THRUST NETWORK ANALYSIS: A NEW METHODOLOGY FOR THREE-DIMENSIONAL EQUILIBRIUM

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Summary

This paper presents a new methodology for generating compression-only vaulted surfaces and networks. The method finds possible funicular solutions under gravitational loading within a defined envelope. Using projective geometry, duality theory and linear optimisation, it provides a graphical and intuitive method, adopting the same advantages of techniques like graphic statics, but offering a viable extension to fully three-dimensional problems. The proposed method is applicable for the analysis of vaulted historical structures, specifically in unreinforced masonry, as well as the design of new vaulted structures. This paper introduces the method and shows examples of applications in both fields.

Keywords: Compression-only structures; Unreinforced masonry vaults; Funicular analysis; Thrust Network Analysis; Reciprocal diagrams; Form-finding; Lower-bound analysis; Structural optimization.

1. Introduction

Medieval vault builders created complex forms carefully balanced in compression. The structural properties of these sophisticated forms are still poorly understood because of a lack of appropriate analysis methods, i.e. methods relating stability and form. Understanding the mechanics of these vaulted structures leads to new insights for both analysis and design.

Thrust Line Analysis is a powerful graphical method for calculating the range of lower-bound equilibrium solutions of compression-only systems, such as unreinforced masonry structures (Fig. 1). It visualises the stability of these structures and suggests possible collapse mechanisms [1]. Unfortunately, thrust line analysis is primarily suitable for 2-D cases and this limitation has prevented it from being used for the assessment of complex 3-D structures. While numerical methods based on elastic solutions give one possible answer, they no longer suggest better form as was inherent to the more holistic graphical methods.
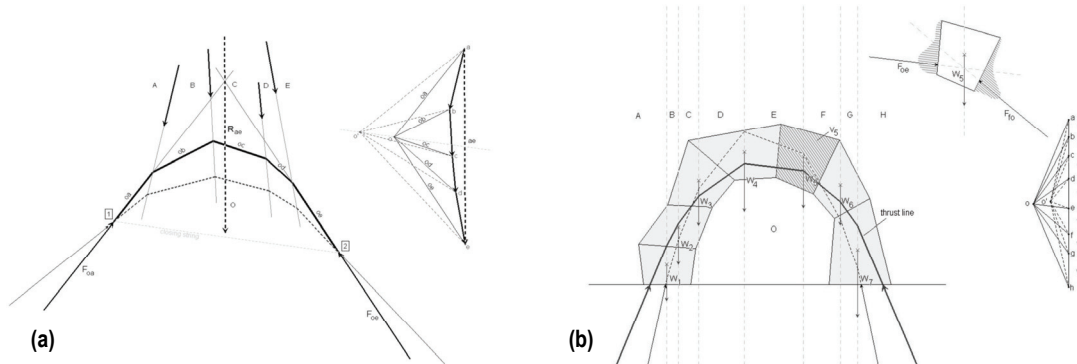

(a)   (b)

Fig. 1 (a) Two possible compression-only equilibrium shapes for a random set of loads, and (b) an interactive thrust-line application developed in [1]: the user can adapt the geometry by dragging control points and the structural feedback, in the form of a thrust-line, is updated in real-time. The magnitudes of the forces in the system are visualized in the accompanying funicular polygon (right).

There is a real need for tools to better understand and visualise the stability of compression-only structures, such as historic unreinforced masonry structures, as well as design tools that suggest better form. Both problems are related to finding axial force structures in equilibrium acting only in compression or tension. Currently, graphic statics provides a holistic analysis and design tool for two-dimensional structures. With today's availability of powerful virtual 3-D and parametric environments, the following question arises: can a fully three-dimensional version of thrust-line analysis provide the same freedom to explore the infinite equilibrium solutions for a certain loading condition?

2. Methodology

The *Thrust-Network Analysis* method presented in this paper is inspired by O'Dwyer's work on funicular analysis of vaulted masonry structures [2]. It is extended by adding the concept of duality between geometry and the in-plane internal forces of networks [3].

