

## REPLICA BETHE ANSATZ STUDIES OF TWO-DIMENSIONAL INTERFACES WITH QUENCHED RANDOM IMPURITIES

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The statistical mechanics of interfaces subject to quenched impurities is studied in two dimensions. The presence of randomness changes the scaling of domain wall fluctuations, and modifies critical behavior at interface-driven depinning (wetting), and commensurate-to-incommensurate phase transitions. All these problems are examined by combining the replica method with Bethe ansatz calculations. Results include expressions for quench-averaged free energies, their cumulants, expectation values, distribution functions, in addition to a number of new critical exponents. The intermediate results include some novel Bethe ansatz solutions, such as the ground state energy of a system of  $n$  attracting fermion species.

### 1. Introduction and summary

One of the most challenging topics of current research is the physics of disordered systems [1]. It was recognized early on that the statistical mechanics, and critical behavior, of random systems is in general different from the corresponding pure ones [2]. However, understanding this behavior, for example in the contexts of random fields [3,4], spin glasses [5,6], or polymers in random media [7], has almost always proved controversial. In this paper I examine the behavior of interfaces in two-dimensional disordered systems. This is a non-trivial example that can be solved exactly, and sheds some light into the complex behavior of random materials.

An important tool for examining properties of quenched random systems is the replica method [5]. In principle all quantities of interest can be obtained from the  $n \rightarrow 0$  limit of an  $n$  times replicated version of the original problem, with interactions between the replicas. In practice this method is hampered by the difficulties associated with solving the interacting problem, and taking the  $n \rightarrow 0$  limit. The complexity of the mean field solution to the spin-glass problem [6], with its broken replica symmetry, attests to these difficulties. These problems are circumvented for the 2d interface problem, where the interacting system can be solved exactly for all  $n$ , and the limit straightforwardly taken. Yet another recent exact solution is that of

the two-dimensional random bond Ising model [8]. This solution, obtained by a renormalization group treatment of the replicated model is somewhat unsatisfactory, as it can not be easily extended beyond an analysis of the critical regime. By contrast in the interface problem, having exact analytic expressions for all  $n$ , allows the full power of the replica formalism to be utilized.

Exact solutions for the replicated interface problems are obtained by using Bethe ansatz methods [9,10]. This is indeed a common tool for solving statistical mechanics of pure two-dimensional models, and is here extended to random systems. Brief reports of the results for commensurate-to-incommensurate [11], and depinning transitions [12] have already appeared. But the complexity of the calculations, and the intricacy of the  $n \rightarrow 0$  limit, can only be appreciated from a fuller account. This paper presents the details of these calculations, along with a number of new results, and suggestions for further studies. The organization of the paper is as follows:

The general philosophy of this technique is outlined in sect. 2 which deals with the properties of a *single interface* in the random bond Ising model. After obtaining a continuum version of this system, a transfer matrix method is used to reduce the problem to the quantum mechanics of a single particle. The free energy of the replicated interface is then related to the ground state energy of a collection of interacting bosons. The Bethe ansatz solution of this problem is actually very simple, and requires no more than rudimentary quantum mechanics. However, even this simple solution contains many non-trivial results for the random interface problem, such as the complete distribution function for the quench-averaged free energy [ $f$ ]. In particular fluctuations in [ $f$ ] are found to scale with the length  $T$  of the interface as  $T^{1/3}$ , implying a scaling of  $T^{2/3}$  for transverse fluctuations [13].

The depinning transition, which describes the delocalization of an interface from an attractive surface [12], is examined in sect. 3. In the absence of randomness the transfer matrix formalism reduces this problem to the quantum mechanics of a particle in an attractive potential with hard core. With randomness, the replicated model corresponds to  $n$  mutually attracting particles in such a potential. Guided by simple ideas, it is possible to write down an exact wave function for the ground state of these particles. The  $n \rightarrow 0$  limit of the ground state energy provides an expression for the interface free energy, and its singularities at the depinning transition. Higher order cumulants of the free energy can also be obtained, and are found to have *distinct* singularities at the transition. This interesting result is probably an example of a more general phenomenon in random systems. The quench-averaged localization length is also calculated, and diverges at the transition with an exponent of 2. This exponent is different from the non-random result, and indicates the relevance of impurities at the transition.

