Wealth condensation in a simple model of economy

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Abstract

We introduce a simple model of economy, where the time evolution is described by an equation capturing both exchange between individuals and random speculative trading, in such a way that the fundamental symmetry of the economy under an arbitrary change of monetary units is insured. We investigate a mean-field limit of this equation and show that the distribution of wealth is of the Pareto (power-law) type. The Pareto behaviour of the tails of this distribution appears to be robust for finite range models, as shown using both a mapping to the random 'directed polymer' problem, as well as numerical simulations. In this context, a phase transition between an economy dominated by a few individuals and a situation where the wealth is more evenly spread out, is found. An interesting outcome is that the distribution of wealth tends to be very broadly distributed when exchanges are limited, either in amplitude or topologically. Favouring exchanges (and, less surprisingly, increasing taxes) seems to be an efficient way to reduce inequalities. © 2000 Published by Elsevier Science B.V. All rights reserved.

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It is a well-known fact that the individual wealth is a very broadly distributed quantity among the population. Even in developed countries, it is common that 90% of the total

wealth is owned by only 5% of the population. The distribution of wealth is often described by 'Pareto'-tails, which decay as a power law for large wealths [1–5].

$$\mathscr{P}_>(W) \sim \left(\frac{W_0}{W}\right)^\mu , \tag{1}$$

where $\mathscr{P}_>(W)$ is the probability to find an agent with wealth greater than W, and μ is a certain exponent, of order 1 both for individual wealth or company sizes (see however [6]).

Here, we want to discuss the appearance of such Pareto tails on the basis of a very general model for the growth and redistribution of wealth, that we discuss in some simple limits. We relate this model to the so-called 'directed polymer' problem in the Physics literature (for a review, see Ref. [7]), for which a large number of results are known, that we translate into the present economical framework. We discuss the influence of simple parameters, such as the connectivity of the exchange network, the role of income or capital taxes and of state redistribution of wealth, on the value of the exponent μ. One of the most interesting output of such a model is the generic existence of a *phase transition*, separating a phase where the total wealth of a very large population is concentrated in the hands of a finite number of individuals (corresponding, as will be discussed below, to the case $\mu < 1$), from a phase where it is shared by a finite fraction of the population.

The basic idea of our model is to write a stochastic dynamical equation for the wealth $W_i(t)$ of the ith agent at time t, that takes into account the exchange of wealth between individuals through trading, and is consistent with the basic symmetry of the problem under a change of monetary units. Since the unit of money is arbitrary, one indeed expects that the equation governing the evolution of wealth should be invariant when all W_i's are multiplied by a common (arbitrary) factor. The evolution equation that we consider is therefore the following:

$$\frac{\mathrm{d}W_i}{\mathrm{d}t} = \eta_i(t)W_i + \sum_{j(\neq i)} J_{ij}W_j - \sum_{j(\neq i)} J_{ji}W_i , \tag{2}$$

where $\eta_i(t)$ is a Gaussian random variable of mean m and variance $2\sigma^2$, which describes the spontaneous growth or decrease of wealth due to investment in stock markets, housing, etc., while the terms involving the (asymmetric) matrix J_{ij} describe the amount of wealth that agent j spends buying the production of agent i (and vice versa). It is indeed reasonable to think that the amount of money earned or spent by each economical agent is proportional to its wealth. This makes Eq. (2) invariant under the scale transformation $W_i \rightarrow \lambda W_i$. Technically, the above stochastic differential equation is interpreted in the Stratonovich sense [8]. [3]

The simplest model one can think of is the case where all agents exchange with all others at the same rate, i.e., $J_{ij} \equiv J/N$ for all $i \neq j$. Here, N is the total number of agents, and the scaling J/N is needed to make the limit $N \rightarrow \infty$ well defined. In this

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[3] With the Ito convention, our evolution equation would have an extra factor of $\sigma^2 W_i$ on the right-hand side.

