THE FINITE ELEMENT METHOD WITH "OVERLAPPING FINITE ELEMENTS"

Klaus-Jürgen Bathe
Massachusetts Institute of Technology, Cambridge, Massachusetts, USA

ABSTRACT: We review our efforts to arrive at a finite element scheme in which the elements can overlap. This property of the elements removes many of the meshing difficulties. The scheme we have been working on for some years to render competitive is the 'method of finite spheres' (where the elements overlap) but the same concepts can also be used for other types of element geometries. We review our latest results in static and dynamic analyses using the method of finite spheres, and propose a general scheme for using overlapping finite elements with traditional finite elements for CAD driven simulations.

1 INTRODUCTION

The finite element method is now established as an effective procedure to simulate on the computer the behavior of structures. Quite general structures can be analyzed, from large scale to very small scale structures, such as long and large bridges, to motor cars, to DNA structures (Bathe 2014, Bathe 2014a). However, in all finite element simulations, it is necessary to establish an appropriate and effective mesh of elements, which may require a large effort for the analyst. Since also quite some experience is needed to construct an adequate mesh, we see that mostly only experienced analysts can perform an effective simulation, even in linear analysis. The difficulties of obtaining an adequate and hence good mesh should ideally be removed from the analysis process.

These meshing considerations are quite different from establishing in the first instance a proper mathematical model and thus finite element model for simulating an event. For complex analyses, the proper modeling can be regarded to be an art because of the creativity, imagination and skill needed (Bathe 2014). In this paper we assume that the mathematical model is relatively simple and that the task of analysis is given by creating an adequate finite element mesh. In fact if a good mesh could be easily created, the finite element method would be much more employed, notably by designers in the CAD environment.

Since there are the difficulties of meshing, many meshless methods have been designed, see Nayroles, Touzot & Villon 1992, Belytschko, Krongauz, Organ, Fleming & Krysl 1996, Duarte & Oden 1996, Atluri & Zhu 1998, De & Bathe 2000, De & Bathe 2001a, Liu 2002 and the references therein. Much research effort has been expended to develop an effective meshless method. Nevertheless, all meshless methods have been identified to be numerically expensive for practical use when compared with the traditional finite element method (Dolbow & Belytschko 1999, De & Bathe 2001b, De & Bathe 2001c, Mazzia, Ferronato, Pini & Gambolati 2007, Babuška, Banerjee, Osborn & Li 2008, Babuška, Banerjee, Osborn & Zhang 2009). Here we focus only on methods that do not entail the adjustment of numerical factors (such adjustments are undesirable in practice) (Bathe 2014). Hence, while the overall aim of using meshless methods is very attractive, such methods have not yet found broad use in engineering practice.

The objective in this paper is to review our efforts in establishing a meshless method, the 'method of finite spheres', in which the spheres are in fact 'overlapping finite elements', and propose how this concept can be used effectively in engineering practice. The key aspect is that we no longer have the restriction of traditional finite elements that they must abut each other and can not overlap.

We first review the 'method of finite spheres', presenting briefly the theory and some solution results in static and dynamic analyses. We only consider in this paper linear analysis conditions of solids. While the concepts can also be used for the analysis of shells, fluids and nonlinear analysis, such applications require further research.

We then present how overlapping finite elements may be used efficiently together with traditional finite elements to remove meshing difficulties in CAD driven analyses while at the same time not

adding an undue amount of computational effort for the system matrices.

The presentation in this paper is forward-looking which also means some reasonable conjectures are given.

2 THE METHOD OF FINITE SPHERES

The solution procedure was designed in an attempt to establish a truly meshless method (a scheme that does not use a spatial mesh and does not use a background mesh for numerical integration). The method was proposed in De & Bathe 2000 and De & Bathe 2001a but was originally only tested in two-dimensional static analyses. The scheme is closely related to other meshless methods (Liu 2002).

The objective in this section is to briefly review the theory of the method of finite spheres and then give some example solutions that demonstrate how the method performs when compared with the use of traditional finite element discretizations.

