\Phi_{\mathbf{F}_2}(t_2,t_1)\|\|\Phi_{\mathbf{F}_1}(t_1,t_o)\|\|\mathbf{\Theta}_1(t_o)\|\end{aligned}$$

Now contraction of the individual systems implies $\|\Phi_{\mathbf{F}_i}(t_i,t_{i-1})\| \leq e^{\alpha_i \Delta t_{si}}$ where $\alpha_i < 0\ \forall\, i$. Thus

$$\|\Phi_J(t,t_o)\| \ \leq\ \|\mathbf{\Theta}_n(t)^{-1}\|\|\mathbf{\Theta}_1(t_o)\| \prod_{i=1}^{i^*} e^{\alpha_i \Delta t_{si}}$$

where $\|\mathbf{\Theta}_n(t)^{-1}\|\|\mathbf{\Theta}_1(t_o)\| \leq c$, with c some constant by boundedness of the inverse of each metric transformation $\mathbf{\Theta}_i$. In here, $\prod_{i=1}^{i^*} e^{\alpha_i \Delta t_{si}}$ will vanish exponentially if $\alpha_i < 0\ \forall\, i$ as $t \to +\infty$, i.e, $i^* \to +\infty$. The above result shows that the switched system verifies the definition of contraction. For the discrete-time case, the proof is identical by just replacing t by k and α by $\ln\beta$ following the unified compositional approach of section 3.2. □

The next theorem is a general condition for switched systems stability with multiple *constant* metrics for a wide class of systems. The result reduces the conservatism in the common Lyapunov function condition and is much simpler to compute than multiple Lyapunov function conditions that only conclude stability in the sense of Lyapunov. The result represents the trade-off in the design of a hybrid switched system between the key properties of the individual systems and the switching speed.

Theorem 6 *A continuous-time (or discrete-time) switched system is contracting if the individual systems are contracting with respect to constant metrics* $\mathbf{M}_i = \mathbf{\Theta}_i^{'}\mathbf{\Theta}_i$ *and* $\eta_i \Delta t_{si} + \ln\gamma(\mathbf{\Theta}_i) < 0$ *(or* $\eta_i \Delta k_i + \ln\gamma(\mathbf{\Theta}_i) < 0$*)* $\forall\, i$. *Here, the contraction rate* $\eta_i \equiv \alpha_i$, *where* $\overline{\lambda}(\mathbf{F}_{si}) \leq \alpha_i$ *for the* i^{th} *continuous-time system operating over a period* Δt_{si}. *Equivalently for the discrete-time case,* $\eta_i \equiv \ln\beta_i$, *where* $\overline{\sigma}(\mathbf{F}_i) \leq \beta_i$ *for the* i^{th} *system operating over a period* Δk_i.

Proof: Again $\Phi_{J_i}(t_i,t_{i-1})$ is the transition matrix for the i^{th} system associated with Jacobian J_i and active during the period $\Delta t_{si} = t_i - t_{i-1}$. The generalized transition matrix is Φ_{Fi} associated with a generalized Jacobian $\mathbf{F}_i$. Then the overall transition matrix $\Phi_J(t,t_o)$ can be written as the composition:

$$\begin{aligned}\|\Phi_J(t,t_o)\| &= \|\Phi_{J_n}(t,t_n)\Phi_{J_{n-1}}(t_n,t_{n-1})\ldots \\ &\quad \ldots\Phi_{J_3}(t_3,t_2)\Phi_{J_2}(t_2,t_1)\Phi_{J_1}(t_1,t_o)\| \\ &= \|\mathbf{\Theta}_n(t)^{-1}\Phi_{\mathbf{F}_n}(t,t_n)\mathbf{\Theta}_n(t_n)\mathbf{\Theta}_{n-1}(t_n)^{-1}\ldots \\ &\quad \ldots\mathbf{\Theta}_2(t_1)\mathbf{\Theta}_1(t_1)^{-1}\Phi_{\mathbf{F}_1}(t_1,t_o)\mathbf{\Theta}_1(t_o)\|\end{aligned}$$

Now if $\mathbf{\Theta}_i$ are constant then

$$\begin{aligned}\|\Phi_J(t,t_o)\| &= \|\mathbf{\Theta}_n^{-1}\Phi_{\mathbf{F}_n}(t,t_n)\mathbf{\Theta}_n\mathbf{\Theta}_{n-1}^{-1}\Phi_{\mathbf{F}_{n-1}}(t_n,t_{n-1})\ldots \\ &\quad \ldots\mathbf{\Theta}_2^{-1}\Phi_{\mathbf{F}_2}(t_2,t_1)\mathbf{\Theta}_2\mathbf{\Theta}_1^{-1}\Phi_{\mathbf{F}_1}(t_1,t_o)\mathbf{\Theta}_1\| \\ &\leq \|\mathbf{\Theta}_n^{-1}\|\|\Phi_{\mathbf{F}_n}(t,t_n)\|\|\mathbf{\Theta}_n\|\|\mathbf{\Theta}_{n-1}^{-1}\|\ldots \\ &\quad \ldots\|\mathbf{\Theta}_2\|\|\mathbf{\Theta}_1^{-1}\|\|\Phi_{\mathbf{F}_1}(t_1,t_o)\|\|\mathbf{\Theta}_1\|\end{aligned}$$

