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Inf–sup testing of upwind methods

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SUMMARY

We propose inf–sup testing for finite element methods with upwinding used to solve convection–diffusion problems. The testing evaluates the stability of a method and compactly displays the numerical behaviour as the convection effects increase. Four discretization schemes are considered: the standard Galerkin procedure, the full upwind method, the Galerkin least-squares scheme and a high-order derivative artificial diffusion method. The study shows that, as expected, the standard Galerkin method does not pass the inf–sup tests, whereas the other three methods pass the tests. Of these methods, the high-order derivative artificial diffusion procedure introduces the least amount of artificial diffusion. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: convection-dominated flow; upwinding; inf–sup condition

1. INTRODUCTION

Finite element methods perform best in solving elliptic problems. When the methods are used for hyperbolic problems, difficulties are encountered. Here, the one-dimensional convection–diffusion equation is used, as a model problem, to study these difficulties. When the Peclet number is small, the elliptic part of the convection–difffusion equation is dominant; on the other hand, when the Peclet number is large, the hyperbolic part of the equation is dominant. In solving the convection–diffusion problem, the finite element method based on the standard Galerkin formulation gives an excellent solution when the Peclet number is low but gives artificial oscillations in the solution when the Peclet number is high. These oscillations show that the method is unstable in solving the hyperbolic type of problem. Upwind methods have been developed to overcome this difficulty and various finite element discretizations using upwinding are stable in solving convection–diffusion problems with high Peclet numbers.

The finite element procedure with upwinding should be stable and accurate to solve high Peclet number problems. However, no upwind method gives as yet totally satisfactory results [1]. The accuracy is not satisfactory because either the results contain oscillations or they are too diffusive.

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In general, the inf–sup condition is a crucial requirement to be satisfied for the stability of a finite element method. The inf–sup condition has been extensively used to analyse the stability of finite element formulations in solid mechanics and for Stokes flow [2, 3]. Here, we extend the use of the inf–sup condition to the stability analysis of finite element formulations for convection–diffusion problems.

The major difficulty in testing an upwind method for the solution of convection-dominated problems lies in that the test has to measure the solution errors in the interior of the domain and near the boundary. Whereas the solution is smooth in the interior, it is highly non-smooth near the boundary. Ideally, we would use norms that can accurately measure errors in the interior and near the boundary. We did not find a norm that does so and leads to tractable computations in the numerical evaluation of the inf–sup condition. For this reason, we propose in this paper a testing which considers first the whole domain using the H^1-norm modified by the Peclet number, and then considers a reduced domain (disregarding the boundary layer) using the original H^1-norm. The testing is employed to study the effectiveness of an upwind method and is therefore useful in research to establish more efficient techniques.

Usually, the performance of an upwind method is evaluated by solving an example problem and evaluating the solution of the problem. If the solution contains some oscillations, the upwind method is considered not to perform well. The inf–sup testing proposed herein evaluates the performance of an upwind method in a more comprehensive manner than to just measure the oscillations in the solutions. The test compactly describes the stability of an upwind method as the Peclet number and element size are varied.

In this study, we consider four discretization schemes; the standard Galerkin procedure, the full upwind method, the Galerkin least-squares method and a high-order derivative artificial diffusion method. First, we briefly review the inf–sup condition and develop the governing equations of the numerical inf–sup testing. Then we choose a one-dimensional test problem, derive appropriate norm definitions for each discretization method and apply the testing to the solution schemes.

2. THE INF–SUP CONDITION AND INF–SUP TESTING

Consider a general problem in given Hilbert spaces U and W with a bilinear form $a(\phi,\psi)$ defined on $U \times W$. The first argument in the bilinear form $a(\cdot\,,\cdot)$ is a solution function and the second argument is a weighting function. We define the following spaces:[‡]

$$U = \left\{ u \mid u \in L^2(\mathrm{Vol});\ \frac{\partial u}{\partial x_k} \in L^2(\mathrm{Vol}), k = 1,2,3;\ u = g \text{ on } S_u \right\}$$

$$W = \left\{ u \mid u \in L^2(\mathrm{Vol});\ \frac{\partial u}{\partial x_k} \in L^2(\mathrm{Vol}), k = 1,2,3;\ u = 0 \text{ on } S_u \right\}$$

[‡]Actually, to be precise, U is not a linear space, but an affine manifold that can be thought of as obtained by translating the linear space W.

where g is the Dirichlet boundary condition function applied on S_u and $L^2(\text{Vol})$ is the space of square integrable functions in the volume, 'Vol', of the body considered,

$$L^2(\text{Vol}) = \left\{ u \mid u \text{ is defined in Vol and } \int_{\text{Vol}} u^2 \, \text{dVol} = \|u\|^2_{L^2(\text{Vol})} < +\infty \right\}$$

Given a linear functional $b(\psi)$ from W to $\mathscr{R}$, we have for the continuous problem:
Find $\phi \in U$ such that

$$a(\phi, \psi) = b(\psi) \quad \forall \psi \in W \tag{1}$$

with $b(\psi) = (f, \psi)$, where f is the forcing term. The finite-dimensional subspaces§ of U and W are defined as follows:

$$U_h = \left\{ u_h \mid u_h \in L^2(\text{Vol}); \frac{\partial u_h}{\partial x_k} \in L^2(\text{Vol}), k = 1, 2, 3; u_h \in Q_n(\text{Vol}^{(m)}); u_h = g \text{ on } S_u \right\}$$

$$W_h = \left\{ u_h \mid u_h \in L^2(\text{Vol}); \frac{\partial u_h}{\partial x_k} \in L^2(\text{Vol}), k = 1, 2, 3; u_h \in Q_n(\text{Vol}^{(m)}); u_h = 0 \text{ on } S_u \right\}$$

where $Q_n(\text{Vol}^{(m)})$ denotes the nth-order polynomial function in element m. An approximate solution of Equation (1) is obtained by solving the following finite-dimensional problem:
Find $\phi_h \in U_h$ such that

$$a(\phi_h, \psi_h) = b(\psi_h) \quad \forall \psi_h \in W_h \tag{2}$$

with $b(\psi_h) = (f, \psi_h)$. Let us introduce a norm $\|\cdot\|_S$ for measuring the size of the solution functions and a norm $\|\cdot\|_T$ for measuring the size of the weighting functions.

