

## I. CONSTRUCTION OF THE GREEN'S FUNCTION

The Helmholtz equation in 4 dimensions is

$$-\left(\nabla_{(4)}^2 + k^2\right) G^{(4)}(x, x') = \delta^4(x - x'). \quad (1)$$

In this equation,  $G$  is the Green's function and 4 refers to the dimensionality. In the very end, we will set the frequency to zero to obtain the Green's function of the Laplacian in 4d. In spherical coordinates,  $x = (r, \theta, \psi, \phi)$  where  $r$  is the radius and the rest are the angular coordinates. A constant value of the polar angle  $\theta$  defines a cone in 4d. The cone constrains  $\theta$  but is *spherically* symmetric with respect to  $\psi$  and  $\phi$  (just like the azimuthal symmetry of 3d cone). So we can expand the Green's function as

$$G^{(4)}(x, x') = \sum_{lm} \frac{1}{\sqrt{rr'}} \frac{1}{\sqrt{\sin \theta \sin \theta'}} g_{lm}^{(4)}(r, \theta, r', \theta') Y_{lm}(\psi, \phi) Y_{lm}^*(\psi', \phi') \quad (2)$$

The renormalization is introduced to use the analogy with the Laplacian in 3d. Then, Eq. (1) takes the form

$$-\left(\nabla_{(2)}^2 - \frac{(l+1/2)^2}{r^2 \sin^2 \theta} + k^2\right) g_{lm}^{(4)} = \frac{1}{r^2} \delta(r - r') \delta(\cos \theta - \cos \theta') \quad (3)$$

Note that the Laplacian in the last equation is defined in 2d:  $\nabla_{(2)}^2 = \frac{1}{r^2} \partial_r r^2 \partial_r + \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta$ .

We should break up the Green's function in the coordinate  $\theta$ , not  $r$  because the boundary condition is defined on a surface of constant  $\theta$  (*i.e.*, cone). So we need a completeness relation in the function space of the other variable  $r$ . The latter is known by the Kontrovich-Lebedev transform and is given by

$$\frac{1}{\pi} \int_0^\infty d\lambda \lambda \sinh \lambda k_{i\lambda-1/2}(r) k_{i\lambda-1/2}(r') = \delta(r - r'),$$

where  $k_\nu$  is the spherical Bessel function of the order  $\nu$ . Using this relation, it is not hard to show that the following solves Eq. (3)

$$g_{0lm}^{(4)} = \frac{\kappa}{2} (-1)^l \int_0^\infty d\lambda \lambda k_{i\lambda-1/2}(\kappa r) k_{i\lambda-1/2}(\kappa r') \frac{\Gamma(i\lambda + l + 1)}{i\Gamma(i\lambda - l)} P_{i\lambda-1/2}^{-l-1/2}(\cos \theta_<) P_{i\lambda-1/2}^{-l-1/2}(-\cos \theta_>). \quad (4)$$

Here,  $\kappa$  is the imaginary frequency ( $k = i\kappa$ ),  $\theta_< = \min(\theta, \theta')$  and similarly defined  $\theta_>$ . We have not introduced the boundaries yet, hence the subscript 0 in  $g$ . The completeness relation together with a useful form of the Wronskian of the Legendre functions can be used to justify the last equation. Note that we cannot take the limit  $\kappa$  at this point. However, by a construction similar to Ref. [1], we can analytically continue the complex order of the Bessel and Legendre functions back to the real axis. A careful implementation of this method gives

$$g_{0lm}^{(4)} = \kappa \sum_{n=l+1}^\infty i_{n-1/2}(\kappa r_<) k_{n-1/2}(\kappa r_>) \frac{n \Gamma(n + l + 1)}{\Gamma(n - l)} P_{n-1/2}^{-l-1/2}(-\cos \theta) P_{n-1/2}^{-l-1/2}(-\cos \theta'). \quad (5)$$

Now the asymmetry arises in the radial variable while the angular coordinates are treated on equal footing. I used the symmetry  $P_{n-1/2}^{-l-1/2}(-x) = (-1)^{l+n+1} P_{n-1/2}^{-l-1/2}(x)$  for integer  $l, n$  with  $l < n$ , to restore the symmetry in the angles. Only now I can take the limit  $\kappa \rightarrow 0$ . Putting all the pieces in Eq. (2) while taking this limit, we find

$$G_0^{(4)}(x, x') = \sum_{n=1}^\infty \sum_{l=0}^{n-1} \sum_{m=-l}^l \frac{1}{2} \frac{\Gamma(n + l + 1)}{\Gamma(n - l)} \frac{r_{<}^{n-1}}{r_{>}^{n+1}} \frac{P_{n-1/2}^{-l-1/2}(-\cos \theta)}{\sqrt{\sin \theta}} \frac{P_{n-1/2}^{-l-1/2}(-\cos \theta')}{\sqrt{\sin \theta'}} Y_{lm}(\psi, \phi) Y_{lm}^*(\psi', \phi'). \quad (6)$$