The most technically complex calculation arises in the context of the commensurate-to-incommensurate [11] transition (CIT), as described in sect. 4. This transition occurs for a system of many non-crossing interfaces. The non-crossing

constraint is most easily implemented in the transfer matrix by regarding the interfaces as world lines of one-dimensional fermions. In the replica formalism, we have to find the ground state energy of  $n$  attracting fermion species.

The behavior of interacting fermions has considerable intrinsic interest. In principle it can be solved by the Bethe ansatz method [10], as formulated by Yang [14] and Sutherland [15]. In practice, however, their solution involving coupled integral equations, obscures most of the interesting physics of the problem. For random interfaces, the number of fermions is the same in all replicas. This leads to considerable simplifications, and in some cases, closed form expressions for the energy. Ground state solutions for attractive interactions are given in sect. 4. Taking the  $n \rightarrow 0$  limit in this case is non-trivial, and again leads to modified exponents at the CIT.

## 2. A single interface

### 2.1. MODEL AND REPLICAS

To describe the behavior of an interface interacting with impurities, we start with the two-dimensional random-bond Ising model. A domain wall is introduced by the choice of (+) and (-) boundary conditions at opposite edges of the system. This interface is treated by a series of approximations, the first of which is the restriction to solid-on-solid configurations, which exclude islands and overhangs [13]. This is formally exact in the extreme anisotropic limit (strong bonds parallel to the interface, and weak bonds perpendicular to it). Each interface configuration is now described by a single-valued function  $\{x(t)\}$ , where  $t$  is the coordinate parallel to the wall, and  $x$  measures its transverse fluctuations. For simplicity the bonds perpendicular to the interface are taken to be uniform (all equal to  $K$ ), while the bonds  $\mu(x, t)$  parallel to the interface are assumed to be independent random variables. The partition function for the domain wall can be obtained from a product of transfer matrices  $\mathcal{T}(t)$ , evaluated at step  $t$ , where

$$\langle x | \mathcal{T}(t) | x' \rangle = \exp[-\mu(x, t) - K|x - x'|]. \quad (2.1)$$

For large  $K$ , transverse steps are infrequent, and  $\mathcal{T}(t)$  can be expanded in  $\gamma = e^{-K}$ :

$$\begin{aligned} \langle x | \mathcal{T}(t) | x' \rangle &= e^{-\mu(x, t)} [\delta_{x, x'} + \gamma(\delta_{x, x'+1} + \delta_{x, x'-1}) + O(\gamma^2)] \\ &\approx \exp[-\mu(x, t) + 2\gamma + \gamma \partial^2 / \partial x^2] = \exp[-\mathcal{H}(t)], \end{aligned} \quad (2.2)$$

where the last step involves going to a continuum limit in the  $x$  direction. The overall weight of paths connecting  $(0, 0)$  to  $(x, t)$  is calculated using transfer matrices as  $W(x, t) = \langle 0 | \mathcal{T}(0) \mathcal{T}(1) \dots \mathcal{T}(t-1) | x \rangle$ . After taking a continuum

limit in the  $t$  direction, this weight is found to evolve according to a  $t$ -dependent hamiltonian  $\mathcal{H}(t)$ , as

$$\frac{\partial W}{\partial t} = -\mathcal{H}(t)W, \quad \text{with } \mathcal{H}(t) = \left[ \mu(x, t) - 2\gamma - \gamma \frac{\partial^2}{\partial x^2} \right]. \quad (2.3)$$

This is a non-markovian diffusion equation, with dynamically random sources and sinks. In the absence of impurities,  $W(x, t)$  behaves asymptotically as  $\exp\{-ft - x^2/4\gamma t\}$ , where  $f = \mu - 2\gamma$  is the interface free energy, and  $\langle x^2 \rangle = 2\gamma t$  measures interface fluctuations in the transverse direction. The interface free energy vanishes linearly at the pure Ising transition temperature ( $\mu = 2\gamma$ ), indicating a Widom interfacial tension exponent [16] of unity. Similarly, with randomness present,  $W(x, t) \sim \exp\{-[f]t - g(x/t^\nu)\}$ , where  $[f]$  is the quench-averaged free energy, and the exponent  $\nu$  describes the scaling of transverse fluctuations.

