

# The Inverted Pendulum System

The inverted pendulum system is a popular demonstration of using feedback control to stabilize an open-loop unstable system. The first solution to this problem was described by Roberge [1] in his aptly named thesis, “The Mechanical Seal.” Subsequently, it has been used in many books and papers as an example of an unstable system.

Siebert [2, pages 177–182] does a complete analysis of this system using the Routh Criterion, by multiplying out the characteristic equation as a polynomial of  $s$  and studying the coefficients. Although correct, this approach is unnecessarily abstruse. This system is the ideal root-locus analysis example.

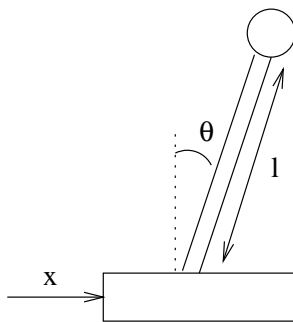


Figure 1: Geometry of the inverted pendulum system

Consider the inverted pendulum system in Figure 1. At a pendulum angle of  $\theta$  from vertical, gravity produces an angular acceleration equal to  $\ddot{\theta}_g = (g/l) \sin \theta$ , and a cart acceleration of  $\ddot{x}$  produces an angular acceleration of  $\ddot{\theta}_x = -(\ddot{x}/l) \cos \theta$ . Writing these accelerations as an equation of motion, linearizing it, and taking its Laplace Transform, we produce the plant transfer function  $G(s)$ , as follows:

$$\begin{aligned} \ddot{\theta} &= \ddot{\theta}_g + \ddot{\theta}_x = (g/l) \sin \theta - (\ddot{x}/l) \cos \theta \\ l\ddot{\theta} - g\theta &= -\ddot{x} \\ G(s) = \frac{\Theta(s)}{X(s)} &= \frac{-s^2}{ls^2 - g} = \frac{-s^2/g}{(\tau_L s + 1)(\tau_L s - 1)} \end{aligned}$$

where the time constant  $\tau_L$  is defined as  $\tau_L = \sqrt{l/g}$ . This transfer function has a pole in the right half-plane, which is consistent with our expectation of an unstable system.

We start the feedback design by driving the cart with a motor with transfer function  $M(s)$  and driving the motor with a voltage proportional to the angle  $\theta$ . Including the familiar motor transfer function

$$M(s) = \frac{X(s)}{V(s)} = \frac{k_M}{s(\tau_M s + 1)}$$

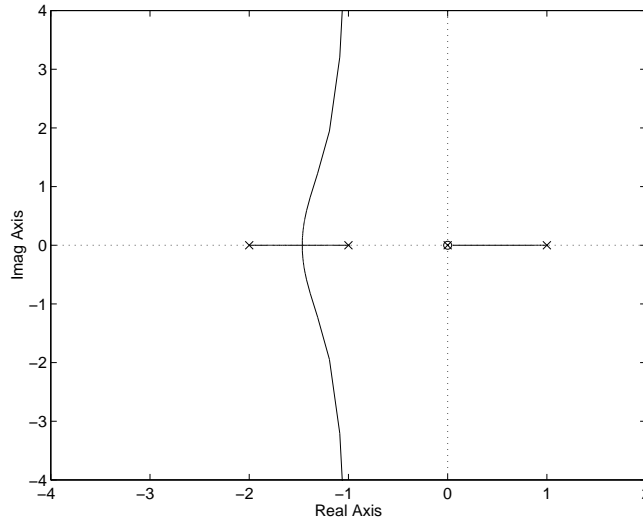


Figure 2: Root-locus plot of pendulum and motor,  $L(s) = M(s)G(s)$

with the plant  $G(s)$ , we get a root locus with one pole that stays in the right half-plane. Using normalized numbers, we get the root-locus plot as is seen in Figure 2.

