Stability and Robustness Analysis of Nonlinear Systems via Contraction Metrics and SOS Programming

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Abstract

A wide variety of stability and performance questions about linear dynamical systems can be reformulated as convex optimization problems involving linear matrix inequalities (LMIs). These techniques have been recently extended to nonlinear systems with polynomial or rational dynamics through the use of sum of squares (SOS) programming.

In this paper we further extend the class of systems that can be analyzed with convexity-based methods. We show how to analyze the robust stability properties of *uncertain* nonlinear systems with polynomial or rational dynamics, via contraction analysis and SOS programming. Since the existence of a global contraction metric is a sufficient condition for global stability of an autonomous system, we develop an algorithm for finding such contraction metrics using SOS programming. The search process is made computationally tractable by relaxing matrix definiteness constraints, the feasibility of which indicates the existence of a contraction metric, to SOS constraints on polynomial matrices. We illustrate our results through examples from the literature and show how our contraction-based approach offers advantages when compared with traditional Lyapunov analysis.

Key words: Robust stability, nonlinear systems, sum of squares, contraction analysis.

1 Introduction

Computational methods have become increasingly important in linear and nonlinear system analysis. One important example is the use of interior point methods to solve the linear matrix inequalities (LMIs) that arise in a variety of system stability and performance problems. Arguably, LMIs first appeared in systems theory in the 1890s with Lyapunov's characterization of stability of a linear system. Prior to the development of interior-point methods, a few types of LMIs could be solved analytically or by simple graphical criteria, but only within the last two decades have general LMIs become computationally tractable.

Originally, LMI-based stability and performance formulations applied to linear systems and the class of nonlinear systems representable as an interconnection of a linear system with bounded uncertainty blocks (e.g., Boyd et al. (1994)). Recently, Parrilo (2000) extended the LMI framework to stability and performance methods for nonlinear systems with polynomial or rational dynamics by using sum of squares (SOS) programming. In this work we extend the class of systems that can be treated within the SOS framework by showing how to analyze stability of *uncertain* nonlinear systems via SOS programming and contraction analysis techniques.

Contraction analysis is a stability theory for nonlinear systems where stability is defined incrementally between two arbitrary trajectories (Lohmiller and Slotine, 1998). The existence of a contraction metric for a nonlinear system ensures that a suitably defined distance between nearby trajectories is always decreasing. If a system is globally contracting, all trajectories converge exponentially to a single trajectory. One application of contraction analysis given in Wang and Slotine (2005) is its use in studying the synchronization of nonlinear coupled oscillators. Recent related work to contraction analysis can be found in Angeli (2002); Fromion, Monaco,

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and Normand-Cyrot (1996); Pogromsky, Santoboni, and Nijmeijer (2002); Nguyen and Egeland (2004); Pavlov, Pogromsky, van de Wouw, and Nijmeijer (2004); Aghannan and Rouchon (2003); Pavlov, van de Wouw, and Nijmeijer (2005). Conceptually, approaches closely related to contraction, although not based on differential analysis, can be traced back to Hartmann (1964) and even to Lewis (1949).

In the autonomous case, contraction analysis is closely related to Krasovskii's Theorem, since one can interpret the search for a contraction metric as the search for a Lyapunov function with a certain structure. We explain this connection further in Section 2. There are certain advantages to searching for a contraction metric instead of a generic Lyapunov function. In general, nonlinear systems with uncertain parameters can prove quite troublesome for Lyapunov techniques, since the uncertainty can change the location of the equilibrium point in complicated ways. This forces the use of parameter-dependent Lyapunov functions to prove stability for a range of the uncertain parameter values. However, in general it may be impossible to obtain any kind of closed form expression of the equilibria in terms of the parameters. This makes the direct parametrization of suitable Lyapunov functions difficult if not impossible.

Contraction analysis provides a framework for studying the stability behavior of uncertain nonlinear systems, which can eliminate some of the restrictions and problems encountered with traditional linearization techniques or Lyapunov methods. If a nominal system is contracting with respect to a contraction metric, it is often the case that the uncertain system with additive or multiplicative uncertainty is still contracting with respect to the same metric, even if the perturbation has changed the position of the equilibrium. This makes it possible to determine if the system is stable for ranges of values of an uncertain parameter without explicitly tracking how the uncertainty changes the location of the equilibrium.

To translate the theoretical discussion above into effective practical tools, it is necessary to have efficient computational methods to obtain contraction metrics numerically. Sum of squares (SOS) programming provides one such method. SOS programming is based on techniques that combine elements of computational algebra and convex optimization; see Prajna, Papachristodoulou, Seiler, and Parrilo (2004a) for a number of control-related applications and references to the literature. In this paper we show how SOS programming enables the search for contraction metrics for the class of autonomous nonlinear systems with polynomial dynamics. We further discuss how to use SOS methods to find bounds on the maximum allowable uncertainty such that the system remains contracting with respect to the same contraction metric as the unperturbed system.

