Mean-field limit of systems with multiplicative noise

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A detailed study of the mean-field solution of Langevin equations with multiplicative noise is presented. Three different regimes depending on noise intensity (weak, intermediate, and strong noise) are identified by performing a self-consistent calculation on a fully connected lattice. The most interesting, strong-noise, regime is shown to be intrinsically unstable with respect to the inclusion of fluctuations, as a Ginzburg criterion shows. On the other hand, the self-consistent approach is shown to be valid only in the thermodynamic limit, while for finite systems the critical behavior is found to be different. In this last case, the self-consistent field itself is broadly distributed rather than taking a well defined mean value; its fluctuations, described by an effective zero-dimensional multiplicative noise equation, govern the critical properties. These findings are obtained analytically for a fully connected graph, and verified numerically both on fully connected graphs and on random regular networks. The results presented here shed some doubt on what is the validity and meaning of a standard mean-field approach in systems with multiplicative noise in finite dimensions, where each site does not see an infinite number of neighbors, but a finite one. The implications of all this on the existence of a finite upper critical dimension for multiplicative noise and Kardar-Parisi-Zhang problems are briefly discussed.

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I. INTRODUCTION

Problems susceptible to being mathematically represented by stochastic (Langevin) equations including a multiplicative noise abound not only in physics, but also in biology, ecology, economy, or social sciences. In a broad sense a Langevin equation is said to be multiplicative if the noise amplitude depends on the state variable(s) itself (themselves) [1]. In this sense, problems exhibiting absorbing states, i.e., fluctuationless states in which the system can be trapped, are described by equations whose noise amplitude is proportional to the square root of the (space and time dependent) activity density, vanishing at the absorbing state [2]. Systems within this class are countless: propagating epidemics, autocatalytic reactions, reaction-diffusion problems, self-organized criticality, pinning of flux lines in superconductors, etc. [2].

In a more restrictive sense, the one we will use from now on, it is customary to restrict the term *multiplicative noise* (MN) to noise amplitudes *linear* in the activity density [3–6]. Such equations appear ubiquitously in economics, optics, population dynamics, study of instabilities, etc. In all these cases, multiplicative noise terms appear rather straightforwardly when constructing (more or less rigorously) stochastic representative equations. In their spatially extended version, Langevin equations including a MN [see Eq. (1) below] were first proposed, to the best of our knowledge, in the context of synchronization of coupled map lattices [7], and soon after studied in Refs. [8–12]. Such equations describe different situations; in particular, there has been a recent interest in their application to nonequilibrium wetting [13] and also to synchronization problems in extended systems [14];

see Ref. [15] for a recent review. In all these cases, there is a phase transition between an active phase in which the system has a stationary nonvanishing activity and another one in which the density field falls continuously toward zero without ever reaching it in finite time. The analogies and differences between this family of models and the one with a square-root type of noise, in which the absorbing state is reachable within a finite time, have been discussed in [16,17].

It is remarkable that by using the so-called Cole-Hopf transformation, the spatially extended MN Langevin equation can be mapped into a nonequilibrium interface, as represented by the Kardar-Parisi-Zhang (KPZ) equation [18,19] in the presence of a bounding wall [15]. In this language, the active phase describes interfaces pinned by the wall while the absorbing one corresponds to depinned KPZ-like interfaces moving away from the wall.

In recent years many aspects of systems with MN have been elucidated. For example, a renormalization group approach has been constructed, scaling relations derived, and the large-N limit (where N is the number of components) studied [8]. It is well established that above two dimensions there are two different regimes; a weak-noise and a strong-noise one, in full analogy with the known phenomenology of KPZ interfaces [19]. There is however a crucial point which remains to be fully understood: the mean-field behavior of such systems. Another interesting mapping is that in the absence of the nonlinear term, the MN equation corresponds to the equation governing the evolution of directed polymers in random media (see [19] and references therein, as well as [15]).

In Langevin equations with additive noise, such as for instance the model *A* describing the universality class of Ising-like transitions [20], the *mean-field* approximation can be obtained in a number of equivalent ways, all of them leading to the same results with different degrees of sophistication. For example, mean-field critical exponents can be obtained (i) by discarding the noise and solving the remaining deterministic equation, (ii) as the most probable solution in a path integral formulation, (iii) self-consistently by assuming that each site sees the average of the remaining, (iv) by naive power counting in the corresponding action, (v) as the lowest order in a perturbative loop expansion, etc.

Contrarily, in systems with MN the meaning of the mean-field solution is much less clear. Indeed, early studies [9] led to conflicting results depending on the considered approximation method. For example, by removing the noise, much of the physics is lost and trivial results (identical to those for additive noise) are obtained. Also, if all spatial dependence is eliminated (by removing the Laplacian term in the MN equation), one is left with a solvable zero-dimensional equation which does not reproduce faithfully the rich mean-field phenomenology. Therefore, contrarily to more standard problems, both space dependence and noise have to be retained in order to construct a sound mean-field solution.

If the bounding wall is eliminated then one is left with the *directed polymer in random media* equation, for which many results are available. In particular, Derrida and collaborators worked out a solution on the Cayley tree, mean-field results, $1/d$ expansions, and solutions on hierarchical lattice exits [21]. Also, Mézard and Parisi derived a variational approach in replica space [22], and also Fisher and collaborators reached interesting results on these issues [23]. For a more detailed review on results for directed polymers, see [19] and references therein.