2.1 *The theory of 'overlapping sphere elements'*

Consider the body shown in Figure 1 discretized using spheres. For illustrative purposes we show in fact disks as used in two-dimensional solutions, but in three-dimensional analysis, we would have spheres. As indicated, the spheres overlap each other, overlap the boundary of the body, and together cover the complete body.

Of course, we recall that in a traditional finite element mesh, the elements need to abut each other and must not overlap, also not the boundary of the analysis domain. Hence the only difference in the discretization used in Figure 1 is that the overlapping is present. For this reason, we call the spheres simply 'overlapping finite elements' and we mention in section 3 that the same concept can also be used to construct other than spherical 'overlapping finite elements'.

Using the general principle of virtual work, as in traditional finite element analysis, we have:
Find $\mathbf{u} \in H^1(\Omega)$ such that

$$\int_\Omega \boldsymbol{\varepsilon}^{\mathrm{T}}(\mathbf{v})\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u})\,\mathrm{d}\Omega = \int_\Omega \mathbf{v}^{\mathrm{T}}\mathbf{f}^B\,d\Omega + \int_{S_f} \mathbf{v}^{\mathrm{T}}\mathbf{f}^S\,\mathrm{d}S + \int_{S_u} \bar{\boldsymbol{\lambda}}^{\mathrm{T}}(\mathbf{u}-\mathbf{u}^S)\,\mathrm{d}S + \int_{S_u} \boldsymbol{\lambda}^{\mathrm{T}}\mathbf{v}\,\mathrm{d}S \quad \forall \mathbf{v} \in H^1(\Omega) \tag{1}$$

where $\mathbf{u}$ is the unknown displacement field, $\boldsymbol{\varepsilon}$ is the strain vector, $\mathbf{C}$ is the elasticity matrix, $\mathbf{v}$ is the virtual displacement field, $\mathbf{f}^S$ is the prescribed surface traction vector on the boundary S_f, $\mathbf{u}^S$ is the prescribed displacement vector on the boundary S_u, $\mathbf{f}^B$ is the body force vector (including inertia forces) and H^1 is the first order Hilbert space. The last two terms are Lagrange multiplier terms that impose the displacement boundary condition with

$$\boldsymbol{\lambda} = \mathbf{N}\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u}), \quad \bar{\boldsymbol{\lambda}} = \mathbf{N}\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{v}) \tag{2}$$

where $\mathbf{N}$ is the direction matrix, in two-dimensional analysis

$$\mathbf{N} = \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix},$$

and where the overbar signifies a virtual quantity. We note that these terms furnish a symmetric contribution to the stiffness matrix.

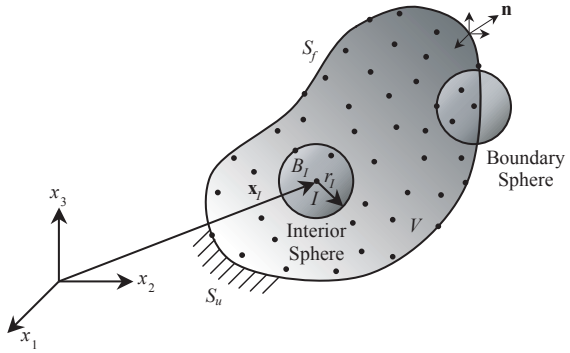

Figure 1 General problem domain V with domain boundary $S = S_u \cup S_f$

The stiffness and mass matrices and the load vector are evaluated from Equation (1) using numerical integration. Much effort has been expended to obtain a reliable and efficient scheme, where we mean by reliability that the integration ensures that important strain terms included in the interpolation are also contained in the numerically evaluated matrices. Unfortunately, a high order numerical integration is needed which is expensive.

For lack of a better scheme, we use a simple standard Gauss numerical integration scheme (Ham, Lai & Bathe 2014, Lai & Bathe 201x).

