Reprinted from May 1987, Vol. 109, Journal of Heat Transfer

J. H. Lienhard V
Department of Applied Mechanics and Engineering Sciences
and
Project in Nonlinear Science,
University of California at San Diego,
La Jolla, CA 92093
Student Mem. ASME

An Improved Approach to Conductive Boundary Conditions for the Rayleigh-Bénard Instability

A technique is developed for predicting the stability limit of conductively coupled horizontal fluid layers heated from below and cooled above. The approach presented gives exact solutions of the stability problem and is numerically much simpler than previous multilayer solutions. Critical Rayleigh numbers are obtained for the case of three and four fluid layers separated by equally spaced identical midlayers of various thicknesses and conductivities with isothermal outer walls and for the symmetric two-layer problem with outer walls of finite thermal conductivity. Other configurations are considered briefly.

Introduction

Natural convection in parallel arrays of fluid layers is a problem of current technical interest. The most outstanding example of a coupled fluid layer array is the set of cover layers on a flat plate solar collector. In this example, as in many others, the basic design objective is convection suppression, to reduce heat loss. A necessary first step toward suppressing convection is prediction of the stability limit of the quiescent state of the fluid.

To date only the two-layer problem has been studied. Gershuni and Zhukhovitskii [1] considered the stability of a pair of identical fluid layers separated by a conducting midlayer, allowing the outer boundaries to be either isothermal or of the same conductivity as the midlayer. They found approximate solutions using the Galerkin method. Catton and Lienhard [2] generalized the problem to allow for fluid layers of differing heights, considering isothermal outer boundaries; their solutions were obtained using a higher order Galerkin approximation.

Here, I consider the stability of the one-dimensional conductive state of an array of horizontal fluid layers heated from below and cooled above. A general technique is formulated for obtaining exact solutions of the stability problem in an arbitrary number of layers when the thermal coupling is by conduction through intermediate solid layers (the important problem of radiative coupling is not treated here). The method follows standard techniques of stability analysis and relies on manipulation of the boundary conditions to simplify the equations to be solved. The resulting computational procedure is much simpler and yields higher accuracy than the previous multilayer solutions.

The heart of the procedure is the formulation of a third kind thermal boundary condition dependent upon the wavenumber and, more generally, a coupling parameter which must be iterated. While third kind conditions were first discussed for the single layer more than fifty years ago by Low [8], only in 1968 was a wavenumber-dependent condition considered by Nield [11], in the context of a conductive boundary slab. Apparently, no subsequent work has dealt with conductive boundaries.

The Effect of Midlayer Conduction on Multilayer Stability

The physical behavior of multilayer instability is discussed in detail by Catton and Lienhard [2] in the context of a two-layer system. The general features of the instability are, at the simplest level, just like those of the ordinary Bénard instability. Once the temperature gradient in a fluid layer becomes large enough, the release of buoyant energy by fluid motion exceeds the associated viscous dissipation and convection ensues. However, heat conduction through adjacent solid layers modifies the stability limit of a layered system and allows thermal interactions between adjacent fluid layers.

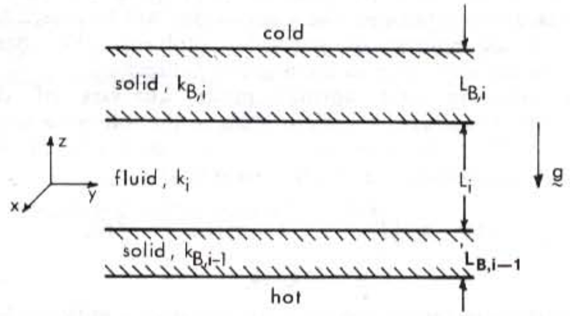

Fig. 1 Archetypical configuration for coupled layers

When fluid layers are coupled by a conducting midlayer, a thermal disturbance in one layer of wavelength λ is present in the adjacent layers at the same wavelength. Thus, if one layer has a higher Rayleigh number than an adjacent layer, we may think of it as becoming unstable at some critical Ra and driving convection in the adjacent layer, even if the latter might otherwise be stable. In no case can one layer in a thermally interacting group be quiescent while convection occurs in the others. If the thermal coupling is weak, however, the amplitude of convection in one layer may be much less than in others.

The thermal conductivity and thickness of the midlayers are the dominant parameters affecting the stability of a layered system. Thicker, more conductive layers tend to damp out thermal disturbances, resulting in weaker coupling between the layers. Thin midlayers allow less damping of thermal disturbances, resulting in greater thermal coupling. In general, when the thermal interaction of the layers is stronger, the stability of an individual layer is lower. Poorly conducting midlayers allow less dissipation of thermal disturbances and result in hot and cold spots along the midlayer, lowering the stability of individual fluid layers.

When considering the stability limit of a layered system, one must distinguish between the overall Rayleigh number of the layered system and the Rayleigh number of an individual fluid layer. The overall critical Rayleigh number may be increased indefinitely by adding more layers (for a fixed overall temperature difference) and by making the midlayers less con-

Contributed by the Heat Transfer Division and presented at the 23rd National Heat Transfer Conference, Denver, Colorado, August 4-7, 1985. Manuscript received by the Heat Transfer Division November 5, 1985.

ductive. The individual layer critical Rayleigh number always varies between 1708 (isothermal boundaries) and 720 (fixed heat flux boundaries), with associated critical wavenumbers of 3.12 and 0, respectively. One other important critical Rayleigh number is 1296 (with wavenumber 2.56) which occurs for a single layer with one isothermal and one fixed heat flux wall.

The results presented here focus on the individual layer critical Rayleigh number. The critical wavenumber is largely ignored, both for brevity and because it varies between the above limits, more or less in tandem with the critical Rayleigh number and in a fairly regular fashion.

Governing Equations

Our analysis begins with the usual disturbance equations for horizontally unbounded, plane solid and fluid layers. The archetypical geometry is illustrated in Fig. 1. A fluid layer is bounded by rigid, conducting solids above and below. The thermal conditions imposed at the outer boundaries of the solid layer are arbitrary. These conditions will be prescribed later in the context of particular problems. The basic temperature gradient is in the negative z direction.

Perturbation and normal mode analysis of the Oberbeck–Boussinesq equations leads to the following sixth-order O.D.E. for the normal mode amplitude of the temperature disturbance, Θ_i, in a fluid layer i [1, 3]

$$\left\{-\frac{s_i}{\Pr}+(D_i^2-a_i^2)\right\}\{-s_i+(D_i^2-a_i^2)\}(D_i^2-a_i^2)\Theta_i = -a_i^2\mathrm{Ra}_i\Theta_i \quad (1)$$

where the dimensional temperature disturbance in layer i is

$$T_i' = \Delta T_i \cdot \Theta_i(z_i)\exp[s_i t_i + i(a_{x_i}x_i + a_{y_i}y_i)]$$

Here all quantities (except T_i') are cast in the scales of layer i. In particular, Pr is the Prandtl number, $\mathrm{Ra}_i = (g\beta/\nu\alpha)L_i^3\Delta T_i$ is the Rayleigh number of layer i, $a_i^2 = a_{x_i}^2 + a_{y_i}^2 = 2\pi L_i/\lambda$ is the wavenumber for layer i, and ΔT_i—the temperature scale for this layer—is the temperature difference across layer i.

