Advances and Trends in Structural Engineering, Mechanics and Computation – Zingoni (Ed.)

Challenges and advances in the analysis of structures

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ABSTRACT: The objective in this presentation is to briefly present major challenges in the analysis of structures and some recent developments in finite element procedures that we have pursued. The general challenges are to solve problems more reliably, accurately and efficiently, and to solve problems that so far cannot be analyzed. We present our recent developments for – the finite element analysis of shells, the solution of wave propagation problems, the time integration in long-time large deformation analyses, the analysis of large deformations of beam structures, and the simulation of fluid flow-structure interactions including various physical phenomena. This paper is an updated version of Bathe (2009a).

1 INTRODUCTION

The analysis of structures is largely performed using finite element procedures. These are now widely used in engineering and the sciences and we can expect a continued growth in the use of these methods (Bathe (ed.) 2009, Zienkiewicz & Taylor 2005, Bathe 1996a, Bathe 2009b).

Considering the analysis and design of civil and mechanical engineering structures, we can categorize the analyses into two broad groups. In the first group, it is quite possible to perform physical tests and thus compare the analysis results with laboratory test data. The analysis of motor cars falls into that group. In the second group, physical tests can only be performed to a very limited extent. The analysis of long-span bridges under earthquake loadings falls into that group. For this second group of analyses it is most important to use reliable finite element methods in order to have the highest possible confidence in the computed results. In addition, of course, the finite element procedures should be efficient. While in the first group, confidence in the safety of a design can be reached by evaluating the analysis results against physical test data, of course, both these requirements – reliability and efficiency – are very important in all analyses, even when physical test data can be obtained – if only to reduce the number of tests to be performed (Bathe 1996b, Elabbasi et al. 2004).

When assessing the current use of finite element methods for the analyses of structures, we naturally find that there is also a great interest in solving structural problems that heretofore could not be tackled. Hence, clearly, novel finite element procedures need to be researched, established, and eventually offered widely in computer programs.

The objective in this paper is to briefly discuss major challenges in the development of methods for the analysis of structures, and briefly focus on our recent developments to advance the state of the art. In our research we have continuously focused on the efficiency and reliability of the methods. Of course, any simulation starts with the selection of a mathematical model, and this model must be chosen judiciously (Bathe 1996a). However, once an appropriate mathematical model has been selected, for the questions asked, the finite element solution of that model needs to be obtained reliably, effectively, and ideally to a controlled accuracy.

In the following sections we first summarize, in a short section, the challenges, and then briefly present our recent developments in the finite element analysis of shells, wave propagation problems, highly nonlinear dynamic long-time duration events, beam structures, and general practical fluid-structure interactions, that can include electro-magnetic effects. This paper is an updated version of Bathe (2009a).

2 CHALLENGES

The challenges in the development of structural analysis procedures can be broadly categorized into those that advance the reliability and efficiency of finite element methods, and those that can be used to solve new classes of problems. But, of course, even when developing methods to solve new classes of problems the issues of reliability and efficiency are always important. The challenges that we consider were given in the Preface of Bathe (ed.) (2005).

Considering the analysis of "new" problems – or rather problems that could not be tackled before – these are abundant in the areas of fracture, wave propagation, composite structures, large strain responses of novel materials in solids and shells, large deformation and long time duration nonlinear dynamic responses, and fluid-structure interactions including electro-magnetic effects. These problems are found in mechanical, civil, aeronautical, earthquake and biological engineering. Frequently, multi-physics and multi-scale effects need to be represented, and stochastic descriptions of geometries, loads, and material data may also be important.

If a design optimization of a structure is to be performed, the difficulties can significantly increase in establishing a model that adequately represents the response and that can be solved in a reasonable amount of computing time.

For the analysis of these "new" problems, many so-called novel methods can be proposed, and simple illustrative problems can be solved. However, the difficulties are that these methods should be reliable, efficient and generally applicable – such that they can be implemented in general computer programs useable by analysts that are not familiar in detail with the computational procedures. As long as the analyst has a good fundamental knowledge of mechanics, and the basic assumptions of the methods are described, these methods should be useable to obtain reliable solutions.