2.1 Reciprocal figures

The duality between the geometry of a network and its internal forces is an old concept, first explained by Maxwell [4]. He called this relationship *reciprocal* and defined it as follows: "*Two plane figures are reciprocal when they consist of an equal number of lines, so that corresponding lines in the two figures are parallel, and corresponding lines which converge to a point in one figure form a closed polygon in the other.*" This means that the equilibrium of a node in the first diagram is represented by a closed polygon in the second diagram and vice versa (Fig. 2). Graphic statics is based on this principle [5].
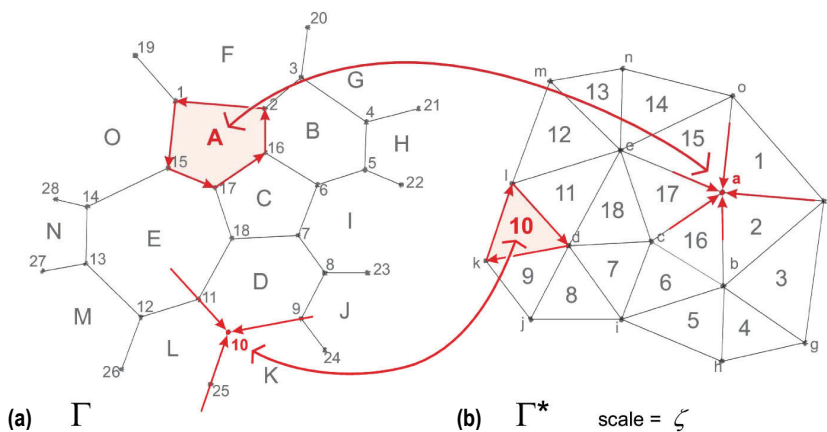

Fig.2 (a) The primal grid Γ and dual grid Γ^* are related by a reciprocal relationship. The equilibrium of a node in one of them is guaranteed by a closed polygon in the other and vice versa. The labelling uses Bow's notation [6]. This illustration demonstrates the idea on a piece of the networks in Fig. 3.

2.2 Assumptions

The proposed method produces funicular (compression-only) solutions for loading conditions where all loads are applied in the same direction, as is the case for gravitational loading. Since the solutions are compression-only this also means that the vaults can never curl back onto themselves, which would demand some elements to go into tension. The resulting three-dimensional networks can represent load paths throughout a structure. There is no constraint on the length of the branches or the planarity of the facets of the solution.

2.3 Thrust Network Analysis

Key elements in the proposed process are (1) force networks, representing possible forces in equilibrium in the structure; (2) interactive reciprocal diagrams, visualising the proportional relationship of the horizontal forces in the network and providing a high level of control for the user to manipulate the force distributions in the system; (3) the use of envelopes defining the solution space; and (4) linear optimisation, resulting in fast computation of results.

