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LARGE EIGENVALUE PROBLEMS IN DYNAMIC ANALYSIS

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INTRODUCTION

Problem Definition.—The equations of motion for a system of structural elements can be written as

$$M\ddot{u} + C\dot{u} + Ku = P \quad \ldots\ldots (1)$$

in which M = the mass, C = the damping, and K = the stiffness matrix of the system; vectors u, $\dot{u}$, $\ddot{u}$, and P are the displacements, velocities, accelerations and loads, respectively (4). The matrices M, C, and K are obtained in the analysis of building structures idealized as an assemblage of beam elements and in the analysis of continuums using a discretization technique such as the finite element method (13).

Assume that the elements in the stiffness, mass, and damping matrix are constant and that a mode superposition analysis is considered to be most economical. The first step in this analysis is to consider free vibration conditions

$$M\ddot{u} + Ku = 0 \quad \ldots\ldots (2)$$

Substituting $u = \phi \sin \omega(t - t_o)$ $\ldots\ldots (3)$

the generalized eigenvalue problem

$$K\phi = \omega^2 M\phi \quad \ldots\ldots (4)$$

is obtained. The n eigenvalues give the natural frequencies of the system and the eigenvectors are the corresponding vibration modes. The complete solu-

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tion to Eq. 4 can be written as

$$K\Phi = M\Phi\Omega^2 \quad \ldots\ldots (5)$$

in which the columns in Φ are the eigenvectors ϕ_i and $\Omega^2 = \text{diag}\,(\omega_i^2)$.

The basis is now changed from the physical coordinate basis which was used to establish Eq. 1 to the M-orthonormal basis of eigenvectors. Using $u = \Phi X$ with the vector, X, listing the coordinates in the new basis, the equilibrium Eqs., Eq. 1, become

$$\ddot{X} + \Delta\dot{X} + \Omega^2 X = \Phi^T P \quad \ldots\ldots (6)$$

in which $\Delta = \Phi^T C \Phi$ and is assumed to be diagonal. This requires that the damping matrix is of a restricted form as described in Ref. 12; Eq. 6 then consists of n uncoupled equations, which can readily be solved (4).

The most time consuming step in the analysis can be the solution of the eigenvalue problem. If the order of the matrices is large, the computer time required to solve for all eigenvalues and vectors can be enormous. However, the structure may respond primarily in a few modes and the contribution of the other modes may be negligible. In particular, in earthquake response analysis it is often sufficiently accurate to consider only the lowest frequencies and corresponding vibration modes. In this case a solution routine is needed which calculates only the required frequencies and vectors with optimum efficiency.

For the eigenvalue solution it is of particular importance that for most structural systems both matrices K and M are banded, i.e., $k_{ij} = 0$ for $j > i + m_K$ and $m_{ij} = 0$ for $j > i + m_M$ in which $(2m_K + 1)$ and $(2m_M + 1)$ are the bandwidths of the matrices. If all rigid body modes have been removed from the system, K is positive definite. If a consistent mass formulation is used, M is also positive definite and $m_M = m_K$. But experience has shown that accurate results may often be obtained using a lumped mass formulation in which M is diagonal with m_{ii} positive or zero.

Rayleigh-Ritz Method.—For large structural systems the order of the matrices can be several thousand; therefore, approximate techniques have been developed to calculate the lowest eigenvalues and vectors. A very general technique is the Rayleigh-Ritz analysis (5). Let V_n denote the n-dimensional space in which the operators K and M are defined. The Rayleigh minimum principle states that

$$\omega_1^2 = \min\ \rho(v) \quad \ldots\ldots (7)$$

in which the minimum is taken over all vectors v in V_n and $\rho(v)$ is the Rayleigh quotient defined as

$$\rho(v) = \frac{v^T K v}{v^T M v} > 0 \quad \ldots\ldots (8)$$

In the Ritz analysis all vectors $\bar{v}$ in a q-dimensional subspace of V_n are considered. A typical element in the subspace is given by

$$\bar{v} = \sum_{i=1}^{q} a_i x_i \quad \ldots\ldots (9)$$

in which the x_i = the Ritz basis vectors and the a_i = the Ritz coordinates. Substituting Eq. 9 into Eq. 8,

$$\rho(\bar{v}) = \frac{\sum_{j=1}^{q}\sum_{i=1}^{q} a_j\, a_i\, \tilde{k}_{ij}}{\sum_{j=1}^{q}\sum_{i=1}^{q} a_j\, a_i\, \tilde{m}_{ij}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

is obtained with $\tilde{k}_{ij} = x_i^T\, K\, x_j$ ..... (11)

$$\tilde{m}_{ij} = x_i^T\, M\, x_j \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

The necessary condition for a minimum of $\rho(\bar{v})$ is $\partial\rho(\bar{v})/\partial a_i = 0$, $i = 1, \ldots q$. This gives

$$\tilde{K}a = \rho\tilde{M}a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

in which a = a vector listing the Ritz coordinates; $\tilde{K}$ and $\tilde{M}$ = the generalized stiffness and the generalized mass matrix with typical elements given in Eqs. 11 and 12.