Contraction of the individual systems implies that $\|\Phi_{\mathbf{F}_i}(t_i,t_{i-1})\| \leq e^{\alpha_i \Delta t_{si}}$, where $\alpha_i < 0\ \forall\, i$. Furthermore, since $\|\mathbf{\Theta}_i\| = \overline{\sigma}(\mathbf{\Theta}_i)$, $\|\mathbf{\Theta}_i^{-1}\| = 1/\underline{\sigma}(\mathbf{\Theta}_i)$ and $\gamma(\mathbf{\Theta}_i) = \overline{\sigma}(\mathbf{\Theta}_i)/\underline{\sigma}(\mathbf{\Theta}_i)$,

$$\|\Phi_J(t,t_o)\| \ \leq\ \prod_{i=1}^{i^*} \gamma(\mathbf{\Theta}_i)\, e^{\alpha_i \Delta t_{si}} \ \leq\ \prod_{i=1}^{i^*} e^{\ln\gamma(\mathbf{\Theta}_i) + \alpha_i \Delta t_{si}}$$

In here, $\prod_{i=1}^{i^*} e^{\alpha_i \Delta t_{si} + \ln\gamma(\mathbf{\Theta}_i)}$ will vanish exponentially if $\ln\gamma(\mathbf{\Theta}_i) + \alpha_i \Delta t_{si} < 0\ \forall\, i$ as $t \to +\infty$, i.e., as $i^* \to +\infty$. Therefore, the switched system is indeed contracting. Again the proof is identical for the discrete-time case by just replacing t by k and α by $\ln\beta$. □

Note that the average dwell time property, see [27], is no other than a conservative special case of the condition in Theorem 6. The average dwell time property uses an *average* Δt_s and a *single* quantity based on the Lyapunov functions, which can be seen as a gross estimate of the conditioning from the different metrics.

Contraction immediately extends to cases where individual systems are *not* all contracting if one can *group* terms in the composition as:

$$\sum_{i=1}^{\infty} \alpha_i \Delta t_{si} \ =\ \sum_{j=1}^{\infty} a_j \qquad \text{(Theorem 5)}$$

$$\sum_{i=1}^{\infty} \alpha_i \Delta t_{si} + \ln\gamma(\mathbf{\Theta}_i) \ =\ \sum_{j=1}^{\infty} a_j \qquad \text{(Theorem 6)}$$

where $a_j < 0,\ \forall\, j$, and similarly for the discrete-time case. This is consistent with the expectation that activating an unstable system or a system that is not contracting in the same metric can preserve stability if we switch away from it quickly enough. A simple example is measurement sampling rate specification in observer designs for unstable continuous-time systems [31,49,63].

4.4 Resetting Hybrid Systems

Resetting hybrid systems, also known as impulsive systems [62,14,15], are defined as systems combining continuous state variables, governed by differential equations for which some or all states are being reset at discrete time instances via a resetting law, i.e., a difference equation. The discrete states' evolution governs such resets.

In this paper, the case of infinitely fast resetting is not considered. We assume that the set of resets is countable, i.e., the resets are distinct. Under these assumptions resets based on time or continuous state values are treated identically.

Definition 2 *A hybrid resetting system is defined by*

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t), \qquad t \neq t_q$$
$$\mathbf{x}(\mathrm{t})^+ = \mathrm{h}(\mathbf{x}(\mathrm{t}), \mathrm{t}), \quad t = t_q$$
$$\mathrm{q}(\mathrm{t})^+ = \mathrm{g}(\mathbf{x}(\mathrm{t}), \mathrm{t}, \mathrm{q}(\mathrm{t}))$$

Here $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$ is the continuous state and $q \in \mathbb{N}_*$ is the discrete state, which is the resetting index. The discrete state can be triggered by a state event, a time event or discrete state history, i.e, memory.

We state two theorems giving two sufficient conditions for contraction of resetting systems, analogous to Theorems 5 and 6 for switched systems. The reader is referred to [7] for proofs and discussion.

Theorem 7 *A resetting system is contracting if $\alpha\Delta t_{ri} + \ln\beta < 0\ \forall i$. Where $\overline{\sigma}(H) \leq \beta$ and H is generalized Jacobian associated with $h(\mathbf{x}, t)$ in a metric $\mathbf{M}_d = \mathbf{\Theta}_d^{'}\mathbf{\Theta}_d$. While, $\overline{\lambda}(\mathbf{F}_s) \leq \alpha$, where $\mathbf{F}_s$ is the symmetric part of the generalized Jacobian associated with $\mathbf{f}(\mathbf{x}, t)$ in the metric $\mathbf{M}_c = \mathbf{\Theta}_c^{'}\mathbf{\Theta}_c$. The metrics need to be equal at the resetting times, which means that $\mathbf{\Theta}_c(t_i) = \mathbf{\Theta}_d(t_i)$ where t_i is the i^{th} resetting time $\forall i$. Here, Δt_{ri} is the period between two resets following the i^{th} reset.*

The next theorem gives a less restrictive condition for contraction of state resetting systems, which applies to systems with constant metrics.

