The method of finite spheres: a summary of recent developments

S. De *, K.J. Bathe

Massachusetts Institute of Technology, Mechanical Engineering Department, 36-766 RLE, Cambridge, MA 02139, USA

Abstract

We summarize the recent developments in the method of finite spheres focusing on the issues of computational efficiency and reliability.

Keywords: Method of finite spheres; Meshless technique; Incompressible analysis

1. Introduction

A review of the literature on meshless techniques reveals that the current trend is towards application of the techniques to solve diverse problems in science and engineering and none of the methods is computationally as efficient as the traditional finite element/finite volume techniques. However, for a meshless technique to be eventually successful and generally applicable, it must also be reasonably efficient compared with the traditional finite element techniques. With this goal in mind we developed the method of finite spheres [1]; a truly meshless technique.

The key to a computationally efficient meshless technique lies in the choice of effective interpolation functions, efficient ways of performing the numerical integration and effectiveness in the incorporation of the boundary conditions.

In the following sections we summarize the method of finite spheres and report upon some efficiency improvements. We illustrate that a pure displacement-based formulation 'locks' in the limit of incompressible deformation and present a mixed formulation based on displacement and pressure interpolations.

2. The displacement-based method of finite spheres

In the method of finite spheres the discretized equations of the governing differential equations are formed by integrating the Galerkin weak form over d-dimensional spheres (d = 1, 2 or 3) centered around nodes and forming a covering for the analysis domain, see Fig. 1. The spheres may lie entirely within the domain (interior spheres) or may have nonzero intercepts with the domain boundary (boundary spheres).

The interpolation uses the Shepard partition of unity functions, $\varphi_I^0(\mathbf{x})$, multiplied by polynomials (or other functions), $\mathrm{p_m}(\mathbf{x})$. Therefore, the global approximation space, V_h, contains functions like

$$\mathrm{v_h} = \sum_{I=1}^{N} \sum_{m} h_{\mathrm{Im}} \alpha_{\mathrm{Im}} \tag{1}$$

where N spheres are used for interpolation. The subscript 'h' represents a measure of the size of the spheres and h_{Im} is the basis function corresponding to the m-th degree of freedom α_{Im} at node I,

$$h_{\mathrm{Im}} = \varphi_I^0(\mathbf{x}) \mathrm{p_m}(\mathbf{x}). \tag{2}$$

Some shape functions at a node are shown in Fig. 1(b). The approximation properties of these shape functions are described in detail in reference [1].

The Shepard functions are generated using a simple 'normalization' scheme

$$\varphi_I^0(\mathbf{x}) = \frac{\mathrm{W_I}(\mathbf{x})}{\sum_{J=1}^{N} \mathrm{W_J}(\mathbf{x})} \tag{3}$$

where $\mathrm{W_I}(\mathbf{x})$ represents a radial function compactly supported on the sphere centered at node I. The choice of the weighting function is very important to ensure that low cost partitions of unity are obtained without sacrificing the accuracy of the solution. In our work we have chosen cubic spline functions.

* Corresponding author. Tel.: + 1 (617) 253-8503;
E-mail: suvranu@mit.edu

Computational Fluid and Solid Mechanics
K.J. Bathe (Editor)

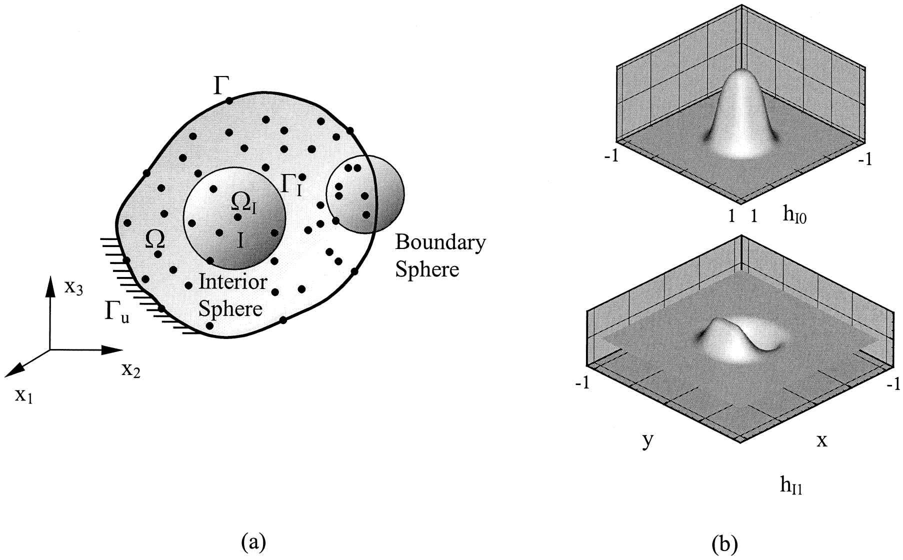

Fig. 1. The method of finite spheres. (a) Domain discretized using spheres. (b) Some interpolation functions (h_{I0} and h_{I1}) on a square (two-dimensional) domain. h_{I0} is the Shepard function at node I, while $h_{I1} = h_{I0}(x - x_I)/r_I$. x_I is the x-coordinate of node I and r_I is the radius of the sphere centered at node I.