On the other hand, restoring the bounding wall (which is the case we are interested in) Birner *et al.* [12] (see also [9–11]) performed a self-consistent calculation and reported on the existence of two different mean-field behaviors: a weak-noise regime (in this particular case, the noise can be completely disregarded) and a strong-noise one exhibiting nonuniversal exponents depending on the noise amplitude. This is in agreement with the field theoretic expectation of two different behaviors for MN-like and KPZ-like equations in high dimensional systems (where mean-field results are expected to be valid) [15,19]. These results are difficult to compare with the above-mentioned ones obtained for directed polymers in random media, which correspond to the depinned phase of the full problem, while the self-consistent approach is intrinsically devised for the pinned phase.

Let us also underline that a full understanding in terms of path integrals and extremal paths is still missing despite some efforts in this direction [24,25].

Together with the determination of the right mean-field theory, another relevant and highly debated issue is to establish the upper critical dimension d_c, above which mean-field results hold. As some exponents, like the dynamical one z, have been claimed [8,15] to coincide for MN and KPZ, both problems are expected to have the same upper critical dimension. While there is consensus that the mean-field weak-noise regime should be valid above $d_c=2$ (coinciding with

the critical dimension for weak-noise Edwards-Wilkinson interfaces [19]) there exist highly conflicting results (some pointing to $d_c=4$ [26] and some supporting $d_c=\infty$ [27]) for the strong-noise regime.

In any generic problem, for sufficiently high dimensions every site in a spatially extended system "sees" the average of its neighbors (assuming the space has been discretized) which can be correctly approximated by its mean value (the distribution of the average values is well described by the standard central limit theorem above the upper critical dimension) defining in this way a sound mean-field solution. In the case of MN, as we will show, the situation is somehow anomalous: in the strong-noise regime the mean-field solution itself breaks down in the neighborhood of the critical point once fluctuations are taken into account (as a version of the Ginzburg criterion shows). This stems from the primary role played by fluctuations and rare events in multiplicative processes [5,6], and makes one wonder whether a mean-field solution can be valid at all in any finite dimension.

In this paper we revisit the self-consistent mean-field solution of Langevin equations with MN and report on a previously overlooked regime. After that, we discuss under which circumstances such a solution breaks down (Ginzburg criterion [28,29]). Also, we present a fluctuating solution aimed to extend the self-consistent one by allowing for fluctuations of the average field. Finally, we present some speculations on the issue of the existence of an upper critical dimension for this type of systems, i.e., if there is a finite space dimensionality above which the nonfluctuating solution holds or not.

The paper is organized as follows. In Sec. II, we outline a self-consistent solution of the MN equation defined in an infinite fully connected lattice. Different regimes are found, corresponding to weak noise, intermediate noise, and strong noise respectively. By constructing a Ginzburg criterion we will show how the strong noise solution is intrinsically unstable as soon as fluctuations are taken into account. In Sec. III we study the solution in finite lattices (fluctuating solution) and show by means of numerical simulations how the mean-field solution breaks down in the vicinity of the critical point. Section IV is devoted to the study of multiplicative noise on random regular graphs. This is done in order to ascertain the type of behavior in the thermodynamic limit when the connectivity of each site remains finite and, therefore, in order to obtain some insight on the behavior of the MN equation in large but finite space dimensions. The final section contains a summary of the findings along with some concluding remarks. Finally, a general scaling theory is presented in the Appendix.

II. NONFLUCTUATING SOLUTION

We summarize and extend the mean-field solution obtained by Birner *et al.* [12] for the MN equation as described in [8,15]:

$$\dot{\phi}_i = a\phi_i - b\phi_i^{p+1} + D\nabla^2\phi_i + \phi_i\eta_i\sigma, \qquad (1)$$

where η is a Gaussian δ correlated noise, a, b, D, and σ are constants, and ϕ is the local order-parameter field, describing

the physical density under study. In order to study its mean-field solution, we consider the case of global coupling, i.e., define the system on a fully connected (or complete) graph

$$\nabla^2 \phi_i = \frac{1}{N-1} \sum_{j \neq i} (\phi_j - \phi_i) \equiv M_i - \phi_i. \quad (2)$$

The associated Fokker-Planck equation [1] for Eq. (1) in the Ito sense can then be solved in the stationary case [9,11,12], assuming $M_i = M$,

$$R(\phi; M) = \phi^{-2-2(D-a)/\sigma^2} \exp\left(-\frac{2b\phi^p}{p\sigma^2} - \frac{2MD}{\phi\sigma^2}\right). \quad (3)$$

This is a power-law distribution function with two cutoffs: an upper one coming from the nonlinear saturation term and a lower one generated by M which acts as a constant external field.

In order to proceed further M is taken equal to its average over realizations, m, which is determined self-consistently by imposing [9,10,12]

$$m = \langle \phi \rangle = \frac{\int d\phi \, \phi R(\phi; m)}{\int d\phi \, R(\phi; m)}. \quad (4)$$

We will denote in the following this solution, valid for $N = \infty$, as the nonfluctuating (NF) solution. Equation (4) has always the trivial solution $m = 0$, stable for $a < a_c$ with $a_c = 0$. At the critical point, a_c, a stable solution with $m > 0$ appears. Introducing the distance from the critical point $\epsilon = a/(\sigma^2/2)$ and the reduced variable $s = 2D/\sigma^2$, we can rewrite the probability distribution as

$$R(\phi|m) = \phi^{-\alpha} \exp\left(-\frac{bs\phi^p}{pD} - \frac{ms}{\phi}\right), \quad \alpha = 2 + s - \epsilon. \quad (5)$$