case, the equation for $W_i(t)$ becomes

$$\frac{dW_i}{dt} = \eta_i(t)W_i + J(\overline{W} - W_i),\tag{3}$$

where $\overline{W} = N^{-1}\sum_i W_i$ is the average overall wealth. This is a 'mean-field' model since all agents feel the very same influence of their environment. By formally integrating this linear equation and summing over i, one finds that the average wealth becomes deterministic in the limit $N \to \infty$:

$$\overline{W}(t) = \overline{W}(0)\exp((m + \sigma^2)t).\tag{4}$$

It is useful to rewrite Eq. (3) in terms of the normalized wealths $w_i \equiv W_i/\overline{W}$. This leads to

$$\frac{dw_i}{dt} = (\eta_i(t) - m - \sigma^2)w_i + J(1 - w_i),\tag{5}$$

to which one can associate the following Fokker–Planck equation for the evolution of the density of wealth $P(w,t)$:

$$\frac{\partial P}{\partial t} = \frac{\partial[J(w-1) + \sigma^2 w]P}{\partial w} + \sigma^2\frac{\partial}{\partial w}\left[w\frac{\partial wP}{\partial w}\right].\tag{6}$$

The equilibrium, long time solution of this equation is easily shown to be

$$P_{\mathrm{eq}}(w) = \mathscr{Z}\frac{\exp - (\mu - 1)/w}{w^{1+\mu}},\quad \mu \equiv 1 + \frac{J}{\sigma^2},\tag{7}$$

where $\mathscr{Z} = (\mu - 1)^\mu/\Gamma[\mu]$ is the normalization factor. One can check that $\langle w\rangle \equiv 1$, as it should be.

Therefore, one finds in this model that the distribution of wealth exhibits a Pareto power-law tail for large w's. In agreement with intuition, the exponent μ grows (corresponding to a narrower distribution), when exchange between agents is more active (i.e., when J increases), and also when the success in individual investment strategies is more narrowly distributed (i.e., when σ^2 decreases).

Also, one can actually define the above model in discrete time, by writing:

$$W_i(t + \tau) = [J\tau\overline{W} + (1 - J\tau)W_i]\mathrm{e}^{-V(i,t)},\tag{8}$$

where V is an arbitrary random variable of mean $m\tau$ and variance $2\sigma^2\tau$, and $J\tau < 1$. In this setting, this amounts to study the so-called Kesten variable [9] for which the asymptotic distribution again has a power-law tail, with an exponent μ found to be the solution of

$$(1 - J\tau)^\mu\langle\mathrm{e}^{-\mu V}\rangle = \langle\mathrm{e}^{-V}\rangle^\mu.\tag{9}$$

Therefore, this model leads to power-law tails for a very large class of distributions of V, such that the solution of the above equation is non-trivial (that is if the distribution of V decays at least as fast as an exponential). It is easy to check that μ is always greater than one and tends to $\mu = 1 + J/\sigma^2$ in the limit $\tau \to 0$. Let us notice that a somewhat similar discrete model was studied in Ref. [10] in the context of a generalized Lotka–Volterra equation. However, that model has an additional term
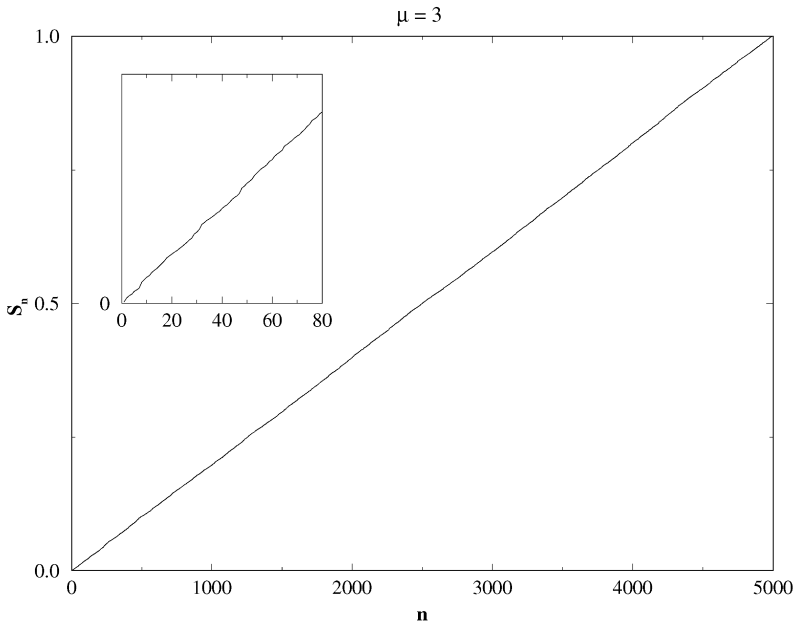
Fig. 1. Fraction of total wealth S_n owned by the first n agents, plotted versus n, in a population of 5000 agents. The wealths have been drawn at random using a distribution with a Pareto tail exponent $\mu = 3$. Inset: detail of the first 80 agents. One sees that S_n grows linearly with n, with rather small fluctuations around the average slope $1/N$.