Theorem 8 *A resetting system is contracting if $\alpha\Delta t_{ri} + \ln(\beta\gamma(\mathbf{\Theta}_c)\gamma(\mathbf{\Theta}_d)) < 0\ \forall i$. Where $\overline{\sigma}(H) \leq \beta$ and H is generalized Jacobian associated with $h(\mathbf{x}, t)$ in a constant metric, $\mathbf{M}_d = \mathbf{\Theta}_d^{'}\mathbf{\Theta}_d$. While, $\overline{\lambda}(\mathbf{F}_s) \leq \alpha$,where $\mathbf{F}_s$ is the symmetric part of the generalized Jacobian associated with $\mathbf{f}(\mathbf{x}, t)$ in a constant metric $\mathbf{M}_c = \mathbf{\Theta}_c^{'}\mathbf{\Theta}_c$. Here, Δt_{ri} is the period between two resets following the i^{th} reset.*

Again contraction can be extended to cases where individual systems are *not* contracting if one can appropriately group terms in the composition, as

$$\sum_{i=1}^{\infty} \alpha\Delta t_{ri} + \ln\beta = \sum_{j=1}^{\infty} a_j \qquad \text{(Theorem 7)}$$
$$\sum_{i=1}^{\infty} \alpha\Delta t_{ri} + \ln(\beta\gamma(\mathbf{\Theta}_c)\gamma(\mathbf{\Theta}_d))) = \sum_{j=1}^{\infty} a_j \qquad \text{(Theorem 8)}$$

where $a_j < 0,\ \forall j$. This can be used for on-line adjustment of the resetting period, e.g., to preserve contraction in the presence of unknown or variable delays.

4.5 Switched-Resetting Hybrid Systems

In this section, extensions to the previous two sections are presented yielding conditions for systems that combine resetting and switching (see [7] for proofs). This class of hybrid systems, which will be referred to as switched-resetting systems has received very little attention in the literature. This may be attributed to the increased complexity due to combining switching and resetting. However, this posses no additional difficulty when exploiting the proposed compositional approach. In fact, the results extend in an analogous manner and are detailed in this section for clarity. Here distinction between continuous-time or discrete-time switched-resetting systems is made based on which dynamics is switched.

Definition 3
(i) A hybrid continuous-time switched-resetting system is defined by the equations:
$\dot{\mathbf{x}}(t) = f_{q1}(\mathbf{x}(t), t), \quad t \neq t_{q2}$
$\mathbf{x}(t)^+ = h(\mathbf{x}(t), t), \quad t = t_{q2}$
$q(t)^+ = g(\mathbf{x}(t), t, q(t))$

(ii) A hybrid discrete-time switched-resetting system is defined by the equations:
$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t), \quad t \neq t_{q2}$
$\mathbf{x}(t)^+ = h_{q1}(\mathbf{x}(t), t), \quad t = t_{q2}$
$q(t)^+ = g(\mathbf{x}(t), t, q(t))$
Here $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$ is the continuous state and $q = [q_1, q_2]^{'}$ is discrete state and $q_i \in \mathbb{N}_*$, which can be triggered by a state event, a time event or discrete state history. In here, q_1 is the switching index and q_2 is the resetting index.

The previous results can be combined as follows.

(i) A continuous-time switched-resetting system is contracting if $\alpha_i\Delta tr_{ij} + \ln\beta < 0\ \forall i, j$. Where $\overline{\sigma}(H) \leq \beta$ and H is generalized Jacobian associated with $h(\mathbf{x}, t)$ in a metric. While, $\overline{\lambda}(\mathbf{F}_{si}) \leq \alpha_i$,where $\mathbf{F}_{si}$ is the symmetric part of the generalized Jacobian associated with $f_i(\mathbf{x}, t)$ all in metrics that are equal at the transition times, i.e., switching times between them and resetting times with metric associated with H. Here, Δt_{rij} is the period between two resets following the j^{th} reset for the i^{th} switched system.

(ii) A discrete-time switched-resetting system is contracting if $\alpha\Delta t_{rij} + \ln\beta_i < 0\ \forall i, j$. Where $\overline{\sigma}(H_i) \leq \beta_i$ and H_i is generalized Jacobian associated with $h_i(\mathbf{x}, t)$ all in the same metric. While, $\overline{\lambda}(\mathbf{F}_s) \leq \alpha$,where $\mathbf{F}_s$ is the symmetric part of the generalized Jacobian associated with $\mathbf{f}(\mathbf{x}, t)$ in the *same* metric. Here, Δt_{rij} is the period between two resets following the j^{th} reset by the i^{th} switched resetting law.

