Lower-bound Analysis of Masonry Vaults

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ABSTRACT: This paper applies *Thrust-Network Analysis*, a three-dimensional computational method for obtaining lower-bound solutions of masonry vaults with complex geometries. The method extends thrust-line analysis to three-dimensional problems by finding equilibrium force networks within the vault's geometry, representing possible paths of the compression forces. Through two case studies, this paper demonstrates the potential of the method as a powerful tool for understanding, visualizing and exploring the equilibrium of compression-only structures. First, an analysis of a series of groin and quadripartite rib vaults investigates the interrelationship between different parameters and the range of possible equilibrium solutions of these vaults. A second case study analyzes the fan vaults of King's College Chapel in Cambridge, England.

1 INTRODUCTION

Medieval vault builders explored three-dimensional equilibrium, creating complex forms carefully balanced in compression. The structural properties of these sophisticated forms are still poorly understood because of a lack of appropriate analysis methods, i.e. methods relating stability and form.

The safety assessment of masonry structures is primarily a problem of geometry and stability, rather than of material stresses (Heyman 1995). Structures in unreinforced masonry work in compression, and the tensile capacity of the stone and mortar can be considered as negligible. These considerations then demand new tools in order to understand how these structures work and why they are able to stand for centuries. Linear elastic analysis using finite element methods is mainly concerned with stresses, and is not appropriate for historic structures in masonry.

1.1 *Lower-bound analysis and the safe theorem*

Heyman (1966) introduced the safe theorem for masonry structures, also known as the lower bound theorem. Put simply, a vault in unreinforced masonry will stand if a network of compression forces in equilibrium with the applied loads can be found which fits within the section of the structure. This solution is a possible lower-bound solution. Although we will never know how exactly the masonry vault is standing, this is not necessary. The safe theorem guarantees that as long as we can demonstrate one way that the structure could stand, i.e. could be in equilibrium with the external forces, then it is safe. This approach initially neglects sliding, which can be checked afterwards to ensure that sufficient friction exists. For further reading on lower-bound analysis for unreinforced masonry structures, see Heyman (1995) and Huerta (2001, 2004).

1.2 *Thrust-line analysis and graphic statics*

Thrust-line analysis is a particularly powerful method for understanding and exploring the range of lower-bound equilibrium solutions of compression-only systems, such as unreinforced masonry structures. It visualizes the relative stability of these structures by showing the paths of the resultant compressive forces throughout the structure and, for two-dimensional problems, suggests possible collapse mechanisms (Ochsendorf 2002, Block et al. 2006b). However, it is primarily a two-dimensional technique and is therefore most appropriate for the analysis of arches, flying buttresses or any structure which can be reduced to a sectional analysis.

Graphic statics can be used to compute thrust lines (Fig. 1). The main advantage of using graphical analysis is that the funicular polygon visually represents the forces in the system. Examples of graphic statics for fully three-dimensional problems were performed and demonstrated by Föppl (1892). These analyses were difficult to perform and limited to statically determinate problems.

2 EQUILIBRIUM ANALYSIS FOR VAULTS IN UNREINFORCED MASONRY

2.1 *Pseudo-3D equilibrium analysis methods*

In order to analyze three-dimensional structures using the same intuitive methods discussed above, the analyst typically must slice the structure, reducing it to a combination of two-dimensional problems. In this way, structural behavior is reduced to a combination of arch actions. This process obviously does not capture the full three-dimensional behavior of the structure and must rely heavily on the chosen discretization. These limitations have been the main reasons why thrust-line analysis has not been used extensively for the assessment of complex 3D structures.

Wolfe (1921) demonstrated how by slicing up a structure a global pseudo-3D analysis can be done by combining local thrust lines (Fig. 1). This methodology was entirely manual and quickly becomes tedious (Boothby 2001). Block et al. (2006b) proposed a method to produce models which contain the graphical construction but which are parametric and interactive, reducing the tedious iterative nature of traditional graphic statics. The models demonstrate the range of possible solutions and can be used to investigate the effects on the global stability of the vault of varying different parameters such as buttress thickness, arch thickness, level of fill, etc. Smars developed in his PhD thesis (2000) computational tools to perform such a pseudo-3D analysis in an automated fashion, starting from the actual measured geometry of the vaults.

2.2 *Fully 3D equilibrium analysis methods*

The main problem with three-dimensional equilibrium analysis for masonry vaults is that they are highly indeterminate structures. Antoni Gaudí's physical form-finding process for the church of the Colonia Guell can be used to explain this. First, before starting to construct a hanging string model, Gaudi had to decide on a suitable force pattern topology to represent the structural action of the vaults. Then, after choosing the structural logic, it is still challenging to control or even predict the final shape, since the equilibrium of each string influences the equilibrium of the entire network. It is a tedious, iterative process of adjusting and refining.