The subscript 0 indicates that this is the free Green's function. In order to impose the boundary condition though, we have to go back to Eq. (4) in which representation, it is not hard to satisfy the boundary condition

$$g_{lm}^{(4)} = \frac{\kappa}{2} (-1)^l \int_0^\infty d\lambda \lambda k_{i\lambda-1/2}(\kappa r) k_{i\lambda-1/2}(\kappa r') \frac{\Gamma(i\lambda + l + 1)}{i\Gamma(i\lambda - l)} \times \left( P_{i\lambda-1/2}^{-l-1/2}(\cos \theta_<) P_{i\lambda-1/2}^{-l-1/2}(-\cos \theta_>) - \frac{P_{i\lambda-1/2}^{-l-1/2}(\cos \theta_0)}{P_{i\lambda-1/2}^{-l-1/2}(-\cos \theta_0)} P_{i\lambda-1/2}^{-l-1/2}(-\cos \theta) P_{i\lambda-1/2}^{-l-1/2}(-\cos \theta') \right). \quad (7)$$

It is immediately clear that this is the Green's function. The first term in the last line of this equation gives the delta function once applied to the Helmholtz equation while the second term does not contribute to the delta function (it has no discontinuity in  $\theta = \theta'$ ) but guarantees that the Green's function vanishes on the cone as  $\theta \rightarrow \theta_0$ . We should implement the same procedure to rotate to the real axis. Taking the limit  $\kappa \rightarrow 0$  then gives

$$G^{(4)}(x, x') = \sum_k \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{\pi}{2} (-1)^l \frac{r_{<}^{\rho_k-1}}{r_{>}^{\rho_k+1}} \frac{\Gamma(\rho_k + l + 1)}{\sin(\rho_k \pi) \Gamma(\rho_k - l)} \frac{P_{\rho_k-1/2}^{-l-1/2}(\cos \theta_0)}{\partial_{\rho_k} P_{\rho_k-1/2}^{-l-1/2}(-\cos \theta_0)} \frac{P_{\rho_k-1/2}^{-l-1/2}(-\cos \theta)}{\sqrt{\sin \theta}} \frac{P_{\rho_k-1/2}^{-l-1/2}(-\cos \theta')}{\sqrt{\sin \theta'}} Y_{lm}(\psi, \phi) Y_{lm}^*(\psi', \phi'). \quad (8)$$

The integer  $k$  represents the  $k$ -th root of the transcendental equation

$$P_{\rho_k-1/2}^{-l-1/2}(-\cos \theta_0) = 0. \quad (9)$$

In general,  $\rho_k$  depends on  $\theta_0$  and  $l$ . This Green's function is the basis for the following computations.

## II. THE LOOP COMPUTATION

The first correction to the Green's function comes from a one-loop diagram. The external legs in this diagram correspond to the Green's function  $G$  defined in the previous section which knows about the boundary condition but computed in the free theory. Naively we might think that the Green's function runs in the loop. However this leads to a divergence in this diagram and must be regularized. It turns out that we should replace the Green's function in the loop by  $G - G_0$  where  $G_0$  is the Green's function in the free space.

While the Green's function in the loop must be summed over all "quantum" numbers  $k, l, m$ , those of the external legs must be evaluated only for the lowest number ( $k = 1, l, m = 0$ ) for a reason similar to Cardy's analysis [2]

$$G|_{k=1, l, m=0} = \frac{1}{8} \frac{r_{<}^{\rho_1-1}}{r_{>}^{\rho_1+1}} \frac{\rho_1}{\sin \rho_1 \pi} \frac{P_{\rho_1-1/2}^{-1/2}(\cos \theta_0)}{\partial_{\rho_1} P_{\rho_1-1/2}^{-1/2}(-\cos \theta_0)} \frac{P_{\rho_1-1/2}^{-1/2}(-\cos \theta)}{\sqrt{\sin \theta}} \frac{P_{\rho_1-1/2}^{-1/2}(-\cos \theta')}{\sqrt{\sin \theta'}}. \quad (10)$$