(the origin of which is unclear in an economic context) which breaks the symmetry under wealth rescaling, and as a consequence the Pareto tail is truncated for large wealths.

In this model, the exponent μ is always found to be larger than one. In such a regime, if one plots the partial wealth $S_n = \sum_{i=1}^{n} w_i$ as a function of n, one finds an approximate straight line of slope $1/N$, with rather small fluctuations (see Fig. 1). This means that the wealth is not too unevenly distributed within the population. On the other hand, the situation when $\mu < 1$, which we shall encounter below in some more realistic models, corresponds to a radically different situation (see Fig. 2). In this case, the partial wealth S_n has, for large N, a devil staircase structure, with a few individuals getting hold of a finite fraction of the total wealth. A quantitative way to measure this 'wealth condensation' is to consider the so-called inverse participation ratio Y_2 defined as

$$Y_2 = \sum_{i=1}^{N} w_i^2 \ . \tag{10}$$

If all the w_i's are of order $1/N$ then $Y_2 \sim 1/N$ and tends to zero for large N. On the other hand, if at least one w_i remains finite when $N \to \infty$, then Y_2 will also be finite. The average value of Y_2 can easily be computed and is given by $\langle Y_2 \rangle = 1 - \mu$ for $\mu < 1$
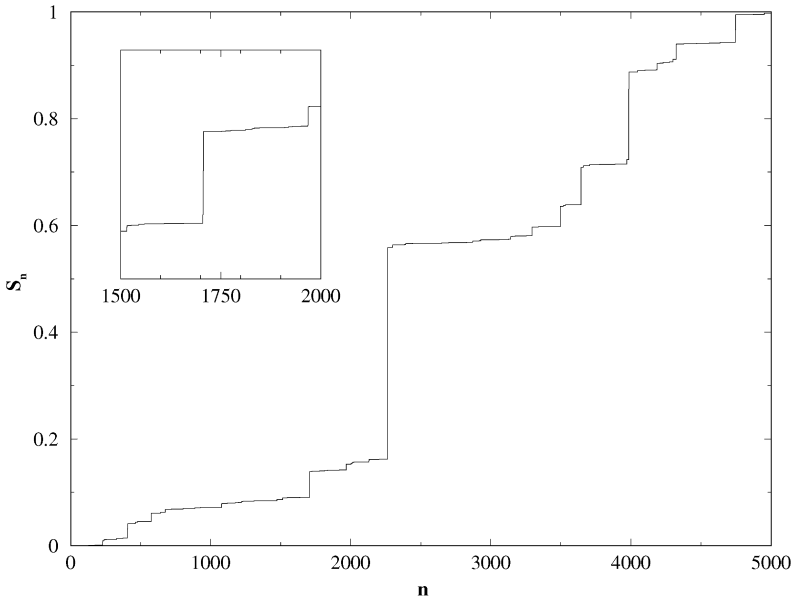
Fig. 2. Fraction of total wealth owned by the first n agents, plotted versus n, in a population of 5000 agents. The wealths have been drawn at random using a distribution with a now Pareto tail exponent $\mu = 0.5 < 1$. Inset: Zoom on finer details of the curve. One clearly sees that the curve is a 'devil' staircase on all scales, with a strong dominance of a few individuals.

and zero for all $\mu > 1$ [11–13]. $\langle Y_2 \rangle$ is therefore a convenient order parameter which quantifies the degree of wealth condensation.

It is interesting to discuss several extensions of the above model. First, one can easily include, within this framework, the effect of taxes. Income tax means that a certain fraction ϕ_I of the income dW_i/dt is taken away from agent i. Therefore, there is a term $-\phi_I \, dW_i/dt$ appearing on the right-hand side of Eq. (2). Capital tax means that there is a fraction ϕ_C of the wealth which is subtracted per unit time from the wealth balance, Eq. (2). If a fraction f_I of the income tax and f_C of the capital tax are evenly redistributed to all, then this translates into a term $+f_I \phi_I \, d\overline{W}/dt + f_C \phi_C \overline{W}$ on the right-hand side of the wealth balance, which now reads:

$$\frac{dW_i}{dt} = \eta_i(t)W_i + J(\overline{W} - W_i) - \phi_I \frac{dW_i}{dt} - \phi_C W_i + f_I \phi_I \frac{d\overline{W}}{dt} + f_C \phi_C \overline{W} \,. \tag{11}$$