In general, we have the following relation [2–4]:

$$\|\phi - \phi_h\|_S \leqslant \left(1 + \frac{k_m}{\gamma}\right) \inf_{\eta_h \in U_h} \|\phi - \eta_h\|_S \tag{3}$$

where k_m is obtained from the continuity equation of the continuous space

$$a(\eta, \psi) \leqslant k_m \|\eta\|_S \|\psi\|_T \quad \forall \eta, \ \psi \in W \tag{4}$$

The continuity equation simply states that the bilinear form $a(\eta, \psi)$ behaves normally. Also, γ is obtained from the inf–sup condition of the finite-dimensional spaces

$$\inf_{\eta_h \in W_h} \sup_{\psi_h \in W_h} \frac{a(\eta_h, \psi_h)}{\|\eta_h\|_S \|\psi_h\|_T} \geqslant \gamma > 0 \tag{5}$$

§See footnote ‡; now U_h is actually an affine manifold.

To prove that inequalities (4) and (5) imply inequality (3), consider the following derivation. From inequality (5) with $\eta_h = \chi_h - \phi_h$, for any $\chi_h \in U_h$ we have

$$\begin{aligned}
\gamma \|\chi_h - \phi_h\|_S &\leqslant \sup_{\psi_h \in W_h} \frac{a(\chi_h - \phi_h, \psi_h)}{\|\psi_h\|_T} \\
&= \sup_{\psi_h \in W_h} \frac{a(\chi_h - \phi, \psi_h) + a(\phi - \phi_h, \psi_h)}{\|\psi_h\|_T} \\
&= \sup_{\psi_h \in W_h} \frac{a(\chi_h - \phi, \psi_h)}{\|\psi_h\|_T} \\
&\leqslant \sup_{\psi_h \in W_h} \frac{k_m \|\chi_h - \phi\|_S \|\psi_h\|_T}{\|\psi_h\|_T} \\
&= k_m \|\chi_h - \phi\|_S
\end{aligned}$$

Using the triangle inequality we thus have

$$\begin{aligned}
\|\phi_h - \phi\|_S &\leqslant \|\phi_h - \chi_h\|_S + \|\chi_h - \phi\|_S \\
&\leqslant \frac{k_m}{\gamma} \|\chi_h - \phi\|_S + \|\chi_h - \phi\|_S \\
&= \left(1 + \frac{k_m}{\gamma}\right) \|\chi_h - \phi\|_S
\end{aligned}$$

which proves inequality (3).

Here k_m is given by the problem considered (and has an upper bound by the given physics), and γ should be independent of critical physical constants (that would make $\gamma \to 0$), the mesh parameter h and the solution of the problem. Note that we use the inequality relations given in References [2–4] with different norms—still to be selected—for the solution and weighting functions. For the moment, let us assume that we have identified appropriate norms and proceed with the evaluation of the inf–sup value.

The value of γ cannot easily be obtained analytically, especially when we consider a sequence of irregular meshes. Here, we evaluate the inf–sup expression (inequality (5)) using a numerical method that is similar to the method given in References [2, 5]. We now need to consider the non-symmetric bilinear form $a(\phi_h, \psi_h)$.

In matrix form, the general Equation (2) can be written as

Find $\mathbf{x} \in \mathscr{R}^n$ such that

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{6}$$

where in general $\mathbf{A}$ is an $n \times n$ non-symmetric matrix and $\mathbf{b} \in \mathscr{R}^n$. Inequality (5) becomes, for a given mesh,

$$\inf_{\boldsymbol{\eta}} \sup_{\boldsymbol{\psi}} \frac{\boldsymbol{\psi}^{\mathrm{T}} \mathbf{A} \boldsymbol{\eta}}{(\boldsymbol{\eta}^{\mathrm{T}} \mathbf{S} \boldsymbol{\eta})^{1/2} (\boldsymbol{\psi}^{\mathrm{T}} \mathbf{T} \boldsymbol{\psi})^{1/2}} = \gamma_n \geqslant \gamma > 0 \tag{7}$$

where $\mathbf{S}$ and $\mathbf{T}$ are symmetric matrices of the norm operators $\|\cdot\|_S$ and $\|\cdot\|_T$; $\boldsymbol{\eta}$ and $\boldsymbol{\psi}$ are vectors that contain the nodal values of η_h and ψ_h and γ is to be independent of h and the aforementioned physical quantities.

To evaluate the left-hand side of inequality (7), let us define

$$\rho(\boldsymbol{\eta},\boldsymbol{\psi}) = \frac{\boldsymbol{\psi}^{\mathrm{T}}\mathbf{A}\boldsymbol{\eta}}{(\boldsymbol{\eta}^{\mathrm{T}}\mathbf{S}\boldsymbol{\eta})^{1/2}(\boldsymbol{\psi}^{\mathrm{T}}\mathbf{T}\boldsymbol{\psi})^{1/2}} \tag{8}$$

and

$$\mathbf{T} = \mathbf{L}^{\mathrm{T}}\mathbf{L}, \quad \boldsymbol{\xi} = \mathbf{L}\boldsymbol{\psi} \tag{9}$$

Hence,

$$\rho(\boldsymbol{\eta},\boldsymbol{\xi}) = \frac{\boldsymbol{\xi}^{\mathrm{T}}\mathbf{L}^{-\mathrm{T}}\mathbf{A}\boldsymbol{\eta}}{(\boldsymbol{\eta}^{\mathrm{T}}\mathbf{S}\boldsymbol{\eta})^{1/2}\,(\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi})^{1/2}} \tag{10}$$

We use the Cauchy–Schwarz inequality

$$|\boldsymbol{\xi}^{\mathrm{T}}\mathbf{L}^{-\mathrm{T}}\mathbf{A}\boldsymbol{\eta}| \leqslant \|\boldsymbol{\xi}\|_2\|\mathbf{L}^{-\mathrm{T}}\mathbf{A}\boldsymbol{\eta}\|_2 \tag{11}$$

with the norm definition:

$$\|\mathbf{v}\|_2 = \left(\sum_{i=1}^{N} v_i^2\right)^{1/2} = (\mathbf{v}^{\mathrm{T}}\mathbf{v})^{1/2}$$

