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Improved stresses for the 4-node tetrahedral element

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ABSTRACT

The objective in this paper is to present the method for the calculation of improved stresses published by Payen and Bathe in [1] for the 4-node three-dimensional tetrahedral element. This element is widely used in engineering practice to obtain, in general, only “guiding” results in the analysis of solids because the element is known to be poor in stress predictions. We show in this paper the potential of this novel approach to significantly enhance the stress predictions with the 4-node tetrahedral element at a relatively low computational cost.

1. Introduction

Most engineering problems in solids and structures are three-dimensional in nature. Since the geometry and other data of the problem are then usually complex, the structure is best analysed using finite element methods. The crucial step in any finite element analysis is to choose an appropriate mathematical model for the physical structure (or more generally the physical phenomenon), since a finite element solution solves only this model, see Ref. [2]. For example, if the structure is thin in one direction and long in the other two directions a shell mathematical model is appropriate, and the problem is solved efficiently using the MITC shell elements, see Refs. [3–6]. However, if the length scales of the structure are similar in all directions, and the loading is general, then there is no option other than to solve the problem using an assemblage of discrete three-dimensional solid elements, see Refs. [2,7].

The simplest three-dimensional solid element available to the finite element analyst is the 4-node constant strain tetrahedral element. This element is used abundantly in practice because the analyst is able to mesh almost any volume regardless of complexity, the element is robust in contact analysis, the element matrices are inexpensive to calculate, and the resulting global stiffness matrix has a relatively small bandwidth. In a typical approach, the analyst would use a mesh of 4-node tetrahedral elements, in a first analysis, to identify the locations of high stress concentrations, and then based upon these results, the analyst would refine the mesh – or, if possible, convert the mesh to 10-node or 11-node tetrahedral elements – in the localised regions of concern, see Ref. [7]. This is necessary, simply because the stresses predicted using the 4-node tetrahedral element are known to be poor, and the lack of accuracy can be seen using stress band plots of unsmoothed stresses, see Refs. [2,8].

Our objective in this paper is to apply the method published by Payen and Bathe in [1] to the 4-node three-dimensional tetrahedral element, and show that by using a simple algorithm, we are able to enhance the stresses in localised regions of concern, without having to refine the mesh or re-analyse the model. While we focus in this paper on linear static analysis and smooth stress conditions, the results are fundamental and might be used also in dynamic analysis and nonlinear solutions [2,9,10].

The theory used for the method has been published in detail in Ref. [1], and hence we shall only summarise the fundamental equations – and their properties – in this paper. The stress prediction is based on the fact that the element nodal point forces are of higher quality than the directly-calculated finite element stresses [2,7]. Hence we use two principle of virtual work statements involving these nodal forces, as summarised in Section 2, to calculate the improved finite element stresses. Indeed, the special properties of the element nodal point forces have been known for many years; however, relatively little attention has been given to their use to improve the finite element stress predictions, see Refs. [1,11,12] and the references therein. In Ref. [1] we mention why our approach is more general and powerful than those previously considered.

When performing a properly formulated finite element solution, two important facts hold, namely, (1) at each node, the sum of the

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element nodal point forces balances the externally applied nodal point loads, and (2) each element is in force and moment equilibrium under the action of its own nodal point forces, *irrespective* of the coarseness of the mesh, see Refs. [2,7]. For this reason, it seems somewhat natural to use these forces to calculate improved stress predictions but the details of establishing a general and effective algorithm are far from apparent. We named the procedure given in Ref. [1] the "nodal point force based stress calculation method" or the "NPF-based method" giving "NPF-based stresses", for short.

In Ref. [1] we showed that the NPF-based method can be used effectively to significantly improve the accuracy of the finite element stress predictions obtained using the 3- and 4-node displacement-based elements in two-dimensional analyses. It is reasonable to expect similar improvements for the 4-node three-dimensional tetrahedral element, and our objective herein is to present a detailed procedure towards that aim. We solve the same set of problems considered in Ref. [1], but of course this time in three-dimensional settings. As expected, we see a significant improvement in the accuracy of the stress predictions for all problems considered. These results are of particular interest, since reliable improvements in stresses for the 4-node tetrahedral element, using incompatible modes or enhanced strains, are difficult to reach in general analyses [13–15].

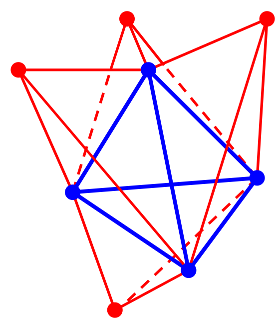

Fig. 1. The stress calculation domain for the 4-node tetrahedral element; element *m* would be the central element or a peripheral element.