Introducing the notation

$$I_{s,k} = \int_0^\infty d\phi \, \phi^k R_s(\phi|m) \quad (6)$$

the first moment $m = I_{s,1}/I_{s,0}$ is easily computed, yielding the relation between m and ϵ in the active region

$$m^{\min[s-\epsilon,p]} \sim \epsilon \quad (7)$$

so that $m \sim \epsilon^{\beta_1}$ with $\beta_1 = \max[1/s, 1/p]$. In this way, two different regimes were found by Birner et al.: one is universal (weak noise) and the other one is not (strong noise) with the order-parameter exponent changing continuously with s. These two regimes are the analogue of the well-known weak-coupling (Gaussian) and strong-coupling (non-Gaussian) regimes of KPZ dynamics [19].

Let us now go beyond the results in [12] by computing higher moments $m_k = \langle \phi^k \rangle = I_{s,k}/I_{s,0}$. The normalization $I_{s,0}$ scales as $m^{-1-s+\epsilon}$, and to leading order
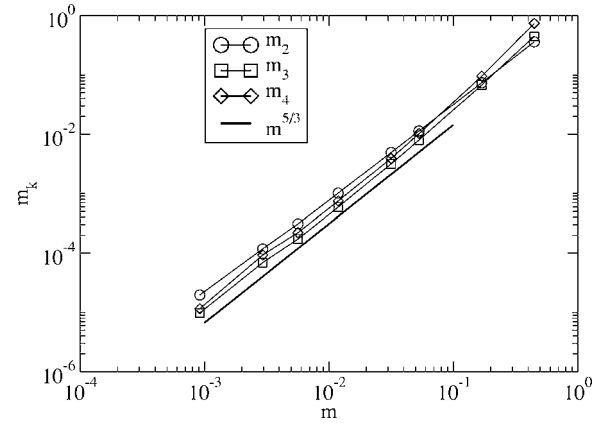

FIG. 1. Plot of the second, third, and fourth moments as a function of the first one in the nonfluctuating strong-noise regime, for a system with $p = 2$, $N = 5000$, and $s = 2/3$. All moments scale in the same way.

$$I_{s,k} = I_{s-k+1,1} \simeq B_k m^{-s+k-1+\epsilon} + C_k + D_k m^{p-s+k-1+\epsilon}. \quad (8)$$

While the third term is always subleading, it depends on k which of the other two dominates. For $s > k - 1 + \epsilon$ the first one is dominant, so that

$$m_k = \langle \phi^k \rangle \sim \frac{m^{k-1-s+\epsilon}}{m^{-1-s+\epsilon}} \sim m^k \quad (9)$$

and for $s < k - 1 + \epsilon$ the second is the leading one, so that

$$m_k = \langle \phi^k \rangle \sim \frac{1}{m^{-1-s+\epsilon}} \sim m^{1+s-\epsilon}. \quad (10)$$

Hence, the regime with nonuniversal first moment is quite rich. For $1/p < 1/s < 1$ the first k moments exhibit standard scaling, as long as $k < s + 1 - \epsilon$, while all others scale with the same exponent $1/s + 1$. We call this regime *intermediate noise*; it does not have a clear analogue in free KPZ-like interfaces.

When $1/s > 1$ full multiscaling occurs: all moments, except the first, scale with the same exponent, $1/s + 1$. This is the *strong-noise* regime; numerical evidence of this multiscaling is provided in Fig. 1, and is similar to the known phenomenology of single-site (zero-dimensional) MN equations [3,8].

From naive power counting (usually expected to reproduce mean-field exponents) performed on the MN equation, the naive time scale T should scale as $T^{-1} \sim a \propto \epsilon$ and, therefore, we expect the critical time-decay exponent of the order parameter θ_1 (or simply θ) defined by $\langle \phi(t, \epsilon = 0) \rangle \sim t^{-\theta_1}$ to coincide with β_1 in the mean field. The same property remains valid for higher moments, so the multiscaling property of the static exponent is translated into multiscaling of the decay exponents [30].

Let us now construct a Ginzburg criterion within the fully connected lattice, to see under which circumstances the previous approximation ceases to be sound. To do so we just need to compute the ratio ($r \sim m_2/m_1^2$) [28,29]. Whenever r diverges fluctuations are expected to play a significant role, breaking down the self-consistent solution, which ignores

them, as soon as they are taken into consideration. From the previously reported scaling for the strong-noise solution one has $r \sim \epsilon^{1-1/s}$, which diverges at the critical point if $1/s > 1$, i.e., in any case [31].

Therefore, the previously reported strong-noise mean-field solution is fully valid only in the strict $N = \infty$ limit, when fluctuations can be safely discarded owing to the law of large numbers. In general, it is expected that such a strong-noise solution will break down as soon as fluctuations are somehow taken into account. If, for example, the MN equation is defined on the top of a d-dimensional lattice (every site sees a finite number of others) the NF strong-noise solution is not expected *a priori* to be valid. We will discuss this aspect in more detail in the forthcoming sections. On the other hand, for the weak- and the intermediate-noise regimes, we have Gaussian scaling of the lowest moments, and therefore r converges to a constant at the critical point.