All these terms can be treated exactly within the above mean-field model allowing for a detailed discussion of their respective roles. The rate of exponential growth of the average wealth $\overline{W}(t)$ becomes equal to

$$\gamma \equiv \frac{m + \sigma^2/(1 + \phi_I) - \phi_C(1 - f_C)}{1 + \phi_I(1 - f_I)} \,. \tag{12}$$

The Pareto tail exponent μ is now given by

$$\mu - 1 = \frac{J(1 + \phi_I)}{\sigma^2} + \frac{1 + \phi_I}{\sigma^2(1 + \phi_I(1 - f_I))}$$
$$\times \left[\phi_I f_I \left(m + \frac{\sigma^2}{1 + \phi_I} \right) + \phi_C(f_C + \phi_I(f_C - f_I)) \right] . \tag{13}$$

This equation is quite interesting. It shows that income taxes tend to reduce the in-equalities of wealth (i.e., lead to an increase of μ), even more so if part of this tax is redistributed. On the other hand, quite surprisingly, capital tax, if used simultane-ously to income tax and not redistributed, leads to a *decrease* of μ, i.e., to a wider distribution of wealth. Only if a fraction $f_C > f_I \phi_I/(1 + \phi_I)$ is redistributed will the capital tax be a truly social tax. Note that in the above equation, we have implicitly assumed that the growth rate γ is positive. In this case, one can check that μ is always $> 1 + (J + \phi_C f_C)(1 + \phi_I)/\sigma^2$, which is larger than one.

Another point worth discussing is the relaxation time associated with the Fokker–Planck equation (6). By changing variables as $w = \xi^{-2}$ and $P(w) = \xi^3 Q(\xi)$, one can map the above Fokker–Planck equation to the one studied in Ref. [14], which one can solve exactly. For large time differences T, one finds that the correlation function of the w's behaves as

$$\langle w(t + T)w(t) \rangle - \langle w(t) \rangle^2 \propto \exp(-(\mu - 1)\sigma^2 T), \quad \mu > 2 \tag{14}$$

and

$$\langle w(t + T)w(t) \rangle - \langle w(t) \rangle^2 \propto \frac{1}{(\sigma^2 T)^{3/2}} \exp(-\mu^2 \sigma^2 T/4), \quad \mu < 2 \tag{15}$$

This shows that the relaxation time is, for $\mu < 2$, given by $4/\mu^2\sigma^2$. Therefore, rich people become poor (and vice versa) on a finite time scale in this model. A reasonable order of magnitude for σ is 10% per $\sqrt{\text{year}}$. In order to get $\mu - 1 \sim 1$, one therefore has to choose $J \sim 0.01$ per year, i.e., 1% of the total wealth of an individual is used in exchanges. [This J value looks rather small, but in fact we shall see below that a more realistic (non-mean field model) allows to increase J while keeping μ fixed.] In this case, the relaxation time in this model is of the order of 100 years.

Let us now escape from the mean-field model considered above and describe more realistic situations, where the number of economic neighbours to a given individual is finite. We will first assume that the matrix J_{ij} is still symmetrical, and is either equal to J (if i and j trade), or equal to 0. A reasonable first assumption is that the graph describing the connectivity of the population is completely random, i.e., that two points are neighbours with probability c/N and disconnected with probability $1 - c/N$. In such a graph, the average number of neighbours is equal to c. We thus scale $\hat{J} = J/c$ in order to compare results with various connectivities (and insure a smooth large connectivity limit). We have performed some numerical simulations of Eq. (2) for $c = 4$ and have found that the wealth distribution still has a power-law tail, with an exponent μ which only depends on the ratio J/σ^2. This is expected since a rescaling of time by a factor α can be absorbed by changing J into αJ and σ into $\sqrt{\alpha}\sigma$; therefore,
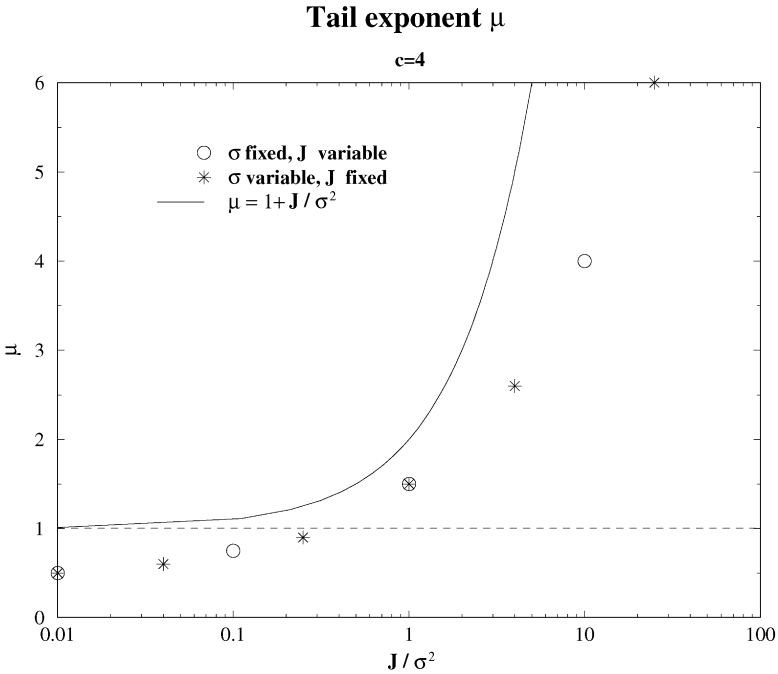
Tail exponent μ