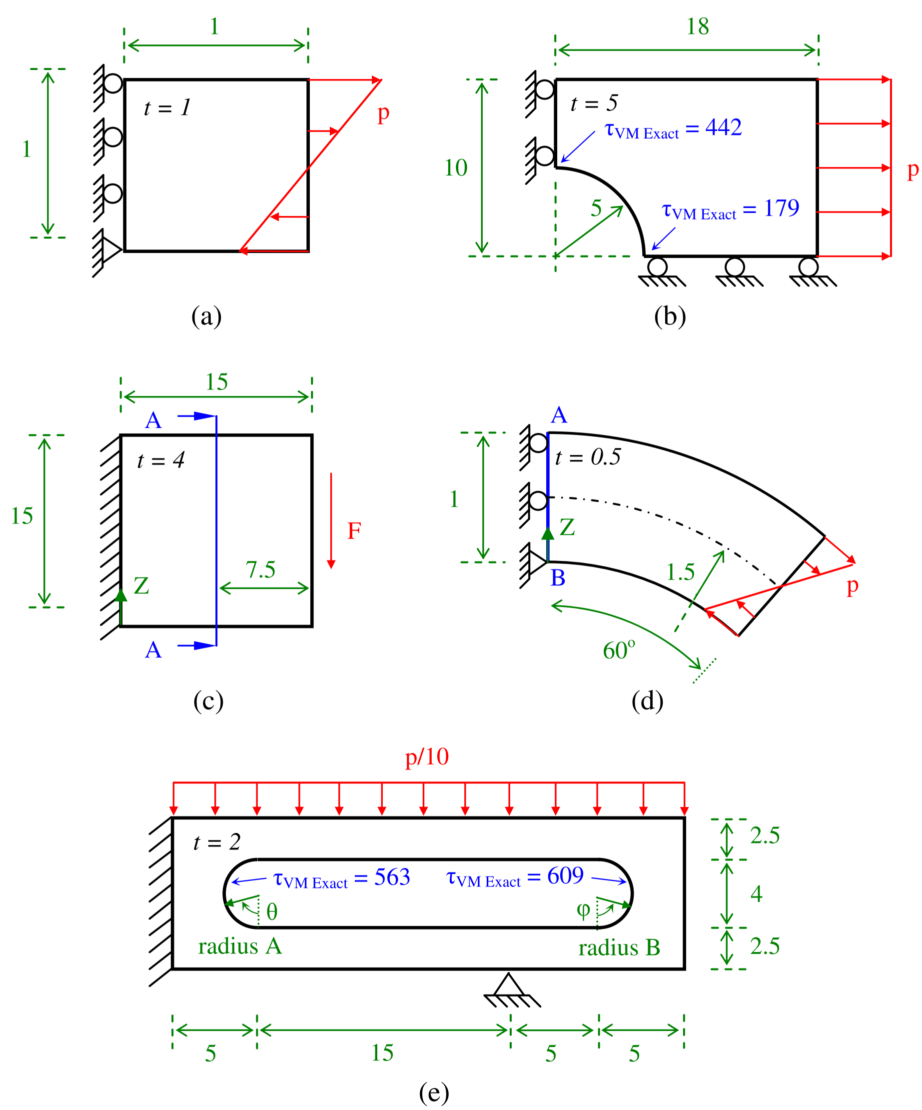

Fig. 2. Five test problems for the 4-node tetrahedral three-dimensional element (E = 72E9, ν = 0.0, p = 100, F = 6,000, t = thickness): (a) the beam in pure bending problem, (b) the finite plate with a central hole under tensile loading problem, (c) the square cantilevered plate under shear loading problem, (d) the curved structure in pure bending problem, and (e) the tool jig problem.

2. Fundamental equations of the method

A detailed review of the general and well-known principles used in the nodal point force based stress calculation method is given in Ref. [1], and hence we shall only summarise here the fundamental equations used. In Section 3 we then focus on the specific details to apply these principles – and their properties – in order to establish improved stress predictions for the 4-node three-dimensional tetrahedral element. We assume linear analysis conditions and use the notation of Ref. [2].

As is standard, we establish the stiffness matrix in the usual manner, solve for the element nodal point displacements $\underline{U}$, and the directly-calculated finite element stresses are then given by

$$\underline{\tau}_h^{(m)} = \underline{C}^{(m)}\underline{\varepsilon}^{(m)} = \underline{C}^{(m)}\underline{B}^{(m)}\underline{U} \tag{1}$$

where $\underline{C}^{(m)}$, $\underline{\varepsilon}^{(m)}$, and $\underline{B}^{(m)}$ are the stress–strain matrix, the finite element strain vector, and the strain–displacement matrix of element m, respectively.

The element nodal point forces $\underline{F}^{(m)}$ corresponding to the directly-calculated finite element stresses $\underline{\tau}_h^{(m)}$ are defined as

$$\underline{F}^{(m)} = \int_{V^{(m)}} \underline{B}^{(m)T}\left\{\underline{\tau}_h^{(m)}\right\} dV \tag{2}$$

where $V^{(m)}$ is the volume of element m .

If we assume that there exists and we can calculate improved finite element stresses $\underline{\tau}^{(m)}$ from these element nodal point forces, we obtain two fundamental equations involving these unknown stresses (and hence the unknown coefficients used to express $\underline{\tau}^{(m)}$). The first fundamental equation states that for any virtual displacement field contained in the element interpolation functions, the virtual work by the element boundary tractions is equal to the virtual work by the element nodal point forces (adjusted for body force effects), and hence we call this equation "the principle of virtual work in the form of boundary tractions"

$$\int_{S_f^{(m)}} \underline{H}^{(m)T}\{\underline{\tau}^{(m)}\underline{n}^{(m)}\} dS = \underline{F}^{(m)} - \int_{V^{(m)}} \underline{H}^{(m)T}\underline{f}^B dV \tag{3}$$

where $\underline{H}^{(m)}$, $S_f^{(m)}$ are the displacement interpolation matrix and the total external surface area of element m, respectively, and $\underline{n}^{(m)}$ is the unit normal to the element boundary. In the absence of body forces $\underline{f}^B$, Eq. (3) reduces to

$$\int_{S_f^{(m)}} \underline{H}^{(m)T}\{\underline{\tau}^{(m)}\underline{n}^{(m)}\} dS = \underline{F}^{(m)} \tag{4}$$