We note that in relation (11) equality holds for $\boldsymbol{\xi} = \mathbf{L}^{-\mathrm{T}}\mathbf{A}\boldsymbol{\eta}$; hence

$$\sup_{\boldsymbol{\psi}} \rho(\boldsymbol{\eta},\boldsymbol{\psi}) = \frac{(\boldsymbol{\eta}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{L}^{-1}\mathbf{L}^{-\mathrm{T}}\mathbf{A}\boldsymbol{\eta})^{1/2}}{(\boldsymbol{\eta}^{\mathrm{T}}\mathbf{S}\boldsymbol{\eta})^{1/2}} \tag{12}$$

Noting that

$$\mathbf{A}^{\mathrm{T}}\mathbf{L}^{-1}\mathbf{L}^{-\mathrm{T}}\mathbf{A} = \mathbf{A}^{\mathrm{T}}(\mathbf{L}^{\mathrm{T}}\mathbf{L})^{-1}\mathbf{A} = \mathbf{A}^{\mathrm{T}}\mathbf{T}^{-1}\mathbf{A} \tag{13}$$

we consider the following eigenproblem:

$$(\mathbf{A}^{\mathrm{T}}\mathbf{T}^{-1}\mathbf{A})\mathbf{x} = \lambda\mathbf{S}\mathbf{x} \tag{14}$$

Therefore,

$$\inf_{\boldsymbol{\eta}} \sup_{\boldsymbol{\psi}} \rho(\boldsymbol{\eta},\boldsymbol{\psi}) = \lambda_{\min}^{1/2} \tag{15}$$

where $\lambda_{\min}$ is the smallest eigenvalue of eigenproblem (14). Hence, for a given formulation, physical constants and finite-dimensional spaces, the value of γ_n is equal to $\lambda_{\min}^{1/2}$.

In the inf–sup testing, we would therefore consider a sequence of meshes and measure $\lambda_{\min}$. If this eigenvalue does not tend to zero, the solution method is stable and optimal in the discretization errors measured in the norm used in Equation (3). The testing is performed like in the inf–sup test for the incompressible problem proposed in References [2, 5].

The key point is that appropriate norms must be selected, for which the matrices $\mathbf{A}, \mathbf{T}$ and $\mathbf{S}$ in Equation (14) are calculated. The requirement for the S-norm is that $\|\phi\|_S$ should be bounded in order for the inequality (3) to make sense, and the norm should be strong enough to measure the errors in the solution. Clearly, the H^1-norm cannot be used because $\|\phi\|_{H^1} \rightarrow \infty$ as $\mathrm{Pe} \rightarrow \infty$, where Pe is the Peclet number, $\mathrm{Pe} = vL/\alpha$, with v the characteristic velocity, L the characteristic length and α the diffusivity of the fluid. Hence, we must modify this norm and we propose two ways to proceed.

In the first approach, we use a modified H^1-norm by introducing the Peclet number such that the norm behaves well even when $\mathrm{Pe} \rightarrow \infty$. For example, for the full upwind method we use for a one-dimensional problem (see Section 3.1)

$$\|\phi\|_S^2 = \int \frac{2}{\mathrm{Pe}} (\phi')^2 \, \mathrm{d}x \tag{16}$$

We refer to testing using this approach as 'testing with a modified H^1-norm'. We shall see that for certain spatial discretizations, however, the norm in Equation (16) does not measure the *accuracy* of solution sufficiently well when coarse meshes are used.

The difficulty in using the H^1-norm stems from the effect of the boundary layers. Hence, our second approach is to simply not include the boundary layer region in the norm and use the true H^1-norm in the rest of the domain. In this case, we cannot claim that Equation (3) is applicable and we are not using Equation (4), but we simply measure the stability of the solution using

$$s = \inf_{(\phi_h, f)} \frac{\|f\|_{L^2}}{\|\phi_h'\|_{L^{2-}}} \tag{17}$$

where (ϕ_h, f) is a pair of (solution, forcing term) as in Equation (2), and L^{2-} denotes the L^2-norm not including the boundary layer region.

The stability of the solution scheme is clearly not affected by the spatial boundary conditions (the effect of which could be subtracted as usual [2]), and hence we use

$$\begin{aligned} \|f\|_{L^2} &= \sup_{\psi_h \in W_h} \frac{b(\psi_h)}{\|\psi_h\|_{L^2}} \\ &= \sup_{\psi_h \in W_h} \frac{a(\eta_h, \psi_h)}{\|\psi_h\|_{L^2}} \end{aligned} \tag{18}$$

and

$$s = \inf_{\eta_h \in W_h} \sup_{\psi_h \in W_h} \frac{a(\eta_h, \psi_h)}{\|\eta_h'\|_{L^{2-}} \|\psi_h\|_{L^2}} \tag{19}$$

Now comparing Equation (19) with the expressions in Equations (5) and (15), we realize that the same eigenvalue problem in Equation (14) should be solved for the smallest eigenvalue $\lambda_{\min}$ to obtain for a given discretization the value of s. That is, denoting by s_n the value of s for a given formulation, physical constants and finite-dimensional spaces, the value of s_n is equal to $\lambda_{\min}^{1/2}$.

The inf–sup testing is performed as in the modified H^1-norm testing, but using Equation (19), we refer to the procedure as 'testing with the H^1-norm excluding the boundary layer'.

3. MODEL PROBLEM, NORMS AND MATRICES FOR THE INF–SUP TESTING

In this section, we apply the inf–sup testing derived in Section 2 for upwind methods to a convection–diffusion problem. The selection of the norm definitions used for each upwind method is described.

Consider the non-dimensionalized convection–diffusion problem in one dimension (described in Figure 1) with the governing equation

$$-\frac{1}{\mathrm{Pe}}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}+\frac{\mathrm{d}\phi}{\mathrm{d}x}=0 \quad \text{in } 0<x<1 \tag{20}$$

where ϕ is the temperature, Pe is the Peclet number, $\mathrm{Pe}=vL/\alpha$ where L, v, α are the domain length, the given fluid flow velocity and the thermal diffusivity. The boundary conditions are

$$\phi(0)=0 \quad \text{and} \quad \phi(1)=1$$

In this specific case, γ should be independent of Pe and the mesh parameter h. Here, we consider the case when the convective term is dominating, $\mathrm{Pe}>1$, and its limit case when $\mathrm{Pe}\to\infty$.