The paper is organized as follows: in Section 2 we give background material on contraction analysis, Krasovskii's Theorem, and their relations to Lyapunov functions. Section 3 discusses sum of squares (SOS) programming. Section 4 presents an algorithm for computationally searching for contraction metrics for polynomial systems via SOS programming. We discuss why and give an example of how contraction analysis is useful for studying systems with uncertain dynamics in Section 5. Finally, in Section 6 we present our conclusions, and outline possible directions for future work.

2 Contraction Analysis

As mentioned in the introduction, contraction analysis is a stability theory for nonlinear systems analysis. It is an approach where stability is defined incrementally between two arbitrary trajectories, and it attempts to answer the question of whether the limiting behavior of a given dynamical system is independent of its initial conditions. This section summarizes the main elements of contraction analysis for autonomous systems. Additional discussion on non-autonomous systems can be found in Lohmiller and Slotine (1998). Related work and approaches closely related to contraction are summarized in the introduction.

We consider autonomous dynamical systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t)), \tag{1}$$

where $\mathbf{f}$ is a nonlinear vector field and $\mathbf{x}(t)$ is an n-dimensional state vector. In our analysis it is assumed that all quantities are real and smooth and thus that all required derivatives exist and are continuous. This assumption clearly holds for polynomial vector fields.

Under these assumptions, we can obtain the following differential relation from equation (1):

$$\delta\dot{\mathbf{x}}(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}(t)}(\mathbf{x}(t))\delta\mathbf{x}(t), \tag{2}$$

where $\delta\mathbf{x}(t)$ is an infinitesimal displacement at a fixed time. For notational convenience we write $\mathbf{x}$ for $\mathbf{x}(t)$ from here on, but in all calculations it should be noted that $\mathbf{x}$ is a function of time.

The infinitesimal squared distance between two trajectories is $\delta\mathbf{x}^T\delta\mathbf{x}$. Using (2), the following equation for the rate of change of the infinitesimal squared distance between two trajectories is obtained:

$$\frac{d}{dt}(\delta\mathbf{x}^T\delta\mathbf{x}) = 2\delta\mathbf{x}^T\delta\dot{\mathbf{x}} = 2\delta\mathbf{x}^T\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\delta\mathbf{x}. \tag{3}$$

If $\lambda_1(\mathbf{x})$ is the largest eigenvalue of the symmetric part of the Jacobian $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$, then it follows from (3) that

$\frac{d}{dt}(\delta\mathbf{x}^T\delta\mathbf{x}) \leq 2\lambda_1(\mathbf{x})\delta\mathbf{x}^T\delta\mathbf{x}$. Integrating both sides gives

$$||\delta\mathbf{x}|| \leq ||\delta\mathbf{x}_o||\, e^{\int_0^t \lambda_1(\mathbf{x})dt}. \tag{4}$$

If $\lambda_1(\mathbf{x})$ is uniformly strictly negative, i.e., $(\frac{\partial\mathbf{f}}{\partial\mathbf{x}} + \frac{\partial\mathbf{f}}{\partial\mathbf{x}}^T) < -\epsilon||\mathbf{x}||^2\ \forall\ \mathbf{x}$, it follows that any infinitesimal length $||\delta\mathbf{x}||$ converges exponentially to zero as time goes to infinity.

A more general definition of infinitesimal length can be given by

$$\delta\mathbf{x}^T\mathbf{M}(\mathbf{x},t)\delta\mathbf{x} \tag{5}$$

where $\mathbf{M}(\mathbf{x},t)$ is a symmetric, uniformly positive definite and continuously differentiable metric (formally, this defines a Riemannian manifold). In this work, we only consider metrics $\mathbf{M}(\mathbf{x})$ which do not explicitly depend on time. From the definition of infinitesimal length given in (5), the equation for its rate of change is

$$\frac{d}{dt}(\delta\mathbf{x}^T\mathbf{M}\delta\mathbf{x}) = \delta\mathbf{x}^T\left(\frac{\partial\mathbf{f}}{\partial\mathbf{x}}^T\mathbf{M} + \mathbf{M}\frac{\partial\mathbf{f}}{\partial\mathbf{x}} + \dot{\mathbf{M}}\right)\delta\mathbf{x} \tag{6}$$

where $\mathbf{M}$ is shorthand notation for $\mathbf{M}(\mathbf{x})$ [1] .

If $\mathbf{M}(\mathbf{x})$ is a matrix function of $\mathbf{x}$, we use the notation $\mathbf{M}(\mathbf{x}) \succ 0$ to denote uniform positive definiteness, i.e., $\mathbf{M}(\mathbf{x}) \succeq \epsilon I$ for all $\mathbf{x}$ and some positive ϵ. Similarly $\mathbf{M}(\mathbf{x}) \prec 0$ indicates uniform negative definiteness.