The Legendre function of the degree  $-1/2$  can be expressed in terms of elementary functions

$$\frac{P_{\rho-1/2}^{-1/2}(\cos \theta)}{\sqrt{\sin \theta}} = \sqrt{\frac{2}{\pi}} \frac{1}{\rho} \frac{\sin \rho \theta}{\sin \theta}. \quad (11)$$

Then,  $\rho_1$  is determined by the first root of  $P_{\rho_1-1/2}^{-1/2}(-\cos \theta_0)$  which translates to  $\sin \rho_1(\pi - \theta_0) = 0 \rightarrow \rho_1 = \frac{\pi}{\pi - \theta_0}$ . Putting all these together, Eq. (10) becomes

$$G|_{k=1, l, m=0} = \frac{1}{4\pi^2} \frac{r_{<}^{\rho_1-1}}{r_{>}^{\rho_1+1}} \frac{\sin \rho_1(\pi - \theta)}{\sin \theta} \frac{\sin \rho_1(\pi - \theta')}{\sin \theta'}, \quad (12)$$

I define the exponent  $\eta$  according to Cardy [2] as the power of  $r_{<}$ . So,

$$\eta_0 = \rho_1 - 1 = \frac{\theta_0}{\pi - \theta_0}. \quad (13)$$

The subscript 0 on  $\eta$  means RW. Now we can compute the 1-loop diagram

$$G^{1\text{-loop}} = -\alpha \int d^4 x'' G(x, x'')|_{k=1} (G - G_0)(x'', x'') G(x'', x')|_{k=1}. \quad (14)$$

Here  $\alpha$  is the *coupling constant*. For the  $N$ -component  $\phi^4$  field theory, this is  $\alpha = (N+2)u_0 = \frac{(N+2)8\pi^2}{N+8} \epsilon \rightarrow_{N \rightarrow \infty} = 2\pi^2 \epsilon$  [?]. I choose the point  $x$  at the lesser radius  $r$  (close to the tip) and  $x'$  to be at the greater radius  $r'$ . I follow (and justify) Cardy in choosing the domain of integration as  $r < r'' < r'$ . The expected logarithm in  $r$  comes out trivially from the latter domain

$$\int_r^{r'} dr'' r''^3 \frac{r^{\rho_1-1}}{r''^{\rho_1+1}} \frac{1}{r''^2} \frac{r''^{\rho_1-1}}{r''^{\rho_1+1}} = -\frac{r^{\rho_1-1}}{r'^{\rho_1+1}} \log(r/r').$$

The coefficient of this term is determined by the integration of the angular coordinates in the loop and the sum over all “*quantum*” numbers. It is instructive to do this for the special case of an infinite plate but in the representation that we developed here. For an infinite plate, we can directly construct the Green’s function from the free Green’s function by method of images

$$G = G_0(x, x') - G_0(x, x'_T),$$

where  $T$  denotes the mirror image of a certain point. The explicit Green’s function can be obtained from Eq. (6). Again the Green’s function in the external legs is obtained by substituting the lowest number in its expansion

$$G|_{k=1, l, m=0} = \frac{1}{\pi^2} \frac{r_{<}}{r_{>}^3} \cos \theta \cos \theta', \quad (15)$$

and the Green’s function in the loop must be subtracted by  $G_0$ . By noting that  $G(x, x') - G_0(x, x') = G(x, x'_T)$ , we find

$$(G - G_0)^{(4)}(x, x') = \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \sum_{m=-l}^l \frac{r_{<}^{n-1}}{r_{>}^{n+1}} \frac{\Gamma(n+l+1)}{\Gamma(n-l)} P_{n-1/2}^{-l-1/2}(-\cos \theta) P_{n-1/2}^{-l-1/2}(\cos \theta') Y_{lm}(\psi, \phi) Y_{lm}^*(\psi', \phi'). \quad (16)$$

The argument of the second Legendre function in the last equation is  $\cos \theta'$  (not  $-\cos \theta'$ ) due to the mirror imaging. The 1-loop computation gives

$$G^{1\text{-loop}} = -\alpha \frac{1}{\pi^2} \frac{r}{r'^3} (-\log(r/r')) \cos \theta \cos \theta' \times \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \frac{2l+1}{8\pi^2} \frac{\Gamma(n+l+1)}{\Gamma(n-l)} \int_{\pi/2}^{\pi} d\theta'' \sin \theta''^2 \cos \theta''^2 P_{n-1/2}^{-l-1/2}(-\cos \theta'') P_{n-1/2}^{-l-1/2}(\cos \theta''). \quad (17)$$