Thrust Network Analysis has been implemented using Matlab [7] and RhinoScripting in Rhinoceros [8]. The set-up of the program is explained in more detail below:
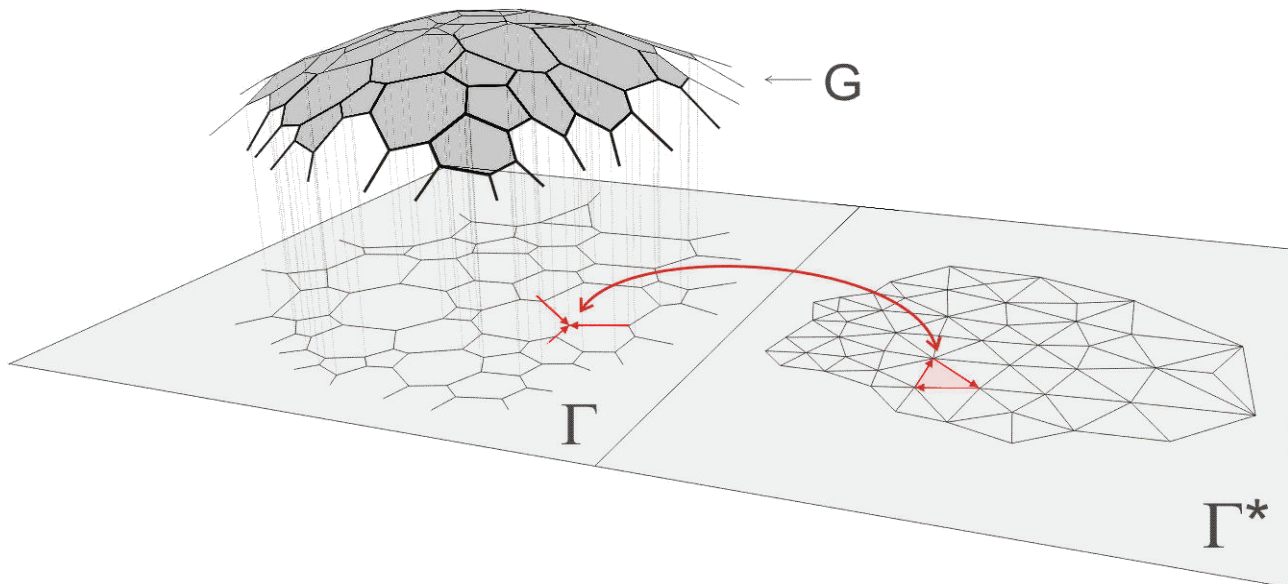

Fig. 2 Relationship between compression shell (G), its planar projection (primal grid Γ) and the reciprocal diagram (dual grid Γ) to determine equilibrium.*

(a) Defining a solution envelope:

The solutions must lie within given boundaries defined by an intrados and an extrados (Fig. 6b). These put height constraints on the nodes of the solution. These limits can be the design envelope or the actual vault geometry for the analysis of existing masonry vaults.

(b) Constructing the primal grid Γ:

In plan, a possible force pattern topology is constructed. This is the primal grid Γ in Fig. 3. The branches represent possible load paths throughout the structure. The loaded nodes represent the horizontal projections of centroids (cf. step c). These force patterns can be drawn by the user or generated automatically. The primal grid Γ is the horizontal projection of the final solution G.

(c) Attributing weights:

The weights attributed to the loaded nodes come from lumping the dead load of the 3-D tributary area around those nodes. In addition to self weight, loads such as asymmetric live loads can be applied.

(d) Generating the dual grid Γ*:

The dual grid Γ* is produced from the primal grid Γ according to Maxwell's definition of reciprocal figures: corresponding branches stay parallel and nodal equilibrium in the primal grid is guaranteed by closed polygons in the dual grid. This is being solved automatically by two consecutive linear optimisation problems with the constraints coming from Maxwell's definition. The details of this optimisation set-up will be elaborated on in future publications [9]. The applied loads do not appear in the dual grid because they disappear in the horizontal projection. Therefore, the dual grid has an unknown scale ζ since the relation between the primal and dual grid is true regardless of their relative scales (Fig. 4).
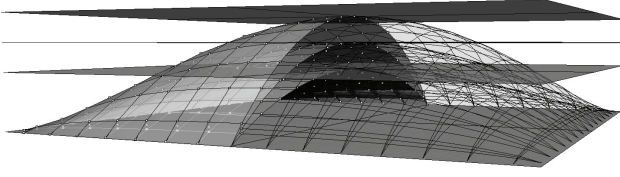
Fig. 4 For a regular square pillow shape with continuous supports on all edges and equal thrust in both main direction, decreasing the scale factor ζ of the dual grid means globally lower horizontal forces in the system and hence a deeper solution which is in equilibrium for the same set of applied loads.