Definition 1 *Given the n-dimensional autonomous system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, a contraction metric *is an $n \times n$ matrix $\mathbf{M}(\mathbf{x})$ that is uniformly positive definite and such that $\frac{\partial\mathbf{f}}{\partial\mathbf{x}}^T\mathbf{M}(\mathbf{x}) + \mathbf{M}(\mathbf{x})\frac{\partial\mathbf{f}}{\partial\mathbf{x}} + \dot{\mathbf{M}}$ is uniformly negative definite. If the system satisfies the stronger condition, $\frac{\partial\mathbf{f}}{\partial\mathbf{x}}^T\mathbf{M}(\mathbf{x}) + \mathbf{M}(\mathbf{x})\frac{\partial\mathbf{f}}{\partial\mathbf{x}} + \dot{\mathbf{M}} \prec -\beta\mathbf{M}(\mathbf{x})$ where $\beta > 0$ then the system is said to be exponentially contracting.*

As the following theorem shows, contraction metrics can be used to prove convergence to a single trajectory, and thus existence and/or uniquess of equilibria.

Theorem 1 *Consider the autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. If a contraction metric exists for the system over the entire state-space and a finite equilibrium exists, then this equilibrium is unique and all trajectories converge to this equilibrium. If the system is exponentially contracting, there exists a unique finite equilibrium, and all trajectories converge to this equilibrium.*

We outline the proof the theorem below. A concise proof of convergence of trajectories for an exponentially contracting system is presented in Aghannan and Rouchon (2003). Existence of a finite equilibrium in this case can be obtained by combining these results with the Contraction Mapping Theorem. First let $\mathbf{X}(\mathbf{x},t)$ be the flow associated with $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. That is,

$$\frac{d}{dt}\mathbf{X}(\mathbf{x},t) = \mathbf{f}(\mathbf{X}(\mathbf{x},t)), \qquad \mathbf{X}(\mathbf{x},0) = \mathbf{x}.$$

Next, consider any two points $\mathbf{x_0}$ and $\mathbf{x_1}$. If a global contraction metric $\mathbf{M}(\mathbf{x})$ exists that satisfies the conditions for exponential contraction in Definition 1, then by the explicit estimates in Theorem 2 of Aghannan and Rouchon (2003),

$$d_M(\mathbf{X}(\mathbf{x_0},t),\mathbf{X}(\mathbf{x_1},t)) \leq e^{(-\beta/2)t}d_M(\mathbf{x_0},\mathbf{x_1}),$$

where d_M is the geodesic distance associated to the metric $\mathbf{M}(\mathbf{x})$. In this case, for any fixed positive t_0 the mapping $\mathbf{x} \mapsto \mathbf{X}(\mathbf{x},t_0)$ is a strict contraction, and thus by the Contraction Mapping Theorem (e.g. Kolmogorov and Fomin (1970)), there is a unique fixed point.

Convergence of trajectories for a contracting, but not exponentially contracting system can be proven by using the positive definiteness of $\mathbf{M}(\mathbf{x})$ and the negative definiteness of $\frac{\partial\mathbf{f}}{\partial\mathbf{x}}^T\mathbf{M}(\mathbf{x}) + \mathbf{M}(\mathbf{x})\frac{\partial\mathbf{f}}{\partial\mathbf{x}} + \dot{\mathbf{M}}$ to obtain the relation

$$\frac{d}{dt}d_M(t) \leq -\epsilon \cdot d_E(t), \qquad \epsilon > 0,$$

where $d_M(t) := d_M(\mathbf{X}(\mathbf{x_0},t),\mathbf{X}(\mathbf{x_1},t))$ is the distance between trajectories in the metric $\mathbf{M}(\mathbf{x})$ and $d_E(t)$ is the distance between trajectories in the Euclidean metric. Since $d_E(t)$ is always nonnegative, $d_M(t)$ is nonincreasing, and by an argument similar to that in LaSalle's Theorem, it follows that the trajectories must converge to the set where $d_E(t)$ vanishes, i.e., their Euclidean distance goes to zero. If we have convergence of trajectories for a system and an equilibrium of that system exists, then it is necessarily unique.

2.1 Relations to Krasovskii's Method and Lyapunov Functions

For a *constant* metric $\mathbf{M}(\mathbf{x}) = M$, this type of analysis reduces to Krasovskii's Method (Khalil, 1992). If additionally the system dynamics are linear, then the conditions above reduce to those in standard Lyapunov analysis. Lyapunov's indirect method shows that the system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$ is stable if and only if there exists a positive definite matrix M such that $A^TM + MA \prec 0$.