4.6 Hybrid Models of Neural Oscillators

Expanding on the discussion of section 3, a spiking neuron is actually best described by resetting dynamics, since it produces precise impulses. Indeed, some of the more recent models of neurons, see [20], use simple resetting logic upon reaching threshold values of the membrane potential. A formal and general hybrid model is developed by considering the extended model of the FitzHugh-Nagumo neuron comprised of the switched continuous-time dynamics and a simple resetting law

$$\begin{aligned} \dot{v} &= c_i(v + w - \frac{1}{3}v^3 + I_i) \\ \dot{w} &= -\frac{1}{c_i}(v - a_i + b_i w) \ , \ t \neq t_j \\ \mathbf{x}^+ &= \mathbf{A}_d\, \mathbf{x} + \mathbf{g}_j \ , \ t = t_j \end{aligned} \tag{9}$$

where $\mathbf{x} = [v, w]^T$, A_d is a constant matrix and $\mathbf{g}_j$ is a discrete input comprised of discrete stimulation and external impulsive effects. The above system is a continuous-time switched-resetting system. In addition to resetting, switching is made between different values of a_i, b_i,c_i, and I_i. Contraction of the overall system is guaranteed if the discrete dynamics is contracting such that $\alpha_i \Delta t_{si} + k_i \ln(\beta\gamma(\mathbf{\Theta}_d)\gamma(\mathbf{\Theta}_i)) < 0, \ \forall i$, where Δt_{si} is the switching period for the i^{th} switched system and $k_i = \Delta t_{si}/\Delta t_{ri}$ is number of resets of that system. $\mathbf{\Theta}_d$ is simply the similarity transformation that transforms A_d to its simplest form and $\beta < 1$ is the modulus of its maximum eigenvalue. Also $\mathbf{\Theta}_i = diag(1, c_i)$ and $\alpha_i = c_i$.

Many important features of neurons [20] which are rarely captured in a single model can be combined here. Examples include *spike frequency adaptation*, where the frequency is relatively high at the onset of stimulation and then decreases, which can be achieved by switching between b_i and c_i values. Other features include *variable interspike frequencies*, which are easily controlled by adjusting Δt_{ri}, as well as *bistability of resting and spiking states* which correspond to switching between stimulations I_i, which can be easily generated in response to the current magnitude or phase of a state as well as time. In fact, *threshold variability*, which suggests that neurons have a variable threshold that depends on their prior activity, is a truly hybrid phenomenon which is simply dealt with here, since any switching or resetting can be triggered by any state and/or time event.

4.7 Floquet Multipliers and Lyapunov Exponents

Finally, let us remark that two widely used tools in nonlinear systems' analysis, Floquet multipliers and Lyapunov exponents, can be easily interpreted in the context of compositional contraction analysis via differential transition operators, suggesting in turn systematic extensions.

Floquet multipliers, commonly used in stability and bifurcations analysis of periodic solutions, are no other than $\sigma_i(\Phi_{\mathbf{F}}(T, 0))$ locally around a periodic trajectory of period T. The Floquet decomposition is given as $\Phi_{\mathbf{J}}(t, t_o) = \mathbf{P}(t)^{-1} e^{\mathbf{R}(t-t_o)} \mathbf{P}(t_o)$. $\mathbf{P}(t) = \mathbf{P}(t + T)$ is a T-periodic transformation defined as $\mathbf{P}(t) = e^{\mathbf{R}t} \Phi_{\mathbf{J}}(t, 0)^{-1}$. In here, the T-periodic metric $\mathbf{M}(t) = \mathbf{\Theta}(t)^{'} \mathbf{\Theta}(t)$ is constructed using $\mathbf{\Theta} = \mathbf{P}(t)\mathbf{S}$, where $\mathbf{S}$ is a similarity transformation matrix that takes $\mathbf{R}$ to its simplest form. The constant (possibly complex) matrix $\mathbf{R}$ is found by $e^{\mathbf{R}T} = \Phi_{\mathbf{J}}(T, 0)$, usually numerically. The periodic solution is strictly stable if and only if $\overline{\sigma}(\Phi_{\mathbf{F}}(T, 0)) < 1$ and unstable if $\overline{\sigma}(\Phi_{\mathbf{F}}(T, 0)) > 1$. This conclusion takes advantage of the composition property of the generalized differential state transition matrix, which gives $\Phi_{\mathbf{F}}(kT + t_o, t_o) = \Phi_{\mathbf{F}}(T, 0)^k$. In order to verify flow contraction to this periodic trajectory one needs that $\exists \ \alpha < 0 : \overline{\sigma}(\Phi_{\mathbf{F}}(kT + t_o, t_o)) \leq e^{\alpha k T}$, see equation (9). It is easy to see that this is the case if an only if $\overline{\sigma}(\Phi_{\mathbf{F}}(T, 0)) < 1$. Whereas a bifurcation is created if $\overline{\sigma}(\Phi_{\mathbf{F}}(T, 0)) = 1$, indicating contraction indifference. This corresponds to a zero eigenvalue of the real matrix $\mathbf{F}_s$. Recall that contraction behavior is only concerned with eigenvalues of $\mathbf{F}_s$ and singular values of $\Phi_{\mathbf{F}}$. However, the classification of different types of bifurcation points is determined by the eigenvalues of $\Phi_{\mathbf{F}}$ or equivalently those of $\mathbf{F}$. In this case, where the real part of eigenvalues of $\mathbf{F}$ is zero, the eigenvalues of $\Phi_{\mathbf{F}}(T, 0)$ are of the form e^{ai}, where i is the imaginary number. The different possible values e^{ai} can take, all of which are of modulus equal to one, yield different types of bifurcations [39]. This approach, which has been widely used to characterize local contraction around a periodic trajectory can be also used *globally* with an appropriate choice of metric transformation $\mathbf{\Theta}(x, t)$.