The step from having a method that can be used by the single researcher to solve a problem to having a method that can quite generally be used to solve problems in a certain category in a reliable and efficient manner is very large – but most important to be achieved if the analysis capabilities are to be advanced in general. To have accomplished that step requires that the method has the required attributes of reliability, efficiency and generality which must be "proven" by as much mathematical analysis as possible, and severe numerical testing with discriminating test problems that are designed specifically to focus on the reliability and efficiency of the method.

A major and quite natural aim in finite element research has been the establishment of 'a posteriori' error measures, which would tell an analyst to what accuracy a chosen mathematical model was solved with the given finite element discretization. While this would clearly be a most desirable feature to have in an analysis, it is a significant question how for general nonlinear analysis, such measures, if to be reliable, inexpensive to use and generally applicable, can be developed. Indeed, considering just general practical linear analyses, such measures, after many years of research, seem to still not be available (Grätsch & Bathe 2005). The basic difficulty is, of course, that the "exact solution' of the mathematical model is unknown (with multiple solutions possible in nonlinear analysis) and yet a comparison of the numerical finite element solution with that unknown exact solution shall be established. Furthermore, considering error measures in practical analyses, the 'actual' error between the real physical response of the structure and the exact solution of the mathematical model is of much interest. While hierarchical modeling is here an important concept, the methods are not easy to apply in advanced nonlinear analyses (Bathe 1996a, Bucalem & Bathe, in prep.).

3 OUR DEVELOPMENTS

Once a mathematical model has been chosen, it is important that well-founded, reliable and of course efficient numerical methods be used for solution. By reliability of a finite element procedure we mean that in the solution of a well-posed mathematical model, the procedure will always, for a reasonable finite element mesh, give a reasonable solution – and if the mesh is reasonably fine, an accurate solution of the chosen mathematical model is obtained (Bathe 1996a).

By reliability of a finite element procedure we also mean that if some analysis conditions are changed, and seemingly only slightly, in the mathematical model, then for a given finite element mesh, time step integration, and so on, the accuracy of the finite element solution does not drastically decrease, unless there are distinct physical reasons. These conditions on analysis methods are very difficult to achieve and require theoretical depth in the understanding of the methods, and thorough and extensive testing based on theoretical insights. These conditions also rule out the use of methods that require the setting and problem-dependent adjusting of numerical parameters to achieve stability of a procedure.

3.1 *Shell elements*

The fundamental requirements in the development of shell elements are that the discretization should satisfy the consistency condition, the ellipticity condition, and ideally the inf-sup condition (Bathe 2009b, Bathe 2001, Chapelle & Bathe 2003, Hiller & Bathe 2003)

$$\sup_{\mathbf{v}_h \in V_h} \frac{b(\boldsymbol{\eta}_h, \mathbf{v}_h)}{\|\mathbf{v}_h\|_V} \geq c \sup_{\mathbf{v} \in V} \frac{b(\boldsymbol{\eta}_h, \mathbf{v})}{\|\mathbf{v}\|_V} \quad \forall\ \boldsymbol{\eta}_h \in E_h \tag{1}$$

where V is the complete (continuous) displacement space, V_h is the finite element displacement space, E_h is the finite element strain space, $b(.,.)$ is the applicable bilinear form, and c is a constant independent of the shell thickness t and the element size h. To show analytically that the inf-sup condition is satisfied for an element formulation is very difficult because it involves the complete space V for any shell geometry.

If an element satisfies these conditions, the discretization is very reliable for all shell analyses, that is, in the analyses of membrane-dominated shells, bending-dominated shells, and mixed behavior shells (Chapelle & Bathe 2003). Displacement-based shell elements do not perform well in bending-dominated cases and mixed elements need to be used.