The second fundamental equation states that for any virtual displacement field contained in the element interpolation functions, the element internal virtual work is equal to the virtual work of

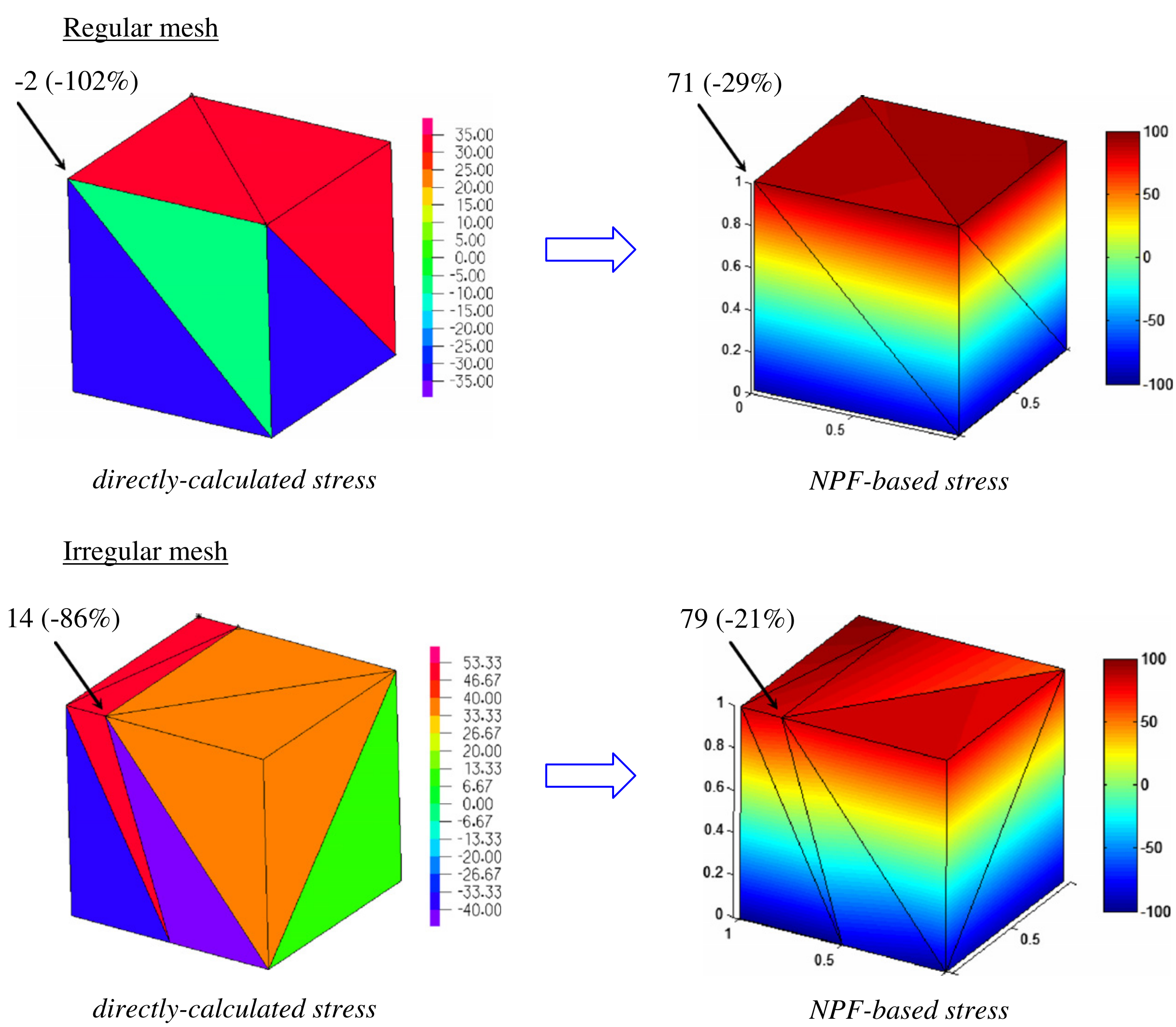

Fig. 3. Longitudinal stress results for the beam in pure bending problem. The solution error is given in the parentheses.

the element nodal point forces, and hence we call this equation "the principle of virtual work in the form of internal stresses"

$$\int_{V^{(m)}} \underline{B}^{(m)T}\{\underline{\tau}^{(m)}\}dV = \underline{F}^{(m)} \tag{5}$$

where, of course, the element nodal point forces $\underline{F}^{(m)}$ always correspond to the directly-calculated finite element stresses $\underline{\tau}_h^{(m)}$, see Eq. (2). We should note that in Eqs. (3) and (5), the $\underline{\tau}^{(m)}$ are assumed stress fields over the element m and hence we do not only use unknown tractions between finite elements.

The NPF-based method uses, as its ingredients, these two fundamental virtual work statements – Eqs. (3) and (5) – to obtain finite element stresses that we can expect to be more accurate than those given by Eq. (1). We expect that, in general, more accurate stresses are predicted because, firstly, the method allows us to assume a richer functional space for the stresses than that implicitly assumed in establishing the stiffness matrix, and, secondly, the nodal point forces are used which always satisfy the above-mentioned important equilibrium requirements, irrespective of the coarseness of the mesh. However, we do not have a proof that the stresses will always be improved at a particular location of the model.