Similarly, *Lyapunov exponents*, commonly used in the study of chaos [13,53], are actually not that different from Floquet multipliers from a contraction perspective, since they are simply $\lambda_i(\mathbf{F}_s)$ *for an identity metric*, as also noted in [29]. In contrast to Floquet multipliers such exponents are indeed used globally. Infinite-time Lyapunov Exponents are commonly used to indicate chaos if $\overline{\lambda}(F_s) > 0$ for $\mathbf{M} = \mathbf{I}$. However, more recent views of Lyapunov exponents, referred to as the direct finite-time Lyapunov exponents (DFTLE) [18] have been used to identify stable invariant manifolds and global domains of attraction of different solutions. In fact, the maximal DFTLE for a trajectory starting at x_o is defined as $DFTLE(x_o, t) = \ln \overline{\sigma}(\Phi_{\mathbf{J}})/(t - t_o)$. Note that in the identification of chaos it is sufficient to find that an averaged maximal exponent is positive without a metric change, since this indicates sensitive dependence on initial conditions. Similarly, to obtain graphical demonstrations of boundaries between different solutions in lower dimensional problems, it is sufficient to find that the largest exponent is positive and maximal, in space, at the boundaries between different solutions upon integrating for long enough time. Again, the use of a metric is needed if more precise and general information is to be obtained.

Global metric analysis for Lyapunov exponents and Floquet multipliers, along the lines of contraction analysis, may lead to significant developments in global numerical analysis of nonlinear systems.

5 Interaction Between Synchronized Groups and Time-Delayed Teleoperation

5.1 Time-Delayed Communications

In some cases, communications between different systems involve non-negligible time-delays, as e.g. when using the Internet. Inspired by the use of passivity [2] and wave (scattering) variables [42] in force-reflecting teleoperation, [30] performed a contraction analysis of the effect of time-delayed transmission channels. As we now discuss, similar results can be obtained using simplified forms of transmitted variables [60], an approach we then apply to group synchronization with communication delays.

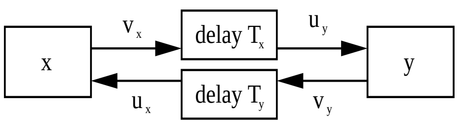

Fig. 9. Two interacting systems with delayed communication

Consider two interacting systems (Figure 9),

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{G}_x \tau_x$$
$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, t) + \mathbf{G}_y \tau_y$$

where $\mathbf{G}_x$, $\mathbf{G}_y$ are constant matrices and τ_x, τ_y have the same dimension. Define the variables to be transmitted as

$$\mathbf{u}_x = \mathbf{G}_x^T \mathbf{x} \ + \ k_x \tau_x \qquad \mathbf{v}_x = \mathbf{G}_x^T \mathbf{x}$$
$$\mathbf{u}_y = \mathbf{G}_y^T \mathbf{y} \ + \ k_y \tau_y \qquad \mathbf{v}_y = \mathbf{G}_y^T \mathbf{y}$$

where k_x and k_y are strictly positive constants. Because of time-delays, we have

$$\mathbf{u}_x(t) \ = \ \mathbf{v}_y(t - T_y) \qquad \mathbf{u}_y(t) \ = \ \mathbf{v}_x(t - T_x)$$

Now consider a differential length similar to that defined in [30]

$$V = \frac{k_x}{2} \delta\mathbf{x}^T \delta\mathbf{x} \ + \ \frac{k_y}{2} \delta\mathbf{y}^T \delta\mathbf{y} \ + \frac{1}{2} \ V_e$$

where

$$\begin{aligned} V_e &= \int_{t-T_x}^{t} \delta\mathbf{v}_x^T \delta\mathbf{v}_x d\tau \ + \ \int_{t-T_y}^{t} \delta\mathbf{v}_y^T \delta\mathbf{v}_y d\tau \\ &= -2 \int_0^t (\ k_x \delta\mathbf{x}^T \mathbf{G}_x \delta\tau_x \ + \ k_y \delta\mathbf{y}^T \mathbf{G}_y \delta\tau_y \ ) \ d\tau \\ &\quad - \int_0^t (\ k_x^2 \delta\tau_x^T \delta\tau_x \ + \ k_y^2 \delta\tau_y^T \delta\tau_y \ ) \ d\tau \end{aligned}$$