The solution of Eq. 13 yields q values $\rho_1, \ldots, \rho_q$ and corresponding vectors $\bar{v}_1, \ldots, \bar{v}_q$, which are obtained using Eq. 9. The values ρ_i, $i = 1, \ldots, q$ are upper bound approximations to the exact eigenvalues of Eq. 4, (5), i.e.:

$$\omega_1^2 \leq \rho_1;\ \omega_2^2 \leq \rho_2; \ldots;\ \omega_q^2 \leq \rho_q \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

The actual error in the solution is, in general, not estimated. It depends on the Ritz basis vectors chosen because the approximate eigenvectors $\bar{v}_1, \ldots, \bar{v}_q$ are elements of the subspace.

The Ritz analysis has been implemented in various ways (9). Ritz basis vectors can be obtained from a static analysis in which q load patterns are specified in P_D(4); then

$$K\, X_D = P_D \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

and $\tilde{K} = X_D^T P_D$; $\tilde{M} = X_D^T M X_D$ ..... (16)

The obvious difficulty is in selecting good load patterns.

Static Condensation Method.—In another scheme which is known as static condensation of the massless degrees-of-freedom, it is assumed that all mass can be lumped at q degrees-of-freedom. Therefore, as an approximation to Eq. 4

$$\begin{bmatrix} K_{aa} & K_{ac} \\ K_{ca} & K_{cc} \end{bmatrix} \begin{bmatrix} \psi_a \\ \psi_c \end{bmatrix} = \omega^2 \begin{bmatrix} M_a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_a \\ \psi_c \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

is obtained with q finite and $(n - q)$ infinite eigenvalues. A reduced eigenvalue problem for the finite eigenvalues is obtained by using static condensation on the ψ_c degrees-of-freedom, then

$$K_a\, \psi_a = \omega^2\, M_a\, \psi_a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

in which $K_a = K_{aa} - Z^T Z$; $K_{cc} = \tilde{L}\tilde{L}^T$; $\tilde{L}Z = K_{ca}$ ..... (19)

This solution is actually a Ritz analysis of the lumped mass model considered

in Eq. 17. The Ritz basis vectors are the displacement patterns associated with the ψ_a degrees-of-freedom when the ψ_c degrees-of-freedom are released. Solving

$$\begin{bmatrix} K_{aa} & K_{ac} \\ K_{ca} & K_{cc} \end{bmatrix} \begin{bmatrix} f_a \\ f_c \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

in which $f_a = K_a^{-1}$,

$$X_D = \begin{bmatrix} I \\ f_c \quad K_a \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

and in Eq. 16 $\tilde{K} = K_a$ and $\tilde{M} = M_a$. In this analysis the lowest q eigenvalues of the lumped mass model in Eq. 17 are calculated exactly, because the Ritz basis vectors span the q-dimensional subspace corresponding to the finite eigenvalues. The accuracy with which the eigenvalues of the lumped mass model approximate the q lowest eigenvalues of the original model in Eq. 4 depends on how well mass was lumped.

In both analyses, the Rayleigh-Ritz and the static condensation method, the smallest eigenvalues are usually approximated best; however, nothing can be said about the accuracy of the eigenvalue approximations obtained. In fact, an approximation to an important eigenvalue may be missed completely.

Accurate Calculation of Required Eigensystem.—Various solution routines have been established to calculate the lowest eigenvalues and corresponding vectors in the generalized eigenvalue problem, Eq. 4, exactly (10). The problem may be transformed into a standard eigenvalue problem and then a Rayleigh quotient iteration with matrix deflation can be used (6). This has the disadvantage that M must be diagonal and positive definite. It is more efficient to solve Eq. 4 directly without a transformation. This is done in Ref. 7. However, the algorithm uses only the Sturm sequence property and is therefore costly unless the bandwidth of the system is very small. A very efficient solution routine has been developed for systems with small to medium bandwidth by combining a determinant search technique with the information obtained from the Sturm sequence property and vector inverse iteration (1). But for systems with large bandwidth and which cannot be taken into high speed storage of the computer, this solution becomes also expensive, mainly because many triangular factorizations are required. The most promising approach for the solution of systems of large order and large bandwidth is the simultaneous iteration with a number of vectors (2,3,8).