3. Improving the stresses of the 4-node tetrahedral element

In order to establish improved stress predictions for a *general* finite element m, the NPF-based stress calculation algorithm employs four basic steps:

1. Solve, in the usual manner, for the element nodal point displacements $\underline{U}$, and the element nodal point forces $\underline{F}^{(m)}$, in accordance with Eq. (2).
2. Assume appropriate functions for $\underline{\tau}^{(m)}$ across a predetermined patch of elements; we call this patch of elements "the stress calculation domain".
3. Use the two principle of virtual work statements – Eqs. (3) and (5) – to solve for the unknown stress coefficients in $\underline{\tau}^{(m)}$.
4. Finally, to establish the improved stresses for an individual element m, the stress coefficients corresponding to all possible element combinations to obtain stress calculation domains that contain element m are calculated using the above steps, and the results are averaged for element m.

Of course, it is important to select appropriate functions for the stress fields in $\underline{\tau}^{(m)}$, since we aim to have a sufficiently rich

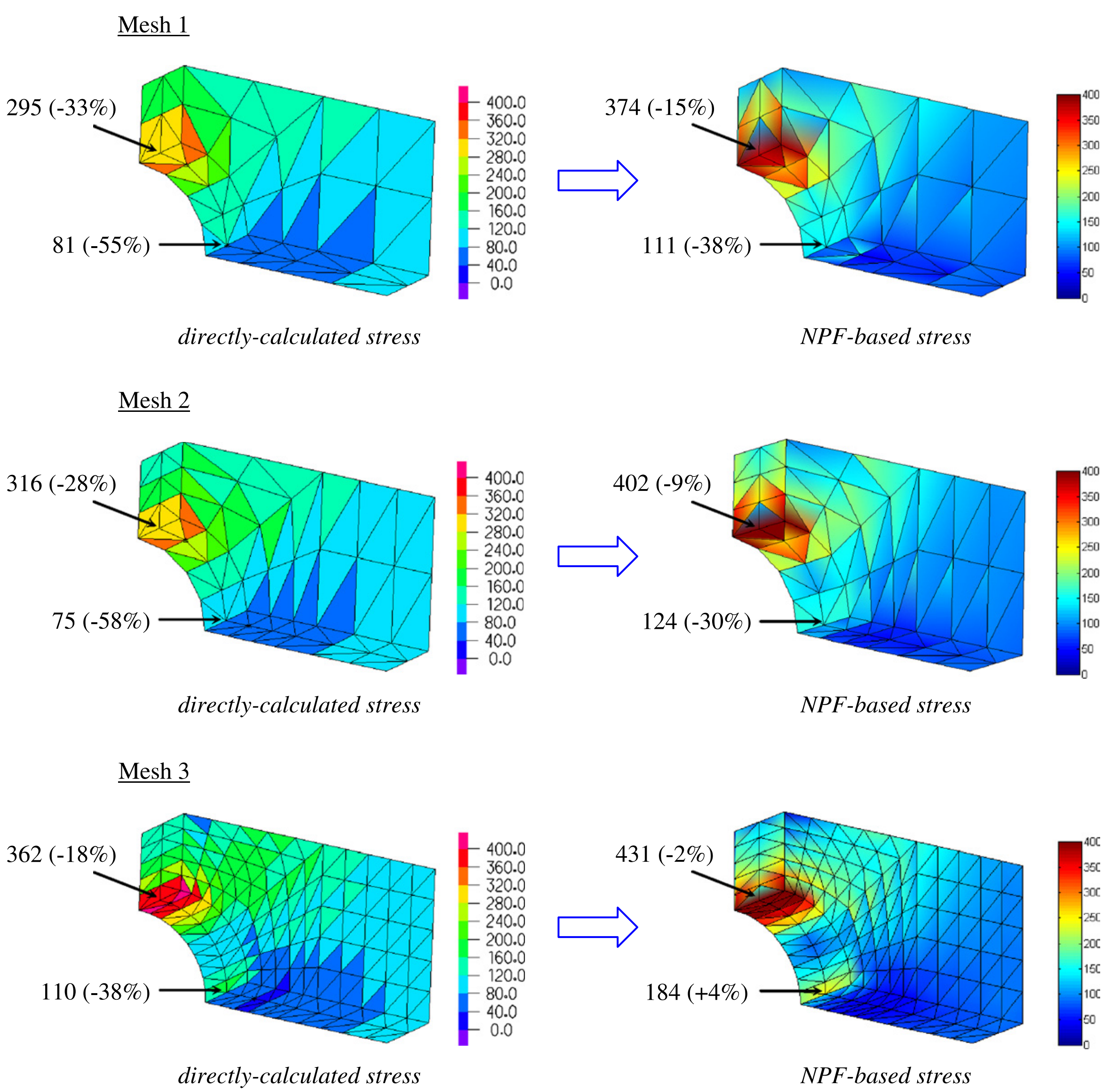

Fig. 4. von Mises stress results for the finite plate with a central hole problem. The solution error is given in the parentheses.

assumed stress space for the stress calculation domain. Clearly, the dimension of the assumed stress space must depend on the number of elements used within the stress calculation domain. That is, for a given dimension of assumed stress space, we must have that the domain contains a sufficient number of elements, such that the problem solution for the unknown stress coefficients is well-posed for all possible domain geometries that might be used.