The important quench averaged quantities, such as  $[f]$ , appear in the exponent. Hence the logarithm of the partition function  $Z = W(x, 0)$ , and not  $Z$  itself, has to be averaged over the distribution of random impurities. In the replica method this is accomplished by using the simple identity

$$[\ln Z] = \lim_{n \rightarrow 0} ([Z^n] - 1)/n, \quad (2.4)$$

where  $[\dots]$  denotes impurity averaging. In the original lattice problem,  $Z^n$  is the partition function of  $n$  copies of the original interface. The averaging over randomness is done by noting that each bond  $\mu(x, t)$  is an *independent* random variable. In any given configuration of interfaces, the bond  $\mu(x, t)$  contributes a factor of  $e^{-m\mu(x, t)}$  to  $Z^n$ , where  $m$  is the number of interfaces crossing that bond ( $m = 0, 1, 2, \dots, n$ ). Assuming a gaussian distribution of mean  $[\mu]$  and variance  $\sigma^2$  for each bond leads to

$$[\exp(-m\mu)] = \exp\left\{-([\mu] - \frac{1}{2}\sigma^2)m + \frac{1}{2}\sigma^2 m(m-1)\right\}. \quad (2.5)$$

This important result is interpreted as follows. The averaging over random bonds results in (i) a uniform effective value of  $[\mu] - \frac{1}{2}\sigma^2$  for each bond and (ii) a pairwise attraction of magnitude  $\sigma^2$  between interfaces on contact. (Note that  $m$  interfaces through a bond correspond to  $\frac{1}{2}m(m-1)$  pairs.) Thus the replicated model describes a collection of  $n$  attracting interfaces on a *uniform* two-dimensional lattice.

To calculate the partition function  $[Z^n]$  for these attracting interfaces, the steps taken previously for a single interface are repeated. In the solid on solid limit the interface configurations are described by the world-lines  $\{x_1(t), x_2(t), \dots, x_n(t)\}$ , and treated by the  $n$ -body transfer matrix  $\langle \{x\} | \mathcal{T}_n | \{x'\} \rangle$ . As in eq. (2.2) after taking a continuum limit (also assuming  $\gamma \ll 1$ ), the transfer matrix can be written as  $\mathcal{T}_n = \exp[-\mathcal{H}_n]$ . The  $n$ -body hamiltonian  $\mathcal{H}_n$  is now  $t$ -independent, and takes

the form

$$\mathcal{H}_n = ([\mu] - \frac{1}{2}\sigma^2 - 2\gamma)n - \gamma \sum_{\alpha=1}^n \frac{\partial^2}{\partial x_\alpha^2} - \sigma^2 \sum_{\alpha < \beta} \delta(x_\alpha - x_\beta). \quad (2.6)$$

The last term represents the attractive interaction between interfaces on contact. The replica method thus converts a product of random,  $t$ -dependent transfer matrices to that of uniform,  $t$ -independent ones, at the expense of introducing interactions between the different replicas.

## 2.2. THE BETHE ANSATZ

As usual, the partition function is related only to the largest eigenvalue of the transfer matrix. Thus, for a system of length  $T$ ,  $[Z^n] = \exp(-E_n T)$ , where  $E_n$  is the ground-state energy of the hamiltonian  $\mathcal{H}_n$  in (2.6). In almost all replica formulations of random systems, the interacting problem cannot be solved exactly, and various approximations have to be made. In this case however, finding the ground state energy of  $\mathcal{H}_n$  is reasonably straightforward. For  $n=2$ , the ground state wave function is clearly  $\Psi_0 \sim \exp(-\kappa|x_1 - x_2|)$ . This is easily generalized to an  $n$ -particle bound state of the form

$$\Psi_0 \sim \exp\left(-\kappa \sum_{\alpha < \beta} |x_\alpha - x_\beta|\right). \quad (2.7)$$

To ensure that  $\Psi_0$  is an eigenstate of  $\mathcal{H}_n$ , the discontinuity in the wave function as two particles cross has to be matched to the strength of the attractive potential. This requires  $4\gamma\kappa = \sigma^2$ , and introduces a new length scale

$$l_d = \kappa^{-1} = 4\gamma/\sigma^2 \quad (2.8)$$

into the problem. The crossover from pure to random behavior for interfaces occurs at length scales larger than  $l_d$ . For the continuum approximation to be valid  $l_d$  must be much larger than the lattice spacing. This condition breaks down at very low temperatures. Since  $\Psi_0$  is an eigenstate of  $\mathcal{H}_n$  with no zeros, it must be the (non-degenerate) ground state of this hamiltonian.