The aim of this paper is to present the simultaneous iteration with p vectors in Eq. 4 as a subspace iteration which then leads to a very effective implementation. The algorithm was developed as part of an automatic package for the calculation of eigenvalues and eigenvectors in a general structural analysis program (1,11). The mass matrix can be diagonal with zero elements or may be banded as in a consistent mass formulation. Operation counts are given for both cases in order to enable solution cost estimates. The algorithm is particularly suited for the solution of systems which are too large for the high speed storage of the computer. A program which was written for systems of practically any order and bandwidth is described herein. Two example solutions, namely, the eigensolution of a plane frame and of a complex three-

dimensional building frame, are given in order to show typical convergence characteristics and iteration times needed.

SUBSPACE ITERATION ALGORITHM

Basic Theory.—The objective is to solve for the p lowest eigenvalues and associated eigenvectors satisfying

$$K\Phi = M\Phi\Omega^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (22)$$

in which Φ stores the p eigenvectors and Ω^2 the corresponding eigenvalues. The eigenvectors are an M-orthonormal basis of the p-dimensional, least dominant subspace of the operators, which will now be called E_∞.

Before developing the algorithm, it may be pointed out that the essential idea is to iterate simultaneously with p linearly independent vectors which initially span the starting subspace E_0, until E_∞ is spanned. The required eigenvectors are then computed without further iteration. The total number of required iterations depends, of course, on how close E_0 is to E_∞. But the effectiveness of the algorithm lies in that it is much easier to establish a p-dimensional starting subspace which is close to E_∞ than to find p vectors which each are close to a required eigenvector. Also, convergence of the subspace is all that is required and not of individual iteration vectors to eigenvectors.

To present the algorithm let X_0 store the p starting vectors which span E_0. Consider simultaneous inverse iteration with the vectors, expressed as

$$KX_k = MX_{k-1} \quad k = 1, 2, \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (23)$$

The iteration vectors in X_k span a p-dimensional subspace E_k and the sequence of subspaces generated converges to E_∞. This seems to contradict the fact that in this iteration each column in X_k is known to converge to the least dominant eigenvector unless the column is deficient in ϕ_1. Actually, there is no contradiction. Although in exact arithmetic the vectors in X_k span E_k, they do become more and more parallel and therefore a poorer and poorer basis. One way to preserve numerical stability is to generate orthogonal bases in the subspaces E_k using the Gram-Schmidt process. In this case the iteration is

$$KX_k = MX_{k-1}R_k^{-1} \quad k = 1, 2, \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (24)$$

in which R_k = an upper triangular matrix; provided the starting vectors in X_0 are not deficient in the eigenvectors corresponding to $\omega_1^2, \omega_2^2, \ldots, \omega_p^2$ in which $\omega_1^2 \leq \omega_2^2 \leq \omega_3^2 \ldots \leq \omega_p^2 < \omega_{p+1}^2$ it holds $X_k \to \Phi$; $R_k^{-1} \to \Omega^2$ as $k \to \infty$. Apart from round-off, this iteration generates the same sequence of subspaces as Eq. 23, but in this case the i'th column in X_k converges to ϕ_i with a rate of $\max\{\omega_{i-1}^2/\omega_i^2, \omega_i^2/\omega_{i+1}^2\}$. Essentially, this poor convergence rate results from the orthogonalization of the iteration vectors from the left to the right. For example, no advantage is taken if the third column in X_k is much closer to ϕ_1 than the first column. In particular, assume that the vectors in X_k span E_∞ but are not eigenvectors; then, although the subspace already converged, many more iterations may be needed in order to turn the orthogonal basis of iteration vectors into the basis of eigenvectors.

The following algorithm finds an orthogonal basis of vectors in E_k, thus preserving numerical stability in the iteration of Eq. 23, and also calculates

in one step the required eigenvectors when E_k converged to E_∞.

For $k = 1, 2, \ldots$ iterate from E_{k-1} to E_k

$$K\overline{X}_k = MX_{k-1} \quad \ldots\ldots (25)$$

Find the projections of the operators K and M onto E_k

$$K_k = \overline{X}_k^T K \overline{X}_k \quad \ldots\ldots (26)$$

$$M_k = \overline{X}_k^T M \overline{X}_k \quad \ldots\ldots (27)$$

Solve for the eigensystem of the projected operators

$$K_k Q_k = M_k Q_k \Omega_k^2 \quad \ldots\ldots (28)$$

Find an improved approximation to the eigenvectors

$$X_k = \overline{X}_k Q_k \quad \ldots\ldots (29)$$

Then provided E_0 is not orthogonal to one of the required eigenvectors,

$$\Omega_k^2 \to \Omega^2; \; X_k \to \Phi \text{ as } k \to \infty$$

is obtained.

Assuming that E_k is close to E_∞ the convergence rate of the i'th column in X_k to ϕ_i is $\omega_i^2/\omega_{p+1}^2$. Although this is an asymptotic convergence rate, it indicates that the lowest eigenvalues converge fastest. Also, faster convergence can be obtained by using q iteration vectors, with $q > p$. In the implementation $q = \min\{2p, p + 8\}$ has been used. Note that then multiple eigenvalues do not decrease the convergence rate provided $\omega_{q+1}^2 > \omega_p^2$.