2.2 *Evaluation in static three-dimensional analyses*

Some evaluations for two-dimensional solutions have been presented in De & Bathe 2000, De & Bathe 2001a, De & Bathe 2001b, De & Bathe 2001c, Hong & Bathe 2005, Macri & De 2005, however a more severe comparison is established in three-dimensional analyses. Such analyses are more complex and numerically more intensive, also in the required numerical integrations. Furthermore, static solutions provide a good evaluation because the stiffness matrix calculation frequently corresponds to a large part of the solution effort. For the comparison, the method was implemented in the user-supplied element routine of ADINA in order to be able to use the same sparse solver for all analyses (Lai & Bathe 201x).

A cantilever beam of square hollow cross-section subjected to a tip load is considered. Figure 2 shows two finite sphere discretizations used. We note that for a coarser discretization, the centers of all spheres

are located on the outside surface of the beam. For spherical domains that are geometrically equal, the numerical integration is only performed once, and the result is then reused in the element assemblage process. This approach can save considerable computational time with overlapping finite elements but is, in general, only possible to some extent.

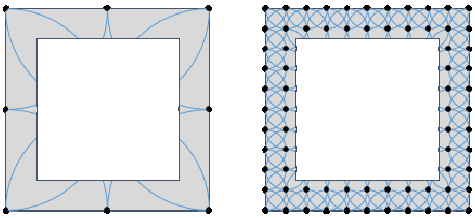

Figure 2. MFS1 and MFS3 discretizations at a section for a short cantilever beam of square hollow section

For the traditional finite element solutions, we use a sequence of compatible uniform meshes consisting of eight-node brick elements. The mesh refinement involves subdividing each brick element into eight brick elements, so that the coarser mesh is embedded in the finer mesh and we can expect monotonic convergence.

Figure 3 gives the convergence of the solutions obtained and Table 1 gives more details on these data. Here the error is calculated by comparison with the solutions obtained using the finest traditional finite element mesh. We deem this solution to be quite close to the unknown mathematically exact solution. The time multiplier gives how much faster (or slower) the solution is when compared to the solution using the finest 8-node brick element mesh. However, when studying the time multipliers for comparisons, we need to take into account the solution accuracy obtained, so here the MFS3 solution time might be approximately compared with the FEM2 solution time. More details on these solutions are given in Lai & Bathe 201x.

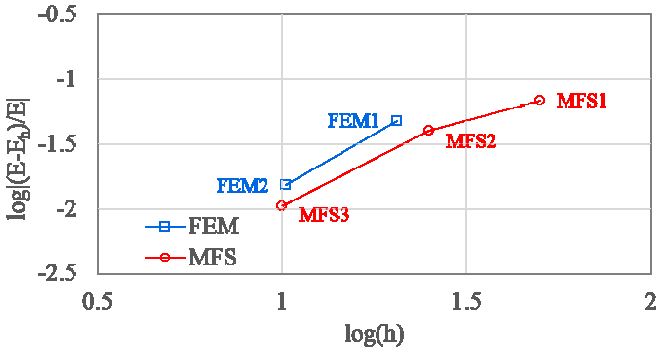

Figure 3 Convergence of strain energy for the method of finite spheres (MFS) and the finite element method (FEM)

Table 1 Strain energy errors and computational time multipliers for MFS and FEM discretizations (as compared to the FEM3 reference solution)

	Number of nodes	Strain energy error (%)	Time multiplier
MFS1	80	6.87	0.68
MFS2	288	4.04	2.38
MFS3	3024	1.04	6.54
FEM1	840	4.81	0.01
FEM2	4920	1.52	0.05
FEM3	32400	*	*

*FEM3: Strain energy (N·mm) = 4096.9; Time (s) = 11.64

Studying further analyses, see Lai & Bathe 201x, we see that using the overlapping spheres, the solution is between 1 to 2 orders of magnitude more expensive than the traditional finite element solution using 8-node brick elements. This conclusion becomes more favorable towards the finite sphere method when traditional 27-node brick elements are used (Lai & Bathe 201x). However, here considering a single load case, the major numerical expense is in establishing the stiffness matrix by numerical integration. In practice, many load cases are solved for (indeed a hundred load cases may be considered) and in such cases, the stiffness matrix is only established once, factorized once, and then forward-reductions and back-substitutions are carried out on the load vectors. A large solution effort is then expended to solve for the different load cases and the comparison using overlapping finite elements with traditional finite elements will be more favorable for the overlapping finite element scheme.