We state without proof that instability will occur by passage through a marginal state in which disturbances neither decay nor grow characterized by $s_i = 0$. This assertion, known as exchange of stabilities, has been proven for various boundary conditions on a single layer (see, e.g. [1, 3]) and for a pair of coupled fluid layers [4]. The proof for an arbitrary number of solid and fluid layers involves only more algebra than the proof in [4]. To find the critical Rayleigh number, we therefore need only consider the case $s_i = 0$ in (1)

$$(D_i^2 - a_i^2)^3\Theta_i = -a_i^2\mathrm{Ra}_i\Theta_i \quad (2)$$

The general solution of the above equation was found long ago by Pellew and Southwell [5] and is

$$\Theta_i = A\cosh(qz_i) + A^*\cosh(q^*z_i) + A_0\cos(q_0z_i) + B\sinh(qz_i) + B^*\sinh(q^*z_i) + B_0\sin(q_0z_i) \quad (3)$$

where $(\ )^*$ denotes a complex conjugate and

$$q_0 = a_i(\tau-1)^{1/2}$$

$$q^2 = a_i^2\left(1+\frac{1}{2}\tau(1\pm i\sqrt{3})\right)$$

with

$$\mathrm{Ra}_i = a_i^4\tau^3$$

Here we may take $z_i \in (-1/2, 1/2)$. The coefficients in (3) will be chosen to satisfy the six boundary conditions on Θ which we shall develop presently.

We are interested in situations in which the thermal disturbance will be carried into the solid layers above and below the fluid layer. In these layers

$$\frac{\partial T_B}{\partial t_B} = \nabla_B^2 T_B \quad (4)$$

where $(\ )_B$ denotes a value in the solid. This equation has been nondimensionalized with the scales: length—L_B, the thickness of the solid layer; time—L_B^2/α_B; temperature—ΔT_B, the temperature difference across the solid layer. Perturbing

Nomenclature

A = midlayer to fluid layer aspect ratio = $L_B/2L_i$; $A_{ij} = L_{B_j}/2L_i$
a_i = dimensionless wavenumber in layer $i = 2\pi L_i/\lambda$
D_i = dimensionless z derivative in layer i
h = heat transfer coefficient, assumed spatially uniform
k = thermal conductivity (of fluid unless subscripted with B or 0)
L_i, L_{B_j} = thickness of fluid layer i and thickness of midlayer j, respectively
M, N = coefficients of even, odd portions of midlayer thermal disturbance
Ra_i = Rayleigh number of fluid layer $i = (g\beta/\nu\alpha)L_i^3\Delta T_i$
Ra_c = critical Rayleigh number of layer i
Ra_T = overall Rayleigh number
Re = real part of a complex number
s_i = dimensionless disturbance growth rate in layer i
T_i' = temperature disturbance in layer i
X_B = fluid to solid layer conductivity ratio = k/k_B; $X_{B_{ij}} = k_i/k_{B_j}$
X_0 = fluid layer to outer wall conductivity ratio = k/k_0

Greek Letters

α = even/odd disturbance amplitude ratio = M/N; α_B is the thermal diffusivity of a solid layer
γ = fluid layer Biot number = (hL_i/k_i)
γ_e = cover plate Biot number = (hL_B/k_B)
$\Delta T_i, \Delta T_{OA}$ = temperature difference across layer i and across entire multilayer array, respectively
Θ = dimensionless normal mode thermal disturbance amplitude
Λ = coefficient of third kind boundary condition: $\Theta' = \Lambda\Theta$; definition varies by context
$\Lambda_{Ui}, \Lambda_{Li}$ = coefficient at upper and lower boundaries of fluid layer i, respectively
λ = dimensional disturbance wavelength

Subscripts and Superscripts

$(\ )_i$ = a variable in layer i
$(\ )_B, (\ )_{B_j}$ = a variable in a solid layer or boundary, in barrier j
$(\ )'$ = differentiation with respect to z_i

equation (4) and analyzing the disturbance into normal modes yields

$$\left\{-s_B+(D_B^2-a_B^2)\right\}\Theta_B=0 \tag{5}$$

The physical wavelength and growth rate (λ and σ) are common to the fluid and solid layers so that

$$\lambda=\frac{2\pi}{a_i}L_i=\frac{2\pi}{a_B}L_B \Rightarrow a_B=\frac{L_B}{L_i}a_i$$

and, with exchange of stabilities,

$$s_i=s_B=0$$

in the marginal state. The solution of equation (5) with $s_B = 0$ is

$$\Theta_B=M\cosh(a_Bz_B)+N\sinh(a_Bz_B),\quad z_B\epsilon\left(-\frac{1}{2},\frac{1}{2}\right) \tag{6}$$

with the constants M and N to be found from the thermal boundary conditions on the solid layer. Commonly, this requires simultaneous solution for all the constants arising in the entire set of fluid and solid layers.

Boundary Conditions

Boundary conditions are required for the vertical velocity disturbance w_i and the fluid and solid temperature disturbances T_i and T_B.

At a rigid, horizontal boundary we have

$$\mathbf{v}=(u,v,w)=0$$

and, with the continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\frac{\partial w}{\partial z}=0$$

at the boundaries. By (2) these are equivalent to

$$(D_i^2-a_i^2)\Theta_i=(D_i^2-a_i^2)D_i\Theta_i=0 \tag{7}$$

The thermal boundary conditions on a fluid layer match temperature and heat flux to those of the bounding solids. In physical variables

$$T_i=T_B,\quad -k_i\frac{dT_i}{dz}=-k_B\frac{dT_B}{dz}$$

or, in terms of the nondimensional disturbances

$$\Theta_i=\left(\frac{k_i}{k_B}\right)\left(\frac{L_B}{L_i}\right)\Theta_B$$

and

$$D_i\Theta_i=D_B\Theta_B \tag{8}$$

Note that the thermal boundary conditions are generally not symmetric about the fluid layer and the even and odd functions in Θ_i cannot be separated as in the symmetric case.

We may use equations (7) to reduce the number of free constants in equation (3). Conditions (7) apply at each boundary of the fluid layer ($z_i = \pm 1/2$), and, by taking advantage of the even/odd behavior of the cos/cosh and sin/sinh functions in the expression for Θ_i, we find

$$\begin{bmatrix}\cosh(q/2) & \cosh(q^*/2)\\ q\sinh(q/2) & q^*\sinh(q^*/2)\end{bmatrix}\begin{bmatrix}(q^2-a^2)A\\ (q^{*2}-a^2)A^*\end{bmatrix}=A_0(q_0^2+a^2)\begin{bmatrix}\cos(q_0/2)\\ -q_0\sin(q_0/2)\end{bmatrix} \tag{9}$$

and

$$\begin{bmatrix}\sinh(q/2) & \sinh(q^*/2)\\ q\cosh(q/2) & q^*\cosh(q^*/2)\end{bmatrix}\begin{bmatrix}(q^2-a^2)B\\ (q^{*2}-a^2)B^*\end{bmatrix}=B_0(q_0^2+a^2)\begin{bmatrix}\sin(q_0/2)\\ +q_0\cos(q_0/2)\end{bmatrix} \tag{10}$$

These equations may be solved directly (e.g., by Cramer's rule) to find A/A_0, B/B_0, and their conjugates, cutting the number of unknown coefficients from six to two.

Finally, we observed that if no forcing of the system's thermal disturbance is imposed the boundary conditions will be homogeneous, and we must choose particular values of Ra to obtain nontrivial solutions for Θ which satisfy equations (7) and (8). The smallest of the eigenvalues Ra must be minimized as a function of wavenumber to obtain the critical Rayleigh number.