The exact solution for the problem is

$$\phi=\frac{\exp(\mathrm{Pe}\,x)-1}{\exp(\mathrm{Pe})-1}$$

For the Galerkin method, the full upwinding and the Galerkin least-squares method [2, 6], we discretize the domain uniformly using linear elements. Therefore, we have the spaces

$$U_h=\left\{u_h \mid u_h\in L^2(\mathrm{Vol});\ \frac{\partial u_h}{\partial x}\in L^2(\mathrm{Vol});\ u_h\in Q_1(\mathrm{Vol}^{(m)});\ u_h=g \text{ on } S_u\right\}$$

$$W_h=\left\{u_h \mid u_h\in L^2(\mathrm{Vol});\ \frac{\partial u_h}{\partial x}\in L^2(\mathrm{Vol});\ u_h\in Q_1(\mathrm{Vol}^{(m)});\ u_h=0 \text{ on } S_u\right\}$$

where $Q_1(\mathrm{Vol}^{(m)})$ denotes the linear function in element m.

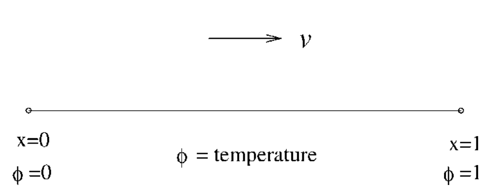

Figure 1. Domain and boundary conditions for the test problem.

For the high-order derivative artificial diffusion method, we discretize the domain uniformly using quadratic elements [1, 7]. Hence, we have the spaces

$$U_h = \left\{ u_h \mid u_h \in L^2(\mathrm{Vol});\ \frac{\partial u_h}{\partial x} \in L^2(\mathrm{Vol});\ u_h \in Q_2(\mathrm{Vol}^{(m)});\ u_h = g \ \text{on}\ S_u \right\}$$

$$W_h = \left\{ u_h \mid u_h \in L^2(\mathrm{Vol});\ \frac{\partial u_h}{\partial x} \in L^2(\mathrm{Vol});\ u_h \in Q_2(\mathrm{Vol}^{(m)});\ u_h = 0 \ \text{on}\ S_u \right\}$$

where $Q_2(\mathrm{Vol}^{(m)})$ denotes the quadratic function in element m.

3.1. *Modified H^1-norm testing*

We derive in this section the norms and matrices for the modified H^1-norm testing.

3.1.1. Standard Galerkin method. The standard Galerkin method for the convection–diffusion Equation (20) is [2]:

Find $\phi_h \in U_h$ such that

$$\int \left(\frac{\mathrm{d}\psi_h}{\mathrm{d}x} \frac{1}{\mathrm{Pe}} \frac{\mathrm{d}\phi_h}{\mathrm{d}x} + \psi_h \frac{\mathrm{d}\phi_h}{\mathrm{d}x} \right) \mathrm{d}x = 0 \quad \forall \psi_h \in W_h \tag{21}$$

where the integration sign shall denote from now on the integration over the fluid domain.

The norm definitions are determined by the continuity equation in the continuous space. Hence, we have

$$\begin{aligned} \int \left(\psi' \frac{1}{\mathrm{Pe}} \phi' + \psi\, \phi' \right) \mathrm{d}x &\leqslant \int \left(\frac{2}{\mathrm{Pe}} \phi'^2 \right)^{1/2} \left(\frac{1}{\mathrm{Pe}} \psi'^2 + \mathrm{Pe}\, \psi^2 \right)^{1/2} \mathrm{d}x \\ &\leqslant \left[\int \frac{2}{\mathrm{Pe}} \phi'^2 \,\mathrm{d}x \right]^{1/2} \left[\int \left(\frac{1}{\mathrm{Pe}} \psi'^2 + \mathrm{Pe}\, \psi^2 \right) \mathrm{d}x \right]^{1/2} \end{aligned}$$

therefore

$$\begin{aligned} k_m &= 1 \\ \|\phi\|_S^2 &= \int \frac{2}{\mathrm{Pe}} \phi'^2 \,\mathrm{d}x \\ \|\psi\|_T^2 &= \int \left(\frac{1}{\mathrm{Pe}} \psi'^2 + \mathrm{Pe}\, \psi^2 \right) \mathrm{d}x \end{aligned}$$

The norm value of the exact solution ϕ in the S-norm is

$$\begin{aligned} \|\phi\|_S^2 &= \int_0^1 \frac{2}{\mathrm{Pe}} \phi'^2 \,\mathrm{d}x \\ &= \frac{\exp(2\,\mathrm{Pe}) - 1}{(\exp(\mathrm{Pe}) - 1)^2} \end{aligned}$$

Therefore, as $\mathrm{Pe} \to \infty$,

$$\|\phi\|_S^2 \to 1$$

and the norm value is bounded.

The element matrices of the standard Galerkin method for the inf–sup test are therefore

$$\mathbf{A} = \int \left(\frac{1}{\mathrm{Pe}} \mathbf{H}_{,x}^{\mathrm{T}} \mathbf{H}_{,x} + \mathbf{H}^{\mathrm{T}} \mathbf{H}_{,x} \right) \mathrm{d}x \tag{22}$$

$$\mathbf{S} = \int \frac{2}{\mathrm{Pe}} \mathbf{H}_{,x}^{\mathrm{T}} \mathbf{H}_{,x} \, \mathrm{d}x \tag{23}$$

$$\mathbf{T} = \int \left(\frac{1}{\mathrm{Pe}} \mathbf{H}_{,x}^{\mathrm{T}} \mathbf{H}_{,x} + \mathrm{Pe}\, \mathbf{H}^{\mathrm{T}} \mathbf{H} \right) \mathrm{d}x \tag{24}$$

where $\mathbf{H}$ is the vector containing the interpolation functions.