Sometimes it is useful to consider the Lyapunov-like function

$$V(\mathbf{x}) = \mathbf{f}^T(\mathbf{x})\mathbf{M}(\mathbf{x})\mathbf{f}(\mathbf{x}),$$

where $\mathbf{M}(\mathbf{x})$ is a contraction metric for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. If an equilibrium for this system exists and the

[1] Note that $\dot{\mathbf{M}}_{ij} = \dot{\mathbf{M}}_{ij}(\mathbf{x}) = \left(\frac{\partial\mathbf{M_{ij}}(\mathbf{x})}{\partial\mathbf{x}}\right)^T\mathbf{f}(\mathbf{x})$.

function $V(\mathbf{x})$ is radially unbounded, this system is globally stable and all trajectories converge to this equilibrium point. If $\mathbf{M}(\mathbf{x})$ is a contraction metric, $V(\mathbf{x})$ is positive for all points in the state space where $\mathbf{f}(\mathbf{x}) \neq 0$,

$$\dot{V}(\mathbf{x}) = \mathbf{f}^T(\mathbf{x})(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{M}(\mathbf{x}) + \mathbf{M}(\mathbf{x})\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \dot{\mathbf{M}})\mathbf{f}(\mathbf{x}) < 0$$

for all points where $\mathbf{f}(\mathbf{x}) \neq 0$, and Lyapunov's Theorem shows the system is globally stable. [2] When dealing with systems with uncertainty it is often convenient to search for a contraction metric $\mathbf{M}(\mathbf{x})$ instead of a Lyapunov function of a fixed form. We further expand on these reasons in Section 5.

The problem of finding a time-invariant contraction metric thus reduces to the search for a matrix function $\mathbf{M}(\mathbf{x})$. SOS techniques will provide a computationally convenient approach to this task.

3 Sum of Squares Programming

The main computational difficulty of problems involving multivariate nonnegativity conditions is the lack of efficient computational methods. A convenient approach for this, originally introduced in Parrilo (2000), is the use of sum of squares relaxations as a suitable replacement for nonnegativity. We present below the basic elements of these techniques.

A multivariate polynomial $p(x_1, x_2, ..., x_n) = p(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$, where $\mathbb{R}[\mathbf{x}]$ is the set of all polynomials in $x_1, ..., x_n$, is a sum of squares (SOS) if there exist polynomials $f_1(\mathbf{x}), ..., f_m(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ such that

$$p(\mathbf{x}) = \sum_{i=1}^{m} f_i^2(\mathbf{x}). \qquad (7)$$

The existence of an SOS representation for a given polynomial is a sufficient condition for its global nonnegativity, i.e., equation (7) implies that $p(\mathbf{x}) \geq 0 \; \forall \, \mathbf{x} \in \mathbb{R}^n$. The SOS condition (7) is equivalent to the existence of a positive semidefinite matrix Q such that

$$p(\mathbf{x}) = Z^T(\mathbf{x}) Q Z(\mathbf{x}) \qquad (8)$$

where $Z(\mathbf{x})$ is a vector of monomials of degree less than or equal to $\deg(p)/2$. This equivalence of descriptions makes finding an SOS decomposition a computationally tractable procedure because, as shown by Parrilo (2000), finding a symmetric positive semidefinite Q subject to the affine constraint (8) is a semidefinite programming problem. There has recently been much interest in SOS programming and SOS optimization as these techniques provide convex relaxations for various computationally hard optimization and control problems; see e.g. Parrilo (2000, 2003); Lasserre (2001); Prajna et al. (2004a) and the volume Henrion and Garulli (2005).

An SOS decomposition provides an explicit certificate of the nonnegativity of a scalar polynomial for all values of the indeterminates. In order to design an algorithmic procedure to use SOS techniques to search for contraction metrics with polynomial entries, we need to introduce a similar idea to ensure that a polynomial matrix is positive definite for every value of the indeterminates. A natural definition is as follows:

Definition 2 (Gatermann and Parrilo (2004)) *Consider a symmetric matrix with polynomial entries $\mathbf{S}(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]^{m \times m}$, and let $\mathbf{y} = [y_1, \ldots, y_m]^T$ be a vector of new indeterminates. Then $\mathbf{S}(\mathbf{x})$ is a* sum of squares matrix *if the scalar polynomial $\mathbf{y}^T\mathbf{S}(\mathbf{x})\mathbf{y}$ is a sum of squares in $\mathbb{R}[\mathbf{x}, \mathbf{y}]$.*

For notational convenience, we define a stricter notion:

Definition 3 *A matrix $\mathbf{S}(\mathbf{x})$ is* a strict SOS matrix *if $\mathbf{S}(\mathbf{x}) - \epsilon \mathbf{I}$ is an SOS matrix for some $\epsilon > 0$.*

Thus, a strict SOS matrix is a matrix with polynomial entries that is strictly positive definite for every value of the indeterminates. An equivalent definition of an SOS matrix can be given in terms of the existence of a polynomial factorization: $\mathbf{S}(\mathbf{x})$ is an SOS matrix if and only if it can be decomposed as $\mathbf{S}(\mathbf{x}) = \mathbf{T}(\mathbf{x})^T\mathbf{T}(\mathbf{x})$ where $\mathbf{T}(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]^{p \times m}$. SOS matrices have also been used recently by Hol and Scherer (2005) and Kojima (2003) to produce relaxations of polynomial optimization problems with matrix definiteness constraints.