New form-finding programs which explore hanging models in the virtual world, based on dynamic relaxation, such as Kilian's CADenary tool have to deal with the same issues (Kilian and Ochsendorf 2005). It is very hard to control and predict how the final shape of the compression network will look like if local changes are being made or a string model is being assembled and hung under gravity. This

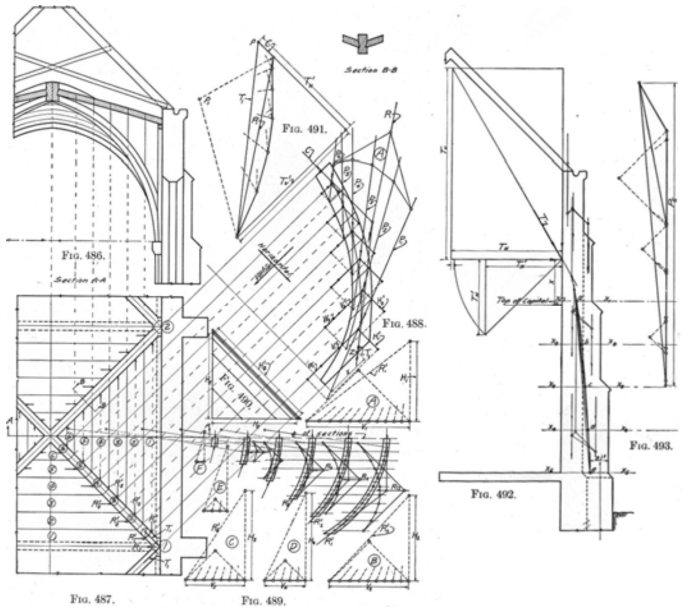


Figure 1. A pseudo-3D analysis of a gothic rib vault using graphic statics (Wolfe 1921). The web of the vault is cut into strips which are analyzed as 2D arches. The main ribs bring the forces from those arches down to the supports.

is true for both physical and virtual string models as for graphical methods.

In order to analyze a three-dimensional indeterminate system these unknowns need to be understood and controlled. This can be achieved by describing the problem as an optimization problem. The equilibrium requirements then are formulated as a set of constraints which have to be satisfied while optimizing a certain objective function.

O'Dwyer (1999) implemented optimization methods to investigate masonry vaults by finding possible compression-only force networks which are entirely contained within the boundaries of the vault. Recently, the problem of controlling a virtual hanging strings network has been elegantly and efficiently been implemented within a similar optimization framework by Andreu et al. (2007). A hanging string network is found which fits within the inverted geometry of the vault to be analyzed.

The following section briefly summarizes *Thrust-Network Analysis*, a new fully three-dimensional extension to thrust-line analysis based on projective geometry, duality theory and linear optimization (Block and Ochsendorf 2007). Examples in Sections 4 and 5 will demonstrate its value for the analysis of vaulted structures in unreinforced masonry.

3 THRUST NETWORK ANALYSIS

3.1 *Motivation*

Our goal was to develop a three-dimensional version of thrust-line analysis similar to previous applications using interactive graphic statics. This means that the following features should be preserved:

- a graphical and intuitive representation of the forces in the system; and
- an interactive exploration of the range of equilibrium solutions bounded by a minimum and maximum thrust.

In order to cope with the challenges of the high degree of indeterminacy of three-dimensional problems, as discussed in section 2.2, we want to be able to:

- identify and control the many unknowns (degrees of freedom); and
- negotiate between the unknowns by formulating an optimization problem with different objective functions.

Therefore we want to explore the impact of different assumptions about the force patterns, internal force distributions, boundary conditions or loading conditions.

3.2 *Methodology*

Thrust-network analysis extends O'Dwyer's (1999) work on funicular analysis of vaulted masonry structures by adding the reciprocal relationship between the geometry and the in-plane internal forces of networks (Williams 1986), which was first described by Maxwell (1864). Figure 2 demonstrates this relationship: the internal force equilibrium of one grid is represented by the geometry of the other grid and vice versa.

Thrust-network analysis is developed for loading conditions where all forces are parallel to each other, such as gravitational loading. Note that this method is therefore appropriate for historic structures in unreinforced masonry since the dominant loading is self-weight. It is important to note that in this case the external forces do not appear in the projection of the system on the plane perpendicular to the direction of the forces. This means that a plane force diagram can be produced which represents the equilibrium in that plane of the system independent of the externally applied loads. As a result the force diagram is scale-less since the external forces which typically give scale to the force diagram are missing.

3.3 *Overview of the main steps*

The set-up of the program is summarized below. Block and Ochsendorf (2007) presents details on the problem formulation and solving procedures.