The Appearance and Use of Third Kind Thermal Boundary Conditions

We are interested in developing a technique for collapsing the coupling conditions (8) into a single condition applied directly to the fluid layer, thereby avoiding the need to solve simultaneously for the constants and Rayleigh numbers in all fluid and solid layers. The simplest example of such an approach was given by Sparrow et al. [6] in 1964. They considered the case in which the medium adjacent to the fluid layer could be characterized by a spatially uniform heat transfer coefficients as

$$-k_i\frac{dT}{dz}=\pm h(T-T_\infty) \tag{11}$$

where the minus sign applies when $T_\infty > T$. Perturbing this condition and scaling with ΔT_i, L_i produces

$$\Theta_i'=\mp\left(\frac{hL_i}{k_i}\right)\Theta_i=\mp\Lambda\Theta_i \tag{12}$$

a third kind condition depending on a constant Λ (which is a Biot number here). For Bénard-type problems the plus sign applies at the (hot) lower surface if the direction of increasing z is taken to be vertically upward. Equation (12) is applied directly to the fluid layer and allows us to ignore the details of the disturbance to the external medium.

Of particular interest are the two limiting cases:

$\Lambda\to\infty\Rightarrow\Theta_i=0$ isothermal wall

$\Lambda\to 0\Rightarrow\Theta_i'=0$ fixed heat flux wall

One may show analytically that the critical Rayleigh number will increase monotonically as Λ is increased, in agreement with the results of Sparrow et al.

A somewhat more advanced example is obtained when the bounding surface is a semi-infinite wall of thermal conductivity k_B. Here the appropriate solution of equation (5) is

$$\Theta_B=M\exp(-a_Bz_B),\quad z_B\epsilon(0,\ \infty) \tag{13}$$

if $z_B = 0$ at the boundary of the fluid and z_B is scaled with L_i. Substitution into condition (8) gives

$$\Theta_i=\left(\frac{k_i}{k_B}\right)Me^{-a_Bz_B}$$

$$\Theta_i'=-M\,a_Be^{-a_Bz_B}=-M\,a_ie^{-a_iz_B}$$

in light of the scaling. We may form a single third kind condition by eliminating M between these equations

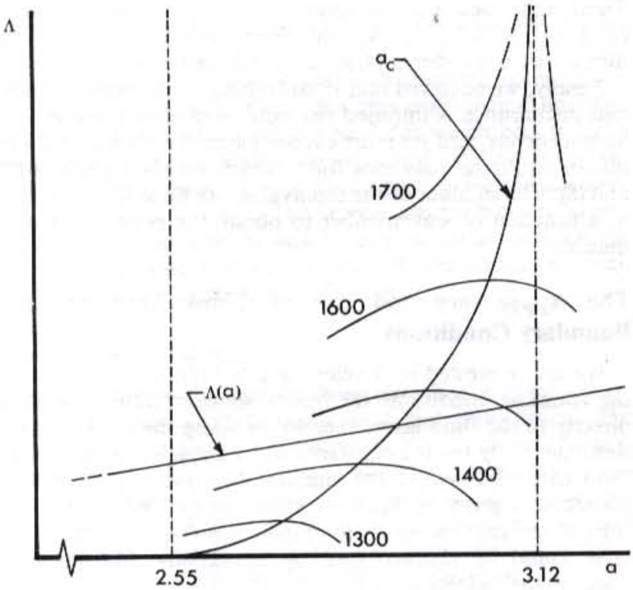

Fig. 2 Level curves of the Rayleigh number as a function of Λ and *a*

$$\Theta_i' = -\left(\frac{a_i}{k_i/k_B}\right)\Theta_i = -\left(\frac{a_i}{X}\right)\Theta_i \quad (14)$$

with X defined as indicated. At a lower surface, z_B will have opposite orientation and the minus sign in equation (1) becomes a plus sign.

For this problem $\Lambda = (a_i/X)$ is wavenumber dependent, and the calculation of the critical Rayleigh number will include a constrained variation of the parameter Λ with a_i. Note, however, that simultaneous solution for M, the coefficients of Θ, and Ra is no longer required. We need consider only the fluid layer and the disturbance in the wall requires no better specification that equation (13).

The wavenumber dependence of Λ has a marked effect on the stability limit of the fluid layer. The cases of equations (12) and (14) are contrasted in Fig. 2, assuming for example that Λ applies on top with the lower surface isothermal. The figure shows schematically the level curves of Ra(a, Λ). The curve a_c is the locus of critical wavenumbers for given $\Lambda \neq \Lambda(a)$, viz., the result of using equation (12). The critical Rayleigh numbers for fixed Λ lie on this curve. On the other hand, when $\Lambda = a/X$ the critical Rayleigh number for a given X corresponds to the lowest Ra level curve intersecting the line $\Lambda = (1/X)a$. Of particular importance is the observation that these points do not generally lie on the curve a_c for constant Λ. Hence, there is not a unique curve for $\text{Ra}_c = \text{Ra}_c(a, \Lambda(a))$ if $\Lambda(a)$ is arbitrary.

To this point our remarks have been organizational. We are now in a position to tackle some previously unsolved problems.

The Two-Layer Problem With Finite Outer Wall Conductivity

We consider here the problem of two identical fluid layers separated by a finite thickness and conductivity midlayer and bounded by identical outer walls of finite conductivity (Fig. 3). This configuration is symmetric about the centerline of the midlayer. Lienhard and Catton [7] found that disturbances of the two layer system with isothermal outer walls are either even or odd about the midlayer centerline when the fluid layers are of equal height. This will also be true when the outer walls are not isothermal.

For even disturbances, the midlayer thermal perturbation must be

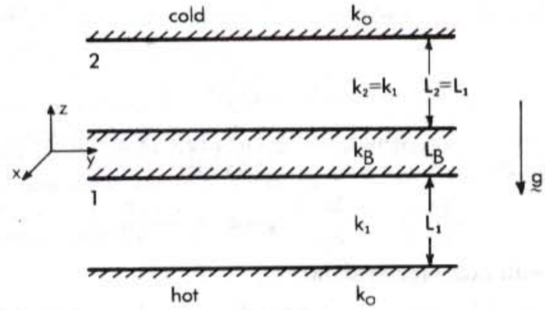

Fig. 3 The symmetric two-layer configuration

$$\Theta_B(z_B) = M \cosh(a_B z_B) \quad (15)$$

and for odd modes

$$\Theta_B(z_B) = N \sinh(a_B z_B) \quad (16)$$

We may substitute these general forms into equations (8) and eliminate M and N. The result for even disturbances is

$$\left.\begin{aligned}\Theta_1'(1/2) &= -\Lambda_e\Theta_1(1/2)\\ \Theta_2'(-1/2) &= \Lambda_e\Theta_2(-1/2)\end{aligned}\right\} \quad (17)$$

with

$$\Lambda_e = \left(\frac{a_1}{X_{B_1}}\right)\tanh(a_1 A) \quad (18)$$

and $X_{B_1} = k_1/k_B$, $A = L_B/2L_1$. For the odd modes we find

$$\Lambda_0 = \left(\frac{a_1}{X_{B_1}}\right)\coth(a_1 A) \quad (19)$$

Observe that $\Lambda_e \leq \Lambda_0$ for a given two-layer system; as noted above Ra_c increases monotonically with Λ, so even modes will always occur before odd modes. This agrees with the results of [7].

The thermal conditions at the outer walls were developed in the preceding section and are

$$\left.\begin{aligned}\Theta_1'(-1/2) &= \Lambda_w\Theta_1(-1/2)\\ \Theta_2'(1/2) &= -\Lambda_w\Theta_2(1/2)\end{aligned}\right\} \quad (20)$$

with $\Lambda_w = a_1/X_{0_1}$, $X_{0_1} = k_1/k_0$. The eigenvalue problems for the two layers are the same. In agreement with physical intuition, the layers must become unstable simultaneously (at the same Rayleigh number). Accordingly, we need solve the problem only for one layer, say the first.