In practice, it is of interest to know what happens in the first few iterations when E_k is not yet close to E_∞. Eqs. 26 to 29 are identified as a Ritz analysis with the vectors in $\overline{X}_k$ as the Ritz basis vectors. Therefore, the eigenvalues in Ω_k^2 are stationary points in conformity with the Rayleigh minimum principle, Eq. 7, and they are upper bound approximations on the eigenvalues sought, Eq. 14.

Starting Subspace.—The number of subspace iterations required for convergence depends on how close the starting subspace is to E_∞. Whenever starting vectors approximate quite well the eigenvectors sought, these vectors should be used in X_0. In this case, the algorithm is ideally suited for solution. For instance, in dynamic optimization, as the structure is modified in small steps, the eigensystem of the previous structure would be a good approximation to the eigensystem of the new structure. Sometimes it may be difficult to judge if the transformation vectors at hand can be considered to be good. In particular, it is important to note that in exact arithmetic, convergence to an eigenvector is not possible, if the starting vectors in X_0 are all orthogonal to the eigenvector. But the conventional Ritz analysis, if excellent load patterns are known, the component mode synthesis and related methods summarized by Uhrig can all be good first subspace iterations (9).

Assume that good transformation vectors are not present and that it is desirable to establish X_0 from the elements in K and M only. It is not necessary to use vectors in X_0 which are close to the required eigenvectors, but only vectors which span a subspace close to E_∞.

The following scheme has been found very effective. The first column in MX_0 is simply the diagonal of M. This assures that all mass degrees-of-freedom are excited in order not to miss a mode. The other columns in MX_0

are unit vectors with + 1 at the coordinates with largest ratios m_{ii}/k_{ii}. This scheme is used because for the special case when K and M are diagonal, these vectors span E_∞. In actual analysis, the same or nearly the same ratio m_{ii}/k_{ii} may occur at many coordinates and it is equally important that the unit entries in the vectors are well spaced for better convergence.

Convergence.—It is apparent that the closeness of E_k to E_∞ and thus convergence, can be measured by the eigenvalue (or eigenvector) approximations calculated using Eqs. 26 to 29. Assuming that in the iterations $(k - 1)$ and k eigenvalue approximations $\omega_i^{2(k-1)}$ and $\omega_i^{2(k)}$ are calculated, then the ratio $\left|\omega_i^{2(k)} - \omega_i^{2(k-1)}\right| / \omega_i^{2(k)}$ may be used as a measure of convergence. For example, for the eigenvalues to be accurate to about 5 digits, it is necessary to iterate until this tolerance is less than 10^{-6}.

Verification of Results.—The starting subspace previously described has proven to be very satisfactory. However, the resulting eigenvalues and eigenvectors may be checked by using the Sturm sequence property (10). This states that in Gauss elimination to evaluate $LDL^T = K - \mu M$, in which μ = the shift, the number of negative elements in D = the number of eigenvalues smaller than μ. In order to use a meaningful μ, it is necessary to find bounds for the exact eigenvalues ω_i^2 using the calculated values $\omega_i^{2(k)}$. A conservative estimate for a region in which the exact eigenvalues lie is given by

$$0.99\ \omega_i^{2(k)} < \omega_i^2 < 1.01\ \omega_i^{2(k)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (30)$$

TABLE 1.—OPERATION COUNT FOR SUBSPACE ITERATIONS

Method (1)	Calculation (2)	Number of Operations $m = m_K = m_M$ (3)	Number of Operations $m = m_K$; $m_M = 0$ (4)
Simple inverse iteration	$K\bar{X}_k = Y_{k-1}$	$nq(2m + 1)$	$nq(2m + 1)$
	$Y_k = M\bar{X}_k$	$nq(2m + 1)$	nq
Total		$2nq(2m + 1)$	$2nq(m + 1)$
Inverse iteration with Gram-Schmidt orthogonalization	$K\bar{X}_k = Y_{k-1}$	$nq(2m + 1)$	$nq(2m + 1)$
	$\bar{Y}_k = M\bar{X}_k$	$nq(2m + 1)$	nq
	$Y_k = \bar{Y}_k R_k^{-1}$	$nq/2\,(3q + 3)$	$nq/2\,(3q + 3)$
Total		$2nq(2m + 3/4\,q + 7/4)$	$2nq(m + 3/4\,q + 7/4)$
Inverse iteration with calculation of operator projections	$K\bar{X}_k = Y_{k-1}$	$nq(2m + 1)$	$nq(2m + 1)$
	$K_k = \bar{X}_k^T Y_{k-1}$	$nq/2\,(q + 1)$	$nq/2\,(q + 1)$
	$\bar{Y}_k = M\bar{X}_k$	$nq(2m + 1)$	nq
	$M_k = \bar{X}_k^T \bar{Y}_k$	$nq/2\,(q + 1)$	$nq/2\,(q + 1)$
	$K_k Q_k = M_k Q_k \Omega_k^2$	$0(q^3)$ neglected . . .	
	$Y_k = \bar{Y}_k Q_k$	nq^2	nq^2
Total		$2nq(2m + q + 3/2)$	$2nq(m + q + 3/2)$

in which only the lowest eigenvalues, which all converged to a tolerance of 10^{-6}, should be included. Eq. 30 can be used to establish bounds on eigenvalue clusters, at which a meaningful Sturm sequence check can be applied.