The integral over  $\psi''$  and  $\phi''$  is trivial as nothing else depends on them. Furthermore, the sum over  $m$  can be explicitly done to find the factor  $2l+1$  in the last equation. The full Green function including the 1-loop correction then becomes

$$G + G^{1\text{-loop}} = \frac{1}{\pi^2} \frac{r}{r'^3} \cos \theta \cos \theta' \times \left( 1 + \alpha \log(r/r') \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \frac{2l+1}{8\pi^2} \frac{\Gamma(n+l+1)}{\Gamma(n-l)} \int_{\pi/2}^{\pi} d\theta'' \sin \theta''^2 \cos \theta''^2 P_{n-1/2}^{-l-1/2}(-\cos \theta'') P_{n-1/2}^{-l-1/2}(\cos \theta'') \right). \quad (18)$$

Surprisingly, we find that only the first term in the sum gives the total contribution. Other terms are not zero but vanish once summed over all values of  $l$  for any  $n \neq 1$ . I do not understand this completely but it can be due to the huge symmetry of an infinite plate. The last equation then becomes

$$G + G^{1\text{-loop}} = \frac{1}{\pi^2} \frac{r}{r'^3} \cos \theta \cos \theta' \left( 1 + \frac{\alpha}{16\pi^2} \log(r/r') \right) = \frac{1}{\pi^2} \frac{r^{1+\epsilon/8}}{r'^{3+\epsilon/8}} \cos \theta \cos \theta'. \quad (19)$$

In the last line, we substituted  $\alpha = 2\pi^2\epsilon$  and exponentiated the logarithm. The exponent  $\eta$  up to 1-loop correction is

$$\eta = \eta_0 + \eta_{1\text{-loop}} = 1 + \epsilon/8. \quad (20)$$

**I have not checked this against the literature.**

### A. Loop Comp. for a Single Needle

The coefficient of the  $\epsilon$ -expansion can be formally written for a general (half-)opening angle. But I will do this for small angles in which case I can find analytical results.

The main challenge lies in computing  $G - G_0$  which runs in the loop. At small angles, there are some analytic expressions for the roots of the Legendre functions. We do not need these formulae but only use the fact that at small

angles and at the leading order in  $\theta_0$ , we can set  $l$  to zero: we only need the  $l = 0$  channel of  $G - G_0$ . Note that the Legendre function of the degree  $-1/2$  are given by Eq. (11). By doing some algebra, we get

$$G|_{l=0} - G_0|_{l=0} = \frac{1}{4\pi r'^2} \sum_{k=1}^{\infty} \frac{1}{k} \frac{\sin^2 k\pi \frac{\pi-\theta''}{\pi-\theta_0} - \sin^2 k\theta''}{\sin^2 \theta''}. \quad (21)$$

Now we should compute  $\int G(G - G_0)G$ . A careful algebra gives

$$G + G^{1-loop} \approx \frac{1}{4\pi^2} \frac{r^{\rho_1-1}}{r'^{\rho_1+1}} \left( 1 + \frac{\alpha}{16\pi^3} \log(r/r') \left( .99\theta_0 + \frac{1}{2}\theta_0 \log \theta_0 \right) \right). \quad (22)$$

The coefficient of the logarithm is analytical while that of the linear term is found numerically. We can then read off the 1-loop correction to the Green's function (keeping in mind that  $\alpha = 2\pi^2\epsilon$ )

$$\eta = \eta_0(\theta_0, \epsilon) + \left( .16\theta_0 + \frac{1}{4\pi}\theta_0 \log \theta_0 \right) \epsilon. \quad (23)$$

The function  $\eta_0$  is given in Eq. (37) of Ref. [3] (assuming that  $\eta$  in my notes is their  $2\beta$ )

$$\eta_0(\theta_0, \epsilon) = \frac{\Gamma(1 - \epsilon/2)}{\Gamma(1/2)\Gamma(1/2 - \epsilon/2)} \theta_0^{1-\epsilon}. \quad (24)$$

Expanding the latter in  $\epsilon$  and replacing in the equation for  $\eta$ , we find

$$\eta \approx \frac{\theta_0}{\pi} + \left( -.22\theta_0 - \frac{1}{\pi}\theta_0 \log \theta_0 \right) \epsilon + \left( .16\theta_0 + \frac{1}{4\pi}\theta_0 \log \theta_0 \right) \epsilon. \quad (25)$$