4 Computational Search for Contraction Metrics via SOS Programming

As explained in Section 2, given a autonomous dynamical system, the conditions for global contraction (and thus stability of a single equilibrium point) are given by a pair of matrix inequalities. For systems with polynomial metrics and polynomial dynamics, relaxing the matrix definiteness conditions of Theorem 1 to SOS matrix-based tests makes this search computationally tractable. More specifically, the matrix definiteness constraints on $\mathbf{M}(\mathbf{x})$ (and $\mathbf{R}(\mathbf{x})$) can be relaxed to SOS matrix constraints by changing the inequality $\mathbf{M}(\mathbf{x}) - \epsilon\mathbf{I} \succeq 0$ to the weaker condition that $\mathbf{M}(\mathbf{x})$ be a strict SOS matrix. We formalize this as follows:

Lemma 1 *If there exist strict SOS matrices $\mathbf{M}(\mathbf{x})$ and $-\mathbf{R}(\mathbf{x}) = -(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T\mathbf{M} + \mathbf{M}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \dot{\mathbf{M}})$, then all trajectories of the autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ converge to a finite equilibrium point.*

[2] If the vector field $||\mathbf{f}(\mathbf{x})|| \to \infty$ as $||\mathbf{x}|| \to \infty$, then $V(x)$ is radially unbounded.

This lemma can easily be extended to contraction with convergence rate β, by considering instead the expression $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{M} + \mathbf{M}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \dot{\mathbf{M}} + \beta\mathbf{M}$.

Using the characterization in Lemma 1, we can use SOS optimization to algorithmically search for time-invariant contraction metrics for nonlinear systems with polynomial dynamics. After relaxing the matrix definiteness constraints to SOS, the corresponding feasibility problem can then be formulated as finding $\mathbf{M}(\mathbf{x})$ and $-\mathbf{R}(\mathbf{x})$ that are strict SOS matrices. The detailed steps in the algorithmic search of contraction metrics for systems with polynomial dynamics are as follows:

(1) Choose the degree of the polynomials in the contraction metric, and write an affine parametrization of symmetric matrices of that degree. For instance, if the degree is two and the dynamical system is two-dimensional, the general form of $\mathbf{M}(\mathbf{x})$ is
$$\begin{bmatrix} \sum a_{ij}x_1^i x_2^j & \sum b_{ij}x_1^i x_2^j \\ \sum b_{ij}x_1^i x_2^j & \sum c_{ij}x_1^i x_2^j \end{bmatrix},$$
where $0 \leq i+j \leq 2$ and a_{ij}, b_{ij}, and c_{ij} are unknown coefficients.
(2) Calculate $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ and define $\mathbf{R}(\mathbf{x}) \triangleq \frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{M} + \mathbf{M}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \dot{\mathbf{M}}$. $\mathbf{R}(\mathbf{x})$ is a symmetric matrix with entries that depend affinely on the same unknown coefficients a_{ij}, b_{ij}, and c_{ij}.
(3) Impose strict SOS constraints on the scalar polynomials $\mathbf{y}^T\mathbf{M}(\mathbf{x})\mathbf{y}$ and $-\mathbf{y}^T R(\mathbf{x})\mathbf{y}$, and solve the associated SOS feasibility problem. If a solution exists, the SOS solver will find values for the unknown coefficients which satisfy the constraints.
(4) Use the obtained coefficients a_{ij}, b_{ij}, c_{ij} to construct the contraction metric $\mathbf{M}(\mathbf{x})$ and the corresponding $\mathbf{R}(\mathbf{x})$.
(5) If the system converges exponentially, an explicit lower bound on an exponential convergence rate may be found by using bisection to compute the largest β for which $\mathbf{M}(\mathbf{x}) \succ 0$ and $\mathbf{R}_\beta(\mathbf{x}) \triangleq \frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{M} + \mathbf{M}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \dot{\mathbf{M}} + \beta\mathbf{M} \prec 0$.

For the specific examples presented later in the paper, we have used SOSTOOLS, an SOS toolbox for MATLAB. developed by Prajna, Papachristodoulou, Seiler, and Parrilo (2004b) for the specification and solution of SOS programs. We present next an example of the application of this procedure. The system studied is a model of a jet engine with controller.

4.1 Example: Moore-Greitzer Jet Engine Model

The algorithm described was tested on the following dynamics, corresponding to the Moore-Greitzer model of a jet engine with stabilizing feedback operating in the no-stall mode (Krstić, Kanellakopoulos, and Kokotović (1995)). In this model, a desired no-stall equilibrium is translated to the origin. The state variables correspond to $\phi = \Phi - 1$, $\psi = \Psi - \Psi_{co} - 2$, where Φ is the mass flow, Ψ is the pressure rise and Ψ_{co} is a constant. The dynamic equations take the form:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 \\ 3\phi - \psi \end{bmatrix} \tag{9}$$

The only real-valued equilibrium of the system is $\phi = 0$, $\psi = 0$. This equilibrium is stable.

Table 1
Contraction metric search results of jet engine dynamics.