Operation Count and Summary of Algorithm.—In order to obtain an estimate of the cost to solve an eigenvalue problem consider the number of Central Processor (high speed storage) operations required for solution. The actual cost includes, of course, the cost of the Peripheral Processor (tape and disc reading) time. However, this time is very system and programming dependent and is therefore not considered in this investigation.

Let one operation equal one multiplication which nearly always is followed by an addition. Assume that the half bandwidths of K and M, i.e., m_K and m_M, are full, and neglect terms which involve the bandwidths and number of iteration vectors only.

Before the subspace iteration is started K is factored into LDL^T using Gauss elimination without interchanges. In particular

$$L_{n-1}^{-1} \ldots L_2^{-1} L_1^{-1} K = U \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (31a)$$

in which $L_i^{-1} =$

$$\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & -l_{i+1,i} & & & \\ & & -l_{i+2,i} & \ddots & & \\ & & \vdots & & \ddots & \\ & & -l_{i+m_K,i} & & & 1 \end{bmatrix};$$

$$l_{i+j,i} = \frac{k_{i+j,i}^{(i)}}{k_{ii}^{(i)}}; \; u_{ii} = k_{ii}^{(i)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (31b)$$

and $k_{lm}^{(i)}$ denotes the (l, m) element after the first $i - 1$ row reductions have been carried out. Writing $L = L_1 \ldots L_{n-1}$, $U = DL^T$ and it is only necessary to store the upper band of K. This factorization requires $(1/2)\, n\, m_K^2 + (3/2)\, n\, m_K$ operations.

A summary of the steps in a subspace iteration together with the corresponding number of operations is given in Table 1. Referring to the table, $\overline{X}_k$ is calculated using

$$LDL^T \overline{X}_k = Y_{k-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (32)$$

The reduction gives

$$L^T \overline{X}_k = D^{-1} L^{-1} Y_{k-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (33)$$

and then $\overline{X}_k$ is obtained by a back-substitution.

Table 1 shows that for $m >> q$ about the same number of operations are needed in all three iteration schemes. Note that about twice as many operations are needed when M is banded.

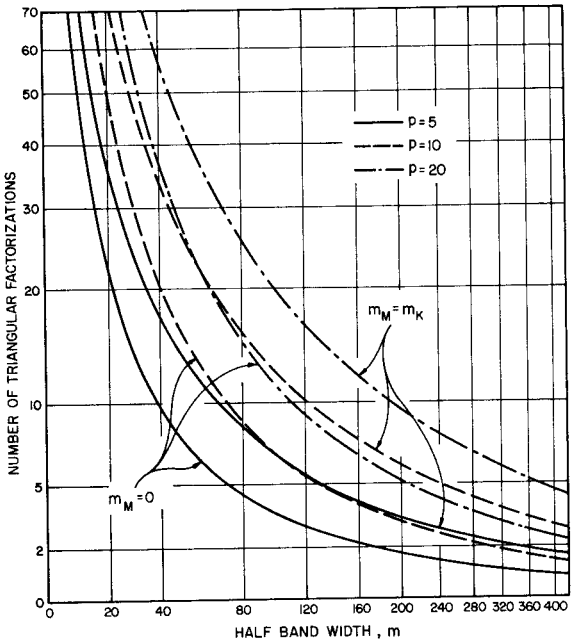


FIG. 1.—NUMBER OF TRIANGULAR FACTORIZATIONS EQUIVALENT TO SUBSPACE ITERATIONS

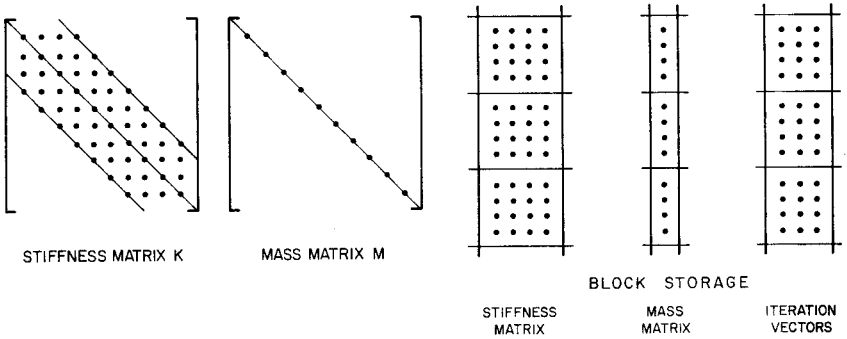


FIG. 2.—BLOCK STORAGE OF STIFFNESS MATRIX, MASS MATRIX, AND ITERATION VECTORS

Let the eigenvalues be required to about five digit precision and let $q = \min\{2p, p + 8\}$, then, with the starting subspace described, by experience about eight subspace iterations are needed. Assume that the projections of K and M