The above solution represents the simplest example of a Bethe ansatz. The general idea is quite simple: For particles interacting in one dimension via a contact potential the phase space is divided into segments depending on their ordering. In each segment the particles are "free", and the wave function is written as a product of plane waves. The allowed "momenta" are then determined by requiring different segments to match, and by using appropriate boundary conditions. In the above example there are  $n!$  permutations (orderings) of particles. For each permutation  $P$ , the absolute values  $|x_\alpha - x_\beta|$  in eq. (2.7) can be expanded as  $+$  or  $-(x_\alpha - x_\beta)$ ,

resulting in the wave function  $\Psi_0 \sim \prod_{\alpha} \exp(-\kappa_{\alpha} x_{p\alpha})$ , which is an explicit product of exponentials. The “momenta”  $\kappa_{\alpha} = (n + 1 - 2\alpha)\kappa$  (for  $\alpha = 1, 2, \dots, n$ ) form an  $n$ -string along the imaginary  $k$ -axis [10] (using the notation of  $\exp(ikx)$  for plane waves). The different segments are matched, as  $\Psi_0$  is continuous and its derivatives have the discontinuity required by the attractive interaction. As two particles go through each other, they simply exchange their momenta (an elastic scattering process), and hence the kinetic energy  $\gamma \sum_{\alpha} \kappa_{\alpha}^2$  is the same in all segments. The total energy, obtained from  $\mathcal{H}_n \Psi = E_n \Psi_0$ , follows from (2.6) as

$$E_n = \left( [\mu] - \frac{1}{2} \sigma^2 - 2\gamma \right) n - \frac{1}{3} \gamma \kappa^2 n (n^2 - 1). \tag{2.9}$$

### 2.3. THE $n \rightarrow 0$ LIMIT

From eqs. (2.4) and (2.9), the quench averaged free energy is obtained as the  $n \rightarrow 0$  limit of  $E_n/n$ , and hence

$$[f]/T = [\mu] - 2\gamma - \frac{1}{2} \sigma^2 - \sigma^4/48\gamma. \tag{2.10}$$

Setting  $[f]$  to zero leads to the transition temperature of this random bond Ising model. Since  $\gamma$  plays the role of temperature, eq. (2.10) suggests a reentrant disordered phase at low temperatures. (Note that  $[f]$  is a non-monotonic function of  $\gamma$ , and is negative for both small and large  $\gamma$ .) Although this is probably correct for a range of parameters, it should be noted that  $l_d \sim \gamma$  implies the breakdown of the underlying continuum approximation at very low temperatures. It would be interesting to repeat this analysis in the original lattice problem. The surface tension  $[f]$  again vanishes linearly at the transition, indicating a Widom exponent of unity [16]. For the pure 2d Ising problem, the correlation length exponent ( $\nu = 1$ ) equals the Widom exponent from a hyperscaling identity [16]. Since such rules are not valid in random systems,  $[f]$  contains no information about the random bond value of  $\nu$ .