This yields

$$\dot{V} = k_x \delta\mathbf{x}^T \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \delta\mathbf{x} + k_y \delta\mathbf{y}^T \frac{\partial \mathbf{g}}{\partial \mathbf{y}} \delta\mathbf{y} - \frac{k_x^2}{2} \delta\tau_x^T \delta\tau_x - \frac{k_y^2}{2} \delta\tau_y^T \delta\tau_y$$

Applying Barbalat's lemma [50] shows that $\delta\mathbf{x}$ and $\delta\mathbf{y}$ will both tend to zero asymptotically for contracting $\mathbf{f}$ and $\mathbf{g}$. This implies that contraction can be preserved through time-delayed interactions. In the case that both $\mathbf{f}$ and $\mathbf{g}$ are time-invariant, the whole system tends towards to an equilibrium point, which is independent of the explicit values of the delays [49]. The result can be extended directly to study more general connections between groups, such as bidirectional meshes or webs of arbitrary size, and parallel unidirectional rings of arbitrary length. Inputs to the overall system can be provided through any of the subgroup dynamics.

It will not have escaped the reader that the whole system's dynamics is in fact equivalent to

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \ + \ \frac{1}{k_x} \ \mathbf{G}_x \ (\ \mathbf{G}_y^T \mathbf{y}(t - T_y) - \mathbf{G}_x^T \mathbf{x}(t) \ )$$
$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, t) \ + \ \frac{1}{k_y} \ \mathbf{G}_y \ (\ \mathbf{G}_x^T \mathbf{x}(t - T_x) - \mathbf{G}_y^T \mathbf{y}(t) \ )$$

i.e., to diffusion-like time-delayed interactions. Assuming that $\mathbf{x}$ and $\mathbf{y}$ have the same dimension, and choosing $\mathbf{G}_x = \mathbf{G}_y = \mathbf{G}$, we get the equations of two coupled systems with standard diffusion couplings

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \ + \ \frac{1}{k_x} \ \mathbf{G}\mathbf{G}^T \ (\ \mathbf{y}(t - T_y) - \mathbf{x}(t) \ )$$
$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, t) \ + \ \frac{1}{k_y} \ \mathbf{G}\mathbf{G}^T \ (\ \mathbf{x}(t - T_x) - \mathbf{y}(t) \ )$$

Thus, *contraction as a generalized form of stability is preserved through time-delayed diffusion couplings.*

This conclusion does not contradict the well-known fact in teleoperation, that even small time-delays in feedback PD controllers may create stability problems for coupled second-order systems [41,42], which motivates approaches based on passivity and wave variables. In fact, a key condition for contraction to be preserved is that the diffusion coupling gains be positive semi-definite (or positive definite) *under the same metric*. Indeed consider for instance (with $b > 0,\ \omega > 0$)

$$\ddot{x} + b\dot{x} + \omega^2 x = k_d(\dot{y}(t-T_2) - \dot{x}(t)) + k_p(y(t-T_2) - x(t))$$
$$\ddot{y} + b\dot{y} + \omega^2 y = k_d(\dot{x}(t-T_1) - \dot{y}(t)) + k_p(x(t-T_1) - y(t))$$

If $T_1, T_2 = 0$, partial contraction analysis shows that x and y converge together exponentially regardless of initial conditions, which makes the origin a stable equilibrium point. If $T_1, T_2 > 0$, we perform a coordinate transformation and get new equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \omega x_2 - b x_1 \\ -\omega x_1 \end{bmatrix} + \mathbf{K}(\begin{bmatrix} y_1(t-T_2) \\ y_2(t-T_2) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix})$$
$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \omega y_2 - b y_1 \\ -\omega y_1 \end{bmatrix} + \mathbf{K}(\begin{bmatrix} x_1(t-T_1) \\ x_2(t-T_1) \end{bmatrix} - \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix})$$

where $\mathbf{f} = \begin{bmatrix} \omega x_2 - b x_1 \\ -\omega x_1 \end{bmatrix}$ is contracting with an identity metric [58], but the transformed coupling gain $\mathbf{K} = \begin{bmatrix} k_d & 0 \\ \frac{k_p}{\omega} & 0 \end{bmatrix}$ is no longer positive semi-definite, for any $k_p \neq 0$. Contraction cannot be guaranteed in this case, and coupled systems turn out to be unstable for large enough delays (Figure 10). This unstability mechanism is actually very similar to that of Smale's model [52,58] (which we alluded to in section 3.1) in which two or more identical biological cells, inert by themselves, tend to self-excited oscillations through diffusion interactions. In both cases, the unstability is caused by a non-identity metric, which makes the transformed coupling gains lose positive semi-definiteness. Note that the relative simplicity with which these two subtle phenomena can be analyzed makes fundamental use of the notion of a *metric*, central to contraction theory. It also points out further directions for designing feedback controllers robust to time-delays.