Indeed, we show this fact in the next section in which dynamic solutions are considered.

2.3 *Evaluations in dynamic analyses*

Dynamic solutions are generally obtained using mode superposition or direct time integration. We consider here first a direct time integration solution of a wave propagation problem in a two-dimensional domain, the data of which are taken from Ham, Lai, & Bathe 2014. A pre-stressed membrane, with a Ricker wavelet applied at its center is considered.

We use the implicit Bathe method for the time integration because it provides more accurate solutions than the trapezoidal rule, both schemes not using any solution factor to be adjusted (Bathe & Noh 2012).

Table 2 gives the error of displacements, measured in the L^2 norm for the overlapping and traditional finite element discretizations. We observe that in this analysis, for an error of less than 4% , the approach of using spheres gives a solution time close

to the one used with the traditional finite element scheme. This is clearly due to the fact that the **K** and **M** matrices are only calculated once, and the time stepping, using a major part of the solution effort, only requires vector forward-reductions and back-substitutions.

Table 2. Percent relative errors and computational times using (a) the traditional finite element method, 4-node elements, and (b) the method of finite spheres with order p harmonics

Discretization	(a) 160x160		(b) 9x9, p=3	
Timestep size (s)	0.003125 (CFL=0.5)	0.00625 (CFL=1)	0.003125	0.00625
Relative error (%)	5.59	3.27	3.55	10.37
Solution time (s)	110.06	59.75	48.37	36.99

Furthermore, there is an important additional observation. Wave propagations in traditional finite element analyses are difficult to compute because an optimal time step has to be selected, and this step depends on the speed of the wave (Bathe 2014, Noh, Ham & Bathe 2013). If the time step is larger than the optimal time step, the solution is unstable when using conditionally stable schemes (e.g. the central difference method) and loses accuracy when using unconditionally stable schemes (e.g. the trapezoidal rule). Moreover, in all time integrations using traditional finite element discretizations, if a time step *smaller* than the optimal time step is used, the solution accuracy is *worse*. This effect can be clearly seen in Table 2 where with a smaller CFL than the optimal one (here = 1.0) the solution error is larger. In practical analyses, there are multiple wave speeds, (e.g. compression, shear and Rayleigh waves) of which only one can be chosen for the optimal time step selection. Therefore, the other waves will not be accurately solved for.

However, Table 2 also shows that a *decrease* in the time step size using the overlapping finite elements leads to an *increase* in solution accuracy. This is an important fact and an analyst may expect this intuitively. With this characteristic, in practical analyses, the largest wave speed c might be chosen to establish the time step size, with the other waves then automatically being solved for more accurately.

We can analyze the numerical dispersion error occurring when using a direct time integration scheme for the traditional and the overlapping finite element discretizations, see Noh, Ham & Bathe 2013, Kim & Bathe 201x.

Here we should first note that in the method of finite spheres, an individual sphere has the same stiffness and mass properties in any direction, whereas the traditional finite elements have different 'effective' lengths depending on the direction chosen through the element. Therefore, the direction of wave travel through a traditional uniform 4-node element mesh will significantly affect the solution error, but not so for the method of finite spheres.

Figure 4 shows the error in the wave speed obtained as the wave travels in the x-direction through a homogeneous arrangement of spheres with the bi-linear polynomial and one harmonic in the "basis". The radius of the spheres is h, and the centers are spaced the distance h apart. A small error is obtained at CFL = 0.5 for $h \leq \lambda_h / 6$, and (as shown for CFL = 1) when the spatial discretization is too coarse to represent the wave, the Bathe method cuts out the wave response from the response prediction. This property of the time integration scheme is very useful in practical analyses because waves that cannot be resolved are not included in the solution. Similar observations hold for the traditional finite element solutions (see Noh, Ham & Bathe 2013).