The first term in this equation is  $\eta$  as computed in the free theory in 4d, then comes the  $\epsilon$  expansion in dimension but in the free theory, finally the last parentheses is the  $\epsilon$  expansion in the interaction but in fixed (four) dimensions. The appearance of logarithms is suggestive of another exponentiation; this time in  $\theta_0$

$$\eta \approx \left( \frac{1}{\pi} - .06\epsilon \right) \theta_0^{1-\frac{3}{4}\epsilon} \rightarrow_{\epsilon=1} .26\theta_0^{1/4}. \quad (26)$$

### B. Loop Comp. for a Single Needle Attached to a Plate

We can carry out the same computation for the case that the cone is attached to a plate. Our first task is to obtain the free Green's function which solves the boundary conditions. We cannot use the method of images because it does not lead to a symmetric Green's function. Let's setup the boundary problem as follows. I consider an *inner* cone defined by  $\theta < \theta_1$  attached to an *outer* cone defined by  $\theta > \theta_2$ . The region  $\theta_1 < \theta < \theta_2$  defines the free space. Sparing the technical steps, we should change the second line of Eq. (7) by

$$\left( 1 - \frac{P_{i\lambda-1/2}^{-l-1/2}(\cos \theta_1) P_{i\lambda-1/2}^{-l-1/2}(-\cos \theta_2)}{P_{i\lambda-1/2}^{-l-1/2}(-\cos \theta_1) P_{i\lambda-1/2}^{-l-1/2}(\cos \theta_2)} \right)^{-1} \times \left( P_{i\lambda-1/2}^{-l-1/2}(\cos \theta_<) - \frac{P_{i\lambda-1/2}^{-l-1/2}(\cos \theta_1) P_{i\lambda-1/2}^{-l-1/2}(-\cos \theta_<)}{P_{i\lambda-1/2}^{-l-1/2}(-\cos \theta_1)} \right) \left( P_{i\lambda-1/2}^{-l-1/2}(-\cos \theta_>) - \frac{P_{i\lambda-1/2}^{-l-1/2}(-\cos \theta_2) P_{i\lambda-1/2}^{-l-1/2}(\cos \theta_>)}{P_{i\lambda-1/2}^{-l-1/2}(\cos \theta_2)} \right). \quad (27)$$

We are interested in a cone attached to a plate. So we choose  $\theta_1 \equiv \theta_0$  and  $\theta_2 = \pi/2$ . A little bit of algebra gives

$$g_{lm}^{(4)} = \frac{\kappa}{2} (-1)^l \int_0^\infty d\lambda \lambda k_{i\lambda-1/2}(\kappa r) k_{i\lambda-1/2}(\kappa r') \frac{\Gamma(i\lambda + l + 1)}{i\Gamma(i\lambda - l)} \frac{1}{P_{i\lambda-1/2}^{-l-1/2}(-\cos \theta_0) - P_{i\lambda-1/2}^{-l-1/2}(\cos \theta_0)} \times \left( P_{i\lambda-1/2}^{-l-1/2}(-\cos \theta_0) P_{i\lambda-1/2}^{-l-1/2}(\cos \theta_<) - P_{i\lambda-1/2}^{-l-1/2}(\cos \theta_0) P_{i\lambda-1/2}^{-l-1/2}(-\cos \theta_<) \right) \left( P_{i\lambda-1/2}^{-l-1/2}(-\cos \theta_>) - P_{i\lambda-1/2}^{-l-1/2}(\cos \theta_>) \right). \quad (28)$$

By rotating back to the real axis, the full Green's function becomes

$$G^{(4)}(x, x') = \sum_k \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{\pi}{2} (-1)^l \frac{r_{<}^{\rho_k-1}}{r_{>}^{\rho_k+1}} \frac{\Gamma(\rho_k + l + 1)}{\sin(\rho_k \pi) \Gamma(\rho_k - l)} \frac{P_{\rho_k-1/2}^{-l-1/2}(\cos \theta_0)}{\partial_{\rho_k} \left( P_{\rho_k-1/2}^{-l-1/2}(-\cos \theta_0) - P_{\rho_k-1/2}^{-l-1/2}(\cos \theta_0) \right)} \\ \frac{P_{\rho_k-1/2}^{-l-1/2}(-\cos \theta) - P_{\rho_k-1/2}^{-l-1/2}(\cos \theta)}{\sqrt{\sin \theta}} \frac{P_{\rho_k-1/2}^{-l-1/2}(-\cos \theta') - P_{\rho_k-1/2}^{-l-1/2}(\cos \theta')}{\sqrt{\sin \theta'}} Y_{lm}(\psi, \phi) Y_{lm}^*(\psi', \phi'). \quad (29)$$