Synchronization can also be preserved through time-delayed diffusion couplings. Consider a power-based leader-followers network

$$\dot{\mathbf{x}}_i = \sum_{j \in \mathcal{N}_i} \mathbf{K}_{ji}\ (\mathbf{x}_j(t - T_{ji}) - \mathbf{x}_i(t)) + \gamma_i\ \mathbf{K}_{0i}\ (\mathbf{x}_0 - \mathbf{x}_i)$$

where $i = 1, \ldots, n+m$, $\mathbf{x}_0$ is constant, and time-delays are involved in communication channels between the followers. For notational simplicity, assume there is only one time-delayed link, which connects nodes n and $n+1$, and separates the whole network into two subgroups $\mathbf{x} = (\mathbf{x}_1\ ,\ \ldots\ ,\ \mathbf{x}_n)$ and $\mathbf{y} = (\mathbf{x}_{n+1}\ ,\ \ldots\ ,\ \mathbf{x}_{n+m})$. The leader $\mathbf{x}_0$ does not connect directly to the nodes in the

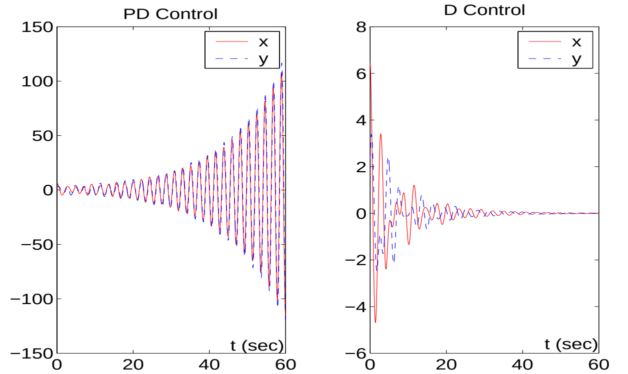

Fig. 10. Simulation of two coupled mass-spring-damper systems with PD control (left) and D control (right). Parameters are $b = 0.5$, $\omega^2 = 5$, $T_1 = 2s$, $T_2 = 4s$, $k_d = 1$, $k_p = 5$(left)/0(right). Initial conditions, chosen randomly, are identical for the two plots.

group $\mathbf{y}$. Thus, $\dot{V}$ now contains

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \ = \ -\mathbf{L}^1_{\mathbf{K}} - \mathbf{I}^n_{\gamma_i \mathbf{K}_{0is}} < 0$$

and

$$\frac{\partial \mathbf{g}}{\partial \mathbf{y}} \ = \ -\mathbf{L}^2_{\mathbf{K}}$$

Applying Barbalat's lemma, $\dot{V}$ will tend to zero, which implies that $\delta\mathbf{x}, \delta\tau_x$ and $\delta\tau_y$ will tend to zero, while $\delta\mathbf{y}$ will verify $\delta\mathbf{x}_{n+1} = \ldots = \delta\mathbf{x}_{n+m}$. Since

$$\mathbf{G}_y^T \delta\mathbf{y} \ = \ \delta\mathbf{u}_y - k_y \delta\tau_y$$

it can be proved directly that $\delta\mathbf{x}_{n+1}$, and therefore the overall $\delta\mathbf{y}$, will tend to zero as well. All states $\mathbf{x}_i$ will converge to the common value $\mathbf{x}_0$ asymptotically, regardless of time-delays. The result can be extended easily to more complex cases.

Different simplifications of the original wave variable design can be made based on the same choice of V, yielding different qualitative properties. For instance, the transmitted signals can be defined as

$$\mathbf{u}_x = \mathbf{G}_x^T \mathbf{x} \ + \ k_x \tau_x \qquad \mathbf{v}_x = -k_x \tau_x$$
$$\mathbf{u}_y = \mathbf{G}_y^T \mathbf{y} \ + \ k_y \tau_y \qquad \mathbf{v}_y = -k_y \tau_y$$

which leads to

$$\dot{V} = k_x \delta\mathbf{x}^T \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \delta\mathbf{x} + k_y \delta\mathbf{y}^T \frac{\partial \mathbf{g}}{\partial \mathbf{y}} \delta\mathbf{y} - \frac{1}{2} \delta\mathbf{x}^T \mathbf{G}_x \mathbf{G}_x^T \delta\mathbf{x} - \frac{1}{2} \delta\mathbf{y}^T \mathbf{G}_y \mathbf{G}_y^T \delta\mathbf{y} \qquad (10)$$

and thus also preserves contraction through time-delayed connections. If $\mathbf{f}$ and $\mathbf{g}$ are both time-invariant, the whole system tends towards an equilibrium point, with

$$\mathbf{u}_x(\infty) \;=\; \mathbf{v}_y(\infty) \qquad \mathbf{u}_y(\infty) \;=\; \mathbf{v}_x(\infty)$$