The equations to be solved are algebraic. Equation (6) is substituted into equations (17) and (20); equations (9) and (10) are used to eliminate A, A^*, B, and B^* in favor of A_0 and B_0. The result is

$$\begin{pmatrix}\mathcal{C}_1 & \mathcal{C}_2\\ \mathcal{C}_3 & \mathcal{C}_4\end{pmatrix}\begin{pmatrix}A_0\\ B_0\end{pmatrix} = 0$$

where the $\mathcal{C}_i$ are uninteresting functions of Ra, a, Λ_e, and Λ_w which we have relegated to the Appendix. For nontrivial solutions we must have

$$\mathcal{C}_1\mathcal{C}_4 - \mathcal{C}_2\mathcal{C}_3 = 0 \quad (21)$$

For a given a, Λ_e, and Λ_w this equation has roots for varius Ra. To solve for the stability limit Ra_c, the smallest of these roots was located using bisection and then minimized as a function of wavenumber using parabolic fit of Ra(a) and the requirement $d\text{Ra}/da = 0$. Calculations made in double precision FORTRAN on a Vax-11 yielded the results in Table 1. These results are believed accurate to the number of figures shown.

Table 1 Critical Rayleigh numbers for the two-layer system

A = 0.001

X_{B_1} \ X_0	0.0	0.1	0.3	1.0	3.0	10.0	∞
0.0	1707.76	1667.96	1607.18	1492.66	1391.00	1329.56	1295.78
0.0003	1668.56	1627.90	1565.65	1447.85	1342.74	1279.06	1244.04
0.001	1596.97	1554.73	1489.67	1364.98	1251.56	1181.69	1142.89
0.003	1483.85	1439.71	1371.02	1235.88	1105.60	1016.89	956.69
0.01	1374.40	1330.31	1261.32	1123.52	985.59	884.11	792.00
0.1	1304.95	1262.17	1195.12	1060.58	924.45	822.50	727.20
1.0	1296.71	1254.16	1187.44	1053.51	917.83	816.06	720.72
∞	1295.78	1253.25	1186.58	1052.71	917.09	815.35	720.00

A = 0.01

X_{B_1} \ X_0	0.0	0.1	0.3	1.0	3.0	10.0	∞
0.0	1707.76	1667.96	1607.18	1492.66	1391.00	1329.56	1295.78
0.001	1694.03	1653.93	1592.64	1477.02	1374.23	1312.07	1277.90
0.01	1596.95	1554.71	1489.65	1364.96	1251.54	1181.67	1142.87
0.03	1483.83	1439.69	1371.00	1235.86	1105.58	1016.88	956.69
0.1	1374.38	1330.29	1261.30	1123.51	985.58	884.10	792.00
1.0	1304.95	1262.17	1195.12	1060.58	924.45	822.49	727.20
10.0	1296.71	1254.16	1187.44	1053.51	917.83	816.06	720.72
∞	1295.78	1253.25	1186.58	1052.71	917.09	815.35	720.00

A = 0.1

X_{B_1} \ X_0	0.0	0.1	0.3	1.0	3.0	10.0	∞
0.0	1707.76	1667.96	1607.18	1492.66	1391.00	1329.56	1295.78
0.03	1667.45	1626.82	1564.59	1446.84	1341.78	1278.12	1243.12
0.1	1594.60	1552.43	1487.47	1362.96	1249.71	1179.93	1141.18
0.3	1481.23	1437.22	1368.74	1234.03	1104.19	1015.86	956.22
1.0	1372.95	1328.98	1260.16	1122.69	985.08	883.84	792.00
3.0	1324.51	1281.30	1213.60	1077.91	940.97	838.85	744.00
10.0	1304.76	1262.00	1194.98	1060.48	924.39	822.47	727.20
∞	1295.78	1253.25	1186.58	1052.71	917.09	815.35	720.00

A = 0.3

X_{B_1} \ X_0	0.0	0.1	0.3	1.0	3.0	10.0	∞
0.0	1707.76	1667.96	1607.18	1492.66	1391.00	1329.56	1295.78
0.03	1690.44	1650.39	1589.17	1473.68	1371.00	1308.90	1274.76
0.1	1654.49	1613.91	1551.74	1434.01	1328.87	1265.13	1230.07
0.3	1577.79	1536.02	1471.67	1348.32	1236.08	1166.91	1128.48
1.0	1452.47	1409.19	1341.80	1209.03	1080.64	992.82	932.78
3.0	1363.40	1320.07	1252.25	1116.81	981.28	881.74	792.00
10.0	1318.28	1275.40	1208.20	1073.47	937.42	835.90	741.60
∞	1295.78	1253.25	1186.58	1052.71	917.09	815.35	720.00

A → ∞

X_{B_1} \ X_0	0.0	0.1	0.3	1.0	3.0	10.0	100.0	∞
0.0	1707.76							
0.1	1667.96	1628.02						
0.3	1607.18	1566.93	1505.29					
1.0	1492.66	1451.68	1388.53	1267.47				
3.0	1391.00	1349.22	1284.33	1157.50	1037.89			
10.0	1329.56	1287.29	1221.24	1090.02	961.13	872.10		
100.0	1299.41	1256.91	1190.30	1056.72	921.90	821.91	753.34	
∞	1295.78	1253.25	1186.58	1052.71	917.09	815.35	740.99	720.00

Table 2 Rayleigh numbers for A/X_B = 1 at various A

A	Ra_c	a_c	% Error, Ra	% Error, b. c.	Column 5 / Column 4
0.001	1330.31	2.50	0	-	-
0.01	1330.29	2.50	0.0013	0.021	20.7
0.1	1328.98	2.50	0.10	2.04	20.4
0.3	1320.07	2.51	0.77	15.4	20.0

The basic trends in the critical Rayleigh number Ra_c as a function of A and X_{B_1} were discussed in [2] for $X_{0_1} = 0$. The results for finite X_{0_1} are entirely as expected, viz., decreased stability as the outer walls become less conductive and thus less able to dissipate thermal disturbances. Observe that as X_{B_1} and $X_{0_1} \rightarrow \infty$ we recover the limiting case $Ra_c = 720$, $a = 0$ first found in [6].

The limit $A \rightarrow 0$ is most profitably discussed with reference to equations (18) and (19). Edwards and Ulrich (see [9]) found that they could correlate Catton and Lienhard's Ra_c values as a function of the single parameter A/X_{B_1} for smaller values of A. To understand how this parameter (not to be confused with the conductance ratio AX_{B_1}) becomes important, we let $A \rightarrow 0$ in equation (18) and find

$$\Lambda_e \sim a_1^2 \left(\frac{A}{X_{B_1}}\right), \quad A \rightarrow 0 \tag{22}$$

The group A/X_{B_1} is indeed the appropriate single parameter for small A. A calculation shows that this approximation introduces less than 1.0 percent error in the boundary conditions if $a_1 A \leq 0.175$ (or $\leq$ 10 percent error if $a_1 A \leq 0.585$). However, the resultant error in Ra_c is much smaller, as illustrated in Table 2. For the case considered by Edwards and Ulrich $X_{0_1} = 0 \Rightarrow a_1 \geq 2.5 \Rightarrow A \leq 0.23$ for 10 percent boundary conditions error; thus, they were able to collapse data for $A \lesssim 0.3$ onto a single curve with high accuracy.