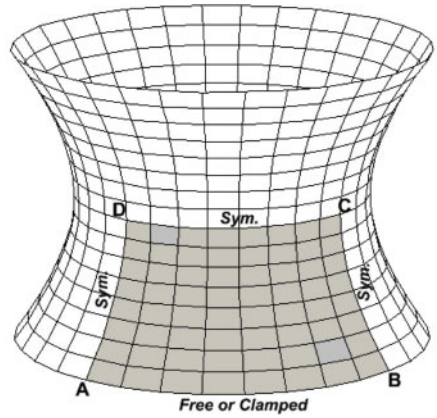
In the development of mixed shell elements, mathematical convergence proofs could so far only be given for certain elements and rather simple shell geometries and boundary conditions, see e.g. Havu & Pitkäranta (2003). However, mathematical analysis has been powerful in guiding how elements should be tested in order to reveal whether the above conditions are satisfied (Chapelle & Bathe 2003, Hiller & Bathe 2003, Chapelle & Bathe 1998, Chapelle & Bathe, in press).

While reasonably effective quadrilateral shell elements are available, also incorporating 3D effects (Kim & Bathe 2008), it is a particularly difficult task to develop a general triangular 6-node shell element that is spatially isotropic, has the same degrees of freedom at every node, does not contain any instability, and converges well in membrane- and bending-dominated problems. The testing of the element should involve specific problems chosen to reveal the element properties, and in particular shell problems based on a hyperbolic shell surface, and appropriate norms to measure the solution errors (Chapelle & Bathe 2003, Hiller & Bathe 2003, Chapelle & Bathe 1998).

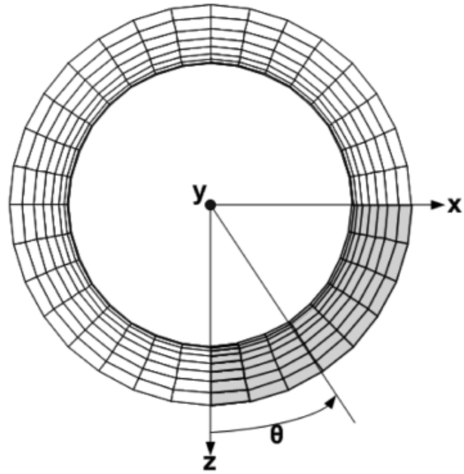
We recently proposed the MITC6 shell element and show the convergence results of the element using the s-norm for hyperboloid shell problems in Figure 1 (Kim & Bathe 2009). This element satisfies all the features sought and converges also almost optimally in the clamped-clamped hyperboloid problem (the membrane-dominated problem) but only reasonably well in the free-free hyperboloid problem (the bending-dominated problem). In the solution of simpler problems the element performs almost optimally. It is one of the remaining challenges of finite element research to develop a still more effective general triangular 6node shell element. In this search, the 'enhanced assumed strain' method could be used (Simo & Rifai 1990) – a method quite different from the MITC approach (Bathe 1996a, Kim & Bathe 2009) – but then no instabilities should arise in nonlinear analysis. Such instabilities have been observed in enhanced assumed strain formulations although the formulations are stable in linear solutions, see for example Wriggers & Reese (1996) and Pantuso & Bathe (1997). The MITC shell elements are also effective in plate solutions (Lee & Bathe 2010).

While therefore advances regarding the linear analysis of shells need still be accomplished, the large strain analysis of shells is considerably more complex, in particular when anisotropic elastoplastic response is to be simulated (Kim et al. 2009, Montans & Bathe 2005). Here too, well-chosen and discriminating test problems should be solved in order to identify the reliability of an element. We use the 3D-Shell MITC4 element in large strain solutions (Bathe et al., in prep).

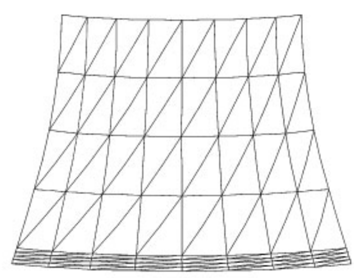
Considering complex beam structures, these can of course be modeled using shell elements, but such solutions can be very expensive to obtain and it is more practical to employ reliable and efficient beam elements. These, however, need to include, in general, the important beam 3D effects for different cross-sections in large deformations with torsion-bending coupling (Mahdavi & Bathe, in prep.).