are calculated in each iteration, then the number of triangular factorizations equivalent in operations to the subspace iterations are

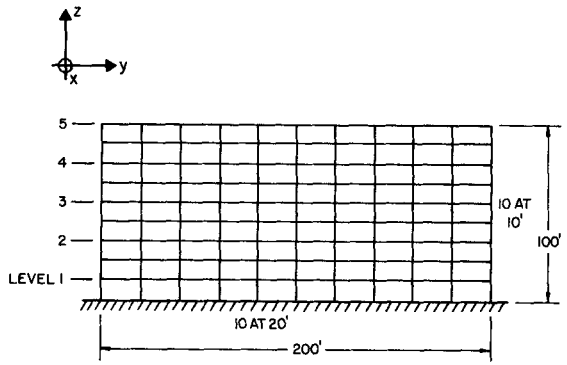
$$\frac{32qm + 32q^2 + 48q}{m^2 + 3m} \quad \text{for } m_M = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (34)$$

and $$\frac{64qm + 32q^2 + 48q}{m^2 + 3m} \quad \text{for } m_M = m_K \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (35)$$

Fig. 1 shows these relations for various values of p. It is seen that with m large the operations required in the subspace iterations are of the order of a triangular factorization. But when m is small, the iterations are equivalent to many factorizations. In this case a determinant search solution algorithm is more efficient (1). Note that at convergence in the iteration a Sturm sequence check is carried out which requires one more triangular factorization.

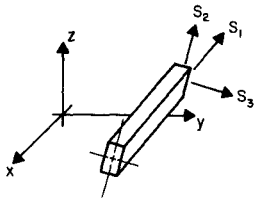
LARGE CAPACITY SOLVER

A large capacity solver was written for the solution of systems which have practically any order and bandwidth (1). The program stores the stiffness



(a) ELEVATION OF FRAME

DATA: YOUNG'S MODULUS = 432000, MASS DENSITY = 1.0
FOR ALL BEAMS AND COLUMNS $A_i = 3.0$, $I_i = I_2 = I_3 = 1.0$
UNITS : FT, KIPS



(b) BEAM ELEMENT DEFINITION

S_1, S_2 AND S_3 = BEAM LOCAL AXES

I_1, I_2 AND I_3 = FLEXURAL INERTIA ABOUT S_1, S_2, AND S_3

A_i = AREA ASSOCIATED WITH S_i

FIG. 3.—PLANE FRAME OF EXAMPLE 1

matrix, the mass matrix, and the iteration vectors in blocks on tape (Fig. 2).

In the solution, the same number of Central Processor operations as in an in-core solution are performed. But in all operations it is now necessary to have at any one time the required matrix elements in high speed storage. In the stiffness factorization, Eq. 31, always only two blocks of K are in high speed core. To perform the reduction and back-substitution of the iteration vectors in Eq. 33, sequentially one block of the factored stiffness matrix and as many vector blocks as are necessary to reduce one block of vectors—or,

TABLE 2.—CALCULATED EIGENVALUES AND SOLUTION TIMES TAKEN IN EXAMPLE ANALYSES

Example	Calculated Eigenvalues ω_i^2				Solution time, central processor sec on CDC 6400
	$i = 1$	$i = 2$	$i = 3$	$i = 4$	
(1)	(2)	(3)	(4)	(5)	(6)
Plane frame	0.589541..	5.52695..	16.5878..		24.48
Building frame	0.41537..	0.54930..	0.78606..	1.0325..	159.59