Deg. of polynomials in $\mathbf{M}(\mathbf{x})$	0	2	4	6
$M \succ 0$, and $R \prec 0$?	no	no	yes	yes
Converg. rate lower bound-β	n/a	n/a	0.78	1.45

The results of the algorithmic search for strict SOS matrices $\mathbf{M}(\mathbf{x})$ and $-\mathbf{R}(\mathbf{x})$ of various orders are given in Table 1. As shown there, for this system no contraction metric with polynomial entries of degree zero or two exists (this is certified by the solution of the dual optimization problem). Increasing the degree of $\mathbf{M}(\mathbf{x})$ to four or higher, we can easily find valid contraction metrics. Explicit lower bounds for the rate of exponential convergence of the trajectories, i.e., the largest value β for which $\mathbf{M}(\mathbf{x}) \succ 0$ and $\mathbf{R}_\beta(\mathbf{x}) \prec 0$, were $\beta = 0.78$ for the 4^{th} degree metric and $\beta = 1.45$ for the 6^{th} degree metric.

For this system, it is also possible to prove stability using standard Lyapunov analysis techniques. However, the contraction viewpoint is interesting because it allows us to study this system when there is parametric uncertainty in the plant dynamics or feedback equations. We elaborate on this in the next section.

5 Uncertainty Analysis with Contraction Metrics and SOS Programming

From the robust control perspective, one of the most appealing features of contraction analysis is that it provides a nice framework in which to study uncertain nonlinear systems where the parametric uncertainty changes the location of the equilibrium points. In general, standard Lyapunov analysis does not handle this situation particularly well, since the Lyapunov function must track the changes in the location of the steady-state solutions. This forces the use of parameter-dependent Lyapunov functions. However, in general it may be impossible to obtain any kind of closed form expression of the equilibria in terms of the parameters, thus complicating the direct parametrization of possible Lyapunov functions.

Much attention has been given to robust stability analysis of linear systems (e.g., Haddad and Bernstein (1995); Gahinet et al. (1996); Feron et al. (1996) Boyd, Ghaoui, Feron, and Balakrishnan (1994); Zhou, Doyle, and Glover (1996)). Less attention, however, has been paid to nonlinear systems with moving equilibria. If a linear model is being used to study a nonlinear system around an equilibrium point, changing the equilibrium of the nonlinear system necessitates relinearization around the new equilibrium. If the actual position of the equilibrium (in addition to its stability properties) depends on the uncertainty, it may be impossible to obtain a of closed-form expression for the equilibrium in terms of the uncertain parameters. Thus, parameterizing the linearization in terms of the uncertainty may not be an option.

Two earlier works addressing nonlinear systems with moving equilibria are by Michel and Wang (1993) and Andersson and Rantzer (1999). The approach in Michel and Wang (1993) is to consider systems described by the equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{h}(\mathbf{x}), \quad (10)$$

where $\mathbf{f}$ and $\mathbf{h}$ are continuously differentiable functions, and $\mathbf{h}(\mathbf{x})$ represents the uncertainties or perturbation terms. Given an exponentially stable equilibrium $\mathbf{x}_e$, Michel and Wang (1993) establish sufficient conditions for existence and local exponential stability of an equilibrium $\tilde{\mathbf{x}}_e$ for (10) with the property $|\mathbf{x}_e - \tilde{\mathbf{x}}_e| < \varepsilon$ where ε is sufficiently small. They do this by using the linearization of the system to produce Lyapunov functions.

Since the method in Michel and Wang (1993) is essentially based on a fixed Lyapunov function, it is more limited than our approach, and can prove stability only under quite conservative ranges of allowable uncertainty. A quantitative measure of this conservativeness will be given in Section 5.3 where we discuss the results of applying both techniques to a specific example.

The method in Andersson and Rantzer (1999) is to first linearize the dynamics around an equilibrium $\mathbf{x}_0$ that is a function of the uncertain parameter, i.e., $\mathbf{x}_0 = \mathbf{g}(\delta),\ \delta \in \Omega$ where Ω is the uncertainty set, and to use structured singular values to determine the eigenvalues of the linearized system $\frac{d\mathbf{z}}{dt} = A(\delta)\mathbf{z}$. In this approach the Jacobian $A(\delta)$ must be rational in δ. If $A(\delta)$ is marginally stable, no conclusions can be made about the stability of the nonlinear system.

The contraction analysis framework eliminates the need for linearization, and even the need to know the exact location of the equilibrium, in order to analyze stability robustness in uncertain nonlinear systems. In contrast to the Lyapunov situation, when certain classes of parametric uncertainty are added to the system, a contraction metric for the nominal system will often remain a contraction metric for the system with uncertainty, even if the perturbation has changed the equilibrium of the nonlinear system.