Note that although taking the  $n \rightarrow 0$  limit of eq. (2.9) is straightforward, the  $n^3$  dependence of  $E_n$  implies that Carlson’s theorem cannot be invoked to deduce the uniqueness of this limit. A different approach is to realize that  $[Z^n] = [e^{n \ln Z}]$  is the characteristic function for the random variable  $\ln Z$ , and as such contains much additional information beyond the quench-averaged free energy. In fact  $[Z^n]$  can be expanded as

$$[Z^n] = \exp \left\{ \sum_{j=1}^{\infty} \frac{n^j}{j!} C_j (\ln Z) \right\}, \tag{2.11}$$

where  $C_j$  is the  $j$ th cumulant of  $\ln Z$ . (The first three cumulants are  $C_1 = [\ln Z] = -[f]$ ,  $C_2 = [\ln Z^2] - [\ln Z]^2$ , and  $C_3 = [\ln Z^3] - 3[\ln Z^2][\ln Z] + 2[\ln Z]^3$ .) Since  $E_n$  in eq. (2.9) only contains terms of order  $n$  and  $n^3$ , the distribution function for  $\ln Z$  is completely characterized by its first and third cumulants.  $C_1$  is given in eq. (2.10),

the second cumulant of  $\ln Z$  is zero to order of  $T$ , and the third cumulant is

$$C_3(\ln Z) = 2\gamma\kappa^2 T = \sigma^4 T / 8\gamma. \quad (2.12)$$

#### 2.4. DISCUSSION AND INTERPRETATION

The vanishing of the second cumulant of  $\ln Z$  is very significant as it implies from eq. (2.12) that typical fluctuations in the interface free energy scale as  $\Delta f \sim T^{1/3}$ , and not as  $T^{1/2}$ , which is the naive expectation [17,18] since each interface encounters  $T$  independent random bonds along its path. The interface also fluctuates in the transverse direction, mostly to take advantage of favorable weak bonds. In a continuum elastic description of the interface, a transverse fluctuation  $x$  results in an energy change proportional to  $x^2/T$  (this is simply the increase in a coarse grained length). Equating this to typical  $\Delta f$  leads to a scaling form  $x \sim T^{2/3}$ . This non-trivial scaling has been observed numerically both at zero [13], and finite temperatures [19]. The exponent of  $\frac{2}{3}$  can also be obtained by a mapping of eq. (2.3) onto the randomly stirred Burger's equation [20,21]. The above calculation is probably the *simplest* derivation of this result. However, the exponent that is obtained directly describes the scaling of free energy fluctuations, and to obtain the scaling of transverse fluctuations the additional assumption of continuum elasticity [13] ( $\Delta f \sim x^2/T$ ) is used. I have not verified this assumption in the replica framework, but as in the pure system, it is probably related to the existence of low-lying excitations in the transfer matrix spectrum. (Due to translational symmetry  $\Psi(k) \sim \Psi_0 \exp(ik \sum_\alpha x_\alpha)$  is also an eigenstate of the hamiltonian  $\mathcal{H}_n$  for all  $k$ , with an energy  $E_n(k) = E_n(0) + n\gamma k^2$ .)

### 3. The depinning transition

#### 3.1. MODEL AND REPLICAS

The critical wetting (or depinning) transition involves the delocalization of an interface from an attractive potential at a surface or a wall [22]. At low temperatures the interface is pinned to the attractive potential from energy considerations, while at high temperatures it moves into the bulk due to entropy effects. The depinning transition has been studied extensively in pure systems [22–24], and more recently in the presence of impurities [12, 25, 26]. The simplest two-dimensional system that exhibits this transition is a semi-infinite Ising model with weak bonds on the surface [23]. In the “solid-on-solid” picture, the interface profile  $\{x(t)\}$  is limited to  $x > 0$ . The new transfer matrix  $\mathcal{T}'$  is the same as the free interface  $\mathcal{T}$  in eq. (2.2) for  $x > 1$ , while at the surface ( $x = 1$ ) it is modified to

$$\langle 1 | \mathcal{T}'(t) | x' \rangle = \exp(-\mu_s) (\delta_{1,x'} + \gamma \delta_{2,x'}). \quad (3.1)$$

The weaker surface bonds  $\mu_s$  are assumed to be uniform for simplicity.