TABLE 3.—CONVERGENCE CHARACTERISTICS OF SUBSPACE ITERATIONS

Analysis	Number of iteration	Eigenvalue approximations							
		(3)							
(1)	(2)								
Plane frame	1	0.5971	6.937	27.30	80.59	101.1	142.3		
	2	0.5895	5.530	16.73	38.44	46.91	75.81		
	3	0.5895	5.527	16.59	35.75	42.85	67.93		
	4	0.5895	5.527	16.59	35.48	41.74	65.41		
	5	0.5895	5.527	16.59	35.43	41.38	64.44		
	6	0.5895	5.527	16.59	35.42	41.26	64.02		
Building frame	1	0.5206	0.9007	1.329	1.869	4.320	7.550	23.67	296.7
	2	0.4177	0.5529	0.7992	1.075	1.676	3.002	4.666	194.9
	3	0.4154	0.5493	0.7864	1.035	1.498	2.210	2.395	3.656
	4	0.4154	0.5493	0.7861	1.033	1.488	2.008	2.293	3.463
	5	0.4154	0.5493	0.7861	1.033	1.487	1.971	2.272	3.432
	6	0.4154	0.5493	0.7861	1.033	1.487	1.962	2.268	3.415
	7	0.4154	0.5493	0.7861	1.033	1.487	1.959	2.266	3.403
	8	0.4154	0.5493	0.7861	1.033	1.487	1.958	2.266	3.391

in the back-substitution, to obtain the new iteration vectors in one block—are taken into high speed storage. In the stiffness factorization and in the vector reductions and back-substitutions, due account is taken of the varying bandwidth of the system and of zero elements within the band.

The starting subspace is established as previously described. In each subspace iteration the projections of K and M are calculated. When m is large, relatively little more operations are required for the vector orthogonalization, and the advantage is that eigenvalue estimates are obtained in each iteration. Also, in some eigenvalue solutions, as in the analysis of frames, even good

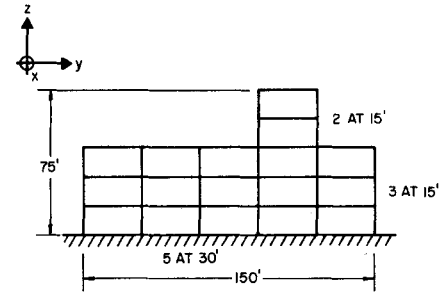
starting vectors may be nearly parallel and should be orthogonalized in the first subspace iteration. With the storage scheme adopted, a Gram-Schmidt orthogonalization would require much tape handling and is not preferable.

For the solution of the eigenvalue problem in Eq. 28, a generalized Jacobi iteration is used. In this iteration K_k and M_k are reduced simultaneously to diagonal form, without a transformation to the standard eigenvalue problem. This avoids numerical difficulties when M_k is ill-conditioned and takes advantage of the fact that K_k and M_k tend towards diagonal form as the number of subspace iterations increases.

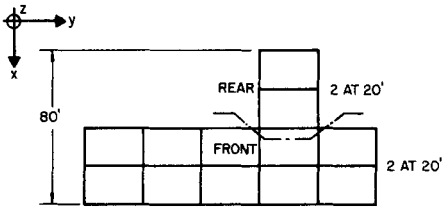
EXAMPLE SOLUTIONS

The program described in the previous section was used for the example analyses. The solution times always include the initial factorization of the stiffness matrix, the subspace iterations and the Sturm sequence check.

Eigensolution of Plane Frame.—The three lowest eigenvalues and corresponding eigenvectors of the 9-story high and 10-bay long frame shown in Fig. 3 were calculated. The stiffness matrix was of order 297, the maximum half



(a) ELEVATION OF BUILDING



(b) PLAN OF BUILDING

DATA:

YOUNG'S MODULUS = 432000 , MASS DENSITY = 1.0
COLUMNS IN FRONT BUILDING : $A_1 = 3.0$, $I_1 = I_2 = I_3 = 1.0$
COLUMNS IN REAR BUILDING : $A_1 = 4.0$, $I_1 = I_2 = I_3 = 1.25$
ALL BEAMS INTO X-DIRECTION : $A_1 = 2.0$, $I_1 = I_2 = I_3 = 0.75$
ALL BEAMS INTO Y-DIRECTION : $A_1 = 3.0$, $I_1 = I_2 = I_3 = 1.0$
UNITS : FT, KIP

FIG. 4.—THREE-DIMENSIONAL BUILDING FRAME OF EXAMPLE 2

bandwidth was 29, and three blocks were used. A lumped mass formulation was employed with zero masses at all rotational degrees-of-freedom. Table 2 gives the final eigenvalues calculated at convergence to a tolerance of 10^{-6} together with the solution time taken. The convergence characteristics of the subspace iteration can be observed in Table 3. Note that the calculated eigenvalues are already accurate to four digits after only three iterations. It is of interest that a Ritz analysis with five transformation vectors obtained by applying unit loads into the y-direction at levels 1, 2, 3, 4, and 5 in Fig. 3 gave as eigenvalue approximations $\omega_1^2 = 0.6113$, $\omega_2^2 = 7.320$ and $\omega_3^2 = 30.08$.