We consider the analysis of linear elastic solids and write the Galerkin weak form as [2]:

Find $\mathbf{u}_h \in V_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) = F(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h \tag{4}$$

where $a(\cdot,\cdot)$ is the bilinear form corresponding to the problem; $F(\cdot)$ is the linear form corresponding to the applied loading; $\mathbf{u}_h$ is the numerical displacement solution in V_h and $\mathbf{v}_h$ is any element in the space V_h.

The discrete set of equations corresponding to the i-th degree of freedom, resulting from the Galerkin weak form, is of the following type:

$$\sum_j K_{ij}\beta_j = f_i + \hat{f}_i \tag{5}$$

where K_{ij} is a stiffness term; β_j is the nodal unknown corresponding to the j-th degree of freedom and f_i is the forcing term due to the applied loading. The term $\hat{f}_i$ vanishes for interior spheres due to the property of compact support of the interpolation functions and is nonzero but 'known' for a sphere on the Neumann boundary.

In reference [1] it is shown that for a node on the Dirichlet boundary, by the chain rule of differentiation,

$$\hat{f}_i = \sum_j \tilde{K}_{ij}\beta_j - \tilde{f}_i \tag{6}$$

where $\tilde{K}_{ij}$ is an additional stiffness term arising at the Dirichlet boundary and $\tilde{f}_i$ is an additional forcing term (this term vanishes if zero displacements are specified on the Dirichlet boundary).

The efficiency of our meshless technique depends largely on the numerical integration scheme used to evaluate the integrals in the Galerkin weak forms. Compared to the finite element/finite volume techniques, the integrands are nonpolynomial (rational) functions, the domains of integration are more complex to deal with and they overlap giving rise to general 'lens' shaped regions [3]. Hence, for two-dimensional problems, we have derived product rules of cubature with arbitrary polynomial accuracy on general annular sectors, on boundary sectors and also on the 'lens' [3].

In Fig. 2 we present the analysis of a square cantilever plate in plane strain conditions with uniformly distributed loading on its top surface (see Fig. 2(a)). In Fig. 2(b) we show the convergence in strain energy when a uniform h-type refinement (with a biquadratic local basis) is performed corresponding to two values of the Poisson's ratio, ν, equal to 0.3 and 0.4999. The strain energy of the reference solution is obtained by solving the same problem using a 50×50 mesh of 9-noded finite elements (u/p elements in the case of the almost incompressible analysis).

We observe that corresponding to a Poisson's ratio of 0.3, we obtain an excellent convergence rate (as the discretization is refined) since C^2 continuous shape functions are used. This implies that fewer nodes are required for the same accuracy in solution. By actually comparing computational costs and performing rough theoretical estimates, we concluded, that for two-dimensional elastostatic problems as these, the method of finite spheres is about five (or say ten) times slower than the traditional finite element techniques, see [3].

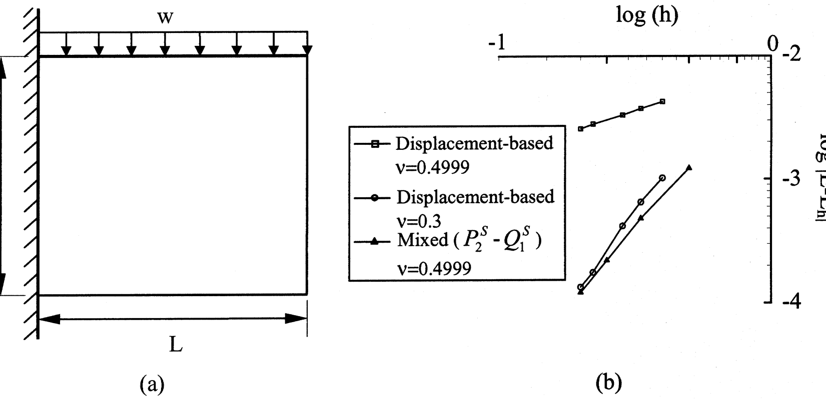

Fig. 2. Analysis of a cantilever plate (L = 2.0) in plane strain. Uniformly distributed load of magnitude w = 1.0 per unit length is applied. Poisson's ratio ν = 0.3 and 0.4999. In (b) the convergence in strain energy (E_h) with decrease in radius of support (h) is shown for two different Poisson's ratios 0.3 and 0.4999. The pure displacement-based formulation is observed to lock when ν = 0.4999. A mixed formulation using both pressure and displacement interpolations remedies locking (refer to the text for an explanation of the symbols used). E is an accurate estimate of the strain energy (reference solution).