Fig. 3. Plot of the numerical values of μ for the model on a random graph with connectivity $c = 4$, as a function of J/σ^2 – we have indeed checked that this scaling holds for our numerical discretization of Eq. (2). The plain line corresponds to the mean-field prediction $\mu = 1 + J/\sigma^2$. For $c = 4$, we find that the condensation transition takes place for $J/\sigma^2 \sim 0.3$.

long time (equilibrium) properties can only depend on the ratio J/σ^2. As shown in Fig. 3, the exponent μ can now be smaller than one for sufficiently small values of J/σ^2. In this model, one therefore expects wealth condensation when the exchange rate is too small. Note that we have also computed numerically the quantity $\langle Y_2 \rangle$ and found very good agreement with the theoretical value $1 - \mu$ determined from the slope of the histogram of the w_i's.

From the physical point of view, the class of models which we consider here belong to the general family of directed polymers in random media. The two cases we have considered so far correspond, respectively, to a polymer on a fully connected lattice, and a polymer on a random lattice. A variant of this model can be solved exactly using the method of Derrida and Spohn [15] for the so-called directed polymer problem on a tree. In this variant, one assumes that at each time step τ the connectivity matrix is completely changed and chosen anew using the same probabilities as above. Each agent i chooses at random exactly c new neighbours $\ell(i, t)$, the wealth evolution equation becomes

$$W_i(t + \tau) = \left[\frac{J\tau}{c} \sum_{\ell=1}^{c} W_{\ell(i,t)} + (1 - J\tau)W_i(t) \right] e^{-V(i,t)} , \qquad (16)$$

where V is a Gaussian random variable of mean zero and variance $2\sigma^2\tau$. One can then write a closed equation for the evolution of the wealth distribution [15]. In this case, the wealth condensation phenomenon takes place whenever $\sigma^2\tau + J\tau \ln(J\tau/c) + (1 - J\tau)\ln(1 - J\tau) > 0$. For $J\tau \ll 1$ the transition occurs for $\sigma^2 = \sigma_c^2 = J(1 + \ln(c/J\tau))$.

For $\sigma > \sigma_c$, one finds that μ is given by

$$\mu \simeq \frac{\ln(c/\sigma^2\tau)}{\ln(c/J\tau)} \tag{17}$$

and is < 1, signalling the onset of a phase where wealth is condensed on a finite number of individuals. This precisely corresponds to the glassy phase in the directed polymer language. The above formula shows that μ depends only weakly on σ or J, in qualitative agreement with our numerical result for the continuous time model (see Fig. 3). Note that in the limit $c \to \infty$, $\sigma_c \to \infty$ and the glassy phase disappears, in agreement with the results above, obtained directly on the mean-field model. Note also that in the limit $\tau \to 0$, where the reshuffling of the neighbours becomes very fast, wealth diffusion within the population becomes extremely efficient and, as expected, the transition again disappears. Finally, in the simple case where $J\tau = 1$ (each agent trading all of his wealth at each time step), the critical value is $\sigma_c^2\tau = \ln c$ and the exponent μ in the condensed phase is simply $\mu = \sigma_c/\sigma$, and $\mu = \sigma^2/\sigma_c^2$ for $\mu > 1$ (see Ref. [15]).