Eigensolution of Three-Dimensional Building Frame.—Fig. 4 shows the building which was analyzed for the four lowest eigenvalues and corresponding vectors. The building was idealized as an assemblage of beam elements only with six degrees-of-freedom at each joint. A lumped mass matrix was used with no mass at all rotational degrees-of-freedom. The order of the system was 468, the maximum half bandwidth was 155, and 13 blocks were used in the solution. Tables 2 and 3 give the eigenvalue approximations calculated in each iteration, the final eigenvalues at convergence to a tolerance of 10^{-6} and the solution time taken. The good convergence of the starting subspace can again be observed.

Note that in both solutions the lowest eigenvalues converge fastest and, that, in each iteration there are upper bounds to the eigenvalues of the discrete element assemblage.

CONCLUSIONS

A very efficient solution technique for large eigenvalue problems in dynamic analysis has been presented. The subspace iteration algorithm solves the eigenvalue problem directly without a transformation to the standard form. The mass matrix may be diagonal with zero elements or banded. The operation counts and the example analyses show the cost effectiveness of the solution technique. A program has been described which solves the eigenvalue problem for any system size and bandwidth. Very large systems, which generally have been analyzed using approximate techniques such as the Rayleigh-Ritz and static condensation method, can economically be solved with this solution routine.

APPENDIX I.—REFERENCES

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APPENDIX II.—NOTATION

The following symbols are used in this paper:

$\mathbf{a}$ = vector of Ritz coordinates;
a_i = Ritz coordinates;
$\mathbf{C}$ = damping matrix;
$\mathbf{D}$ = diagonal matrix;
E_k = subspace spanned by vectors in $\mathbf{X}_k$;
$\mathbf{f}_a$, $\mathbf{f}_c$ = flexibility matrices defined in Eq. 20;
i, j = indices of matrix elements;
$\mathbf{K}$ = stiffness matrix;
$\tilde{\mathbf{K}}$ = generalized stiffness matrix;
$\mathbf{K}_k$ = projection of stiffness matrix, see Eq. 26;
$\mathbf{K}_{aa}$, $\mathbf{K}_{cc}$, $\mathbf{K}_{ac}$, $\mathbf{K}_{ca}$ = submatrices of $\mathbf{K}$, see Eq. 17;
$\mathbf{K}_a$ = stiffness matrix obtained from $\mathbf{K}$ by static condensation of massless degrees-of-freedom, see Eq. 19;
k = subscript indicating number of iteration;
k_{ij} = element of $\mathbf{K}$;
$\tilde{k}_{ij}$ = element of $\tilde{\mathbf{K}}$, see Eq. 11;
$\mathbf{L}$ = lower unit triangular matrix;
$\tilde{\mathbf{L}}$ = Cholesky factor of $\mathbf{K}_{cc}$, see Eq. 19;
l_{ij} = element of $\mathbf{L}$;
$\mathbf{M}$ = mass matrix;
$\tilde{\mathbf{M}}$ = generalized mass matrix;
$\mathbf{M}_a$ = submatrix of $\mathbf{M}$, see Eq. 17;

m = maximum of m_K and m_M;
m_{ij} = element of M;
$\tilde{m}_{ij}$ = element of $\tilde{M}$, see Eq. 12;
m_K = half bandwidth of K;
m_M = half bandwidth of M;
n = order of stiffness and mass matrix;
P = load vector;
P_D = load matrix used in Ritz analysis, see Eq. 15;
p = number of required eigenvalues and eigenvectors;
Q_k = eigenvectors of stiffness and mass matrix projections, see Eq. 28;
q = number of iteration vectors used;
R_k = upper triangular matrix, see Eq. 24;
U = upper triangular matrix;
u, $\dot{u}$, $\ddot{u}$ = displacement, velocity, and acceleration vectors;
u_{ij} = element of U;
V_n = n-dimensional space in which K and M are defined;
$\underline{v}$ = element of V_n;
$\bar{v}$ = element of q-dimensional subspace of V_n, see Eq. 9;
$\bar{v}_i$ = eigenvector approximations calculated in Ritz analysis;
X, $\dot{X}$, $\ddot{X}$ = modal displacement, velocity, and acceleration vectors;
X_D = displacement matrix calculated in Ritz analysis, see Eq. 15;
X_k = iteration vectors;
x_i = Ritz basis vector;
Y_k = iteration vectors weighted with M;
Z = matrix used in the static condensation, see Eq. 19;
Δ = damping matrix;
ρ = Rayleigh quotient, see Eq. 8;
ρ_i = eigenvalue approximations calculated in Ritz analysis;
Φ = matrix of M-orthonormal eigenvectors;
ϕ_i = M-orthonormal eigenvector;
ψ_a, ψ_c = displacements associated with mass and massless degrees-of-freedom, see Eq. 17;
Ω^2 = diagonal matrix storing the eigenvalues;
Ω_k^2 = diagonal matrix storing eigenvalue approximations;
ω_i^2 = eigenvalue and circular frequency squared; and
$\omega_i^{2(k)}$ = approximation to ω_i^2 calculated in iteration k.