(a) Defining a solution envelope:
The compression-only solutions must lie within given boundaries defined by an intrados and an extrados (Fig. 3a). These put height constraints on the nodes of the solution. These limits are obtained from

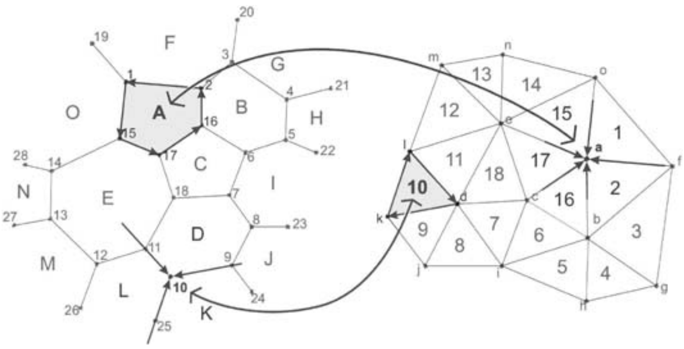


Figure 2. The two plane grids have a reciprocal relationship as defined by Maxwell. The equilibrium of a node in one of them is guaranteed by a closed polygon in the other and vice versa. The labeling uses Bow's notation (Bow 1873).

a three-dimensional model of the actual vault.

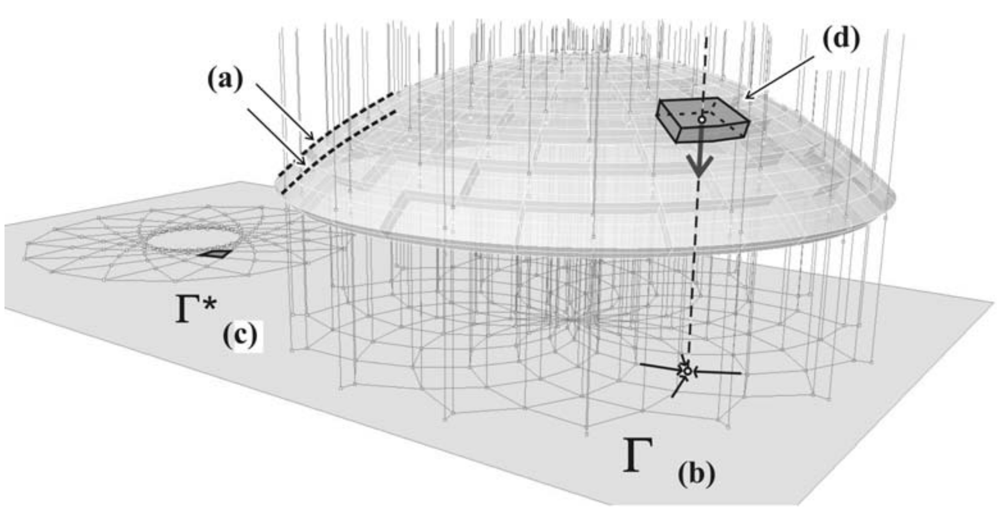
(b) Choosing a force pattern Γ:
In plan, a possible force pattern topology is constructed (Fig. 3b). The branches represent possible load paths throughout the structure. The loaded nodes represent the horizontal projections of centroids (cf. step d). This pattern is the horizontal projection of the final solution.

(c) Generating the reciprocal force diagram Γ^*:
The reciprocal force diagram (Fig. 3c) is produced from the force pattern such that corresponding branches stay parallel and nodal equilibrium in the pattern is guaranteed by closed polygons in the reciprocal diagram (Fig. 2). Note that the applied loads do not appear in the force diagram because they reduce to a single point in the horizontal projection (Fig. 4). This results in a force diagram with an unknown scale since the relation between pattern and diagram is true regardless of their relative scales.

(d) Attributing weights:
The weights attributed to the loaded nodes come from distributing the dead load of the 3-D tributary area to those nodes (Fig. 3d). In addition to self weight, other loads can be applied, such as the level of fill.

(e) Updating the force diagram:
In the case of an indeterminate force pattern containing nodes with more than three bars coming together per node, the user can change the internal force distribution by manipulating the force diagram (Fig. 4). This notion becomes interesting when forces want to be attracted to certain lines in the structure, such as along the ribs.

(f) Solving for the equilibrium solution G:
Using the geometry of the force pattern and diagram, the weights applied at the nodes and the boundary conditions, this problem can be solved using a one-step linear optimization (LO).



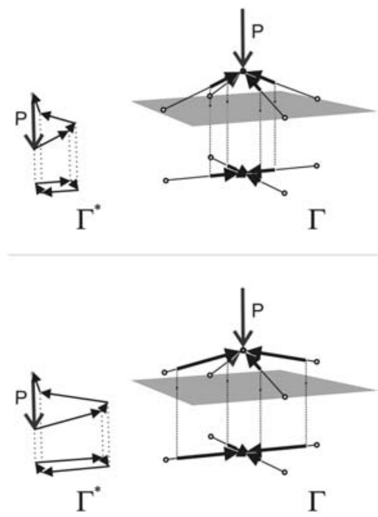


Figure 3. (left image) The input for the *Thrust Network Analysis* method: (a) the boundaries, the intrados and extrados of the vault; (b) a possible force pattern Γ defined on the horizontal plane; (c) the reciprocal force diagram Γ^* automatically produced from Γ; and the weight associated per node coming from the vault's self-weight and other imposed loads.