(a) The hyperboloid shell problems considered (side view)



(b) The hyperboloid shell problems considered (top view)



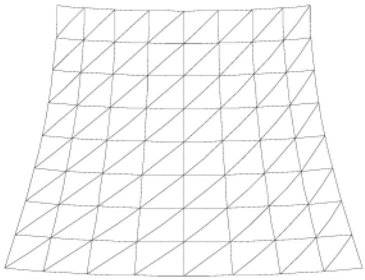
(c) 8x8 element mesh used for the clamped shell (including the boundary layer), modeling 1/8th of the shell

Figure 1. Analysis of the hyperboloid shell problems, t = thickness of shell, L = characteristic length, here L = 1; varying pressure loading is applied (Kim & Bathe 2009).

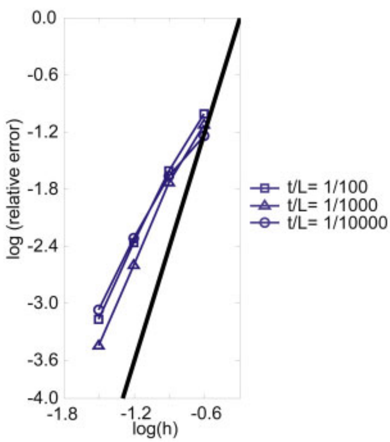
3.2 *Wave propagation problems*

Although, in principle, the finite element method can directly be applied to the solution of wave propagation problems, and indeed has been used abundantly for such analyses, a specific required accuracy in the response may be difficult to reach, and the solutions can computationally be very expensive. Accurate solutions can be particularly difficult to obtain in multi-scale analyses, in which

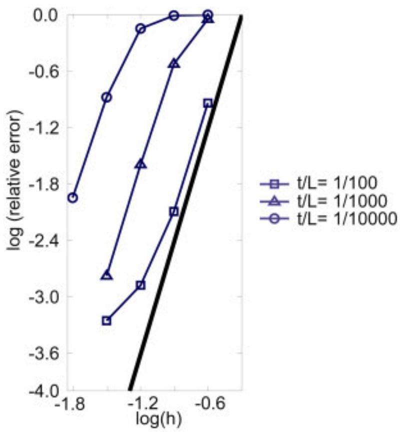
We have worked on the development of a method that, in essence, combines the attractive features of classical finite element discretization and spectral



(d) 8x8 mesh used for free-free shell, modeling 1/8th of the shell



(e)Convergence results using the s-norm: clamped-clamped shell



(f) Convergence results using the s-norm: free-free shell; the slopes of the bold lines correspond to optimal convergence

Figure 1. continued

techniques (Kohno et al. 2010). The key step is that the classical finite element shape functions are enriched with wave patterns. To illustrate the method, let us consider 1D solutions. If a classical linear shape function is used, we assume

$$g_{(\alpha,j)}(\xi)=\frac{1}{2}(1+\xi_\alpha\xi)\exp\left[i2\pi v_j\left(x_e+\frac{\Delta x}{2}\xi\right)\right] \quad (2)$$

where i, xe, Δx and ξ are the imaginary unit, the x-coordinate at the center of the element, the length of the element and the coordinate variable in the calculation space ($-1 \le \xi \le 1$). The physical space is then related to the calculation space by $x = xe + (\Delta x/2)\xi$. The subscript α denotes the local node number, and the coordinates of $x\alpha$ correspond to $\xi 1, 2 = -1, 1$. The wave numbers $2\pi\nu j$ are determined by $\nu j = j\nu$, where ν is the fundamental frequency and j is an integer in the range $-(N\text{F}-1)/2 \le j \le (N\text{F}-1)/2$ with the cutoff wave number (or number of harmonics) NF. The values of ν and NF need to be chosen for the analysis and further research is needed to develop an algorithm that automatically chooses appropriate values, and of course also element sizes.

Interpolations using classical finite element functions combined with analytical solutions have of course been used for a long time, see for example Astley (1983) and Bathe & Almeida (1980). However, in Equation (2) no specific analytical solution is employed. Instead, the waves embedded within the classical finite element functions are to capture the unknown wave propagations. This possibility is important when solving, for example, wave equations governing the response of plasmas, in which waves of different and sometimes widely varying properties and mode conversions are observed.

Figure 2 shows an application of the solution procedure and illustrates the effectiveness of the method, see Kohno et al. (2010), also for further solutions.