III. FLUCTUATING SOLUTION

In the previously reported approach the crucial step is the replacement of M (the mean value seen by any arbitrary single site) by a fixed nonfluctuating m. As said before, owing to the law of large numbers this is exact if $N = \infty$ whatever the probability distribution of the neighboring sites, but one can wonder whether this replacement is acceptable if a finite system size N is considered. In other words, what is the critical behavior for large but finite values of N? This is a perfectly sensible question, since for MN a sharp phase transition is well defined for any value of N, even for $N = 1$ [3].

In this section we relax the condition of a fixed value for M, by allowing the spatial average value of the ϕ field in the stationary state to fluctuate in time, sampling some probability distribution $Q(M)$ to be determined self-consistently.

Assuming that M changes in a time scale much larger than the characteristic time scale for ϕ, we can still solve the Fokker-Planck for a fixed value of M, and therefore Eq. (3) still holds, but now it has to be interpreted as a conditional probability $R(\phi|M)$. The full distribution $P(\phi)$ is given by the convolution

$$P(\phi) = \int_0^\infty dM \, R(\phi|M) Q(M). \tag{11}$$

In this case, the self-consistent equation (4) has to be replaced by a self-consistent functional equation for $Q(M)$,

$$\int dM \, M \, Q(M) = \frac{\int d\phi \, \phi \int dM \, Q(M) R(\phi|M)}{\int d\phi \int dM \, Q(M) R(\phi|M)}. \tag{12}$$

By solving this functional equation one could obtain the full solution as in the NF case, within the slowly changing M approximation.

Instead of solving numerically such a self-consistent equation we now try to write down an evolution equation for $M(t)$, from which $Q(M)$ follows. For that, we sum Eq. (1) over i and divide it by N, giving

$$\dot{M} = aM - b\frac{1}{N}\sum_i \phi_i^{p+1} + \sigma\frac{1}{N}\sum_i \phi_i \eta_i. \tag{13}$$

This turns out to be an effective MN equation for the field M in zero dimensions. Let us justify this statement. The second term on the right-hand side (RHS) is perfectly analogous to the nonlinear term in Eq. (1): it just introduces an upper cutoff M_u in the distribution of the field M. The last term on the RHS is less trivial: it can be rewritten as $\sigma(1/N)\sum_i(\phi_i/M)\eta_i M$; hence in the form of a multiplicative noise $\eta' M$. The average value of η' is clearly zero, since η' has random sign. Its second moment, which we denote as $\sigma^2(N)$, is

$$\sigma^2(N) = \frac{\sigma^2}{N^2}\left\langle \left(\sum_i \frac{\phi_i}{M}\eta_i\right)^2 \right\rangle. \tag{14}$$

The variables η_i and ϕ_i are uncorrelated. The η_i are normally distributed. In practice, to compute $\sigma^2(N)$ we must evaluate the sum of N variables ϕ_i/M, distributed according to $R(\phi/M|M)$ with random signs. The distribution $R(\phi/M|M)$ is a power law with exponent $\alpha = 2 + s - \epsilon$, lower cutoff at 1, and upper cutoff at $1/M$.

We now discuss the properties of this solution depending on whether we are working in the strong-noise regime or not.

A. Weak and intermediate noise

Let us consider first what happens when $1/s < 1$, i.e., in the weak- and the intermediate-noise regimes. In these cases, the exponent α of the distribution R is larger than 3, for a sufficiently small. Adding N such variables with random signs is equivalent to adding Gaussian variables [32]. Then

$$\left(\sum_i \frac{\phi_i}{M}\eta_i\right)^2 \sim N, \tag{15}$$

so that

$$\sigma^2(N) \sim \sigma^2/N. \tag{16}$$

We can conclude that the dynamics of M is governed by an effective equation with MN in zero dimensions with a noise η' with renormalized variance σ^2/N. It is clear that η' is correlated in time, but its finite correlation time can be eliminated by suitably rescaling the time variable. The distribution for M is then given by the solution of the zero-dimensional MN [3],

$$Q(M) = M^{-2[1-a/\sigma^2(N)]} \exp(-M/M_u)^2. \tag{17}$$

As expected, this distribution function is a solution of the previously written functional self-consistent Eq. (12).

For finite N the system undergoes an absorbing phase transition for a finite value of the control parameter

$$a_c(N) = \frac{\sigma(N)^2}{2} \sim \frac{\sigma^2}{2N}. \tag{18}$$

The exponents for the transition with finite N are given by the values for zero-dimensional MN transition [3,8]: $\beta_1 = 1$, and $\theta = 1/2$.
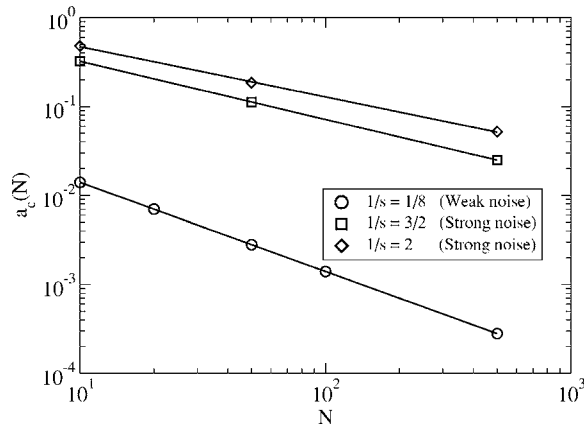
FIG. 2. Plot of $a_c(N)$ versus N for three values of $1/s$. Thick lines are power-law fits with exponents -1 for $1/s=1/8$, -0.65 for $1/s=3/2$, and -0.57 for $1/s=2$ [to be compared with the output of Eqs. (18) and (21)].