Figure 4. (right image) For a simple, but indeterminate, 4-bar structure, keeping the load P, the force pattern Γ, and the depth of the structure the same, this image shows the effect of manipulating the force diagram Γ*, i.e. changing the internal distributions of the forces. Stretching the force diagram to double the size in one direction is equal to doubling the forces in that direction and thus resulting in a structure half as deep in that direction.

The constraint equations have the following form (matrix notation):

$$C^T\left(H^T H^*\right) C\, z - p\, r = 0 \qquad (1)$$

In this equation, the unknowns are linear combinations of the nodal heights *z* and the unknown scale of the force diagram *r*. The coefficients in the equations are functions of the connectivity matrix *C,* which represents the topology of the networks (Schek 1974); the branch lengths *H* and H^* of the force pattern Γ and diagram Γ^* respectively; and *p*, the loading in each node.

We solve simultaneously for the nodal heights of the solution and the scale of the force diagram. The horizontal components of the forces in the solution *G* can easily be found by measuring the lengths of the branches in the force diagram grid and multiplying them by the actual scale.

3.4 *Applications*

Before able to solve the problem using LO, the analyst must choose the objective function of the optimization problem. Examples are (1) increasing the load factor of an imposed load until no solution can be found that fits within the boundaries of the structure (Fig.5a); (2) finding the one solution which maximizes the geometric factor of safety to demonstrate if a structure is safe or not (Fig.5b); or (3) finding the range of thrust, defined by a minimum and maximum thrust solution (Fig.5c), to understand the capacities of the 3D vault.

The first option gives an upper-bound solution. O'Dwyer (1999) demonstrates this for a barrel vault with a point load. Although such an analysis is relevant for bridge structures, such a loading is unlikely for vaults inside of a building. The second option is used by Andreu et al. (2007). If the optimization produces a result, then the safe theorem guarantees that the vault is safe and the geometric factor of safety gives an indication of the relative stability of the vault. The third option is used in this paper. The range of possible thrust values gives a useful characterization of the structural behavior of the vault. The minimum (or passive) thrust state represents the least amount this vault can push horizontally on its neighboring elements, as a function of its self-weight and shape. The maximum (or active) state of thrust on the other hand represents the largest horizontal force this vault can provide. So, this value demonstrates how much horizontal force this vault can safely take from its neighboring elements. For the optimization problem this means that we want to minimize and maximize the scale of the force diagram, resulting in globally the smallest versus the largest horizontal forces in the system or also the deepest versus the shallowest solution which fits

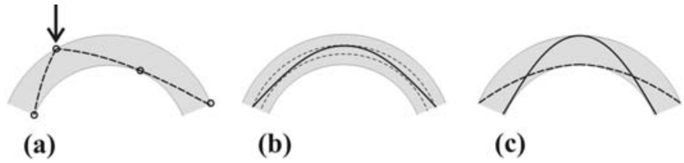


Figure 5. Possible objectives for the LO problem: (a) maximizing the load factor of an applied load; (b) maximizing the geometric safety factor; and (c) finding the range of thrust values, defined by a minimum and maximum thrust value.

within the boundaries of the structure.

The viability of the proposed method is demonstrated through two case studies, highlighting its multiple applications. First, an analysis of a series of groin and quadripartite rib vaults investigates the interrelationship between different parameters and the range of possible equilibrium solutions of these vaults. A second case study looks specifically at the fan vaults of King's College Chapel in Cambridge, England.

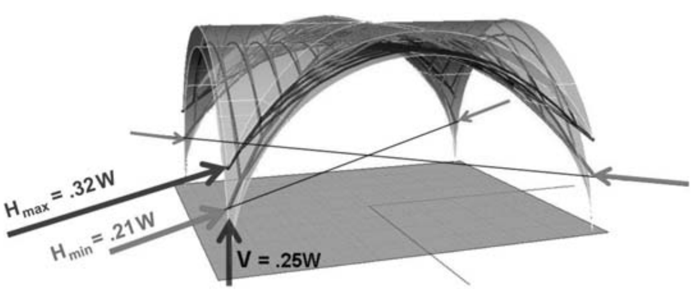


Figure 6. Possible thrust values at the corners for this groin vault range from 21% to 32% of its total weight.

4 GROIN AND QUADRIPARTITE VAULTS

Variables influencing the range of vault thrust include the influence of the web geometry, the boundary conditions, the role and effect of cross-ribs, the assumptions for the internal force patterns, the presence of cracks and other pathologies, and the impact of fill above the haunches.

The groin vault shown in Figure 6 is found to have a range of possible horizontal thrust values at the corners which vary from 21% to 32% of the total weight of the vault. This results from the choice of a pseudo-3D force pattern inspired by Wolfe's analysis (Fig. 1), where arches are assumed to span between the ribs, which carry the loads to the supports. The presence of Sabouret cracks which run parallel to the edges of the vault could justify such a pattern.