Conversely, from equation (19)

$$\Lambda_0 \sim \frac{1}{AX_{B_1}}, \quad A \rightarrow 0$$

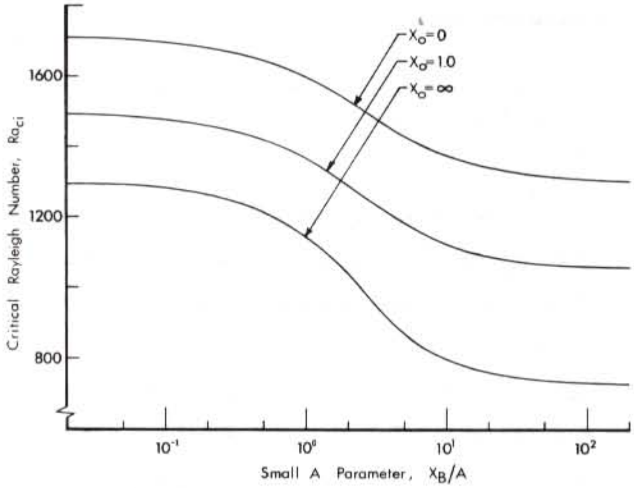

Fig. 4 Critical Rayleigh number of the two-layer problem

Table 3 Single layer with nonisothermal walls; present (PW); Hurle, Jakeman, and Pike (HJP); Gershuni and Zhukhovitskii (GZ)

X_0 \ X_{B_1}		0.0	0.5	1.0	2.0	∞
X_{B_1}	PW	1707.762	1415.093	1267.471	1116.412	720.000
	HJP	1707.762	1415.093	1267.471	1116.412	720
	GZ	1708	1415	1269	1117	720.0

X_{B_1} \ X_0		0.1	1.0	10.0	∞
0.0	PW	1667.96	1492.66	1329.56	1295.78
	GZ	1668	1496	1337	1304
∞	PW	1253.25	1052.71	815.35	720.00
	GZ	1260	1054	815	720.0

which is a midlayer to fluid layer conductance ratio. While $A \to 0$ gives rise to a fixed heat flux boundary conditions for even modes (horizontal dissipation of heat negligible), the limit produces as isothermal wall condition for odd modes (horizontal dissipation dominant). Of particular importance is the observation that A/X_{B_1} is only a meaningful parameter for the special case of even two-layer modes with $A \to 0$. Nonetheless, A/X_{B_1} provides a useful characterization of the stability limit for this case.

The critical Rayleigh number is plotted as a function of A/X_{B_1} for various X_{0_1} in Fig. 4, using values for $A = 0.001$. These universal curves also represent Ra_c for $A = 0.01$, 0.1, and 0.3 to within about 1 percent. We also note that, to within a tenth of a percent, $A = 1.0$ coincides with $A = \infty$.

The case $A \to \infty$ is essentially a single layer with outer walls of different conductivity ratios X_{B_1} and X_{0_1}. The case $X_{B_1} = X_{0_1}$ was solved exactly by Hurle et al. [10] and the general case was solved approximately by Gershuni and Zhukhovitskii [1]. A comparison to those works is made in Table 3.

The case $X_{0_1} = 0$ was approximated in [2]. A comparison of the present exact solutions to the former approximate solutions (Table 4) shows Catton and Lienhard's Ra_c values to be accurate to about 0.6 percent for thin midlayers and 0.02 percent for $A = 1.0$, the worst errors occurring for $X_{B_1} = 0(1)$.

Finally, a few calculations were made for the odd modes of

Table 4 Comparison to Catton and Lienhard (CL), $X_0 = 0$

X_{B_1}	A	0.01	0.1	0.3	1.0
0	PW	1707.76	-	-	-
	CL	1708	-	-	-
0.2	PW	1338.49	1525.71	1612.08	1634.67
	CL	1345.3	1527.9	1612.7	1634.9
1.0	PW	1304.95	1372.95	1452.47	1491.90
	CL	1312.6	1378.5	1454.7	1492.2
100	PW	1295.87	1296.69	1298.12	1299.36
	CL	1299.8	1297.5	1298.5	1299.6

Table 5 Odd modes of the two-layer system, $A = 0.1$

X_0 \ X_B	0.1	1.0	10.0	100.0
0	1694.965	1609.318	1400.371	1309.836
0.1	1655.331	1570.484	1360.753	1267.868
1.0	1408.697	1399.015	1182.588	1072.172
10.0	1318.172	1239.070	1011.321	859.441
∞	1284.511	1205.946	975.458	807.893

the two-layer system. The coefficient Λ_0, however, is exactly the boundary condition employed by Nield [11] in his study of a single layer with one isothermal wall and a finite conductivity, finite thickness slab covering the other isothermal wall. Therefore, since the results for $X_{0_1} = 0$ have already been found by Nield, we give only a brief table (Table 5) illustrating the effect of taking $X_{0_1} > 0$.

General Midlayer Disturbances; Multilayer Arrays

The preceding results were easily obtained because the midlayer disturbances could be found *a priori* by symmetry considerations. More general configurations lack such symmetry, however, and we must develop a more systematic approach for these cases.

Recall the general solution for Θ_B (equation (6)). Defining $\alpha = M/N$ ($-\infty < \alpha < \infty$), we obtain

$$\Theta_{B_j} = N_j(\alpha_j \cosh(a_{B_j} z_{B_j}) + \sinh(a_{B_j} z_{B_j}))$$

where the subscript j denotes the jth solid layer. Elimination of N between conditions (8) as before produces

$$\Theta_i' = \Lambda_{ij}\Theta_i$$

with

$$\Lambda_{ij} = \left(\frac{a_i}{X_{B_{ij}}}\right)\left\{\frac{\cosh(a_i A_{ij}) \pm \alpha \sinh(a_i A_{ij})}{\alpha \cosh(a_i A_{ij}) \pm \sinh(a_i A_{ij})}\right\} \tag{23}$$

wherein the minus sign applies at the upper surface of a fluid layer ($z_{B_j} = -1/2$; $z_i = 1/2$) and

$$A_{ij} = \left(\frac{L_{B_j}}{2L_i}\right), \quad a_{B_j} = 2A_{ij}a_i, \quad \text{and} \quad X_{B_{ij}} = \frac{k_i}{k_{B_j}}$$

We have successfully removed one of the two unknowns. To solve for the remaining unknown, α, we require an additional constraint. At this point, we observe that, for given α, we may solve directly for Ra in a particular fluid layer by substituting the appropriate Λ's into equation (21) and proceeding as before. This suggests an iterative solution for α subject to a

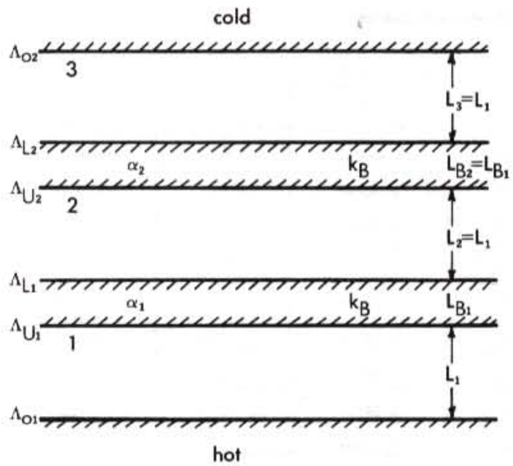

Fig. 5 The symmetric three-layer configuration

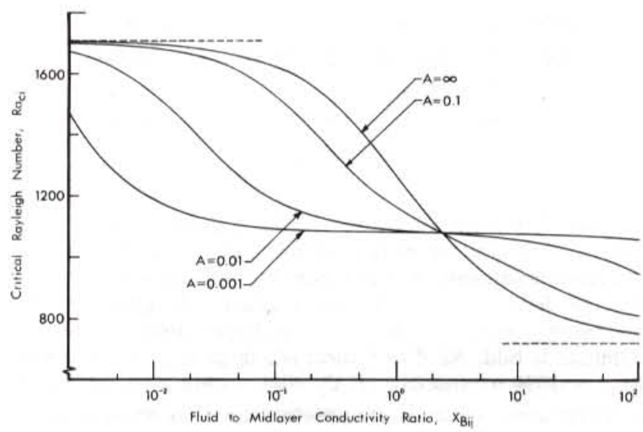

Fig. 6 Critical Rayleigh number of the three-layer problem

matching condition which we now develop for a multilayer array.