FIG. 3. Main: plot of m vs a for $1/s=1/8$ (weak noise). Inset: the same data of the main part with $N=10$, plotted vs $a-a_c(N)$ and showing the NF behavior $\{m\sim[a-a_c(N)]^{1/2}\}$ and the zero-dimensional one $\{m\sim[a-a_c(N)]\}$. Solid lines are guides to the eye.

Notice that this transition is qualitatively different from that occurring in the thermodynamic limit. Here the transition takes place because the distribution of M becomes non-normalizable due to the divergence for $M\to 0$. In the thermodynamic limit, instead, the distribution $Q(M)$ remains normalizable (Gaussian) at the transition and criticality comes from the peak position M_u moving toward zero (see the Appendix, where a coherent general scaling picture is presented).

These two regimes are therefore distinguished by the presence of a (Gaussian) peak for finite M (the NF limit) or a broad distribution with a power-law divergence for $M \to 0$. The crossover occurs where the power-law exponent in Eq. (17) changes sign, i.e., for

$$\tilde{a}(N) = \sigma^2(N) = 2a_c(N). \qquad (19)$$

The behavior for fixed N can then be summarized as follows. For $a \gg \tilde{a}(N)$ the system exhibits the critical behavior of the weak noise $N=\infty$ solution. For $0 < a-a_c(N) \ll a_c(N)$ it behaves as if it was zero dimensional. In the latter case, the crossover can also be observed with fixed a by looking at the temporal evolution of the first moment m: at short times the NF solution is followed, crossing over later to the asymptotic zero-dimensional scaling.

We have checked the correctness of this scenario by means of numerical simulations of Eq. (1) with $p=2$. From the numerical point of view, the first problem is the determination of the critical point $a_c(N)$. The criterion we have chosen is based on the way the first moment $m(t)$ decays in time. $a_c(N)$ is the value separating a concave behavior [for $a > a_c(N)$] from a convex one [for $a < a_c(N)$]. In this way, we obtain the values plotted in Fig. 2. The behavior found in the weak-noise regime is in perfect agreement with the predicted $1/N$ behavior.

With $p=2$, the temporal behavior in the weak-noise case is the same ($\theta=1/2$) both in the thermodynamical limit and for the effective zero-dimensional behavior valid at finite N, not allowing to distinguish between the two regimes. In Fig. 3 we analyze th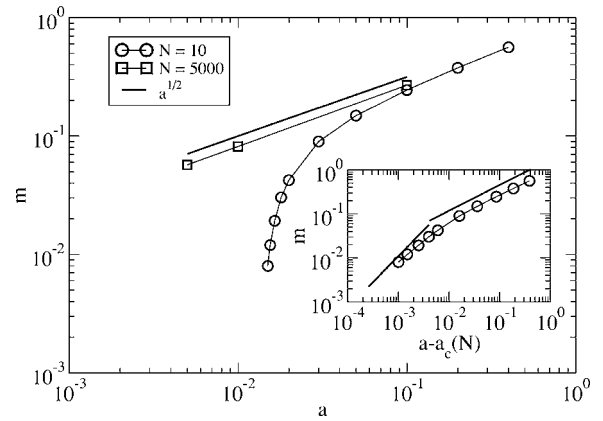e behavior of the stationary value of the first moment m. In the main part of the figure we observe that, for $N=5000$, the system follows very accurately (for the values of a considered) the decay with $\beta_1=1/2$, expected in the thermodynamic limit. For $N=10$, instead, a singularity is observed for finite a. If we plot m as a function of $a-a_c(N)$ (Fig. 3, inset) both behaviors can be observed: for $a-a_c(N) \gg a_c(N) \approx 0.014$ the scaling is of the NF type ($\beta_1=1/2$); for $a-a_c(N) \ll a_c(N)$ we observe the zero-dimensional exponent $\beta_1=1$.

A direct validation of Eq. (17) is provided by Fig. 4, where the distribution of the self-consistent field M exhibits a peak in $M_u(a)$ for $a-a_c(N) \gg a_c(N)$ while a power-law divergence at zero develops for $a-a_c(N) \ll a_c(N)$.

B. Strong noise

Let us consider now the strong-noise case $1/s > 1$. The general picture is similar to the one previously described, with the main difference that the exponent α in the distribu-
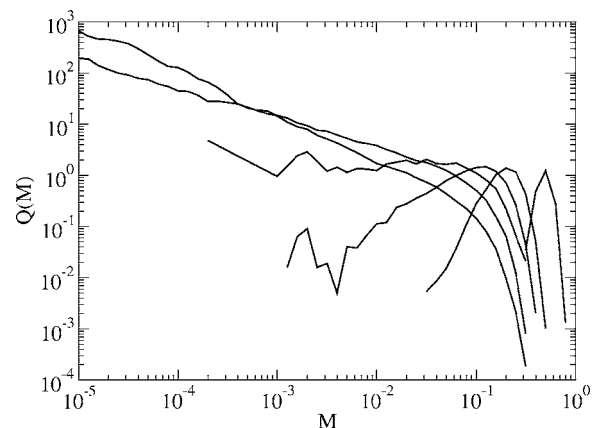