Figure 7 illustrates the relation between the chosen force patterns (i.e. the horizontal projection of the thrust network solution), the reciprocal force diagrams, and the shape of the 3D equilibrium solutions. The image demonstrates how the force diagrams clearly visualize the internal force distribution of the different force path assumptions (e.g. how much more force goes into the diagonals compared to the arches spanning in between them). Not only the proportions inside the force diagrams are useful, but the global scale allows for a direct comparison of the overall magnitude of the forces in the system between the different force pattern assumptions.

The first force pattern and diagram (Fig. 7a) represent the minimum thrust state shown in Figure 6. The second pattern (Fig. 7b) shows a different assumption on how the forces could travel through the structure: all force lines go directly to the corner

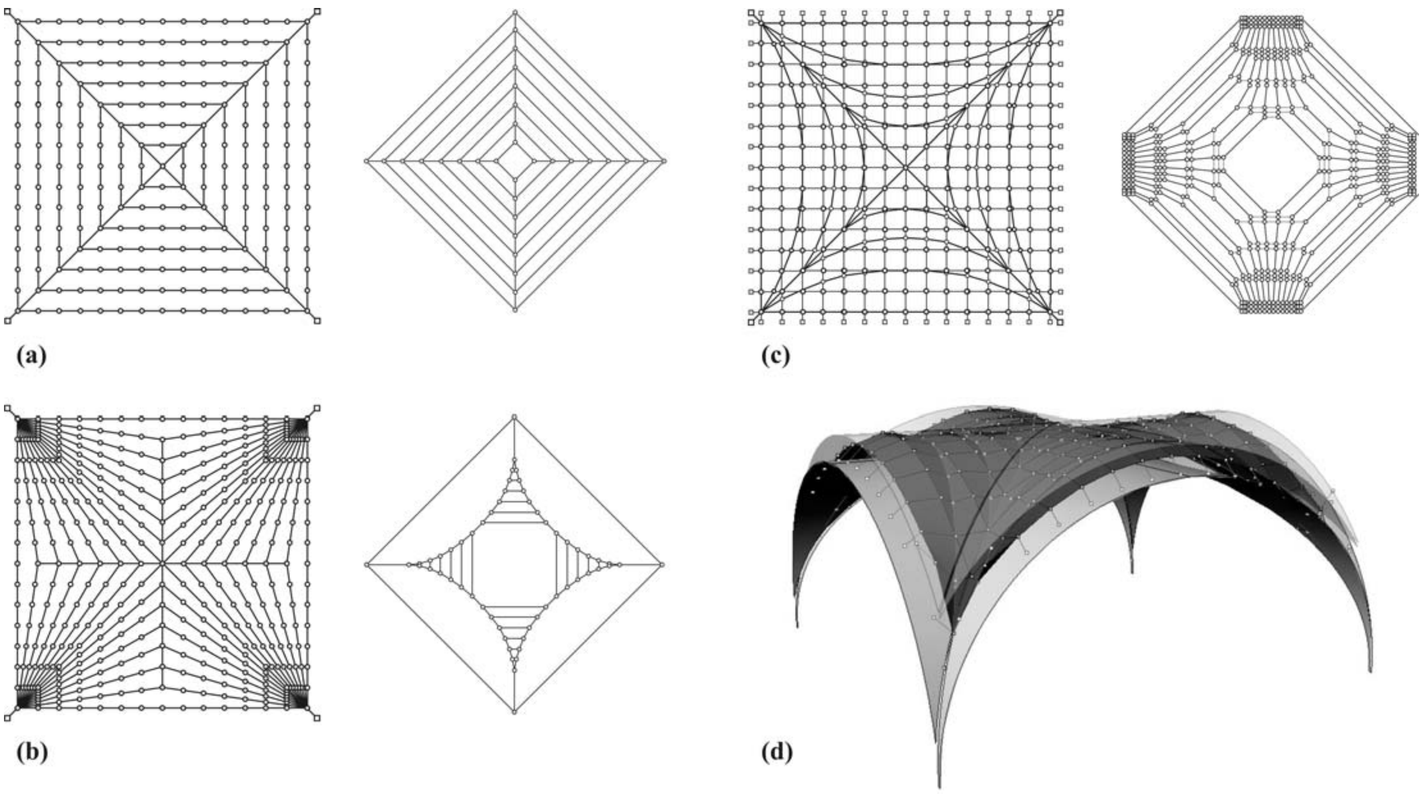


Figure 7. Different possible force patterns for a groin or rib vault: (a) the diagonal ribs bring down the forces to the corner supports, arches in the web span in between the ribs; (b) all force lines go directly to the corner supports; and, (c) primary force lines accumulate towards the supports and a continuous 3-D mesh distributes the loads to these force lines. (d) shows the three-dimensional thrust network, resulting from the assumptions in (c), which fit within the vault's geometry.