Without bulk randomness [24] the lowest energy eigenstate of  $\mathcal{F}'$  is a bound state  $\Psi_0 \sim \exp(-\lambda x)$ , with  $\lambda \approx [(\mu - \mu_s) - \gamma]/\gamma$ . The inverse localization length  $l_p = 1/\lambda$  represents the competition between the pinning energy  $(\mu - \mu_s)$ , and the entropy loss due to elimination of paths to  $x < 1$  by the surface. The entropy term dominates at high temperatures, leading to a depinning transition at  $\gamma = (\mu - \mu_s)$ , with the pinning length diverging as  $l_p \sim (\gamma - \gamma_c)^{-1}$ , i.e. with an exponent  $\nu_{\perp} = 1$ . The free energy (obtained from the largest eigenvalue of  $\mathcal{F}'$ ), is  $f'/T \approx \mu - 2\gamma - \gamma\lambda^2$ . It describes a second order depinning transition (as  $\lambda \rightarrow 0$ ) with a discontinuous interface specific heat. At high temperatures ( $\gamma > \gamma_c$ ) the attractive potential has no bound state, and the free energy is identical to that of the free interface.

With bulk randomness, the replication procedure and the subsequent random averaging is performed [12] as in the previous chapter. The replicated problem describes  $n$  attracting interfaces, each in turn attracted to weak bonds on the surface. The partition function is again calculated via a transfer matrix, which in the continuum limit can be written as  $\mathcal{F}' = \exp[-\mathcal{H}'_n]$ , with

$$\mathcal{H}'_n = ([\mu] - \frac{1}{2}\sigma^2 - 2\gamma)n + \sum_{\alpha=1}^n \left[ -\gamma \frac{\partial^2}{\partial x_{\alpha}^2} + V_s(x_{\alpha}) \right] - \sigma^2 \sum_{\alpha < \beta} \delta(x_{\alpha} - x_{\beta}). \quad (3.2)$$

The weak surface bonds result in an external potential  $V_s(x)$  on the particles. This potential has an attractive part and an impenetrable core, and causes the wave function of a particle immediately next to it to fall off as  $\exp(-\lambda x)$ . The possibility of more than one particle in the potential will be ignored. This assumption, and the continuum limit, are valid if both the disorder length  $l_d$  and the pinning length  $l_p$  are larger than the lattice spacing.

### 3.2. THE BETHE ANSATZ

There is no precedent for a Bethe ansatz solution of interacting particles in an external potential. However, guided by intuition gained from basic quantum mechanics, guessing the wave function is not difficult in this case. For each permutation  $P$  of particles (i.e. with  $0 < x_{P1} < x_{P2} < \dots < x_{Pn}$ ) the wave function is written as a product of exponentials, i.e.

$$\Psi'_0 \sim \exp\left(-\sum_{\alpha} \kappa_{\alpha} x_{P\alpha}\right). \quad (3.3)$$

The choice of  $\kappa_{\alpha} = \lambda + 2(\alpha - 1)\kappa$  ensures the proper derivative discontinuities when two particles are exchanged. Also for the particle closest to the surface, the wave function has the required  $\exp(-\lambda x_{P1})$  fall-off. Again as this eigenfunction has no nodes, it represents the ground state of the hamiltonian  $\mathcal{H}'_n$ . The set of ‘‘momenta’’  $\{\kappa_{\alpha}\}$  is the same for all orderings, and the kinetic energy is  $\gamma \sum \kappa_{\alpha}^2$  as before. The



total energy of this  $n$ -particle bound state is then easily evaluated as

$$E'_n = ([\mu] - \frac{1}{2}\sigma^2 - 2\gamma)n - \gamma[(\lambda^2 - 2\lambda\kappa + \frac{2}{3}\kappa^2)n + 2\kappa(\lambda - \kappa)n^2 + \frac{4}{3}\kappa^2n^3]. \quad (3.4)$$

### 3.3. THE $n \rightarrow 0$ LIMIT

The quench-averaged free energy  $[f']$  of the pinned interface is obtained from the  $n \rightarrow 0$  limit of  $E'_n/n$  in eq. (3.4). More relevant is the difference between this free energy, and that of the unbound interface is eq. (2.10). This pinning free energy equals

$$([f'] - [f])/T = \lim_{n \rightarrow 0} (E'_n - E_n)/n = -\gamma(\lambda - \kappa)^2, \quad (3.5)$$

and decreases with introduction of bulk randomness. It vanishes for  $\lambda = \kappa = \sigma^2/4\gamma$ , indicating a depinning transition. This transition can thus be induced by increasing bond randomness, as well as by raising temperature.