FIG. 4. Plot of $Q(M)$ vs a for $1/s=1/8$ (weak noise), displaying the crossover between a Gaussian around $M_u>0$ and a power-law divergence in 0. The value of the critical point is $a_c(N=10) \approx 0.014$. The values of a are, from right to left, 0.4, 0.1, 0.05, 0.03, 0.02, 0.155.
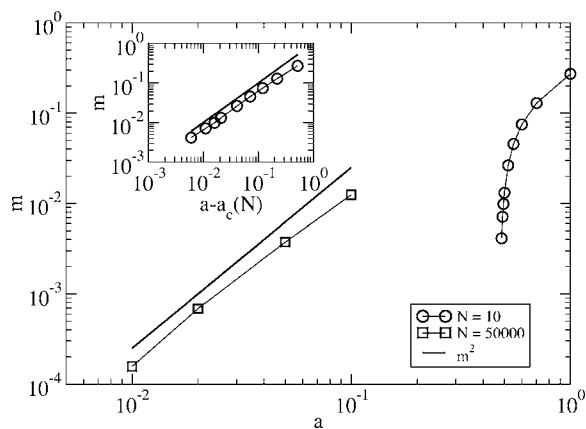
FIG. 5. Main: plot of m vs a for $1/s=2$ (strong noise). Inset: the same data of the main part with $N=10$, plotted vs $a-a_c(N)$ and in agreement with the zero-dimensional linear behavior. Solid lines are guides to the eye.

tion $R(\phi/M|M)$ is now smaller than 3. From the theory of Levy-stable distributions [32] we know that the sum of N variables distributed as a power law with exponent $1+\mu$ with $1<\mu<2$ and an upper cutoff $1/M$, scales as $N^{1/\mu}$ for $N \ll M^{-\mu}$ while it behaves in a Gaussian way, $N^{1/2}$, for $N \gg M^{-\mu}$. This implies that, for fixed N, η' is a power-law distributed noise with exponent $1+\mu$ for $M \ll N^{-1/\mu}$, and a Gaussian noise with $\sigma^2(N) \sim \sigma^2/N$ for $M \gg N^{-1/\mu}$. In the present case $\mu=\alpha-1=1+s-2a/\sigma^2$.

Equation (13) describes now a zero-dimensional MN with a rather exotic noise, whose distribution depends on M. We have no clear theoretical understanding of such a model. In principle it could give rise to completely new and nontrivial critical features. However, as shown numerically below, it turns out to behave asymptotically as the zero-dimensional case with standard noise. The only change is in the position of the critical point $a_c(N)$. Accordingly, we assume, rather crudely, that the power-law noise does not change the behavior of the zero-dimensional MN, except for the form of $\sigma^2(N)$,

$$\sigma^2(N) \sim \sigma^2 N^{2[1/(\alpha-1)-1]} = \sigma^2 N^{2[1/(1+s-2a/\sigma^2)-1]}. \quad (20)$$

The distribution of M is then given again by Eq. (17), with the additional complication that $\sigma^2(N)$ depends on a. The critical point is determined by the implicit condition

$$a_c(N) = \frac{\sigma^2(N)}{2} \sim \sigma^2 N^{2\{1/[1+s-2a_c(N)/\sigma^2]-1\}}. \quad (21)$$

If, as a first approximation, we neglect the dependence of α on a, i.e., we take $\alpha=2+s$, we obtain $a_c(N) \sim N^{-2s/(1+s)}$, not far from the numerical results of Fig. 2.

Again for $a > \tilde{a}(N)=2a_c(N)$ the exponent of the distribution $Q(M)$ becomes negative and there is a crossover to the NF limit. In Fig. 5 we plot m versus a in the strong-noise regime. Again the value $\beta_1=1/s$, valid in the thermodynamic limit, is observed for large N, as the critical point and the crossover are very close to 0. Near the transition the expo-
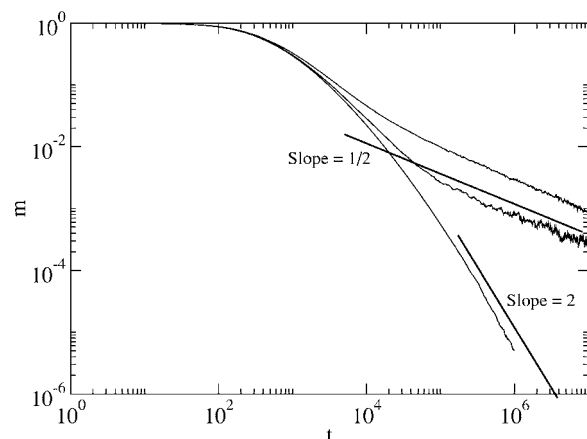

FIG. 6. Plot of m vs t for $1/s=2$ (strong noise) and $a \approx a_c(N)$ for (top to bottom) $N=50$, 500, and 50 000.

nent is the zero-dimensional one, $\beta_1=1$, indicating that the nontrivial noise does not modify the zero-dimensional critical behavior.

As in the other case, it is interesting to look also at the temporal evolution of m. In the strong-noise regime, since the exponent θ is different in the NF solution and in the zero-dimensional regime, the crossover between the two regimes results in a crossover in the decay of $m(t)$. This is evident from Fig. 6. The effective exponent switches from a value close to the prediction $\theta=1/s=2$ for short times, to the zero-dimensional value $\theta=1/2$, for longer times.

C. Discussion

Let us underline the noncommutativity of the limits $a \to a_c$ and $N \to \infty$. It is only when the thermodynamic limit is taken first and a homogeneous mean-field M is considered that the NF solution is recovered. On the other hand, if one takes first the limit $a \to a_c(N)$ for a generic finite value of N, the zero-dimensional solution always dominates the scaling nearby the critical point, no matters how large N. Therefore, the inclusion of fluctuations has a dramatic effect on the NF solution.