supports. The force diagram represents in a clear fashion the equilibrium of the corner nodes and center node. A similar force pattern is used in Section 5 to analyze the fan vaults. The last pattern (Fig. 7c) shows a fully three-dimensional force pattern. An important difference with the previous two assumptions is that forces no longer only go to the corner supports. This network assumes that a part of the vault is carried along the edges. As a result the thrust at the corner supports will be reduced. The main force lines (heavier lines in the force pattern) are distinguished from a continuous, regular grid in between. As can be seen in the force diagram, more force is attracted to the main force lines compared to the grid lines resulting in primary structural action over these lines and three-dimensional vault action between them. The location of these force lines can for example be inspired by the location of ribs in the actual vault. Figure 7d shows a three-dimensional compression-only thrust network which fits within the groin vault's geometry.

5 FAN VAULTS OF KING'S COLLEGE

The fan vaults of King's College Chapel in Cambridge, England were constructed between 1512 and 1515 (Fig. 8a). These double-curvature vaults were first analyzed by Heyman (1977) using membrane theory. These vaults with complex geometries can be analyzed using thrust-network analysis.

From available documentation (Leedy 1980), a detailed three-dimensional model is constructed (Fig. 8b, c). This model is used for obtaining precise nodal height constraints and good approximations of the weights applied at each node.

Since there are cracks between the transverse arches and the conoid fan vaults (Leedy 1980), we can assume that no compressive forces can be transferred between them. The transverse arches and the fan vaults work independently of each other. The chosen force pattern should reflect this, i.e. no branches should cross the interfaces between the fan vault and the transverse arch. Figure 8d shows a thrust network which fits within the vault's section demonstrating that it is stable. The force pattern follows the radial rib pattern.

The fill adds weight and alters the thrust-network but also adds more depth to the section for the thrust-network to travel through. This is very clear if we look at the equilibrium of the main transverse arch (Fig. 9). Without the level of fill, this arch would be too thin to stand under its own weight. The level of fill causes an increase of thrust, i.e. the hori-