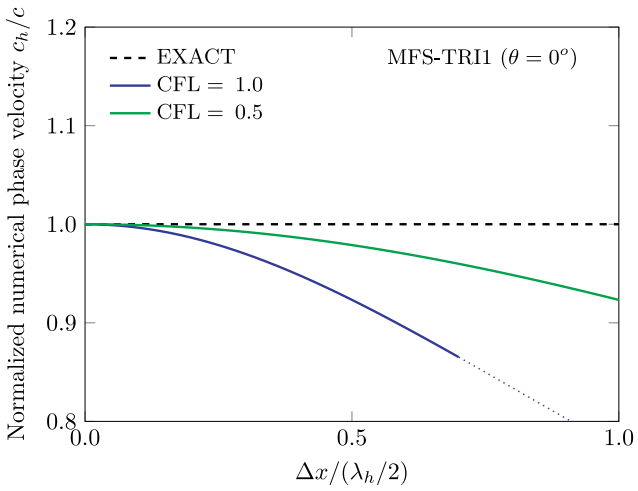

Figure 4. Relative wave speed errors; MFS, Bathe method; discarded wave modes dotted; λ is the wave length; $\Delta x = h$

In the above solutions, direct implicit time integration was performed and we could conclude that the finite element method using overlapping spheres shows some good attributes and in some analyses may even be competitive, compared with the traditional finite element discretizations.

The other approach widely used for dynamic analyses, but largely for structural response solutions, is mode superposition (Bathe 2014). Here the mass and stiffness matrices are established and then the natural frequencies and mode shapes are calculated. The response is obtained by superimposing the modal responses that are excited by the initial conditions and the load vector.

In the traditional finite element discretizations, by far the largest solution effort is in calculating the required frequencies and mode shapes. Considering the method of finite spheres, we may conjecture that the method may, in some analyses, also be good in solution time compared with the use of traditional

finite elements. The reason is that the major solution effort will also be in solving the eigenvalue problem, and not in establishing the **K** and **M** matrices. However, actual numerical comparisons should be established.

3 AN EFFECTIVE WAY OF MESHING USING TRADITIONAL AND OVERLAPPING FINITE ELEMENTS

The method of finite spheres, a meshless method, was designed to reduce the time of preparing a numerical model for a given physical problem; namely the time and effort spend on meshing. In most cases however the method is numerically too expensive to use.

Our objective in this section is to suggest that the overlapping elements can be used effectively with very simple Cartesian meshes of traditional finite elements. This novel approach is in fact a further development of the immersed boundary cut-cell approach used in the FloEFD program of Mentor Graphics for CFD analyses. The meshing is with FloEFD relatively simple and the meshing and solution procedure are tightly coupled into CAD packages.

Consider the geometry of a two-dimensional part generated using a CAD software, like SolidWorks, as shown in Figure 5. The discretization would be performed as follows.

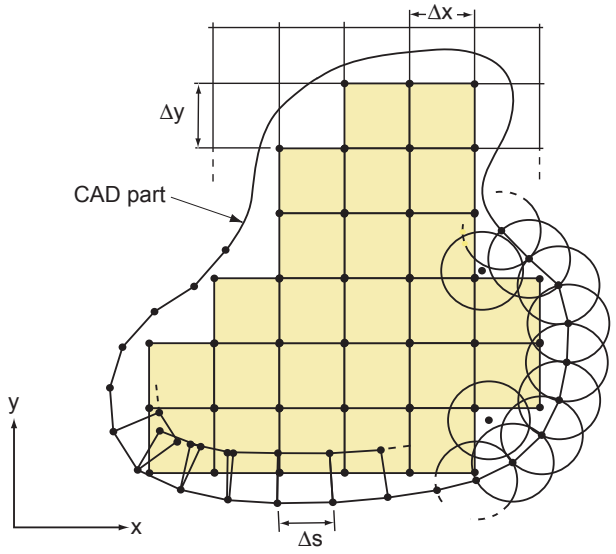

Figure 5. Schematic of two-dimensional CAD part; the (Δx, Δy) grid generated; the internal cells retained and converted to 4-node traditional finite elements; the straight-line Δs-segmentation of the boundary; and some overlapping spheres and quadrilateral elements used along and near the boundary

The *first* step is to generate a two-dimensional grid over the whole part, with Δx and Δy distances between lines, see Figure 5. The part can be thought of as being 'immersed' in the grid of Cartesian cells which can obviously be established with negligible human and computational effort.