We seek a relationship between the Rayleigh numbers of the various fluid layers at onset. In the marginal state, one-dimensional conduction prevails. Accordingly, we may use the "voltage divider" to find ΔT_i

$$\Delta T_i = \frac{(\Delta T_{OA})(L/k)_i}{\sum_j (L/k)_j + \sum_j (L/k)_{B_j}} \tag{24}$$

where ΔT_{OA} is the temperature difference between the outermost boundaries of the fluid layers. From this

$$(\mathrm{Ra}_i/\mathrm{Ra}_j) = (L_i/L_j)^4 \tag{25}$$

if each layer has the same fluid. Thus, we may solve for α_j and Ra_i by iterating guesses for the α's until equation (25) is satisfied throughout. Notice that we need only find the roots of equation (21) corresponding to each layer's Λ's—no more complicated eigenvalue problem need be solved. We have reduced the multilayer problem to a set of one layer problems. Presently, we shall illustrate this approach for the three-layer problem. Before doing so, we note that equation (25) allows us to relate the individual layer critical Rayleigh number to the overall critical Rayleigh number Ra_T which is of interest to the designer. The result is

$$\mathrm{Ra}_T \equiv \left(\frac{g\beta}{\nu\alpha}\right)\left(\sum_j L_j + \sum_j L_{B_j}\right)^3 \Delta T_{OA}$$

$$= \mathrm{Ra}_i \left(\frac{k}{L}\right)_i \left(\sum_j L_j + \sum_j L_{B_j}\right)^3 \left(\sum_j \left(\frac{L}{k}\right)_j + \sum_j \left(\frac{L}{k}\right)_{B_j}\right)$$

We note in passing that convective heat transfer behavior for multilayer problems is best described in terms of Ra_T (see, e.g., [7]). The basic trend in the heat transfer rate is that, when the critical value of Ra_T decreases, the heat transfer at a given supercritical Ra increases. However, from the preceding equation we see that Ra_T and Ra_i need not vary in the same direction as a function of conductivity or aspect ratios (see the discussion in [2]). Thus, some care is required in inferring the effects of midlayer conductivity and thickness on heat transfer from stability results for Ra_i.

Three-Fluid Layer Stability

When a fluid layer is partitioned with two identical midlayers, evenly spaced, the resulting configuration is symmetric about the centerline of the innermost fluid layer (Fig. 5). Symmetry considerations show that $\alpha_1 = -\alpha_2$ and $\Lambda_{U2} = \Lambda_{L1}$; therefore, we need solve only for α_2, say. (In asymmetric situations both α's must be found; this is possible, but more complication than needed for illustration.) The eigenvalue problems in layers one and three are equivalent.

For convenience, we take the outermost boundaries to be isothermal ($\Lambda_{0_i} \rightarrow \infty$). Layer three has the boundary conditions

$$\Theta_3(1/2) = 0$$

$$\Theta_3'(-1/2) = \Lambda_{L2}\Theta_3(-1/2)$$

with

$$\Lambda_{L2} = \left(\frac{a_3}{X_{B_{32}}}\right)\left\{\frac{\cosh(a_3A_{32}) + \alpha_2 \sinh(a_3A_{32})}{\alpha_2 \cosh(a_3A_{32}) + \sinh(a_3A_{32})}\right\} \tag{26}$$

while layer two has

$$\Theta_2'(1/2) = \Lambda_{U2}\Theta_2(1/2)$$

$$\Theta_2'(-1/2) = -\Lambda_{L1}\Theta_2(-1/2) = -\Lambda_{U2}\Theta_2(-1/2)$$

with

$$\Lambda_{U2} = \left(\frac{a_2}{X_{B_{32}}}\right)\left\{\frac{\cosh(a_2A_{22}) - \alpha_2 \sinh(a_2A_{22})}{\alpha_2 \cosh(a_2A_{22}) - \sinh(a_2A_{22})}\right\} \tag{27}$$

Observe that

$$a_3 = a_2 \quad \text{and} \quad A_{32} = A_{22}$$

The matching condition (25) becomes

$$\mathrm{Ra}_2 = \mathrm{Ra}_3$$

Solutions for this geometry were obtained iteratively. With a guess for α_2, Ra_3 was found by solving equation (21) with

$$\Lambda_U = \infty, \quad \Lambda_L = \Lambda_{L2}$$

and Ra_2 from equation (21) with

$$\Lambda_U = \Lambda_L = \Lambda_{U2}$$

The α_2 root of $(\mathrm{Ra}_3 - \mathrm{Ra}_2)(\alpha) = 0$ was then obtained via bisection. (Note that sharp changes in the function $(\mathrm{Ra}_3 - \mathrm{Ra}_2)(\alpha)$ made more sophisticated root-finding procedures, such as the secant method, unsuitable.) After obtaining the root α_2, the function $\mathrm{Ra}_i(a)$ was minimized as before.

The three-layer stability limit is presented in Table 6 and Fig. 6 for various A and X_B. Basic trends in Ra_{c_i} are as expected. When $X_B \rightarrow 0$, $\mathrm{Ra}_{c_i} \rightarrow 1708$ (isothermal midlayers) and when $X_B \rightarrow \infty$, $\mathrm{Ra}_{c_i} \rightarrow 720$ (fixed heat flux midlayers). These limits, together with $A \rightarrow \infty$ (single layer with identical, finite outer wall conductivities), respresent decoupling of the fluid layers.

Table 6 Critical Rayleigh numbers for the three-layer system

$A_{ii} = 0.001$

$X_{B_{ii}}$	$-\alpha_2$	a_i	Ra_i
0.005	365.007	2.27	1270.78
0.01	209.323	2.19	1187.49
0.02	112.572	2.16	1137.41
0.05	47.1581	2.15	1104.24
0.1	23.9591	2.15	1092.61
0.2	12.0781	2.15	1086.67
0.5	4.85546	2.15	1083.03
1.0	2.43182	2.14	1081.72
2.0	1.21718	2.14	1080.92
5.0	0.487393	2.14	1079.95
10.0	0.244104	2.14	1078.82
20.0	0.122277	2.14	1076.74
50.0	0.0492742	2.13	1070.73
100.0	0.0249956	2.11	1061.05

$A_{ii} = 0.1$

$X_{B_{ii}}$	$-\alpha_2$	a_i	Ra_i
0.005	9.04824	3.10	1696.87
0.01	8.96797	3.07	1686.22
0.02	8.79883	3.03	1665.61
0.05	8.29346	2.92	1609.08
0.1	7.47959	2.75	1530.34
0.2	6.05125	2.52	1417.30
0.5	3.56912	2.26	1257.23
1.0	2.08358	2.16	1164.85
2.0	1.16421	2.10	1096.96
5.0	0.555987	1.99	1020.04
10.0	0.347477	1.86	960.88
20.0	0.239858	1.68	904.05
50.0	0.165709	1.42	842.18
100.0	0.132730	1.22	807.62

$A_{ii} = 0.01$

$X_{B_{ii}}$	$-\alpha_2$	a_i	Ra_i
0.005	86.9031	2.92	1612.61
0.01	78.0692	2.76	1536.03
0.02	62.7325	2.53	1425.47
0.05	36.4911	2.27	1270.64
0.1	20.9289	2.19	1187.27
0.2	11.2578	2.16	1136.98
0.5	4.71905	2.15	1103.22
1.0	2.40015	2.14	1090.59
2.0	1.21259	2.14	1082.65
5.0	0.491045	2.13	1073.09
10.0	0.249515	2.11	1062.23
20.0	0.129106	2.08	1043.51
50.0	0.0585810	1.98	999.03
100.0	0.0357651	1.85	950.82