Let us note, *en passant*, that the model considered by Derrida and Spohn has another interesting interpretation if the W_i's describe the wealth of companies. The growth of a company takes place either from internal growth (leading to a term $\eta_i(t)W_i$ much as above), but also from merging with another company. If the merging process between two companies is completely random and takes place at a rate λ per unit time, then the model is exactly the same as the one considered in Section 3 of [15] (see in particular their Eq. (3.2)).

Although not very realistic, one could also think that the individuals are located on the nodes of a d-dimensional hypercubic lattice, trading with their neighbours up to a finite distance. In this case, one knows that for $d > 2$ there exists again a phase transition between a 'social' economy where $\mu > 1$ and a rich dominated phase $\mu < 1$. On the other hand, for $d \leqslant 2$, and for large populations, one is always in the extreme case where $\mu \to 0$ at large times. In the case $d = 1$, i.e., operators organized along a chain-like structure, one can actually compute exactly the distribution of wealth by transposing the results of Ref. [16]. One finds, for example, that the ratio of the maximum wealth to the typical (e.g. median) wealth behaves as $\exp\sqrt{N}$, where N is the size of the population, instead of $N^{1/\mu}$ in the case of a Pareto distribution with $\mu > 0$. The conclusion of the above results is that the distribution of wealth tends to be very broadly distributed when exchanges are limited, either in amplitude (i.e., J too small compared to σ^2) or topologically (as in the above chain structure). Favouring exchanges (in particular with distant neighbours) seems to be an efficient way to reduce inequalities.

Let us now discuss in a cursory way the extension of this model to the case where the matrix J_{ij} has a non-trivial structure. One can always write:

$$J_{ij} = D_{ij} \exp - \frac{F_{ij}}{2}, \qquad J_{ji} = D_{ij} \exp + \frac{F_{ij}}{2}, \tag{18}$$

where D_{ij} is a symmetric matrix describing the frequency of trading between i and j. F_{ij} is a local bias: it describes by how much the amount of trading from i to j exceeds that from j to i. In the absence of the speculative term $\eta_i W_i$, Eq. (2) is actually a Master equation describing the random motion of a particle subject to local *forces* F_{ij}, where J_{ij} is the hopping rate between site j and site i. This problem has also been much studied (for a review, see e.g.: [17]). One can, in general, decompose the force F_{ij} into a potential part $U_i - U_j$ and a non-potential part. For a purely potential problem, the stationary solution of Eq. (2) with $\eta_i \equiv 0$ is the well-known Bolzmann weight:

$$W_{i,\mathrm{eq}} = \frac{1}{Z} \exp(-U_i), \qquad Z = \sum_{i=1}^{N} \exp(-U_i). \tag{19}$$

The statistics of the W_i therefore reflects that of the potential U_i; in particular, large wealths correspond to deep potential wells. Pareto tails correspond to the case where the extreme values of the potential obey the Gumbel distribution, which decays exponentially for large (negative) potentials [13].

The general case where η_i is non-zero and/or F_{ij} contains a non-potential part is largely unknown, and worth investigating. A classification of the cases where the Pareto tails survive the introduction of a non-trivial bias field F_{ij} would be very interesting. Partial results in the context of population dynamics have been obtained recently in Ref. [18]. The case where the i's are on the nodes of a d dimensional lattice should be amenable to a renormalization group analysis along the lines of [19,20], with interesting results for $d \leqslant 2$. Work in this direction is under way [21].

In conclusion, we have discussed a very simple model of the economy, where the time evolution is described by an equation capturing, at the simplest level, exchange between individuals and random speculative trading in such a way that the fundamental symmetry of the economy under an arbitrary change of monetary units is obeyed. Although our model is not intended to be fully realistic, the family of equations given by Eq. (2) is extremely rich, and leads to interesting generic predictions. We have investigated in detail a mean-field limit of this equation and showed that the distribution of wealth is of the Pareto type. The Pareto behaviour of the tails of this distribution appears to be robust for more general connectivity matrices, as a mapping to the directed polymer problem shows. In this context, a transition between an economy governed by a few individuals from a situation where the wealth is more evenly spread out is found. The important conclusion of the above model is that the distribution of wealth tends to be very broadly distributed when exchanges are limited. Favouring exchanges (and, less surprisingly, increasing taxes) seems to be an efficient way to reduce inequalities.

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