In the specific case of the 4-node tetrahedral three-dimensional element, we assume the stresses to be linearly interpolated and continuous across the entire stress calculation domain,

$$\tau_{ij}^{(m)} = \alpha_1^{ij} + \alpha_2^{ij}x + \alpha_3^{ij}y + \alpha_4^{ij}z \quad for \quad m = 1,2,3,4,5 \tag{6}$$

where the (i, j) refer to the coordinate directions, and the α_k^{ij} are the twenty-four unknown stress coefficients to be found. As an aside, we note that for the 3-node constant strain triangle considered in Ref. [1] we instead assumed bilinear interpolations across its stress calculation domain.

With the assumption in Eq. (6), each stress calculation domain for the 4-node tetrahedral element shall contain at least five elements, this way we ensure a well-posed problem for the solution of the coefficients. Although any five adjacent elements could be used, we define a stress calculation domain in a quite natural manner as the unique combination corresponding to a central element surrounded by four peripheral elements, where each peripheral element shares a face with the central element, as shown in Fig. 1. This stress calculation domain allows us also to maximise the accuracy of the stress prediction, since the averaging in step 4 is used, see above and the further comments below.

In general, the algorithm solves for the unknown stress coefficients in $\underline{\tau}^{(m)}$ by imposing Eq. (3) to all possible closed contour boundaries contained within the stress calculation domain, and in addition Eq. (5) to the complete domain. However, in this case, we have assumed the stresses to be linearly interpolated, and hence we need to only apply Eq. (3) in order to solve for the stress coefficients. The reason is that in the absence of body forces, Eq. (5) is not independent of Eq. (3), see Ref. [1]. Furthermore, we assume inter-element stress continuity, and hence Eq. (3) can be imposed to every possible closed contour boundary by simply imposing the equation to the five tetrahedral element boundaries.

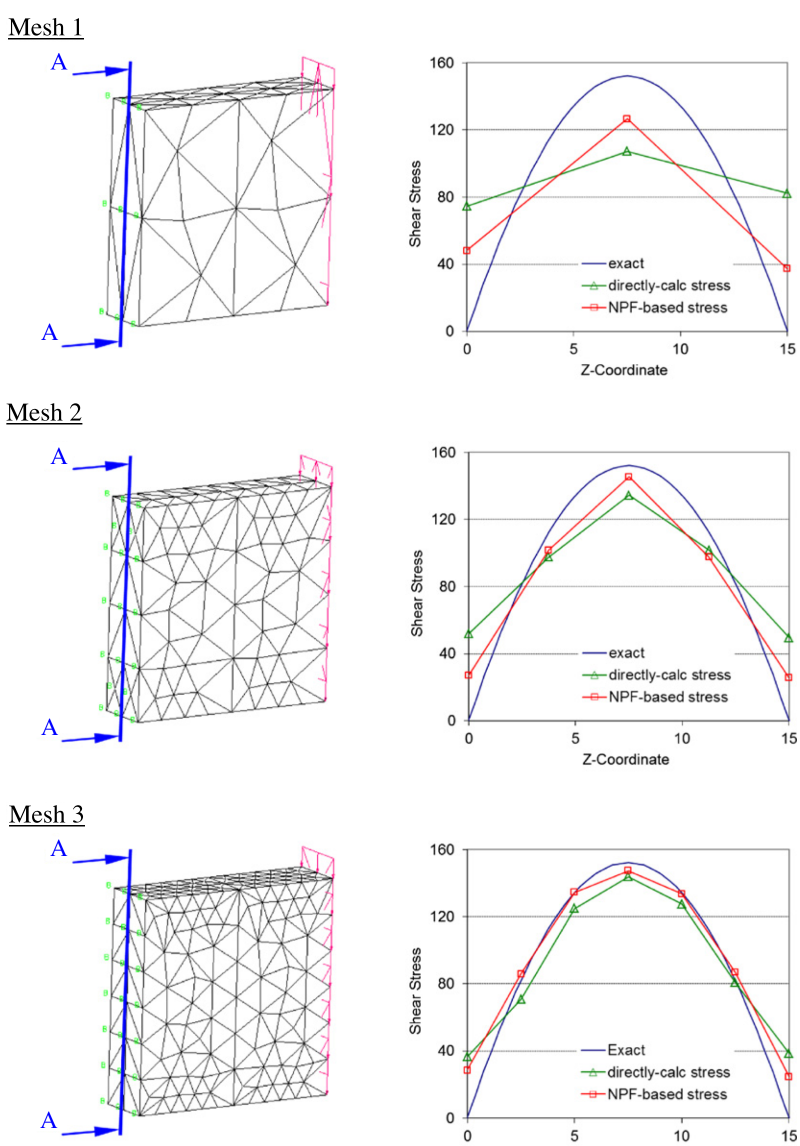

Fig. 5. In-plane shear stress results for the square cantilevered plate problem across section A.

In this way, we generate sixty equations, of which, for the configuration considered in Fig. 1, only thirty-three are linearly independent. Since there are twenty-four unknown stress coefficients, the system of equations is over-determined, and so, in general, a solution which exactly satisfies Eq. (3) does not exist. Hence we use the least squares method to solve for the unknown stress coefficients, with the consequence that the element nodal point forces calculated from the NPF-stresses (see Eq. (4)) will only satisfy the individual element and nodal equilibrium properties mentioned earlier, in a least squares sense.

Finally, to obtain the improved stresses for each tetrahedral element m, we average the stress coefficients corresponding to the possible stress calculation domains that contain element m. Of course, for the chosen geometry there can be no more than five domains that contain element m, that is, respectively, one and four domains for the element taking the position of the central element and the peripheral elements. In the exceptional case that no domain, as described above, exists which contains element m (e.g. in a corner of a meshed geometry), we simply construct the stress calculation domain using four elements that are properly connected to element m, and no averaging is applied.