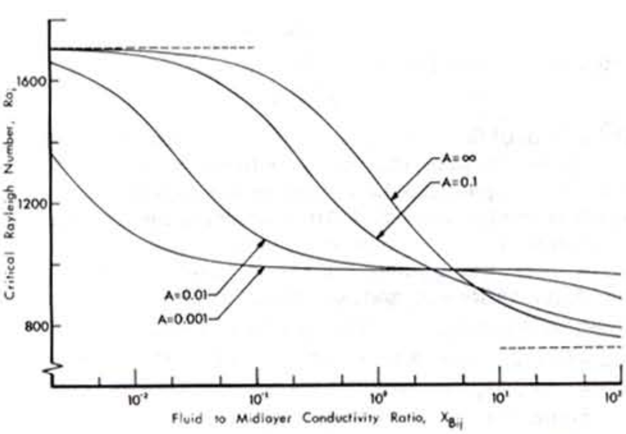

Fig. 7 Critical Rayleigh number of the four-layer problem

Observe that the curves for varius A cross at $X_B \simeq 2.4$. Here $\alpha = -1$ and we see from equations (26) and (27) that the Λ's are independent of A at this point and equal to (a_i/X_B). To the left, $\alpha_2 < -1$ and midlayer disturbances are dominantly even; to the right, midlayer disturbances are dominantly odd. As $A \to 0$ there is a large region over which $Ra_{c_i} \simeq 1080$ irrespective of X_B (viz., within a certain range midlayer conductivity is unimportant for thin midlayers). In this region $-\alpha X_B \simeq 2.4$, as one might deduce by formally setting $A = 0$ in equations (26) and (27); expansion of equations (26) and (27) shows that this region is characterized by $1 \gg |\alpha|\, aA \gg (aA)^2$. Accordingly, we find the [empirical] result that

$$Ra \simeq 1080, \quad a \simeq 2.14$$

when

$$(a_iA)^2 \ll 2.4\left(\frac{a_iA}{X_B}\right) \ll 1$$

However, an attractive conclusion such as (22) is not found for the three-layer problem.

Symmetric Four-Layer Array

If a fluid layer is partitioned with three evenly spaced, identical midlayers we have symmetry about the center of the innermost midlayer. Reductions similar to those used in the two and three-layer cases show that we need use only a single α and a pair of layers having

$$\Theta_4(1/2) = 0 \qquad \text{(say)}$$

$$\Theta_4'(-1/2) = \Lambda_{L4}\Theta_4(-1/2)$$

$$\Theta_3'(1/2) = \Lambda_{U3}\Theta_3(1/2)$$

$$\Theta_3'(-1/2) = \Lambda_{L3}\Theta_3(-1/2)$$

with

$$\Lambda_{L4} = \left(\frac{a_4}{X_{B_{43}}}\right)\left\{\frac{\cosh(a_4A_{43}) + \alpha_3\sinh(a_4A_{43})}{\alpha_3\cosh(a_4A_{43}) + \sinh(a_4A_{43})}\right\}$$

$$\Lambda_{U3} = \left(\frac{a_3}{X_{B_{33}}}\right)\left\{\frac{\cosh(a_3A_{33}) - \alpha_3\sinh(a_3A_{33})}{\alpha_3\cosh(a_3A_{33}) - \sinh(a_3A_{33})}\right\}$$

Table 7 Critical Rayleigh numbers for the four-layer system

A_{ii} = 0.001

$X_{B_{ii}}$	$-\alpha_3$	a_i	Ra_i
0.005	288.969	2.04	1192.28
0.01	161.266	1.94	1096.33
0.02	85.3790	1.91	1040.44
0.05	35.4119	1.89	1004.19
0.1	17.9379	1.89	991.62
0.2	9.02782	1.89	985.22
0.5	3.62590	1.89	981.31
1.0	1.81586	1.89	979.93
2.0	0.909067	1.89	979.08
5.0	0.364419	1.89	978.11
10.0	0.182772	1.88	977.02
20.0	0.0919105	1.88	975.05
50.0	0.0374406	1.87	969.49
100.0	0.0192864	1.84	960.92

A_{ii} = 0.01

$X_{B_{ii}}$	$-\alpha_3$	a_i	Ra_i
0.005	70.5713	2.89	1598.56
0.01	63.8184	2.69	1509.11
0.02	51.3663	2.38	1377.34
0.05	28.9029	2.04	1192.16
0.1	16.1297	1.94	1096.12
0.2	8.54649	1.91	1040.04
0.5	3.55188	1.89	1003.20
1.0	1.80402	1.89	989.68
2.0	0.913255	1.88	981.40
5.0	0.373372	1.87	972.04
10.0	0.192665	1.85	962.18
20.0	0.102307	1.81	946.49
50.0	0.0482971	1.71	912.89
100.0	0.0302173	1.58	879.24

A_{ii} = 0.1

$X_{B_{ii}}$	$-\alpha_3$	a_i	Ra_i
0.005	7.42090	3.09	1695.39
0.01	7.36231	3.07	1683.27
0.02	7.23779	3.02	1659.77
0.05	6.86719	2.89	1594.87
0.1	6.23853	2.68	1503.35
0.2	5.07408	2.38	1369.52
0.5	2.92593	2.02	1180.07
1.0	1.68672	1.90	1076.27
2.0	0.949694	1.83	1005.79
5.0	0.466513	1.71	936.81
10.0	0.296231	1.59	891.11
20.0	0.205361	1.43	850.03
50.0	0.141446	1.20	806.30
100.0	0.112737	1.04	781.94

$$\Lambda_{L3} = \left(\frac{a_3}{X_{B_{32}}}\right) \tanh\,(a_3 A_{32})$$

and

$$\mathrm{Ra}_{c3} = \mathrm{Ra}_{c4}$$

$$a_3 = a_4;\quad X_{B_{32}} = X_{B_{33}} = X_{B_{34}};\quad A_{32} = A_{33} = A_{43}$$

Results of this calculation are given in Table 7 and Fig. 7. The behavior of the critical Rayleigh number is qualitatively very similar to the three layer case, but for each value of A the stability limit is lower. The curves for each A do not have a common intersection point because the third kind condition on the center layer is quite different from those on the outer midlayers. We again find a flattening of the stability curve for thin midlayers in a region characterized by the same inequality as the three-layer flat spot but having instead

$$\mathrm{Ra} \approx 980,\quad a \approx 1.89$$

Comparison to Experiments

At present a very limited amount of experimental data is available for multilayer arrays. Ulrich [9] obtained critical Rayleigh numbers of 1319 for a symmetric two-layer system and of 1154 for a symmetric three-layer system, both having isothermal outer walls and thin Teflon midlayers (A = 0.001, X = 0.01). Hollands and Wright [12] obtained a critical Rayleigh number of ~1220 for the same type of two-layer configuration. All three points are within 6 percent or better of the predictions made herein and thus agree to well within the experimental uncertainty of the data.

Table 8 Comparison of exact solution to Sparrow et al. approximate model

A = 0.01

X_B \ Y	1	3	10
(Ra_{sp})	(1398.508)	(1497.594)	(1607.104)
Exact			
0.01	1648.346	1653.344	1665.496
	3.00	3.02	3.05
0.1	1483.324	1541.551	1617.780
	2.78	2.91	3.02
1.0	1407.859	1497.192	1594.400
	2.75	2.89	3.01

A = 0.1

X_B \ Y	1	3	10
(Ra_{sp})	(1398.508)	(1497.594)	(1607.104)
0.1	1641.724	1645.910	1655.666
	3.00	3.01	3.04
1.0	1460.544	1494.599	1529.101
	2.76	2.85	2.92
10.0	1344.928	1353.619	1357.887
	2.64	2.66	2.67

Limitations of the Sparrow et al. Model

To come full circle, we may apply our generalized approach to the prototype convective condition with which we began (equation (11)). This model, from Sparrow et al. [6], applies a convective thermal boundary condition directly to the upper surface of a rigidly bounded fluid layer, neglecting the possible effects of horizontal conduction in the cover plate. We wish to assess the validity of this assumption.