(e) Updating the dual grid:

In the case of an indeterminate primal grid, i.e. a grid with nodes with a higher valency than 3, the user can manually change the force distribution by manipulating the dual grid (Fig. 5).
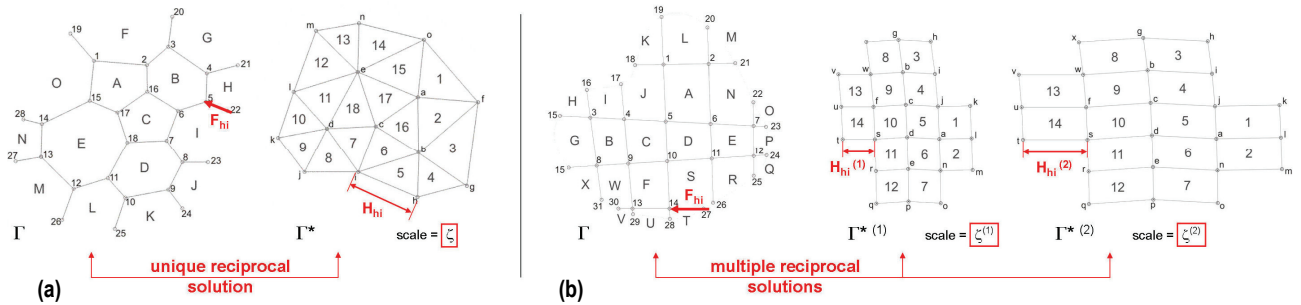

Fig. 5 For a determinate (i.e. 3-valent) primal grid, there is a unique relationship between primal and dual (a). For an indeterminate primal grid, multiple dual grids, which all satisfy Maxwell's definition, are possible (b).

(f) Solving for the result G:

Using the geometry of both primal (Γ) and dual (Γ^*) grid, the weights applied at the nodes and the boundary conditions, this problem can be solved using a one-step linear optimisation. We solve simultaneously for the nodal heights of G and the scale of the dual grid ζ^*. The horizontal components of the forces in the solution G can easily be found by measuring the lengths of the branches in the dual grid and multiplying them by the actual scale ζ^*.

2.3.2 Linear optimisation formulation

The first set of constraints comes from enforcing static equilibrium at all nodes (Fig. 6a). The vertical equilibrium of a typical internal node i gives

$$F_{ji}^{V} + F_{ki}^{V} + F_{li}^{V} = P_i \tag{1}$$

We describe (1) as a function of the horizontal components of the forces

$$F_{ji}^{H} \cdot \frac{(z_i - z_j)}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}} + F_{ki}^{H} \cdot \frac{(z_i - z_k)}{\sqrt{(x_i - x_k)^2 + (y_i - y_k)^2}} + F_{li}^{H} \cdot \frac{(z_i - z_l)}{\sqrt{(x_i - x_l)^2 + (y_i - y_l)^2}} = P_i \tag{2}$$

The lengths of branch ij in the primal and dual grids are defined respectively as $H_{i,j}$ and $H_{i,j}{}^*$. The horizontal components of the forces in the branches, F_{ij}^{H}, can be expressed as a function of the dual branch lengths $H_{i,j}{}^*$, measured from the dual grid Γ^* and multiplied by the as-yet unknown scale factor ζ.

$$F_{ji}^{H} = \zeta \cdot H_{i,j}^{*}, \quad F_{ki}^{H} = \zeta \cdot H_{i,k}^{*}, \quad F_{li}^{H} = \zeta \cdot H_{i,l}^{*} \tag{3}$$
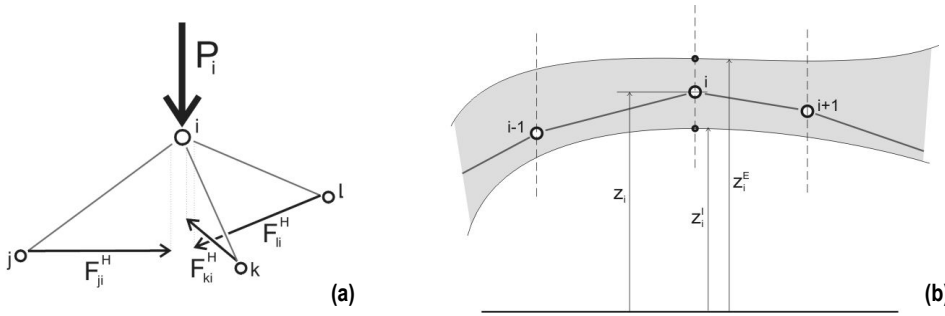
Fig. 6 The constraints come from (a) static equilibrium in every node under the applied loading and (b) the given boundaries, resulting in nodal height constraints.