at steady state, which immediately implies that

$$\mathbf{G}_x^T \; \mathbf{x}(\infty) \;=\; \mathbf{G}_y^T \; \mathbf{y}(\infty)$$

and, in the case that $\mathbf{G}_x = \mathbf{G}_y = \mathbf{G}$ with $\mathbf{G}$ of full rank, that $\mathbf{x}(\infty) = \mathbf{y}(\infty)$. Thus, contrary to the case of diffusion-like coupling, the remote tracking ability of wave variables is preserved. Finally note that if $\mathbf{G}_x$ or $\mathbf{G}_y$ has full rank, the corresponding subsytem needs only have upper bounded Jacobian rather than be contracting, since from (10) $\delta\mathbf{x}$ and $\delta\mathbf{y}$ will both tend to zero asymptotically for appropriate choices of gains.

Extensions to communication with time-varying delays [41,40,5] can be handled in a very similar way.

5.2 Mutual Pertubation

In a large population, groups may not communicate tightly. For instance, the signals between groups may not contain explicit state information, but only information on group synchronization. Consider two such groups, with coupled elements $\mathbf{x}_i$ $(i = 1, \ldots, n)$ and $\mathbf{y}_i$ $(i = 1, \ldots, m)$, respectively. Elements in the different groups could have different dynamics. In each group, we assume there is one (or a few) particular element, node 1 for instance, which has the ability to communicate with other groups (see Figure 11 as an illustration), with dynamics given as

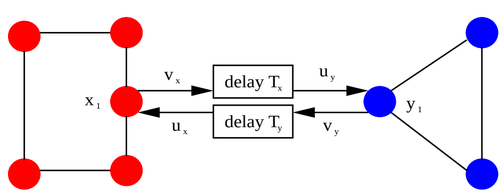

Fig. 11. Synchronized groups perturb each other through mutual communication.

$$\dot{\mathbf{x}}_1 = \mathbf{f}(\mathbf{x}_1, t) + \sum_{j \in \mathcal{N}_1} \mathbf{K}_{j1} \; (\mathbf{x}_j - \mathbf{x}_1) + \mathbf{G}_x \tau_x$$
$$\dot{\mathbf{y}}_1 = \mathbf{g}(\mathbf{y}_1, t) + \sum_{j \in \mathcal{M}_1} \mathbf{H}_{j1} \; (\mathbf{y}_j - \mathbf{y}_1) + \mathbf{G}_y \tau_y$$

where

$$\tau_x = \frac{1}{k_x} \; (\; \mathbf{G}_y^T \mathbf{p}_y(t - T_y) - \mathbf{G}_x^T \mathbf{p}_x(t) \;)$$
$$\tau_y = \frac{1}{k_y} \; (\; \mathbf{G}_x^T \mathbf{p}_x(t - T_x) - \mathbf{G}_y^T \mathbf{p}_y(t) \;)$$

and $\mathbf{p}_x, \mathbf{p}_y$ represent synchronization error in each group, as seen from its node 1,

$$\mathbf{p}_x = \sum_{j \in \mathcal{N}_1} \mathbf{K}_{j1} \; (\mathbf{x}_j - \mathbf{x}_1)$$
$$\mathbf{p}_y = \sum_{j \in \mathcal{M}_1} \mathbf{H}_{j1} \; (\mathbf{y}_j - \mathbf{y}_1)$$

Using a similar proof as in the last section, define a Lyapunov-like function

$$V = \frac{k_x}{2} \mathbf{x}^T \mathbf{L}_{\mathbf{K}} \mathbf{x} \; + \; \frac{k_y}{2} \mathbf{y}^T \mathbf{L}_{\mathbf{H}} \mathbf{y} \; + \; \frac{1}{2} \; (\int_{t-T_x}^{t} \mathbf{v}_x^T \mathbf{v}_x d\tau \; + \; \int_{t-T_y}^{t} \mathbf{v}_y^T \mathbf{v}_y d\tau \;)$$

with

$$\mathbf{v}_x = \mathbf{G}_x^T \mathbf{p}_x \qquad \mathbf{v}_y = \mathbf{G}_y^T \mathbf{p}_y$$

Then

$$\dot{V} \;=\; k_x \mathbf{x}^T \; (\; \mathbf{L}_{\mathbf{K}\mathbf{\Lambda}} - \mathbf{L}_{\mathbf{K}}^2 \;) \; \mathbf{x} \; + \; k_y \mathbf{y}^T \; (\; \mathbf{L}_{\mathbf{H}\mathbf{\Lambda}} - \mathbf{L}_{\mathbf{H}}^2 \;) \; \mathbf{y}$$

and it is negative semi-definite under conditions similar to those derived in Section 3.5. This implies that, for loosely-tied groups, synchronization depends only on each individual group's property, although a small disturbance in a single group may cause big uncertainty in the entire population, so that it may take a long time for the whole population to finally settle down. Again the result can be extended directly to more general connections between groups, such as bidirectional meshes or parallel unidirectional rings. Similar phenomena can be found e.g. in human psychology and macro-economics.

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