3.1.2. Full upwind method. Using the same solution and weighting function spaces as for the standard Galerkin method, the full upwind method for the convection–diffusion Equation (20) is [2]:

Find $\phi_h \in U_h$ such that

$$\int \left\{ \frac{\mathrm{d}\psi_h}{\mathrm{d}x} \left(\frac{1}{\mathrm{Pe}} + \frac{h}{2} \right) \frac{\mathrm{d}\phi_h}{\mathrm{d}x} + \psi_h \frac{\mathrm{d}\phi_h}{\mathrm{d}x} \right\} \mathrm{d}x = 0 \quad \forall \psi_h \in W_h \tag{25}$$

where h is the normalized element length (using $L = 1$). The continuity equation of the full upwind method in the continuous space is

$$\begin{aligned} \int \left\{ \psi' \left(\frac{1}{\mathrm{Pe}} + \frac{h}{2} \right) \phi' + \psi\, \phi' \right\} \mathrm{d}x &\leqslant \int \left[\frac{2}{\mathrm{Pe}} \phi'^2 \right]^{1/2} \left[\left(\frac{1}{\mathrm{Pe}} + \frac{h^2\,\mathrm{Pe}}{4} + h \right) \psi'^2 + \mathrm{Pe}\, \psi^2 \right]^{1/2} \mathrm{d}x \\ &\leqslant \left[\int \frac{2}{\mathrm{Pe}} \phi'^2 \, \mathrm{d}x \right]^{1/2} \left[\int \left\{ \left(\frac{1}{\mathrm{Pe}} + \frac{h^2\,\mathrm{Pe}}{4} + h \right) \psi'^2 + \mathrm{Pe}\, \psi^2 \right\} \mathrm{d}x \right]^{1/2} \end{aligned}$$

Therefore, we have

$$\begin{aligned} k_m &= 1 \\ \|\phi\|_S^2 &= \int \frac{2}{\mathrm{Pe}} \phi'^2 \, \mathrm{d}x \\ \|\psi\|_T^2 &= \int \left\{ \left(\frac{1}{\mathrm{Pe}} + \frac{h^2\,\mathrm{Pe}}{4} + h \right) \psi'^2 + \mathrm{Pe}\, \psi^2 \right\} \mathrm{d}x \end{aligned}$$

The element matrices of the full upwind method for the inf–sup test are

$$\mathbf{A} = \int \left\{ \left(\frac{1}{\text{Pe}} + \frac{h}{2} \right) \mathbf{H}_{,x}^{\text{T}} \mathbf{H}_{,x} + \mathbf{H}^{\text{T}} \mathbf{H}_{,x} \right\} \mathrm{d}x \tag{26}$$

$$\mathbf{S} = \int \frac{2}{\text{Pe}} \mathbf{H}_{,x}^{\text{T}} \mathbf{H}_{,x} \, \mathrm{d}x \tag{27}$$

$$\mathbf{T} = \int \left\{ \left(\frac{1}{\text{Pe}} + \frac{h^2 \, \text{Pe}}{4} + h \right) \mathbf{H}_{,x}^{\text{T}} \mathbf{H}_{,x} + \text{Pe} \, \mathbf{H}^{\text{T}} \mathbf{H} \right\} \mathrm{d}x \tag{28}$$

3.1.3. Galerkin least-squares method. Using the same solution and weighting function spaces as for the standard Galerkin method, the Galerkin least-squares formulation for the convection–diffusion equation is [2, 6]:

Find $\phi_h \in U_h$ such that

$$\int \left\{ \frac{\mathrm{d}\psi_h}{\mathrm{d}x} \left(\frac{1}{\text{Pe}} + \tau \right) \frac{\mathrm{d}\phi_h}{\mathrm{d}x} + \psi_h \frac{\mathrm{d}\phi_h}{\mathrm{d}x} \right\} \mathrm{d}x = 0 \quad \forall \psi_h \in W_h \tag{29}$$

with

$$\tau = \frac{h}{2} \coth \left(\frac{h \, \text{Pe}}{2} \right) - \frac{1}{\text{Pe}} \tag{30}$$

This value of τ gives the nodally exact solution. Substituting Equation (30) into Equation (29), we have

$$\int \left[\frac{\mathrm{d}\psi_h}{\mathrm{d}x} \left\{ \frac{h}{2} \coth \left(\frac{h \, \text{Pe}}{2} \right) \right\} \frac{\mathrm{d}\phi_h}{\mathrm{d}x} + \psi_h \frac{\mathrm{d}\phi_h}{\mathrm{d}x} \right] \mathrm{d}x = 0 \tag{31}$$

The continuity equation of the Galerkin least-squares method in the continuous space is

$$\begin{aligned}
&\int \left[\psi' \left\{ \frac{h}{2} \coth \left(\frac{h \, \text{Pe}}{2} \right) \right\} \phi' + \psi \, \phi' \right] \mathrm{d}x \\
&\quad \leqslant \int \left[\frac{2}{\text{Pe}} \phi'^2 \right]^{1/2} \left[\frac{h^2 \, \text{Pe}}{4} \coth^2 \left(\frac{h \, \text{Pe}}{2} \right) \psi'^2 + \text{Pe} \, \psi^2 \right]^{1/2} \mathrm{d}x \\
&\quad \leqslant \left[\int \frac{2}{\text{Pe}} \phi'^2 \, \mathrm{d}x \right]^{1/2} \left[\int \left(\frac{h^2 \, \text{Pe}}{4} \coth^2 \left(\frac{h \, \text{Pe}}{2} \right) \ \psi'^2 + \text{Pe} \, \psi^2 \right) \mathrm{d}x \right]^{1/2}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
k_m &= 1 \\
\|\phi\|_S^2 &= \int \frac{2}{\text{Pe}} \phi'^2 \, \mathrm{d}x \\
\|\psi\|_T^2 &= \int \left\{ \frac{h^2 \, \text{Pe}}{4} \coth^2 \left(\frac{h \, \text{Pe}}{2} \right) \ \psi'^2 + \text{Pe} \, \psi^2 \right\} \mathrm{d}x
\end{aligned}$$