Rearranging equation (2) and writing it as a function of the branch lengths in both grids using equations (3) gives

$$\left(\frac{H_{i,j}^*}{H_{i,j}} + \frac{H_{i,k}^*}{H_{i,k}} + \frac{H_{i,l}^*}{H_{i,l}}\right) \cdot z_i - \frac{H_{i,j}^*}{H_{i,j}} \cdot z_j - \frac{H_{i,k}^*}{H_{i,k}} \cdot z_k - \frac{H_{i,l}^*}{H_{i,l}} \cdot z_l - P_i \cdot r = 0 \tag{4}$$

where r is the inverse of the unknown scale of the dual grid, ζ. We can write equation (4) as

$$C_i \cdot z_i + C_j \cdot z_j + C_k \cdot z_k + C_l \cdot z_l - P_i \cdot r = 0 \tag{5}$$

The constants C_i of the linear function (5) are a function of the primal and dual branch lengths. The equilibrium constraints of the nodes can be written as a linear combination of z_i, the unknown nodal heights, and r. This emphasises the importance of using the information provided by the dual grid (3). Thanks to this insight, the nonlinear constraints (2) can be made linear by treating r as a variable. Note that, because lengths (absolute values) are used, this formulation guarantees that all solutions G will be compression-only.

A second set of constraints comes from the limits put on the nodal heights (Fig. 6b). We want the solutions to lie within the given boundaries defined by an intrados and an extrados.

$$z_i^I \le z_i \le z_i^E \tag{6}$$

Since we are interested in the range of possible solutions that fit within the given envelope, we want to minimise or maximise r (= 1/ ζ), resulting in respectively the shallowest or deepest solution still contained within the limits, for a chosen combination of primal and dual grid. This then becomes the objective function of the linear optimisation problem.

3. Applications for the analysis of vaulted masonry structures

Using the proposed methodology for the assessment of unreinforced masonry structures fits within the realm of lower-bound analysis. Put simply, if a compression-only network can be found that fits within the boundaries of a vault, then the vault will stand in compression. This is a powerful concept for understanding the stability and proximity to collapse of such structures. Additional reading on this topic can be found in Heyman [10], O'Dwyer [2], Boothby [11] and Block et al. [1].

The method is particularly appropriate for historic masonry structures because their self-weight is the dominant load. The range of possible equilibrium states, bounded by a minimum and maximum thrust, can be produced (Fig. 7a). We are interested in this range of thrust values because it provides the most useful characterization of the structural behaviour of the vault. The minimum (or passive) thrust state represents the least amount this vault will push sideways onto its neighbouring elements, just because of its self-weight and shape. The maximum (or active) state of thrust on the other hand represents the largest horizontal force this vault can provide. So, this value demonstrates how much horizontal pushing this vault can safely take from its neighbours.
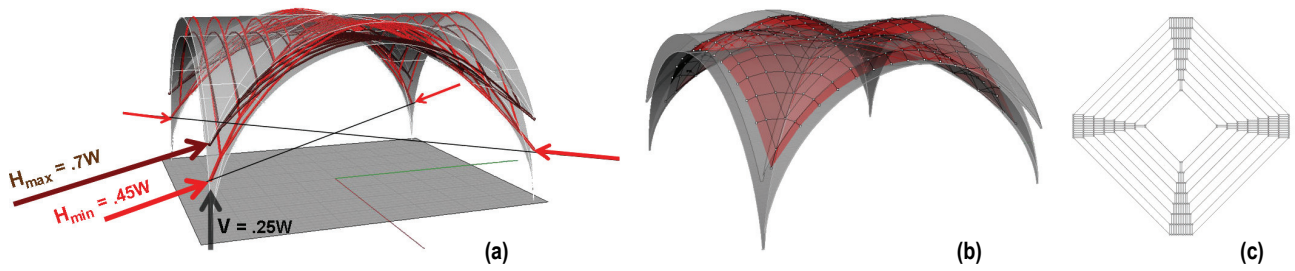
Fig. 7 (a) Possible thrust values for this groin vault range from 45% to 70% of its total weight. (b) 3-D web and rib action with the forces mainly spanning between the ribs as represented in the dual grid (c).