Such a conclusion holds for all the regimes considered: weak, intermediate, and strong noise. This is due to the fact that the two aforementioned limits do not commute, implying the presence of a nonanalyticity of the most general solution in a neighborhood of the critical point. This is analogous to the observation of Gaussian scaling in standard phase transitions whenever the system size is not infinite; it is only when the thermodynamic limit is taken that the true asymptotic scaling emerges. In the case studied here the role of the Gaussian scaling is replaced by a MN zero-dimensional nontrivial scaling. This breaking down of the thermodynamic-limit behavior for finite systems has, in principle, nothing to do with the previously constructed Ginzburg criterion which leads to a breakdown of the mean-field solution only at the strong-noise regime.

In the Appendix we present a general scaling theory accounting in a compact form for all the previously discussed phenomenology.
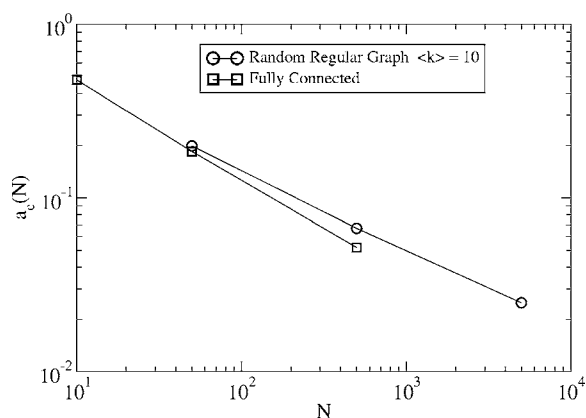
FIG. 7. Plot of $a_c(N)$ versus N for $1/s=2$ (strong noise) for the fully connected and the RRG.

FIG. 8. Plot of m vs t for $1/s=2$ (strong noise) and $a \approx a_c(N)$ for (top to bottom) $N=50$, 500, 5000, and 50 000 on the RRG.

IV. FINITE CONNECTIVITY: RANDOM REGULAR GRAPHS

In order to shed some light on the question of whether the NF solution holds or not for an arbitrarily large space dimensionality d, in which the number of sites seen by any given one is finite ($2d$ for a hypercubic lattice) we should first answer the following question: does the NF behavior emerge because the size of the system goes to infinity or because the number of nearest neighbors diverges? (Note that in the fully connected graph these two limits coincide.) In order to clarify this point we have considered the MN on a connected regular random graph (RRG), where each site has fixed degree $k>2$ and random connections (for $k=2$ we have a one-dimensional lattice). In this case, as the number of neighbors is finite for each site, we expect fluctuations in M and therefore a possible breakdown of the NF solution (at least in the strong-noise limit), even if the large system size limit is taken.

However, numerical results disprove such a conjecture, as we show in what follows. We have performed simulations of a system with $k=10$ and growing N. It turns out that the position of the critical point $a_c(N)$ depends on N and does not reach a finite value dependent only on k (see Fig. 7). Its behavior is not very different from what occurs on the fully connected system [see Eq. (21)].

Moreover, one can monitor the temporal behavior of $m(t)$ for $a \approx a_c(N)$ (Fig. 8). One finds a crossover from a NF behavior at short time to a zero-dimensional behavior at longer time, exactly as for the fully connected graph.

We conclude that the thermodynamical limit is described, also for a regular random graph with finite connectivity, by the NF behavior. The zero-dimensional behavior holds only as long as criticality is studied for finite system sizes. A discussion of these facts is presented in the next section.

V. SUMMARY AND CONCLUSIONS

In this paper we have investigated the properties of the mean-field solution for systems with multiplicative noise. First, we have revisited the self-consistent solution obtained in Ref. [12] by assuming that M is a homogeneous nonfluc-
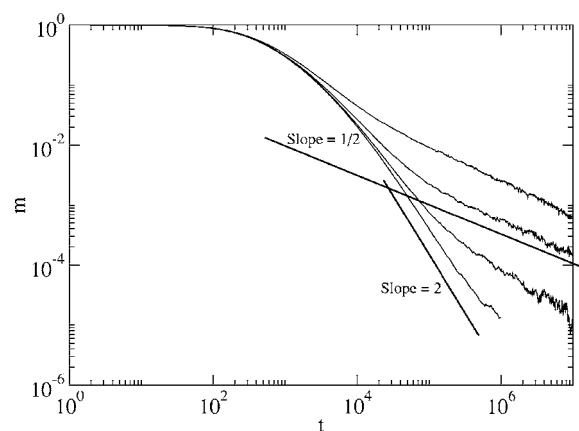
tuating field. We have shown that three regimes can actually be identified depending on the noise amplitude $1/s=\sigma^2/(2D)$. In the weak-noise case ($1/s<1/p$) all moments of the order-parameter distribution obey ordinary Gaussian scaling. In the intermediate regime ($1/p<1/s<1$), Gaussian scaling holds only up to a certain order, with multiscaling emerging for higher moments. In the strong-noise regime ($1/s>1$) all moments scale with exponent $1/s+1$ except for the first one that goes as $\epsilon^{1/s}$.

Complete absence of fluctuations for the spatial average M holds only in the thermodynamic limit, with a strictly infinite number of nearest neighbors. In this paper we have relaxed the condition that M can assume only one (self-consistently fixed) value and let it fluctuate: this is equivalent to considering a fully connected graph with a finite number N of nodes.