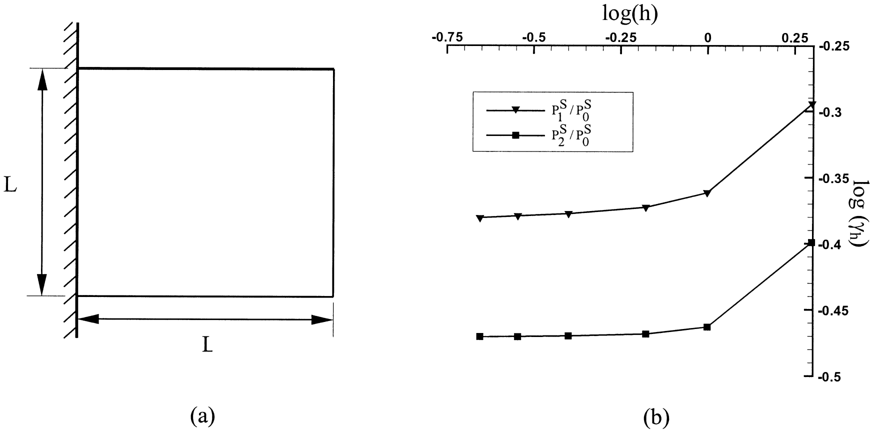

Fig. 3. (a) A square cantilevered plate (L = 2.0) in plane strain was used as a model problem for the numerical inf-sup test (described in the text). In (b) the results of the numerical inf-sup test are shown for two discretization schemes that pass the numerical inf-sup test.

Another important observation from Fig. 2(b) is that (with no surprise) the displacement-based formulation 'locks' when the Poisson's ratio is increased to 0.4999 [2].

3. Displacement/pressure mixed formulation

In order to remedy the problem of locking, we have developed a mixed formulation based on displacement and pressure interpolations [4].

To label the various mixed interpolation function spaces we introduce the following notation. Let P_n and Q_n denote, respectively, the space of complete polynomials of degree 'n' and complete tensor product polynomials of degree 'n' in R^2 (e.g. $P_1 = \mathrm{span}\{1, x, y\}$ and $Q_1 = \mathrm{span}\{1, x, y, xy\}$). In the method of finite spheres we use the following interpolation spaces: $P_n^S = \sum_{I=1}^{N} \varphi_I^0 P_n$ and $Q_n^S = \sum_{I=1}^{N} \varphi_I^0 Q_n$ and refer to a mixed interpolation scheme using, for example, P_1^S for displacement interpolation and P_0^S for pressure interpolation simply as the 'P_1^S–P_0^S interpolation'. In Fig. 2(b) we see that a P_2^S–Q_1^S mixed interpolation scheme alleviates the problem of locking in this example.

To obtain a stable and optimal procedure for the selected interpolation, the mixed formulation should satisfy, among others, the ellipticity condition (readily satisfied in this linear analysis) and the inf-sup condition [2]

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in V_h} \frac{(q_h, \operatorname{div} \mathbf{v}_h)}{\|q_h\|_0 \|\mathbf{v}_h\|_1} = \gamma_h \geq \gamma > 0 \tag{7}$$

where γ is a positive constant independent of h, q_h and $\mathbf{v}_h$ are elements of the pressure and displacement interpolation spaces, Q_h and V_h respectively. Here the inf-sup parameter, γ_h, is computed for a sequence of discretizations as the square root of the smallest eigenvalue of the following generalized eigenvalue problem:

$$\mathbf{G}_\mathrm{h}\boldsymbol{\varphi}_\mathrm{h} = \lambda\mathbf{S}_\mathrm{h}\boldsymbol{\varphi}_\mathrm{h} \tag{8}$$

where $(\mathrm{q_h},\ \mathrm{div}\,\mathbf{v}_\mathrm{h}) = \mathbf{W}_\mathrm{h}^\mathrm{T}\mathbf{G}_\mathrm{h}\mathbf{V}_\mathrm{h},\ ||\mathrm{q_h}||_0^2 = \mathbf{W}_\mathrm{h}^\mathrm{T}\mathbf{G}_\mathrm{h}\mathbf{W}_\mathrm{h}$ and $||\mathbf{v}_\mathrm{h}||_1^2 = \mathbf{V}_\mathrm{h}^\mathrm{T}\mathbf{S}_\mathrm{h}\mathbf{V}_\mathrm{h}$. The vectors $\mathbf{W}_\mathrm{h}$ and $\mathbf{V}_\mathrm{h}$ are the nodal variable vectors for a given discretization. In Fig. 3(b) we show the behavior of the inf-sup parameter with increase in refinement for two schemes that pass the inf-sup test. A square cantilever plate in plane strain condition is used as the model problem. We discuss various other mixed interpolation schemes in [4].

4. Concluding remarks

We have summarized our latest developments in the method of finite spheres in this paper. While considerable advance in computational efficiency has been achieved, the technique is still not as efficient as the traditional finite element techniques. The preprocessing cost is, however, less and therefore there is some advantage. Further improvements in efficiency are still possible, especially if much more efficient integration rules can be developed.

References

[1] De S, Bathe KJ. The method of finite spheres. Comput Mech 2000;25:329–345.

[2] Bathe KJ. Finite Element Procedures. Prentice Hall, 1996.

[3] De S, Bathe KJ. Towards an efficient meshless computational technique: the method of finite spheres. Eng Comput, to appear.

[4] De S, Bathe KJ. Displacement/pressure mixed interpolation in the method of finite spheres. Int J Numer Methods Eng, in press.