The element matrices of the Galerkin least-squares method for the inf–sup test are

$$\mathbf{A}=\int\left\{\left(\frac{1}{\mathrm{Pe}}+\tau\right)\mathbf{H}_{,x}^{\mathrm{T}}\mathbf{H}_{,x}+\mathbf{H}^{\mathrm{T}}\mathbf{H}_{,x}\right\}\mathrm{d}x \tag{32}$$

$$\mathbf{S}=\int\frac{2}{\mathrm{Pe}}\mathbf{H}_{,x}^{\mathrm{T}}\mathbf{H}_{,x}\,\mathrm{d}x \tag{33}$$

$$\mathbf{T}=\int\left\{\frac{h^2\,\mathrm{Pe}}{4}\coth^2\left(\frac{h\,\mathrm{Pe}}{2}\right)\mathbf{H}_{,x}^{\mathrm{T}}\mathbf{H}_{,x}+\mathrm{Pe}\,\mathbf{H}^{\mathrm{T}}\mathbf{H}\right\}\mathrm{d}x \tag{34}$$

3.1.4. High-order derivative artificial diffusion method. The high-order derivative artificial diffusion method for the convection–diffusion equation is [1, 7]

Find $\phi_h\in U_h$ such that

$$\int\left(\frac{\mathrm{d}\psi_h}{\mathrm{d}x}\frac{1}{\mathrm{Pe}}\frac{\mathrm{d}\phi_h}{\mathrm{d}x}+\psi_h\frac{\mathrm{d}\phi_h}{\mathrm{d}x}\right)\mathrm{d}x+\sum_m\int_{l^m}\frac{\mathrm{d}^2\psi_h}{\mathrm{d}x^2}\frac{1}{9}\left|\frac{\mathrm{d}x}{\mathrm{d}r}\right|^3\frac{\mathrm{d}^2\phi_h}{\mathrm{d}x^2}\,\mathrm{d}x=0\quad\forall\psi_h\in W_h \tag{35}$$

For a regular mesh, $|\mathrm{d}x/\mathrm{d}r|=h/2$, so we have

$$\sum_m\int_{l^m}\left(\frac{\mathrm{d}\psi_h}{\mathrm{d}x}\frac{1}{\mathrm{Pe}}\frac{\mathrm{d}\phi_h}{\mathrm{d}x}+\psi_h\frac{\mathrm{d}\phi_h}{\mathrm{d}x}+\frac{\mathrm{d}^2\psi_h}{\mathrm{d}x^2}\frac{h^3}{72}\frac{\mathrm{d}^2\phi_h}{\mathrm{d}x^2}\right)\mathrm{d}x=0 \tag{36}$$

The continuity equation of the high-order derivative artificial diffusion method in the continuous space is

$$\begin{aligned}
&\sum_m\int_{l^m}\left(\psi'\frac{1}{\mathrm{Pe}}\phi'+\psi\,\phi'+\psi''\frac{h^3}{72}\phi''\right)\mathrm{d}x\\
&\leqslant\sum_m\int_{l^m}\left(\frac{h^3}{72\,\mathrm{Pe}^3}\phi''^2+\frac{2}{\mathrm{Pe}^3}\phi'^2\right)^{1/2}\left(\frac{h^3\,\mathrm{Pe}^3}{72}\psi''^2+\mathrm{Pe}\psi'^2+\mathrm{Pe}^3\psi^2\right)^{1/2}\mathrm{d}x\\
&\leqslant\left[\sum_m\int_{l^m}\left(\frac{h^3}{72\,\mathrm{Pe}^3}\phi''^2+\frac{2}{\mathrm{Pe}^3}\phi'^2\right)\mathrm{d}x\right]^{1/2}\left[\sum_m\int_{l^m}\left(\frac{h^3\mathrm{Pe}^3}{72}\psi''^2+\mathrm{Pe}\psi'^2+\mathrm{Pe}^3\,\psi^2\right)\mathrm{d}x\right]^{1/2}
\end{aligned}$$

and we have

$$\begin{aligned}
&k_m=1\\
&\|\phi\|_S^2=\sum_m\int_{l^m}\left(\frac{h^3}{72\,\mathrm{Pe}^3}\phi''^2+\frac{2}{\mathrm{Pe}^3}\phi'^2\right)\mathrm{d}x\\
&\|\psi\|_T^2=\sum_m\int_{l^m}\left(\frac{h^3\,\mathrm{Pe}^3}{72}\psi''^2+\mathrm{Pe}\,\psi'^2+\mathrm{Pe}^3\,\psi^2\right)\mathrm{d}x
\end{aligned}$$

The norm value of the exact solution ϕ in the S-norm is

$$\|\phi\|_S^2 = \int_0^1 \left(\frac{h^3}{72\,\mathrm{Pe}^3} \phi''^2 + \frac{2}{\mathrm{Pe}^3} \phi'^2 \right) \mathrm{d}x$$
$$= \frac{h^3(\exp(2\,\mathrm{Pe}) - 1)}{144(\exp(\mathrm{Pe}) - 1)^2} + \frac{(\exp(2\,\mathrm{Pe}) - 1)}{\mathrm{Pe}^2(\exp(\mathrm{Pe}) - 1)^2}$$

Therefore, as $\mathrm{Pe} \to \infty$,

$$\|\phi\|_S^2 \to \frac{h^3}{144}$$

which is bounded for small element length h.

The element matrices of the high-order derivative artificial diffusion method for the inf–sup test are

$$\mathbf{A} = \int \left(\frac{1}{\mathrm{Pe}} \mathbf{H}_{,x}^{\mathrm{T}} \mathbf{H}_{,x} + \mathbf{H}^{\mathrm{T}} \mathbf{H}_{,x} \right) \mathrm{d}x + \int \frac{h^3}{72} \mathbf{H}_{,xx}^{\mathrm{T}} \mathbf{H}_{,xx} \, \mathrm{d}x \tag{37}$$

$$\mathbf{S} = \int \frac{2}{\mathrm{Pe}^3} \mathbf{H}_{,x}^{\mathrm{T}} \mathbf{H}_{,x} \, \mathrm{d}x + \int \frac{h^3}{72\,\mathrm{Pe}^3} \mathbf{H}_{,xx}^{\mathrm{T}} \mathbf{H}_{,xx} \, \mathrm{d}x \tag{38}$$

$$\mathbf{T} = \int (\mathrm{Pe}\, \mathbf{H}_{,x}^{\mathrm{T}} \mathbf{H}_{,x} + \mathrm{Pe}^3\, \mathbf{H}^{\mathrm{T}} \mathbf{H}) \, \mathrm{d}x + \int \frac{h^3 \mathrm{Pe}^3}{72} \mathbf{H}_{,xx}^{\mathrm{T}} \mathbf{H}_{,xx} \, \mathrm{d}x \tag{39}$$