Since we assume the stresses to be linearly interpolated, the numerical effort involved in improving the stress predictions for each tetrahedral element is given by the effort required to solve for twenty-four unknown stress coefficients at most five times (that is, we must calculate the stress coefficients corresponding to every possible domain which contains element m).

This computational effort is relatively small, but, also, an important feature of the algorithm is that there is no need to apply these stress calculations to all elements in the assemblage, instead only to those elements where improved stresses should be calculated. Indeed, in practice, the finite element analyst is not always able to perform – due to stringent constraints on time and computational resources – a detailed mesh refinement stress convergence study, especially for complex problems that are expensive to solve. Instead, in many cases, the analyst will solve the problem only once, using the finest mesh possible that for the available computational resources still results in a reasonable solution time. Given this solution and the above rather simple algorithm, it is then possible to enhance the stress prediction with relatively little computational effort in only the specific areas of concern.

In addition to enhancing the stress prediction, the results obtained with the algorithm give, of course, also insight into the accu-

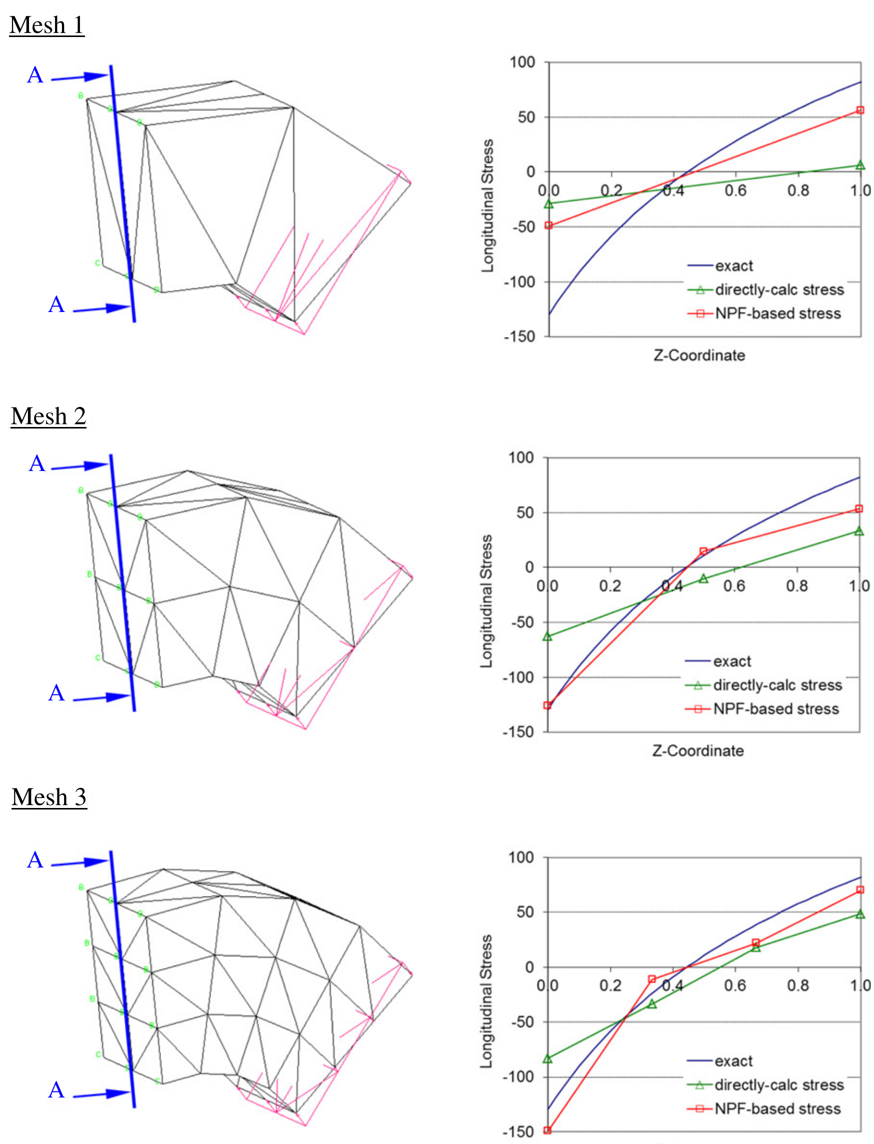

Fig. 6. Longitudinal stress results for the curved structure problem across section A.

racy of the directly-calculated stresses. Namely, if the two stress values are far apart, in important areas of the model, the mesh used is too coarse for the directly-calculated stresses to be sufficiently accurate.

We recognise that we have not mathematically proven stability of the algorithm for *all* possible geometries of the stress calculation domains. Therefore, it is possible, that for certain meshes with grossly distorted elements the algorithm establishes ill-conditioned matrices in which case the solution would have to be abandoned for that particular domain (where the elements are too distorted). However, we have tested the procedure in a large number of domains containing highly distorted elements and have not encountered this difficulty. Hence our experience is that as long as the mesh is reasonable (which is anyways required for the original displacement solution) the algorithm seems to be quite robust and stable.

The effectiveness of the algorithm for the 4-node tetrahedral element is illustrated using the same five test problems as considered in Ref. [1]: a beam in pure bending, a finite plate with a central hole under tensile loading, a square cantilevered plate under shear loading, a curved structure in pure bending, and a tool jig problem (like considered in Ref. [7]). We define these test problems in Fig. 2, and show the results (rounded to full digits) in Figs. 3–8 respectively, where the NPF-based stress refers to the stresses calculated using the proposed nodal point force based stress calculation method.