The order  $\rho_k$  is the  $k$ -th root of the transcendental equation  $P_{\rho_k-1/2}^{-l-1/2}(-\cos \theta_0) - P_{\rho_k-1/2}^{-l-1/2}(\cos \theta_0) = 0$ . Again the main challenge is constructing  $G - G_0$ . By the same line of reasoning, we find an analog of Eq. (21) for a small angle cone attached to a plate

$$G|_{l=0} - G_0|_{l=0} = \frac{1}{4\pi r''^2} \sum_{k=1}^{\infty} \frac{1}{k} \frac{\sin^2 k\pi \frac{\pi/2-\theta''}{\pi/2-\theta_0} - \sin^2 2k\theta''}{\sin^2 \theta''}. \quad (30)$$

Similar computations give the exponent  $\eta$

$$\eta = \eta_0(\theta_0, \epsilon) + \frac{2}{\pi} \left( 1.14 \theta_0 + \frac{1}{2} \theta_0 \log \theta_0 \right) \epsilon. \quad (31)$$

To find  $\eta_0(\theta_0, \epsilon)$ , we have to solve

$$P_{\rho-1+\delta}^{-\delta}(-\cos \theta_0) - P_{\rho-1+\delta}^{-\delta}(\cos \theta_0) = 0 \quad \delta = \frac{d-3}{2}. \quad (32)$$

Using the expansion of the Legendre function for arguments close to 1 and -1

$$P_{\nu}^{\mu}(\cos \theta_0) = \frac{1}{\Gamma(1-\mu)} (\theta_0/2)^{-\mu} (1 + \mathcal{O}(\theta_0^2)), \\ P_{\nu}^{\mu}(-\cos \theta_0) = \frac{\Gamma(-\mu)}{\Gamma(-\mu-\nu)\Gamma(1-\mu+\nu)} (\theta_0/2)^{\mu} (1 + \mathcal{O}(\theta_0^2)) - \frac{\sin(\pi\nu)\Gamma(\mu)}{\pi} (\theta_0/2)^{-\mu} (1 + \mathcal{O}(\theta_0^2)),$$

the condition (32) gives

$$\frac{2}{\Gamma(1+\delta)} (\theta_0/2)^{\delta} - \frac{\Gamma(\delta)}{\Gamma(1-\rho)\Gamma(\rho+2\delta)} (\theta_0/2)^{-\delta} = 0.$$

Note that in the limit  $\theta_0 \rightarrow 0$ , this gives  $\rho = 2$  (so  $\eta = 2\beta = 1$ ) as it should. At small angles, I find

$$\rho \approx 2 + \frac{2\Gamma(2+2\delta)}{\Gamma(\delta)\Gamma(1+\delta)} (\theta_0/2)^{2\delta}. \quad (33)$$

Then,

$$\eta \approx 1 + \frac{4\theta_0}{\pi} + \left( -1.52 \theta_0 - \frac{4}{\pi} \theta_0 \log \theta_0 \right) \epsilon + \left( .76 \theta_0 + \frac{1}{\pi} \theta_0 \log \theta_0 \right) \epsilon. \quad (34)$$

Exponentiating  $\theta_0$ , we get

$$\eta \approx 1 + \left( \frac{4}{\pi} - .76\epsilon \right) \theta_0^{1-\frac{3}{4}\epsilon} \rightarrow_{\epsilon=1} 1 + .51 \theta_0^{1/4}. \quad (35)$$

The exponent  $\gamma$  is related to the other exponents via  $\gamma = \nu(2 - \eta)$ . Equations (26) and (35) then define  $\Delta\gamma$

$$\frac{\Delta\gamma}{\nu} = (2 - \eta_c) - (2 - \eta_{cp}) = 1 + .25 \theta_0^{1/4}. \quad (36)$$

[1] M. F. Maghrebi et al., (2010).

[2] J. L. Cardy, Journal of Physics A Mathematical General **16**, 3617 (1983).

[3] E. Ben-Naim and P. L. Krapivsky, Journal of Physics A Mathematical General **43**, 5007 (2010).