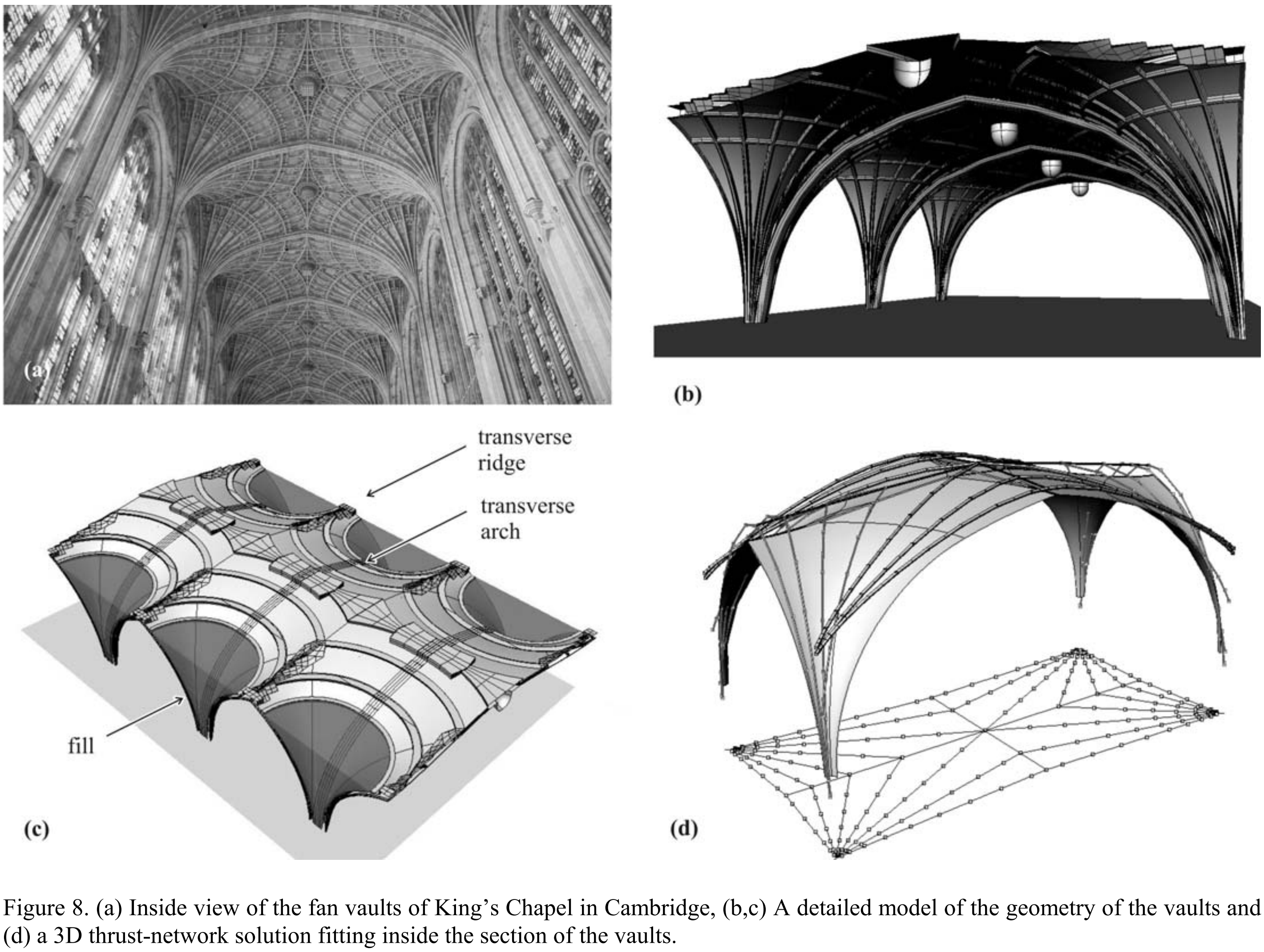


Figure 8. (a) Inside view of the fan vaults of King's Chapel in Cambridge, (b,c) A detailed model of the geometry of the vaults and (d) a 3D thrust-network solution fitting inside the section of the vaults.

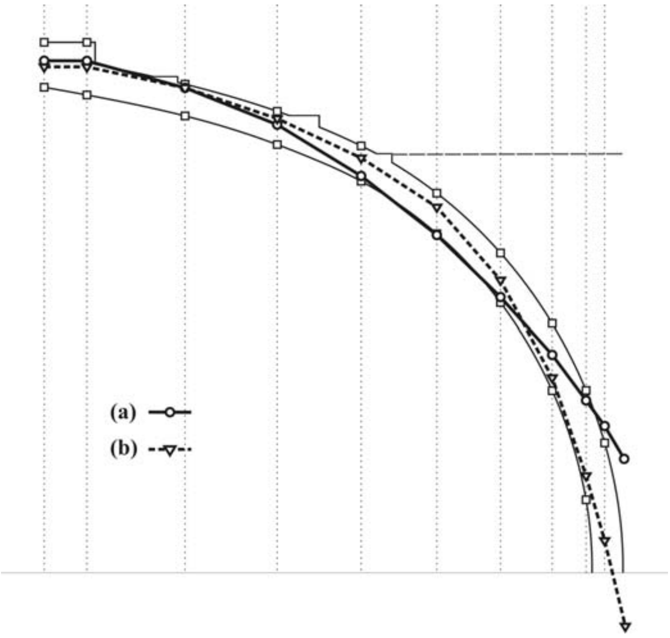


Figure 9. (a) The line of thrust exits the section of the transverse arch if the level of fill is left out. (b) shows the stable situation with fill.

zontal component, of the main arch by less than 25% compared to the case with no fill. On the other hand, the vertical component is more than tripled due to the added weight of the fill. We can conclude from this that the level of fill has a significant stabilizing effect on the buttresses which carry the arches and the vaults.

Furthermore, the level of fill appeared to be of crucial importance in order to find an acceptable thrust-network, which is shown in Figure 8d. In fact, the thrust-lines in the long direction of the vaults only fit within the section due to the fill. Another example illustrating the careful choice of the geometry of the vaults and fill is the added height and weight over the transverse ridges between two adjacent fan vaults (Fig. 8c) which causes a kink in the thrust lines in the short direction of the vaults such that they fit in the very steep section. From this preliminary analysis it seems that each stone, and even the level of fill, has been sculpted carefully to maintain the stability of this thin vault.

The importance of the level of fill could be investigated even further. For a given vault geometry, what would be the optimal level of fill? What loading would cause the thrust-network to lie as close to the middle surface of the vault as possible? We can solve this by using equation (1) differently. For a given choice of force network and force distribution, we now want to find the funicular loading which results in a given shape (Williams 1990). This means that in Equation (1) now C, H, H^* and z are known and that p and r are the unknowns. So, the optimization process now defines the optimal level of fill which would cause the thrust-network to lie as close to the middle surface of the vault as possible.

6 DISCUSSION

The methodology uses existing 3D drawing software as input and output resulting in an interactive tool with a visual representation of results and force distributions. Analysis proceeds from an accurate 3D model of the vault without the need for abstraction or simplification. The computation is done in MatLAB. The number of elements that the implementation can handle is limited to approximately 500 due to computational speed but this could easily be improved.

For the thrust-network analysis of masonry vaults, various parameters can be changed. (a) Different force patterns can be chosen to compare assumptions on how forces may be traveling through the structure. A distinction can be made between primary force lines and secondary force lines. (b) The force diagrams can be manipulated in order to redistribute the internal force distributions. (c) The solution envelopes can be chosen in order to constrain the solutions to the middle third zone of the vault or to exclude the thickness of ribs. (d) Different boundary conditions can be chosen. The vault can have a continuous edge support or only corner supports. This decision can be influenced by the curvatures of the vaults or by the existence of cracks. (e) Level of fill or other imposed loads can easily be integrated by adding load to affected nodes.