In the *second* step a 'characteristic straight line length' Δs is used for discretizing the boundary of the part. This length should be small enough, so that straight lines of this length will represent the complete boundary of the part with sufficient accuracy, see Figure 5.

This first and second step need to be automatized to have (in general varying) Δx, Δy and Δs conform to the part geometry.

The *third* step is that all Cartesian cells that do not cut the boundary are represented by traditional finite elements, like 4-node elements. The other cells are removed.

The *fourth* step is that the boundary is meshed with overlapping finite elements using the characteristic length as spacing. It is important to place the centers of the spheres at these boundary points (the end points of the boundary lines) because then the displacement boundary conditions can be easily imposed. These overlapping finite elements must extend over to the traditional finite elements (established in step 3). Usually the one layer of spheres placed along the boundary does not extend sufficiently into the traditional finite elements and additional spherical elements need to be placed, so that the union of traditional finite elements and overlapping finite elements covers the complete geometric part and displacement continuity is ensured. We illustrate this process in Figure 5 and indicate also how quadrilateral overlapping finite elements could be employed. These quadrilateral elements are formulated in the same way as the spherical elements but now the Shepard functions are tensor products of the one-dimensional functions aligned along the local element directions.

In three-dimensional analysis, the same steps are followed but the grid is for the three Cartesian coordinate directions and a 'characteristic surface' is used in step 2. Then traditional brick finite elements would be employed with overlapping spheres or the three-dimensional generalization of the overlapping quadrilateral elements.

The coupling between the overlapping finite elements and the traditional finite elements is achieved as presented in Hong & Bathe 2005; see also Macri & De 2005.

The effort in meshing using this approach is clearly much smaller than when using traditional finite elements throughout the analysis domain. The accuracy of solution for a given number of elements may in many analyses (like in the solution of fluid flow problems) increase because mostly undistorted elements can be used. The computational time might be in most cases larger than when using the traditional finite element discretizations. However, based on all the experience reviewed above, this new approach will quite likely require much less total engineering time (time of meshing by an engineer + computer solution time of finite element model),

with the actual time gained dependent on the specific analysis performed.

The use of the overlapping finite elements can also be attractive to refine a mesh, and to embed special functions in the approximation spaces, like harmonic functions for wave propagation problems (as for Table 2 solutions), see e.g. Hong & Bathe 2005, Ham, Lai & Bathe 2014, Kim & Bathe 201x.

4 CONCLUDING REMARKS

The objective in this paper was to review our latest developments on the use of the method of finite spheres (overlapping finite elements), and then present a new meshing scheme for CAD driven finite element simulations.

We concluded that the use of overlapping finite elements is, mostly, computationally still very expensive in static analyses when these elements are used for the complete geometric domain. In dynamic analyses, however, the solution times used with overlapping finite elements compare better in expense with traditional finite elements, since the step-by-step solution or the solution of the eigenvalue problem require a considerable part of the computational effort in both approaches.

Based on this experience, we finally proposed a novel scheme for meshing and finite element solutions of solids tightly coupled to the use of CAD programs. A Cartesian mesh of traditional finite elements is used as much as possible but in conjunction with overlapping finite elements near the boundary to properly represent the boundary (and the nearby volume) of the geometric part.

This last section of the paper does not give solutions, but the proposed scheme is further presented in Bathe 2016. The main point of the scheme using overlapping finite elements is that finite element solutions of solids are efficiently and directly embedded into the computer-aided design process.

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