Considering these results, the values given in the band plots are un-averaged, while the given numerical stress values are the averaged nodal point values with the solution error shown in parentheses. This error is measured with respect to the solution (called "exact" in figures) obtained using a very fine mesh of 27-node hexahedral elements.

Note that a given numerical stress value may be outside the scale of the band plot because we selected the scale to reasonably indicate the stress variation over the complete domain.

As expected, we see a significant improvement in the accuracy of the predicted stresses for all problems solved. However, the improvement in stresses is somewhat less than what we have seen for the 3-node constant strain triangle in Ref. [1], which is partly due to the fact that, for the three-dimensional analyses, we are using linear, and not bilinear, stress interpolations, see Eq. (6).

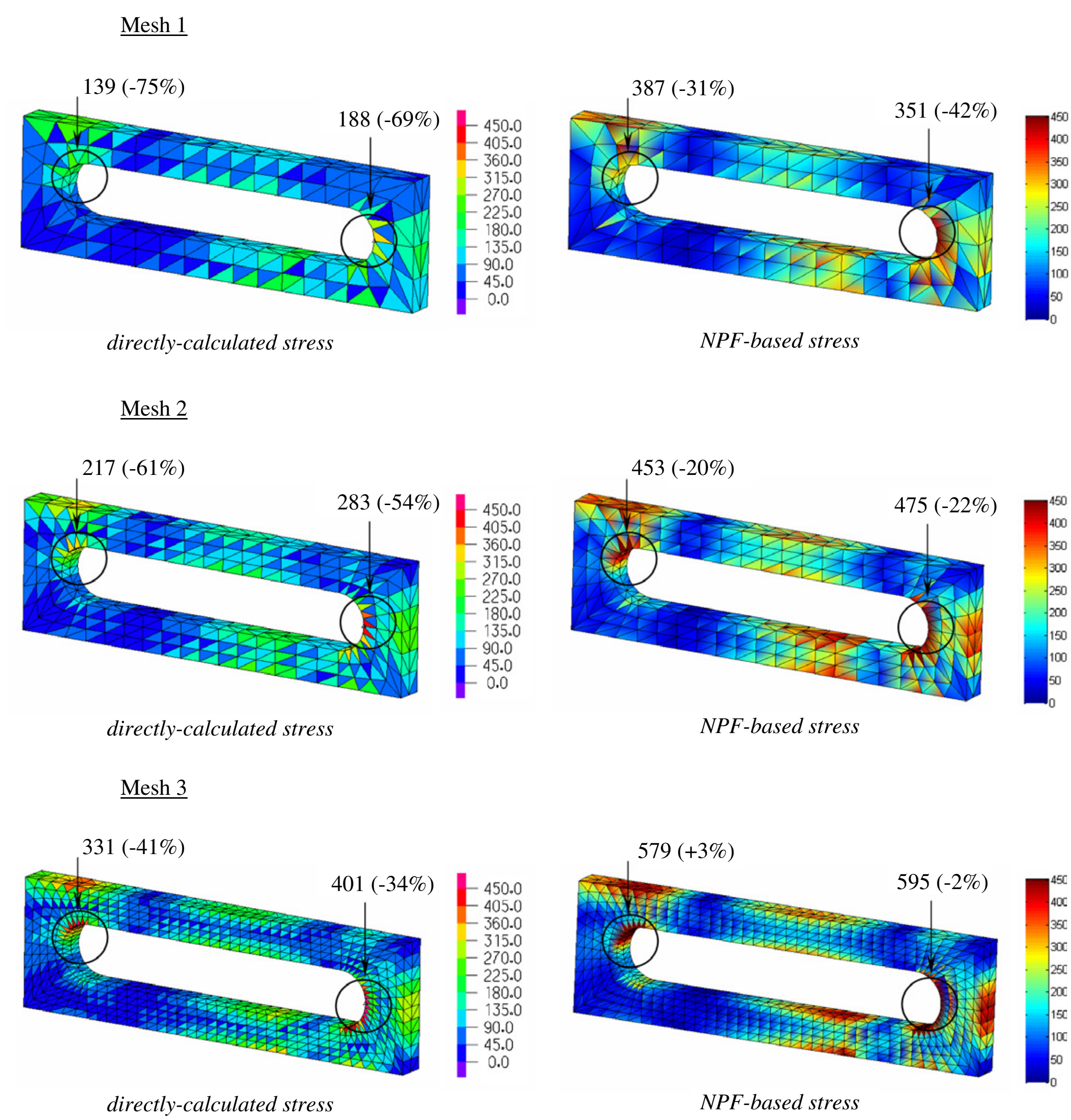

Fig. 7. von Mises stress results for the tool jig problem. The solution error is given in the parentheses.