3.2. H^1-norm testing excluding boundary layer

For the H^1-norm testing excluding the boundary layer, the same element matrices $\mathbf{A}$ as for the modified H^1-norm testing are used (see Section 3.1), but the matrices representing the norms are simpler. Namely, in each case the element matrices are (see Equation (19))

$$\mathbf{S} = \int \mathbf{H}_{,x}^{\mathrm{T}} \, \mathbf{H}_{,x} \, \mathrm{d}x \tag{40}$$

$$\mathbf{T} = \int \mathbf{H}^{\mathrm{T}} \, \mathbf{H} \, \mathrm{d}x \tag{41}$$

Of course, an important point is that the elements in the boundary layer must not be included in the assemblage of the complete system matrix $\mathbf{S}$. This matrix, therefore, has zero rows and columns corresponding to the degrees of freedom in the boundary layer. Each such zero row and column results into an infinite eigenvalue, which however does not affect our result that $\lambda_{\min}$ should be computed, see Equation (15) [2].

We also note that with this choice of the S-norm for the continuous problem

$$\|\phi\|_S \to 0 \tag{42}$$

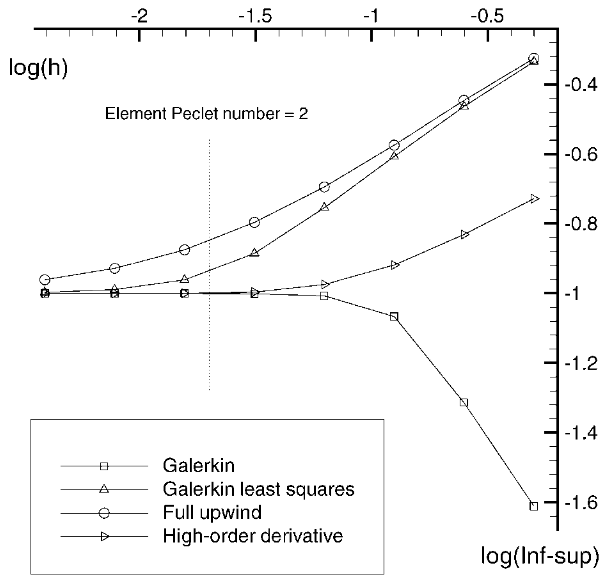

Figure 2. Inf–sup value curves as the mesh is coarsened with $\mathrm{Pe}=100$ for the modified H^1-norm testing.

and

$$\inf_{\eta} \sup_{\psi} \frac{\int_0^1 ((1/\mathrm{Pe})\eta'\psi' + \eta'\psi)\,\mathrm{d}x}{\|\eta\|_S \, \|\psi\|_T} \to 1 \tag{43}$$

as Pe tends to ∞, provided the infimum is taken only over the functions $\eta \in H^1$ such that $-(1/\mathrm{Pe})\eta'' + \eta'$ belongs to $L^2(0,1)$. Indeed, for every fixed smooth η the $\sup_{\psi}$ equals $\| - (1/\mathrm{Pe})\eta'' + \eta'\|_{L^2}/\|\eta\|_S$. Always for smooth η this quotient tends (as Pe tends to ∞) to $\|\eta'\|_{L^2}/\|\eta\|_S$, which is always bigger than or equal to 1, but equals 1 whenever η vanishes identically in the interval $(1-2h, 1)$. This continuous inf–sup property justifies the use of the H^1-norm testing excluding the boundary layer for discrete problems.

4. INF–SUP TEST RESULTS

We consider the model problem of Section 3 and perform the inf–sup tests described in Section 2.

Figures 2 and 3 show the results using the modified H^1-norm test. In Figure 2, the Peclet number of the problem is 100 and the number of elements is increased. In Figure 3, the element length h is 0.0625 (number of elements $=16$) and the Peclet number is increased.

Figure 2 shows that as the mesh is made coarser, the inf–sup value corresponding to the standard Galerkin method decreases. This trend indicates that the method does not pass the inf–sup test which means that the method does not satisfy the inf–sup condition (Equation (5)). The method is predicted to be unstable when we use too coarse a mesh. This instability is displayed by oscillations in the temperature solution. Figure 2 also shows that as the mesh is made finer, the inf–sup value corresponding to the standard Galerkin method approaches a fixed value. Of course, as known, the method is stable when the element Peclet number <2.

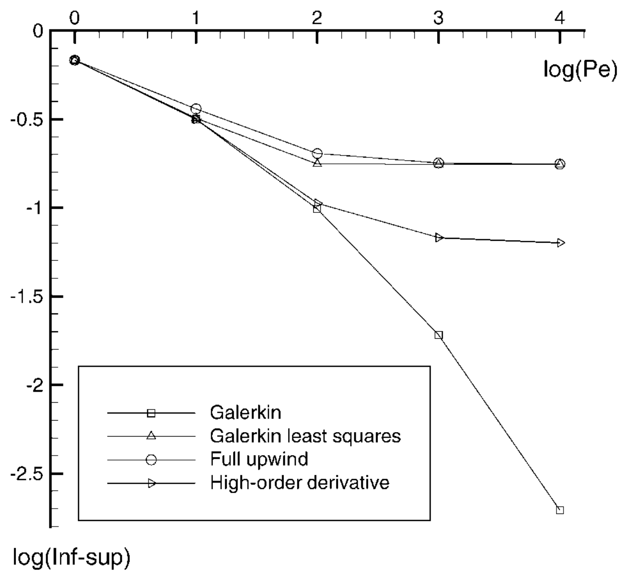

Figure 3. Inf–sup value curves as Pe is increased with $h = 0.0625$ for the modified H^1-norm testing.

Figure 2 shows that as the mesh is coarsened, the inf–sup values corresponding to the full upwind method, the Galerkin least-squares method and the high-order derivative artificial diffusion method are bounded from below. This indicates that these methods pass the inf–sup test and are predicted to be stable. Note that as the mesh is made finer, all curves approach the value of the Galerkin method.