In order to stabilize the system, we need to get rid of the remaining zero at the origin so that the locus from the plant pole on the positive real axis moves into the left half-plane. Thus our compensator must include a pole at the origin. However, we should balance the added compensator pole with an added zero, so that the number of poles less the number of zeros remains equal to two, leaving the root-locus asymptotes at  $\pm 90^\circ$  (otherwise, the asymptotes would be  $180^\circ$  and  $\pm 60^\circ$  which eventually lead the poles into the right half-plane). Thus we use a compensator

$$K(s) = \frac{\tau_K s + 1}{\tau_K s}$$

and we assume that  $\tau_M < \tau_K < \tau_L$ . The block diagram of the system is shown in Figure 3, and the root-locus plot becomes as in Figure 4 (note that since there is an inversion in  $G(s)$ , we draw the block diagram with a positive summing junction).

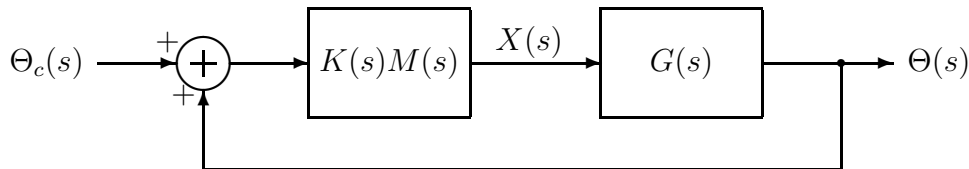


Figure 3: Block diagram of the compensated system

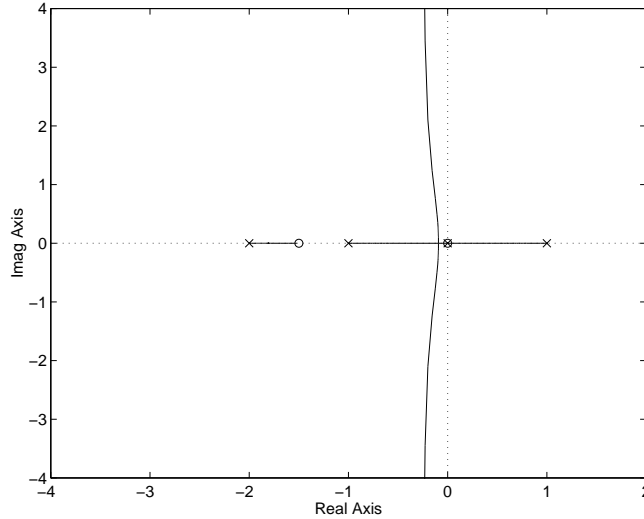


Figure 4: Root-locus plot of pendulum with integrating compensator,  $L(s) = K(s)M(s)G(s)$

Siebert explains that a physical interpretation for the need for this integrator arises from the fact that we are using a voltage-controlled motor. Without the integrator a constant angular error only achieves a constant cart velocity, which is not enough to make the pendulum upright. In order to get “underneath” the pendulum, the cart must be accelerated; therefore, we need the integrator.

This system is now demonstrably stable, however, the root locus is awfully close to the  $j\omega$ -axis. The resulting closed-loop system has a very low margin of stability and would have very oscillatory responses to disturbances. An easy fix to this problem is to decrease the motor time constant with velocity feedback, which moves the centroid of the asymptotes to the left. The root-locus plot of this system is seen in Figure 5.

Unfortunately, there is still a problem with this system, albeit subtle. Consider the closed-loop transfer function from  $\theta_c(t)$  to  $x(t)$  in Figure 3.

$$\frac{X(s)}{\Theta_c(s)} = \frac{K(s)M(s)}{1 - K(s)M(s)G(s)} = \frac{1}{s^2} \left( \frac{k_M(\tau_K s + 1)(\tau_L^2 s^2 - 1)}{\tau_K(\tau_M s + 1)(\tau_L^2 s^2 - 1) + (k_M/g)(\tau_K s + 1)} \right)$$

The poles at the origin makes the system subject to drift. With these integrators, Murphy’s Law guarantees that the time response of  $x(t)$  will grow without bound, and the cart will quickly run out of track.