In addition, the loading cases do not have to be constrained to only gravity loads. An initial measure of the stability of a vault under lateral acceleration can be assessed by applying an equivalent static horizontal force. As shown in Block et al. (2006a) using interactive graphic statics for 2D, this can be simulated by tilting the model.

Future work includes the development of a more sophisticated optimization set-up which takes into account all possible force patterns and diagrams and searches for the absolute minimum and maximum thrust values for the masonry vaults.

7 CONCLUSION

This paper applied *Thrust-Network Analysis*, a fully three-dimensional computational method, to obtain lower-bound solutions for masonry vaults with complex geometries.

Key elements in the proposed process are (1) force networks, representing possible force paths through the structure; (2) interactive reciprocal diagrams, visualizing the proportional relationship of all forces in the force network and providing a high level of control for the user to understand and manipulate the force distributions within the system; (3) the use of envelopes defining the solution space; and

(4) linear optimization, resulting in fast computation and visualization of results.

For the input of the vault geometry and the loading conditions and for the output of the three-dimensional results, implementations are written in existing architectural software. This allows for clear visualizations of the results, and the smooth integration of the form-finding process in the analysis process.

Through several examples, this paper demonstrated the potential of thrust-network analysis as a powerful tool for understanding, visualizing and exploring the equilibrium of compression-only structures such as historic vaults in unreinforced masonry.

REFERENCES

Andreu, A., Gil, L. & Roca, P. 2007. Computational Analysis of Masonry Structures with a Funicular Model. *Journal of Engineering Mechanics* 133(4): 473-480.

Boothby, T.E. 2001. Analysis of masonry arches and vaults. Progress *in Structural Engineering and Materials* 3: 246-256.

Bow, R.H. 1873. *Economics of construction in relation to frames structures*. London: Spon.

Block, P. Dejong, M. & Ochsendorf, J. 2006a. As Hangs the Flexible Line: Equilibrium of Masonry Arches. *The Nexus Network Journal* 8(2): 13-24.

Block, P., Ciblac, T. & Ochsendorf, J. 2006b. Real-time limit analysis of vaulted masonry buildings. *Computers and Structures* 84(29-30): 1841-1852.

Block, P. & Ochsendorf, J. 2007. Thrust Network Analysis: A new methodology for three-dimensional equilibrium. *Journal of the International Association for Shell and Spatial Structures* 48(3).

Cremona, L. 1879. *Le Figure Reciproche nella Statica Grafica*. Milan: Ulrico Hoepli.

Föppl, A. 1892. *Das Fachwerk im Raume*. Leipzig: Verlag von B.G. Teubner.

Heyman, J. 1966. The stone skeleton. *International Journal of Solids and Structures* 2: 249-279.

Heyman, J. 1977. *Equilibrium of Shell Structures.* Oxford: Clarendon Press.

Heyman, J. 1995. *The Stone Skeleton: Structural engineering of masonry architecture*. Cambridge: Cambridge University Press.

Huerta, S. 2001. Mechanics of masonry vaults: The equilibrium approach. In P.B. Lourenço, P. Roca (eds.), *Historical Constructions*: 47-70, Guimarães.

Huerta, S. 2004. *Arcos bóvedas y cúpulas. Geometría y equilibrio en el cálculo tradicional de estructuras de fábrica*. Madrid: Instituto Juan de Herrera.

Leedy, W.C. 1980. *Fan Vaulting: A Study of Form, Technology, and Meaning*. Santa Monica: Arts and Architecture Press.

Maxwell, J.C. 1864. On reciprocal figures and diagrams of forces. *Phil. Mag. Series* 4(27): 250-261.

Ochsendorf, J.A. 2002. *Collapse of masonry structures*. PhD. dissertation, Department of Engineering, Cambridge University.

O'Dwyer, D.W. 1999. Funicular analysis of masonry vaults. *Computers and Structures* 73(1-5): 187-197.

Schek, H.-J. 1974. The Force Density Method for formfinding and computation of general networks. *Computer Methods in Applied Mechanics and Engineering* 3: 115-134.

Smars, P. 2000. *Etudes sur la stabilité des arcs et voûtes*. PhD Thesis, Department of Civil Engineering, Katholieke Universiteit Leuven.

Ungewitter, G. 1890. *Lehrbuch der gotischen Konstruktionen*. Leipzig: Weigel Nachfolger.

Williams, C.J.K. 1986. Defining and designing curved flexible tensile surface structures. In J.A. Gregory (ed.), *The mathematics of surfaces*: 143-177. Oxford: Clarendon Press.

Williams, C.J.K. 1990. The generation of a class of structural forms for vaults and sails. *The Structural Engineer* 68(12): 231-235.

Wolfe, W.S. 1921. *Graphical analysis: a handbook on graphic statics*. New Cork: McGraw-Hill.