As noted in Section 2, if a global time-invariant contraction metric exists for an autonomous system, all trajectories converge to a unique equilibrium point, and we can produce a Lyapunov-like function of the form $V(\mathbf{x}) = \mathbf{f}(\mathbf{x})^T\mathbf{M}(\mathbf{x})\mathbf{f}(\mathbf{x})$. When a system contains parametric uncertainty, this formula yields the parameter-dependent function $V(\mathbf{x},\delta) = \mathbf{f}(\mathbf{x},\delta)^T\mathbf{M}(\mathbf{x})\mathbf{f}(\mathbf{x},\delta)$ for ranges of the uncertainty δ where the contraction metric for the nominal system is still a contraction metric for the system with perturbed dynamics. Thus, if a contraction metric can be found for the system under a range of uncertainty, we can construct a Lyapunov-like function which tracks the uncertainty for that range.

5.1 *Bounds on uncertainty range where the system remains contractive with respect to nominal metric.*

If the uncertainty δ enters the dynamics affinely (i.e., $\mathbf{f}(\mathbf{x}) = \mathbf{f_1}(\mathbf{x}) + \delta\,\mathbf{f_2}(\mathbf{x})$), we can use SOS programming to estimate the range of uncertainty under which the contraction metric for the nominal system is still a contraction metric for the perturbed system. To calculate this range, we can write an SOS program to minimize or maximize the amount of uncertainty allowed subject to the constraint $\frac{\partial \mathbf{f}_\delta}{\partial \mathbf{x}}^T\mathbf{M}_{nom} + \mathbf{M}_{nom}\frac{\partial \mathbf{f}_\delta}{\partial \mathbf{x}} + \dot{\mathbf{M}}_{nom}(\mathbf{f}_\delta(\mathbf{x})) \prec 0$, where $\mathbf{f}_\delta(\mathbf{x})$ are the dynamics for the system with parametric uncertainty and $\mathbf{M}_{nom}$ is the contraction metric for the nominal system.. The uncertainty bound is a decision variable in the SOS program and appears affinely in the constraint above in the $\frac{\partial \mathbf{f}_\delta}{\partial \mathbf{x}}$ and $\mathbf{f}_\delta(\mathbf{x})$ terms.

If we have more than one uncertain parameter in the system, we can also find polytopic inner approximations of the set of allowable uncertainties. For instance, for the case of two uncertain parameters, substitute the four points $(\delta_1, \delta_2) = (\pm\gamma, \pm\gamma)$ into $\mathbf{f}_{\delta=[\delta_1,\delta_2]^T}(\mathbf{x})$ and then maximize γ subject to the metric contractiveness constraint. The resulting values define a polytope over which stability is guaranteed.

5.2 *Largest symmetric uncertainty interval for which the system is contracting.*

Alternatively, rather than use the nominal $\mathbf{M}(\mathbf{x})$, we can find a metric that provides the largest symmetric uncertainty interval for which we can prove the uncertain system is contracting. If the scalar uncertainty δ enters the system dynamics affinely, we can perform this optimization as follows. First write $\mathbf{R}(\mathbf{x},\delta) = \mathbf{R_0}(\mathbf{x}) + \delta\mathbf{R_1}(\mathbf{x})$. To find the largest interval $(-\gamma,\gamma)$ such that for all δ that satisfy $-\gamma < \delta < \gamma$ the system is contracting, introduce the following constraints into an SOS program:

$$\mathbf{M}(\mathbf{x}) \succ 0,\ \mathbf{R_0}(\mathbf{x}) + \gamma\mathbf{R_1}(\mathbf{x}) \prec 0,\ \mathbf{R_0}(\mathbf{x}) - \gamma\mathbf{R_1}(\mathbf{x}) \prec 0.$$

Notice that γ multiplies the scalar decision coefficients a_i, b_i, and c_i in $\mathbf{R_1}(\mathbf{x})$ and thus we must use a bisection procedure to find the maximum value of γ.

Deg. of polys. in $\mathbf{M}(\mathbf{x})$	4	6
δ range	(-0.126,0.630)	(-0.070, 0.635)

Table 2
Range of perturbations for which the system (11) is contracting with respect to the nominal metric.

A similar scheme applies in the case of more uncertain affine parameters. In the two-dimensional case, to find the largest uncertainty square with width and height γ such that for all δ_1 and δ_2 that satisfy $-\gamma < \delta_1 < \gamma$ and $-\gamma < \delta_2 < \gamma$ the system is contracting, first write $\mathbf{R}(\mathbf{x}, \delta_1, \delta_2) = \mathbf{R_0}(\mathbf{x}) + \delta_1 \mathbf{R_1}(\mathbf{x}) + \delta_2 \mathbf{R_2}(\mathbf{x})$. Then, introduce the following constraints into an SOS program: $\mathbf{M}(\mathbf{x}) \succ 0$, $\mathbf{R}(\mathbf{x}, \pm\gamma, \pm\gamma) \prec 0$. As in the scalar uncertainty case, we use bisection procedure to find the maximum value of γ for which the constraints are feasible. In the case of a large number of uncertain parameters, standard relaxation and robust control techniques can be used to avoid an exponential number of constraints.