If a single fluid layer 1 is covered by a conductive slab B having above it a uniform heat transfer coefficient h, we easily find that

$$\alpha_B = -\frac{(\gamma_e S + a_B C)}{(a_B S + \gamma_e C)}$$

where $\gamma_e = (hL_B/k_B)$, $S = \sinh(a_B/2)$, and likewise for C. Substitution into equation (23) and some algebra yield the thermal coefficient at the top of the fluid layer

$$\Lambda_U = \left(\frac{a_1}{X_B}\right)\left\{\frac{1 + (a_B/\gamma_e)\left(\dfrac{2SC}{S^2+C^2}\right)}{(a_B/\gamma_e) + \left(\dfrac{2SC}{S^2+C^2}\right)}\right\} \quad (28)$$

Following [6], we define $\gamma = (hL_i/k_i) = 2AX_B/\gamma_e$. Some analysis then shows that

$$\Lambda \simeq \gamma$$

only if

$$a_1^2\left(\frac{L_B}{L_1}\right)^2 \ll \gamma_e \ll 1 \quad (29)$$

As a typical example for solar collectors, consider a 1/8 in. glass cover plate (mean length: 0.75 m) over an air layer with an 11 mph wind (300 K) above. Then

$$\gamma_e \approx 0.025 \ll 1$$

and, if $a_1 \approx 3$, the first inequality of (29) might be satisfied if $L_1 \approx 7\ 1/2$ in. A more reasonable value would be $L_1 = 1$ in., however, an exact solution for this value using equation (28) yields

$$\mathrm{Ra}_1 = 1687.9, \quad a_1 = 3.08$$

Here $\gamma \cong 10$ and the Sparrow et al. approximation yields

$$\mathrm{Ra}_1 = 1607.1, \quad a_1 = 3.03$$

While the error is only 5 percent of the absolute Ra_i, it represents 20 percent (80/(1708 − 1296)) of the possible variation in Ra_i. Clearly, the Sparrow et al. approximation should not be used indiscriminantly. Some additional values are given in Table 8 to help put this example in perspective.

Finally, we remark that the result of [2] that maximum stability occurs for evenly spaced midlayers will only be true when the outer bounding surfaces are of the same conductivity. In general, maximum stability is achieved by placing the midlayer closer to the less conductive boundary. Exactly how much closer must be determined by further study.

Summary and Conclusions

- The procedure developed herein facilitates simple, exact calculation of the stability limit of arbitrary combinations of fluid layers, conductive midlayers, and conductive boundaries.
- The wavenumber dependence of the fluid layer thermal boundary condition drastically affects the stability limit of that layer.
- The stability limit of the symmetric two-layer system with conductive boundaries is as given in Table 1.
- Previous approximate solutions of the single layer, conductive wall problem and the two-layer problem have been reasonably accurate.
- The stability limit of the three and four-layer systems with identical, evenly spaced midlayers and isothermal boundaries is as given in Tables 6 and 7.
- Cover plate conduction can strongly affect stability with the convective boundary condition of Sparrow et al. [6].
- The present solutions are in good agreement with available experimental data.

References

1 Gershuni, G. Z., and Zhukhovitskii, E. M., *Convective Stability of Incompressible Fluids*, Israel Program for Scientific Translations, Jerusalem, 1976.

2 Catton, I., and Lienhard, J. H. V., "Thermal Stability of Two Fluid Layers Separated by a Solid Interlayer of Finite Thickness and Thermal Conductivity," ASME JOURNAL OF HEAT TRANSFER, Vol. 106, 1984, pp. 605–612.

3 Chandrasekhar, S., *Hydrodynamic and Hydromagnetic Stability*, Clarendon Press, Oxford, 1961.

4 Lienhard, J. H. V., "Thermal Instability and Heat Transfer in a Singly Partitioned Horizontal Fluid Layer," Master's Thesis in Engineering, UCLA, 1984.

5 Pellew, A., and Southwell, R. V., "On Maintained Convective Motion in a Fluid Heated from Below," *Proc. Roy. Soc.* (*London*) *A*, Vol. 176, 1940, pp. 312–343.

6 Sparrow, E. M., Goldstein, R. J., and Jonsson, V. K., "Thermal Instability in a Horizontal Fluid Layer: Effect of Boundary Conditions and Nonlinear Temperature Profile," *J. Fluid Mech.*, Vol. 18, 1964, pp. 513–528.

7 Lienhard, J. H. V., and Catton, I., "Heat Transfer Across a Two-Fluid-Layer Region," ASME JOURNAL OF HEAT TRANSFER, Vol. 108, 1986, pp. 198–205.

8 Low, A. R., "On the Criterion for Stability of a Layer of Viscous Fluid Heated From Below," *Proc. Roy. Soc.* (*London*) *A*, Vol. 125, 1929, pp. 180–195.

9 Ulrich, T. R., "Heat Transfer Across a Multi-Layered Air Enclosure," Master's Thesis in Engineering, UCI, 1984.

10 Hurle, D. T. J., Jakeman, E., and Pike, E. R., "On the Solution of the Bénard Problem With Boundaries of Finite Conductivity," *Proc. Roy. Soc.* (*London*) *A*, Vol. 296, 1967, pp. 469–475.

11 Nield, D., "The Rayleigh-Jeffreys Problem With Boundary Slab of Finite Conductivity," *J. Fluid Mech.*, Vol. 32, 1968, pp. 393–398.

12 Hollands, K. G. T., and Wright, J. L., "Heat Loss Coefficients and Effective Products for Flat-Plate Solar Collectors With Diathermanous Covers," *Solar Energy*, Vol. 30, 1983, pp. 211–216.

APPENDIX

This appendix gives the functions $\mathcal{C}_i$ used in equation (21). Boundary conditions of the form

$$D\Theta + \Lambda_U\Theta = 0, \quad z = 1/2$$

$$D\Theta - \Lambda_L\Theta = 0, \quad z = -1/2$$

applied to equation (3) produce the $\mathcal{C}_i$ indicated in the text as

$$\mathcal{C}_1 = -q_0 \sin(q_0/2) + \Lambda_U \cos(q_0/2) + 2\,\mathrm{Re}\left\{\frac{A}{A_0}(q \sinh(q/2) + \Lambda_U \cosh(q/2))\right\}$$

$$\mathcal{C}_2 = q_0 \cos(q_0/2) + \Lambda_U \sin(q_0/2) + 2\,\mathrm{Re}\left\{\frac{B}{B_0}(q \cosh(q/2) + \Lambda_U \sinh(q/2))\right\}$$

$$\mathcal{C}_3 = q_0 \sin(q_0/2) - \Lambda_L \cos(q_0/2) - 2\,\mathrm{Re}\left\{\frac{A}{A_0}(q \sinh(q/2) + \Lambda_L \cosh(q/2))\right\}$$

$$\mathcal{C}_4 = q_0 \cos(q_0/2) + \Lambda_L \sin(q_0/2) + 2\,\mathrm{Re}\left\{\frac{B}{B_0}(q \cosh(q/2) + \Lambda_L \sinh(q/2))\right\}$$

in which B/B_0 and A/A_0 may be evaluated from equations (9) and (10).