When N is finite M fluctuates and the critical behavior is not described anymore by the NF thermodynamic limit, in none of the three previously described regimes. While in the NF case, criticality is governed by fluctuations of the ϕ field in a single realization around its mean value, for finite N what actually matters are the fluctuations of the self-consistent field M. They are described by an effective equation for a zero-dimensional system with multiplicative noise, which exhibits a critical behavior distinct from the one displayed in the thermodynamic limit.

The crossover between the two types of behavior is well described by the zero-dimensional description: both the crossover point and the critical point location depend on N so that for any finite value of N there is a finite interval of values of the control parameter where the effective zero-dimensional behavior is observed.

Finally, by means of numerical simulations on connected random regular graphs, we have also shown that all these results still hold (at least qualitatively) for random regular graphs with finite connectivity. This is somehow counterintuitive, and seems to contradict the previously introduced Ginzburg criterion for the strong-noise regime. Indeed, given that the fluctuations of M are predicted by the Ginzburg criterion to be relevant in the strong-noise regime of the NF solution when the number of nearest neighbors is not infinity, it is hard to understand, why the solution in the random

regular graph, in which M_i is the fluctuating average taken over the k neighbors of any given site i, behaves so similarly (for any value of N) to the solution in the fully connected network in which $M = M_i$ is fluctuationless.

We do not have a clear explanation of this fact, but we believe that this is so because of the small-world [33] nature of the RRG topology. The small-world property implies that one can reach any arbitrary node starting from a generic site with a small number of steps following network links. In this way, every site is nearby any other one, making it difficult to create "local patches" with an overdensity or subdensity, which would give rise to inhomogeneities and a broad field distribution. At this point, it would be interesting to study the MN equation on a Cayley tree to see if, by introducing well defined spatial neighborhoods, the previous results and interpretation are sustained. We are presently analyzing such a problem, but prefer to leave the delicate issues involved in such a study for a future publication.

Many interesting questions remain to be answered. The main one is whether the obtained NF solution in the strong-noise regime is valid in arbitrarily large but finite physical dimensions, or whether it emerges only in $d = \infty$ (which is obviously related to the existence of a finite upper critical dimension for MN and KPZ problems). The Ginzburg criterion for the strong-noise solution seems to point out to the second possibility, while the fact that in the finite-connectivity random regular graphs the NF solution emerges in the thermodynamic limit, could be interpreted as supporting the first one. Further analysis along these lines is left for future work.

It remains also to be understood which is the fate of the intermediate-noise regime once the MN equation is embedded into a d-dimensional lattice. Does it simply disappear? Does it survive, introducing some type of anomalous effect for high-order moments? Does it have any analogous in KPZ-like systems?

It is our hope that this work will stimulate future research in this exciting and ever surprising field of systems with multiplicative noise.

ACKNOWLEDGMENTS

APPENDIX: GENERAL SCALING OF THE DISTRIBUTIONS

It is possible to summarize all the scaling regimes described in the paper in a general scaling form of the distri-

butions. This describes the crossover between the zero-dimensional case and the thermodynamic limit as N grows. The equations obeyed by the distribution $R(\phi|M;a,N)$ of the ϕ field conditional to a value M and by the full distribution $P(\phi;a,N)$ are

$$R(\phi|M;a) \propto \phi^{-\alpha} \exp\left(-\frac{bs\phi^p}{pD} - \frac{Ms}{\phi}\right),$$

$$P(\phi;a,N) = \int_0^\infty dM \, R(\phi|M;a)Q(M;a,N). \quad (A1)$$

The solution of these equations found assuming no fluctuations for M is

$$R(\phi;M) \propto \phi^{-\alpha} \exp\left(-\frac{bs\phi^p}{pD} - \frac{Ms}{\phi}\right),$$

$$Q(M;a,N) \approx \delta[M - m(a)],$$

$$P(\phi;a,N) = R(\phi|M)_{M=m(a)}. \quad (A2)$$

A more general explicit ansatz for the solution to this set of equations, valid for generic N, is

$$R(\phi|M) \propto M^{\alpha-1}\phi^{-\alpha} \exp\left(-\frac{bs\phi^p}{pD} - \frac{Ms}{\phi}\right),$$

$$Q(M;a,N) \approx M^{-\alpha'} e^{-(M/M_u)^2},$$

$$P(\phi;a,N) \sim \begin{cases} \dfrac{\phi^{-\alpha'}}{\alpha - \alpha'}, & \phi < M_u(a), \\[3mm] \dfrac{\phi^{-\alpha} e^{-bs\phi^p/pD}}{\alpha - \alpha'} M_u^{\alpha-\alpha'}, & \phi > M_u(a), \end{cases}$$

$$(A3)$$

where $\alpha' = 2[1 - a/\sigma^2(N)]$. The form of $Q(M)$ encodes the different critical behaviors. In the NF limit (N going to ∞ first) α' is negative. In this way $Q(M)$ tends to $\delta(M - M_u)$; moments go to zero because M_u goes to zero as a^β. In the opposite nontrivial limit (ϵ going to zero with N fixed) α' is positive and grows up to 1 for $a \to a_c(N)$. In this case, the moments go to zero because the distribution $Q(M)$ develops a peak in zero. The critical behavior of the moments in this limit is governed by the way α' goes to 1. The origin of the two limits ($\epsilon \to 0$ and $N \to \infty$) noncommutativity has its roots in the nonanalytical form of $Q(M)$ at $M = 0$.

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