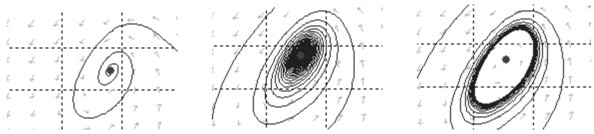
5.3 Ex: Moore-Greitzer Jet Engine with Uncertainty

As described above, SOS programming can be used to find ranges of uncertainty under which a system with uncertain perturbations is still contracting with respect to the original contraction metric. The contraction metric found for the deterministic system continues to be a metric for the perturbed system over a range of uncertainty even if the uncertainty shifts the equilibrium point and trajectories of the system. For the Moore-Greitzer jet engine model, the dynamics in (9) were perturbed by adding a constant term δ to the first equation:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 + \delta \\ 3\phi - \psi \end{bmatrix}. \qquad (11)$$

In Table 2 we display the ranges of δ for which the system was still contracting with the original metric for 4^{th} and 6^{th} degree contraction metrics. Notice that the range of allowable uncertainty is not symmetric.

When we instead optimized the contraction metric search to get the largest symmetric δ interval we obtained the results listed in Table 3. A 6^{th} degree contraction function yields the uncertainty range $|\delta| \leq 1.023$. Because a Hopf bifurcation occurs in this system at $\delta \approx 1.023$, making the system unstable for $\delta > 1.023$, we can conclude that the 6th degree contraction metric is the highest degree necessary to find the maximum range of uncertainty for which the system is contracting. The Hopf bifurcation is shown in Figure 1. Using the techniques in Michel and Wang (1993) we computed the allowable uncertainty range for system (11) to be $|\delta| \leq 5.1 \times 10^{-3}$. [3] The allowable range $|\delta| \leq 1.023$ computed via contraction analysis and SOS programming is much larger.



(a) $\delta = -0.5$ (b) $\delta = -1.01$ (c) $\delta = -1.1$

Fig. 1. Hopf bifurcation in uncertain jet dynamics.

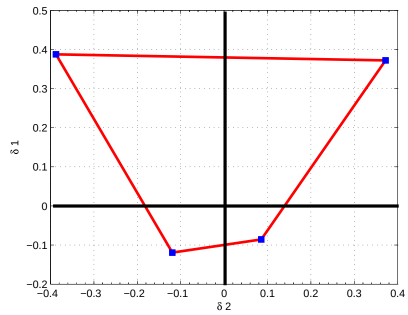


Fig. 2. Polytopic region of uncertainty over which system (9) with additive uncertainty in each equation is contracting with respect to nominal metric.

Deg. of polys. in $\mathbf{M}(\mathbf{x})$	4	6	8
length of allowed uncert. box	0.7093	0.7321	0.7346

Table 4
Symmetric range of perturbations for which system (9) with additive uncertainty in each equation is contracting.

For the case of multiple uncertain coefficients, consider the system that results from introducing an additive uncertainty δ_1 to the top equation and an additive uncertainty δ_2 to the bottom equation of the right hand side of (9). We computed an uncertainty polytope (shown in Figure 2) for which this system is guaranteed to be contracting with respect to the original metric. Alternatively, Table 4 shows the results of optimizing the contraction metric to find the largest uncertainty square with width and height γ such that for all δ_1 and δ_2 that satisfy $|\delta_1| \leq \gamma$ and $|\delta_2| \leq \gamma$, the system is contracting.

6 Conclusions

In this paper we have described how SOS programming enables an algorithmic search for contraction metrics for nonlinear systems with polynomial dynamics. We also have illustrated the results through several examples. These examples also exhibit how contraction analysis

[3] We calculated the other parameters in Assumption 1 of Michel and Wang (1993) as (in their notation): $\mathbf{h} = [\delta, \ \ 0]^T$, $|A^{-1}|_\infty = 1$, $|D\mathbf{h}(\mathbf{x}_e)|_\infty = 0$, $a = \frac{1}{30}$, and $|\mathbf{h}(\mathbf{x}_e)|_\infty = |\delta|$, where δ is the perturbation term in (11).

Degree of polynomials in $\mathbf{M}(\mathbf{x})$	4	6	8
δ range	$\lvert\delta\rvert \leq 0.938$	$\lvert\delta\rvert \leq 1.023$	$\lvert\delta\rvert \leq 1.023$

Table 3
Symmetric range of perturbations for which the uncertain system (11) is contracting.

sometimes offers advantages when compared with traditional Lyapunov analysis. The contraction approach is particularly useful in the analysis of nonlinear systems with uncertain parameters where the uncertainty changes the location of the equilibrium points of the system. In addition, a slightly modified version of the standard algorithmic search allows us to obtain a contraction metric that provides the largest uncertainty interval for which the system is provably contracting. For a globally contracting autonomous system, the system remains globally stable for parameters in the allowable uncertainty range.

Subjects of future research include a careful evaluation of how the computational resources needed by the algorithm scale with system size, as well as the benefits and limitations of this approach in the context of other nonlinear system analysis techniques.

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