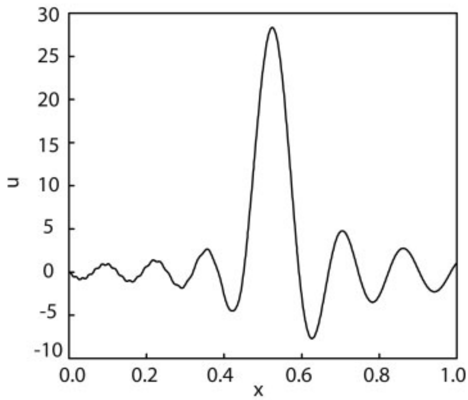
3.3 *Time integration in nonlinear analyses*

Considering transient structural response, a widely used scheme of time integration is the Newmark method trapezoidal rule. However, as well known, if large deformations over long-time durations need be solved, the trapezoidal scheme can become unstable and the generalized alpha method is then frequently used (Bathe 1996a, Chung & Hulbert 1993). With this method stability and accuracy parameters have to be set and depending on the values used, the calculated response can be inaccurate, resulting into experimentation with the parameters. The time integration scheme given in Bathe (2007), which in essence extends a method published much earlier for first-order systems (Bank et al. 1985), can in such cases be considerably more effective.

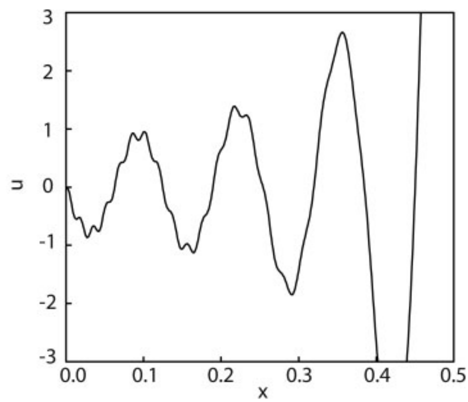
Consider that the solution at time t has been established and that the solution at time $t+\Delta t$ shall be obtained. The governing equations for a structural model to be solved are then

$$\mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{U}}+\mathbf{C}\,{}^{t+\Delta t}\dot{\mathbf{U}}={}^{t+\Delta t}\mathbf{R}-{}^{t+\Delta t}\mathbf{F} \quad (3)$$

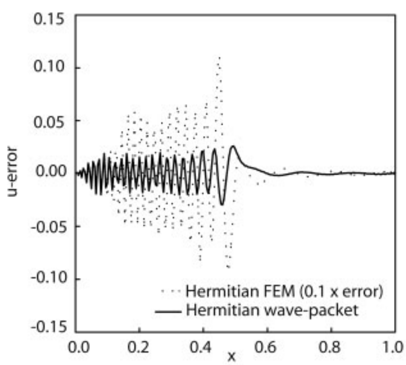
where **M** is the mass matrix, **C** is the damping matrix, **R** is the vector of externally applied loads, **F** is the vector of nodal forces equivalent to the element stresses, and **U** denotes nodal displacements (including rotations). The superscript $t+\Delta t$ denotes of course time and an overdot a time derivative. In the time integration scheme given in Bathe (2007) we consider each time step Δt to consist of two equal sub-steps, each



(a) Response over complete domain



(b) Detail to show short and long wave responses



(c) Comparison using Hermitian classical functions only and with wave packets

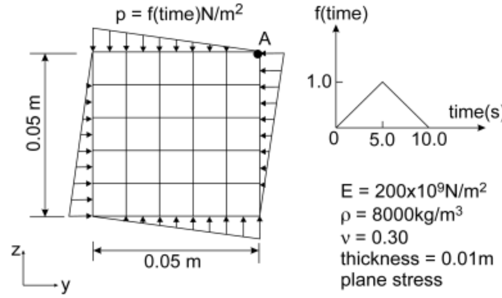
Figure 2. Solution of Wasow equation using wave-packet procedure of Kohno et al. (2010).

solved implicitly. The first sub-step is solved using the trapezoidal rule and the second sub-step is solved using the three-point Euler backward method with the governing equations