Figure 7b shows a solution with three-dimensional web action. The distribution of the horizontal components in the network is represented in its dual grid (Fig. 7c). Such a 3-D equilibrium analysis is valuable because it informs us very clearly about the stability of a vault. A forthcoming publication will show more detailed applications of this methodology to masonry vaults [12].

4. Applications for the design of compression-only structures

Figure 8 gives a series of compression-only solutions for a uniformly applied loading, starting from a regular rectangular grid. It shows the relationship between the dual grid and the corresponding solution. From the dual grids, the internal distribution of all horizontal forces in the networks can be understood in a glimpse, and since all dual grids are drawn at the same scale, the overall magnitude of the forces in the different solutions can be compared directly. Some special features are that force lines do not have to go through loaded nodes (8e-i) or that the edges can be freed, arching in space (8h-i).

The examples in Figure 8 show the potential of the method for design through a series of examples, which were possible because of the flexibility and intuitiveness of the method: forces can be redistributed internally within the network by 'tweaking' the dual grids (e.g. 8a versus 8b); more force can be attracted in primary force lines (e.g. 8e or 8f); and different boundary conditions can be explored (e.g. 8h).
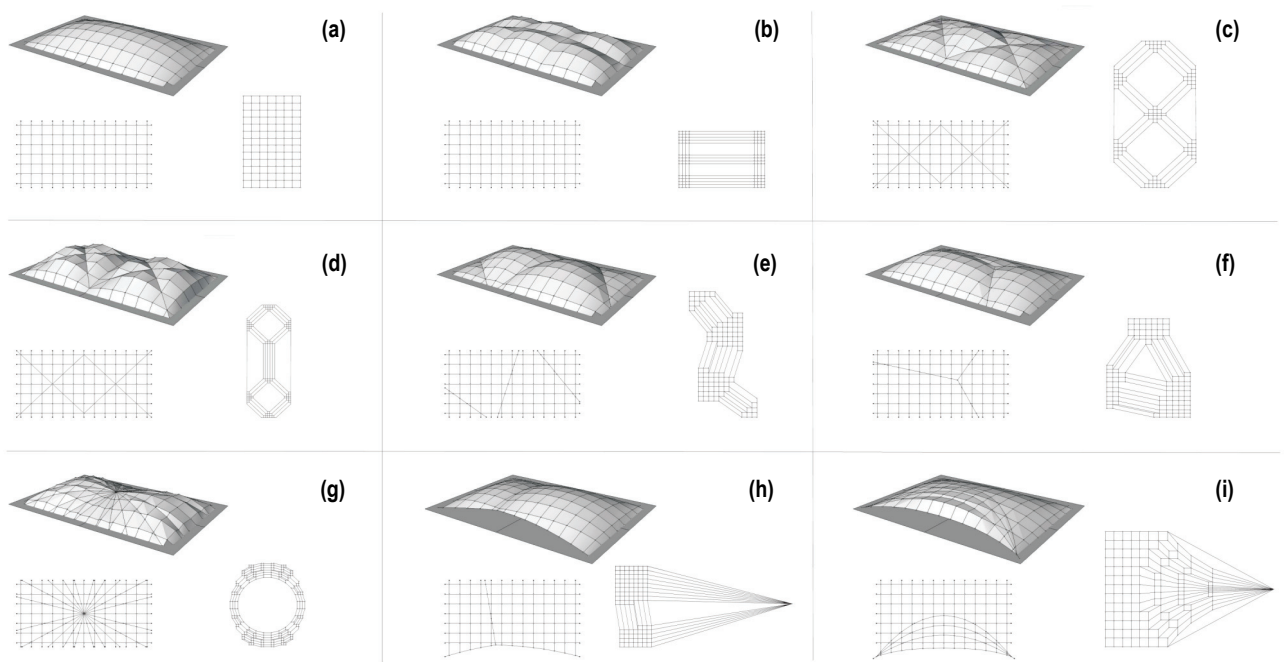

Fig. 8 A series of examples, starting from a regular rectangular grid, showing the relationship between the primal and dual grid and the corresponding solution.