The inf–sup values corresponding to the full upwind method are higher than those of the other curves. This indicates that the method is the most diffusive. The high-order derivative upwind method is stable and yields the smallest artificial diffusion.

Comparing the slopes of the inf–sup value curves in the coarse meshes, we observe that the Galerkin method has the largest (absolute value) slope. This corresponds to the highest convergence rate of the method (being of second order).

Figure 3 shows the inf–sup values as the Peclet number increases. The results in Figure 3 lead to the same conclusions as obtained from Figure 2.

In this study, we have used an even number of elements to discretize the domain. If an odd number of elements is used, the inf–sup value corresponding to the standard Galerkin method is bounded from below as we coarsen the mesh, or as the Peclet number increases. This is because the method is stable when an odd number of elements is used, although highly inaccurate in the interior domain when the mesh is coarse. The finite element solution for a given Peclet number using a coarse mesh is a saw tooth solution for which Equation (3) is still satisfied. However, when the mesh is fine, the right-hand side in Equation (3) is small and the saw tooth response is not satisfying Equation (3) and therefore not a solution. Hence, an even number of elements should be used for the one-dimensional problem in this inf–sup test.

The reason why the saw-tooth solution is not identified as a highly inaccurate solution lies in the norms used. The H^1-norm modified by the Peclet number includes the boundary layer but to include it, the norm contains the factor $(1/\mathrm{Pe})^{1/2}$. The result is that the norm does not provide a

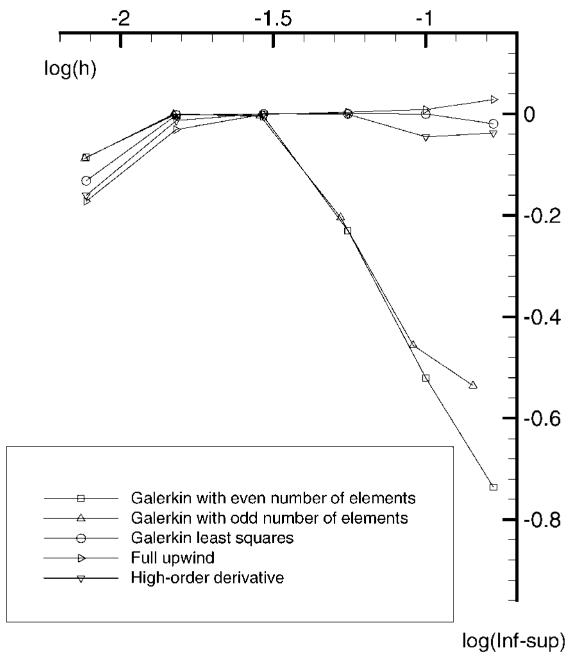

Figure 4. Inf–sup value curves as the mesh is coarsened with Pe = 100 for the H^1-norm testing excluding boundary layer.

sufficiently 'hard' measure for the errors in the numerical saw-tooth solution when a coarse mesh is used. Indeed the S-norm of a basis function with value 1 at one internal node and 0 at the other nodes is $2/(\mathrm{Pe}\, h)^{1/2}$, which is small for coarse grids and large Peclet number.

We next apply the H^1-norm testing excluding the boundary layer. Figure 4 presents the results for Pe = 100 as we coarsen the mesh. We observe that the curves for the three stable methods considered are bounded from below; on the other hand, in the case of the standard Galerkin method the inf–sup value, measured on the value of one, decreases as the mesh becomes coarse. The same observations are valid for Figure 5, where we consider a fixed mesh of 18 elements (19 in the case of the Galerkin method with an odd number of elements) and the Peclet number is increased. Further theoretical and numerical results using the H^1-norm testing excluding the boundary layer are given in Reference [8].

5. CONCLUDING REMARKS

Our objective in this paper was to develop an inf–sup testing procedure for measuring the effectiveness of stabilization methods used in the finite element solution of convection-dominated flows. We first reviewed the inf–sup condition for the problem area considered and then developed the numerical testing procedure. To demonstrate the technique, we applied the testing to a one-dimensional model problem when various well-known finite element discretization techniques are used.

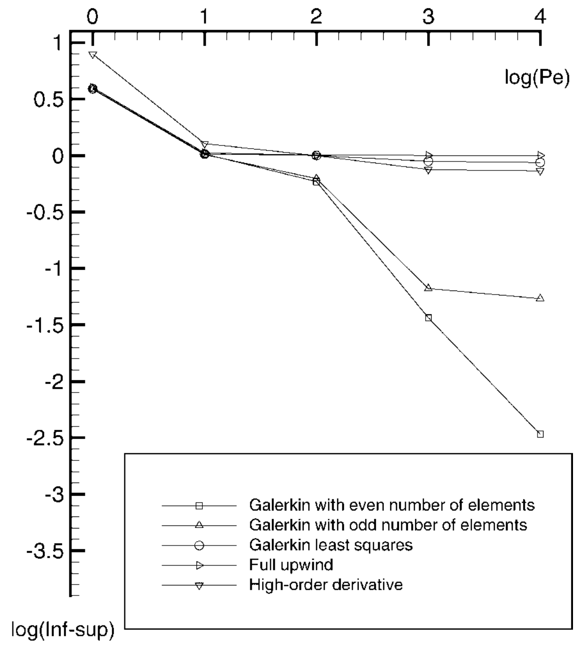

Figure 5. Inf–sup value curves as Pe is increased with $h=0.0556$ for the H^1-norm testing excluding boundary layer ($h=0.0526$ for Galerkin method with odd number of elements).

The procedures developed in this paper are quite general, but the effectiveness of the inf–sup testing depends on the norms used. The difficulty with convection-dominated flow problems is that the solution is smooth in the interior of the domain, but can be highly non-smooth near the boundary. The norm used for the solution function should ideally be able to measure *equally well* any errors in the smooth and non-smooth parts of the solution. We have not succeeded as yet to identify an 'ideal such norm' that can also be employed effectively in the computations. Hence, while we have used adequate norms to perform the inf–sup testing, we leave the search for more effective norms for further research.

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