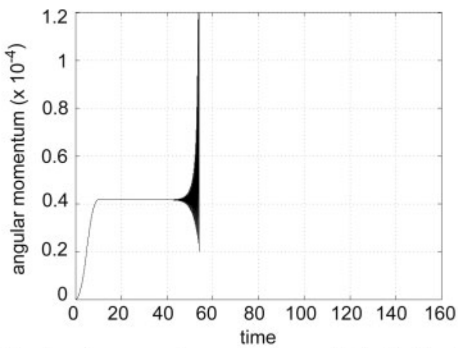
$$^{t+\Delta t}\dot{\mathbf{U}} = c_1{}^{t}\mathbf{U} + c_2{}^{t+\Delta t/2}\mathbf{U} + c_3{}^{t+\Delta t}\mathbf{U} \tag{4}$$

$$^{t+\Delta t}\ddot{\mathbf{U}} = c_1{}^{t}\dot{\mathbf{U}} + c_2{}^{t+\Delta t/2}\dot{\mathbf{U}} + c_3{}^{t+\Delta t}\dot{\mathbf{U}} \tag{5}$$

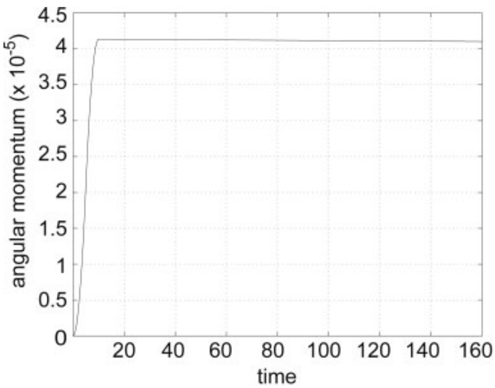
where $c_1 = 1/\Delta t$, $c_2 = -4/\Delta t$, $c_3 = 3/\Delta t$.



(a) The problem considered



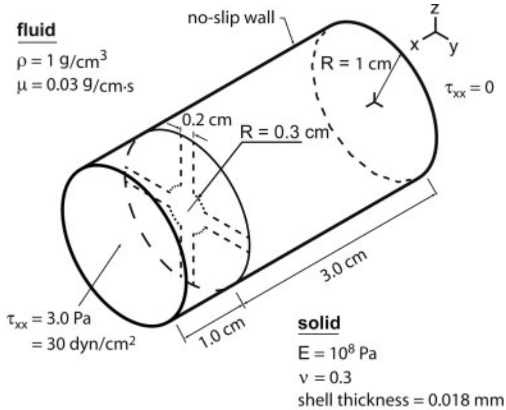
(b) Angular momentum using trapezoidal rule $\Delta t = 0.02$ s



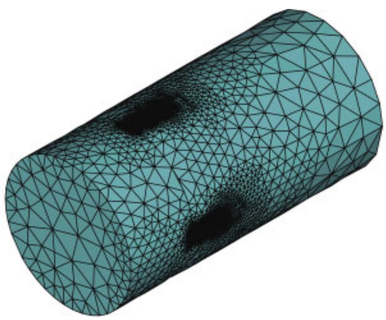
(c) Angular momentum using scheme of Bathe (2007), with the larger time step $\Delta t = 0.4$ s

Figure 3. Solution of rotating plate problem.

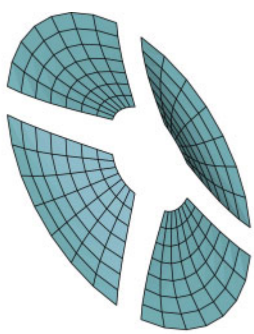
This scheme is a fully implicit second-order accurate method and requires per time step about twice the computational effort as the trapezoidal rule. However, no parameters have to be set, the accuracy per time step is significantly increased, and in particular the method remains stable when the trapezoidal rule fails to give the solution, see Figure 3 for an example. We use this scheme as an option in nonlinear dynamics of solids and structures, and also in fluid-structure interaction solutions. In this case the first-order fluid flow equations and the second-order structural equations are integrated consistently using the same scheme, either staggered or monolithically (Bathe & Zhang 2004).