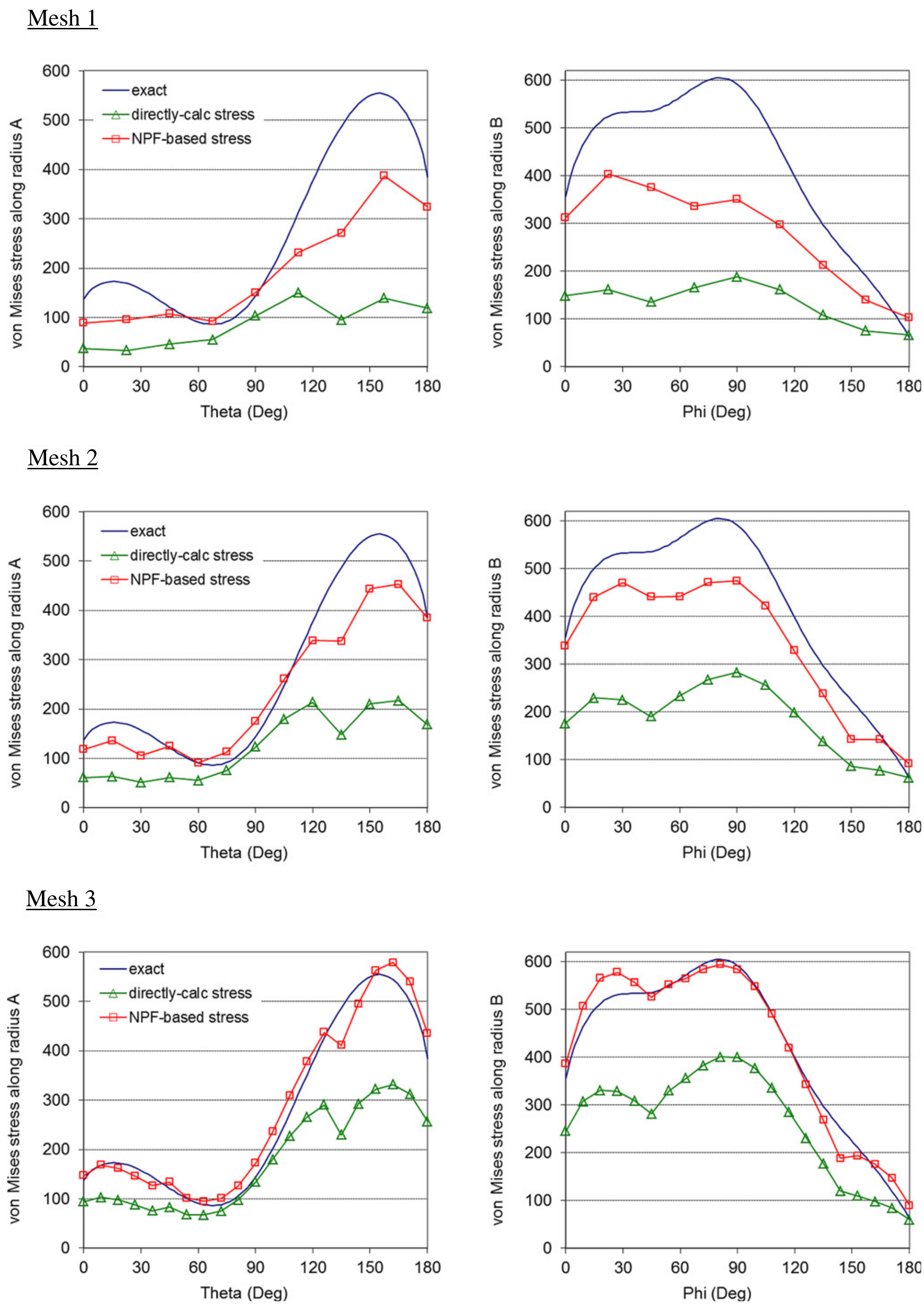

Fig. 8. von Mises stress results for the tool jig problem. Radius A and radius B are defined in Fig. 2, and Mesh 1, Mesh 2 and Mesh 3 are shown in Fig. 7. The figures on the left show the von Mises stress along radius A, whereas the figures on the right show the von Mises stress along radius B.

It is interesting to note that, for the problems considered in Figs. 4 and 7, the percentage improvement in stresses increases as the mesh is refined. Naturally, the improvement is most important in the regions of high stress gradients, which, of course, is due to the fact that the stresses $\underline{\tau}_h^{(m)}$ are constant for the 4-node tetrahedral finite element.

In these problems, we have set the Poisson ratio to zero, to ensure consistency with Ref. [1]; however, the same level of improvement is also observed for non-zero values of Poisson ratio, for example, when $\upsilon = 0.3$, as long as the material is not almost or fully incompressible. When the medium is incompressible, as well-known, the 4-node displacement-based tetrahedral element is not effective because it does not satisfy the inf–sup condition [2,15,16] and is better not used.

4. Concluding remarks

In this paper we applied the approach given in Ref. [1] to establish a procedure for the calculation of improved stresses for the widely-used 4-node three-dimensional displacement-based tetrahedral element. As expected, when we applied the procedure, we have indeed seen a significant improvement in the stress predictions for all problems solved.

These results are quite encouraging, and the simple algorithm might well be attractive in practice (after further studies, see below) – especially, for complex problems that are expensive to analyse – since the procedure allows the analyst to enhance the stress predictions in localised regions of concern without having to refine the mesh or re-analyse the model.

Regarding future research on the NPF-method for stress predictions, as we pointed out already in Ref. [1], a strong mathematical basis of the procedure would be of great value in order to identify the optimal stress assumptions and associated stress calculation domains to use. Given a specific scheme, theoretical studies of convergence and numerical studies on more complex problems are clearly needed to identify how, and how well, the NPF-based stress predictions converge to the solutions sought. Then improvements to the algorithms used in Ref. [1] and herein may well be identified, and limitations may be established. In addition, the use of the NPF-method for stress calculation might be explored in the analysis of shells, dynamic analyses, and the solution of nonlinear problems.

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