The solution is positive feedback around the motor and compensator. This feedback loop has the effect of moving the poles off the origin, thus preventing the pole/zero cancellations that are the source of this uncontrollable mode. The root-locus plot of the corrected system appears in Figure 6.

Siebert notes that this positive feedback causes the motor to initially make deviations in  $x(t)$  worse, but that this behavior is the desired effect. When balancing a ruler in your hand,

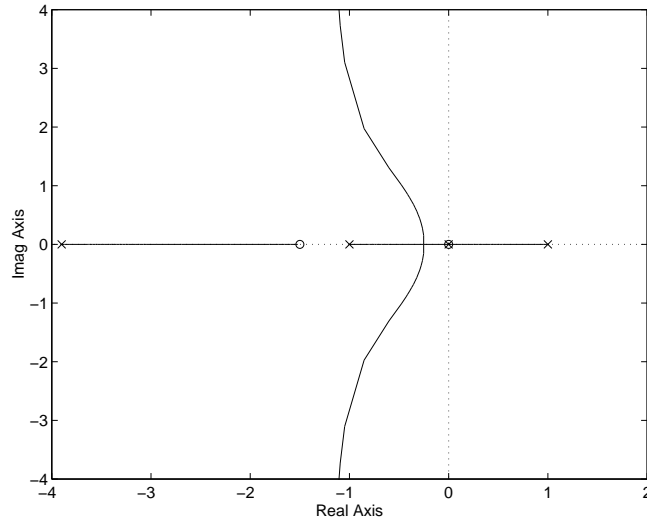


Figure 5: Root-locus plot of pendulum with improved motor time constant

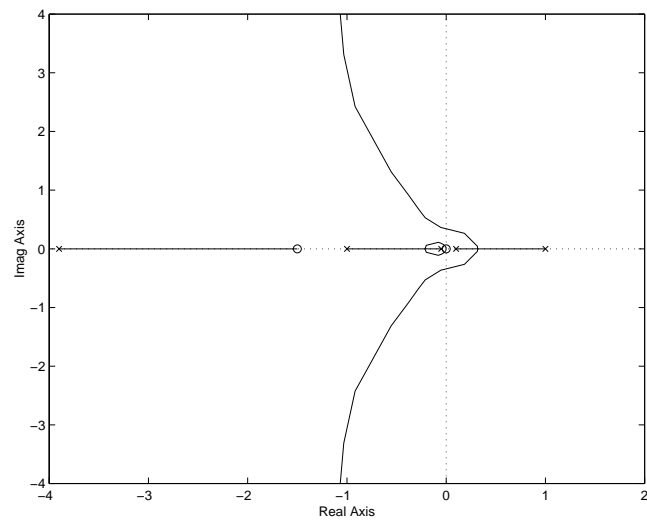


Figure 6: Root-locus plot of pendulum with position compensation

to move the ruler to the right, you must first move your hand sharply to the left, pointing the ruler to the right, so that when you catch the ruler, you have moved both your hand and ruler to the right.

Physically, the pendulum is stabilized at a small angle from vertical, such that it always points toward the center of the track. Thus, the pendulum is always “falling” toward the center of the track, and the only possible equilibrium is a vertical pendulum in the middle of the track. If the cart is to the left of the track center, the control will stabilize the pendulum pointing to the right, such that it then falls a little more to the right. To catch the falling pendulum, the cart must move to the right (back toward the center). That motion is the desired behavior!

## References

- [1] James K. Roberge. The mechanical seal. Bachelor’s thesis, Massachusetts Institute of Technology, May 1960.
- [2] William McC. Siebert. *Circuits, Signals, and Systems*. MIT Press, Cambridge, Massachusetts, 1986.