(a) Problem considered



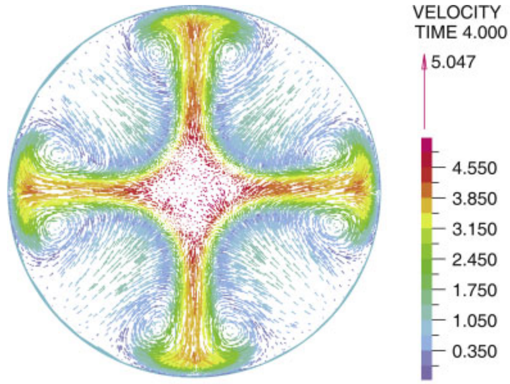
(b) Final CFD mesh reached after 6 mesh adaptations (547,741 elements)



(c) Mesh of shell elements modeling blades, in final configuration (deformations are shown to scale)

Figure 4. Fluid-structure interaction problem of flow around blades (Bathe & Zhang 2009).

Since we use the same finite element discretizations in explicit and implicit dynamic solutions of solids and structures, automatic switching from explicit to implicit integration, and vice versa, is directly possible. This automatic time integration can be very effective when fast and slow transients intermingle, when a dynamic response is followed by an almost static response (like in metal forming problems), and if convergence difficulties are encountered in the implicit time stepping.



(d) Velocities on plane 0.5 cm downstream from the blades, using final mesh

Figure 4. continued

3.4 *Fluid-structure interactions (FSI)*

Considering Navier-Stokes fluids, the FSI solution requires effective discretizations to model the fluid including high Péclet and Reynolds number conditions, effective finite element methods for the structure, and the proper coupling of the discretizations (Rugonyi & Bathe 2001).

We have concentrated our development efforts on establishing, for incompressible fluids, finite element discretization schemes that are stable, even when coarse meshes are used for very high (element) Péclet and Reynolds numbers and show good accuracy (Bathe & Zhang 2002, Bathe & Zhang 2004, Bathe & Zhang 2009). The basic approach is to use velocity-pressure interpolations to satisfy the inf-sup condition for incompressibility, flow-condition-based interpolations (FCBI) in the convective terms of the fluid, and to use element plane stress control volumes (like in the finite volume method) to assure local mass and momentum conservation.

The resulting FCBI elements do not require a tuning of upwind parameters, satisfy the above properties, and also, the interpolations can be used to establish consistent Jacobian matrices for the iterations in the incremental solutions.

We use these elements for incompressible, slightly compressible, and low speed compressible fluids. These flows and highly compressible fluid flows can all be coupled in an arbitrary Lagrangian-Eulerian formulation with general structural models (Bathe & Zhang 2004, Bathe & Ledezma 2007, Degroote et al. 2009). However, as well known, for very large structural deformations the fluid mesh can become much distorted unless an algorithm of remeshing is employed. We have developed an algorithm for mesh repair and adaptation based on element sizes and flow solution variables, like the gradient of pressure, for CFD (highly compressible and incompressible fluids) and general FSI (Bathe & Zhang 2009). As an example, Figure 4 shows the solution obtained of a fluid-structure interaction problem. We see that with the element grading in the mesh, a very fine mesh was established near the blades to capture the secondary flow.

4 CONCLUDING REMARKS

The objective in this paper was to briefly present major challenges in the analysis of structures and some of our recent developments to advance the state of the art. As emphasized above, our research focus is on the development of more efficient and reliable finite element methods, and on novel techniques for the solution of new classes of problems, like those described in Deilmann & Bathe (2009) and Sedeh et al. (2009). Additional areas of our research not explicitly mentioned in this paper are the modeling of electro-magnetic effects based on the full Maxwell's equations, specifically in fluid-structure interactions, multi-scale solutions in bio-mechanics, the speed-up of implicit and explicit solutions through SMP and DMP processing (Bathe et al., in prep., Bathe in prep.), meshless methods (Hong & Bathe 2005) and the more reliable and accurate solution of complex contact problems (Elabbasi & Bathe 2001), including thermo-mechanical effects. Many demonstrative solutions are given at http://www.adina.com/newsgrp.shtml.

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