The following example in Figure 9 shows a possible force pattern for a spherical dome inspired by structures by Pier Luigi Nervi. The generated dual grid (Fig. 9c), which represents the equilibrium of the forces in the primal grid (Fig. 9b), has a particularly beautiful shape and could become more interesting as force pattern than the original primal grid. Since the primal and dual grid have a dual relationship, this means that the primal grid could also be seen as the representation of the forces in the dual grid. So, notice that the force pattern with an oculus in Fig. 9c is a solution with approximately equal horizontal forces in all elements because all branch lengths in Fig. 9b are approximately the same.
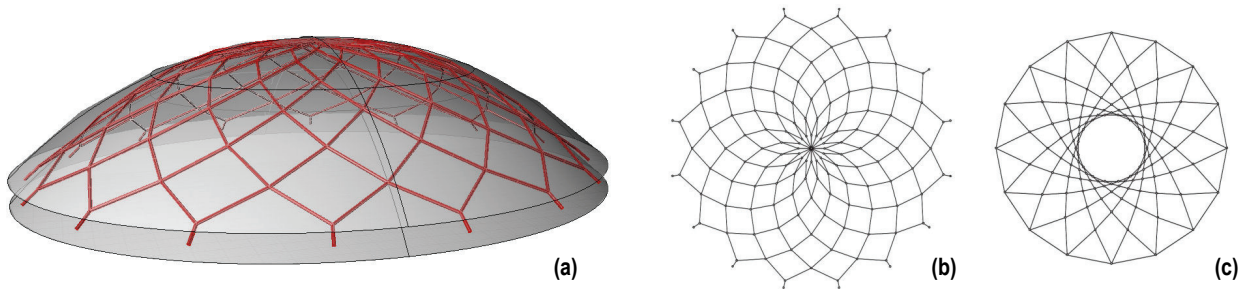

(a)   **(b)**   **(c)**

Fig. 9 For a spherical dome with an angle of embrace of 90°, a Nervi-inspired force pattern is chosen (b). The dual grid (c) represents the forces in the primal grid. (a) shows the deepest solution under its self-weight for this choice of primal and dual grid that fits within the dome's section, representing the minimum state of thrust of this dome.

This example shows that the generation of a dual grid not only allows to represent the forces in the system in a very visual manner, but that it also can surprise and start to inspire new designs. It becomes more interesting if both grids can freely be exchanged and changes can be made in either grid, influencing and updating the other.

5. Discussion and conclusions

This paper has proposed the Thrust Network Analysis method. It provides

- a viable three-dimensional extension for thrust-line analysis;
- a flexible, intuitive and interactive design tool for finding three-dimensional equilibrium of compression-only surfaces and systems; and
- an improved lower-bound method for the assessment of the stability of masonry vaults with complex geometries.

Key features are

- clear graphical representation of forces in the system (through the use of force diagrams, i.e. the dual grids);
- a high level of control, allowing the exploration of different possible equilibrium solutions; and
- fast solving times because of the formulation as a simple linear optimisation problem.

Currently, the number of elements in the network is limited by the implementation in Matlab and the user has to switch between programs. Future work includes going towards true interactivity and bi-directionality between both grids, implemented in a fully parametric environment, and the automatic generation of possible network topologies according to e.g. curvature, openings, support conditions or architectural preferences. We believe that this methodology has great potential for the use in both design and analysis of compression-only vaulted structures.

6. Acknowledgments

7. References

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