The pinning free energy in (3.5) goes to zero quadratically as  $\lambda \rightarrow \kappa$ , indicating a discontinuous interface specific heat ( $\alpha = 0$ ) as in the case of thermal depinning [23]. The second and third cumulants of the free energy in the pinned state can also be read off from eq. (3.4) as

$$\begin{aligned} C_2(\ln Z') &= 4\gamma\kappa(\lambda - \kappa)T, \\ C_3(\ln Z') &= 8\gamma\kappa^2T. \end{aligned} \quad (3.6)$$

For the pinned interface, unlike the free one, the second cumulant is non-zero and typical fluctuations scale as  $T^{1/2}$ , with a coefficient however, that vanishes as  $\lambda \rightarrow \kappa$ . Thus (from eqs. (3.5) and (3.6)) the depinning transition is characterized by a quadratic singularity in the quench-averaged free energy, a linear one in its second cumulant, and a discontinuity of  $6\gamma\kappa^2T$  in the third cumulant. This hierarchy of singularities is reminiscent of the multitude of exponents observed in recent dynamical studies of random systems [27] and is probably symptomatic of a more common underlying characteristic.

Another quantity that can be calculated exactly is the quench-averaged localization length  $[l_p] = [\langle x \rangle]$  where  $\langle \dots \rangle$  denotes thermal expectation values. In the replica formalism this is evaluated from

$$[\langle x \rangle] = \lim_{n \rightarrow 0} \langle x_\alpha \rangle = \lim_{n \rightarrow 0} \left\langle \sum_\alpha x_\alpha / n \right\rangle. \quad (3.7)$$

(Note the equivalence of replicas.) The expectation value in the  $n$ -body problem is calculated from the wave function  $\Psi'_0$ , which determines the asymptotic distribution

probabilities of interfaces, as

$$\left\langle \sum_{\alpha} x_{\alpha} \right\rangle = \int_0^{\infty} \prod_{\alpha} dx_{\alpha} \left( \sum_{\alpha} x_{\alpha} \right) \Psi'_0(\{x_{\alpha}\}) / N(\lambda). \tag{3.8}$$

The normalization factor  $N(\lambda)$  is the sum of  $n!$  equal contributions from the different orderings. Therefore,

$$N(\lambda) = n! \int_0^{\infty} dx_1 \exp[-\lambda x_1] \int_{x_1}^{\infty} dx_2 \exp[-(\lambda + 2\kappa)x_2] \dots \int_{x_{n-1}}^{\infty} dx_n \exp[-(\lambda + 2(n-1)\kappa)x_n], \tag{3.9}$$

where the region of integration reflects the ordering  $0 < x_1 < x_2 < \dots < x_n$ . The exponential integrals are now performed sequentially, starting from the outer one, leading to

$$N(\lambda) = n! \prod_{\alpha=1}^n \{ \alpha\lambda + [2\alpha n - \alpha(\alpha + 1)]\kappa \}^{-1} = \exp\left\{ - \sum_{\alpha} \ln[\lambda + (2n - \alpha - 1)\kappa] \right\}. \tag{3.10}$$

Since for all particle orderings a factor of  $\lambda \sum_{\alpha} x_{\alpha}$  appears in the exponent for  $\Psi'_0$ , the expectation value in (3.8) can be calculated from

$$\begin{aligned} \left\langle \sum_{\alpha} x_{\alpha} \right\rangle &= - \frac{\partial}{\partial \lambda} \ln N(\lambda) = \sum_{\alpha} \frac{1}{\lambda + (2n - \alpha - 1)\kappa} \\ &= \sum_{\alpha} \int_0^{\infty} dy \exp\{-[\lambda + (2n - \alpha - 1)\kappa]y\} \\ &= \int_0^{\infty} dy \exp\{-[\lambda + 2(n-1)\kappa]y\} \frac{\exp(n\kappa y) - 1}{\exp(\kappa y) - 1}. \end{aligned} \tag{3.11}$$

The purpose of the above manipulations is to bring  $\langle \sum_{\alpha} x_{\alpha} \rangle$  into a form that can be analytically continued in  $n$ . Indeed this limit is taken straightforwardly from the final expression, and using eq. (3.7)

$$[l_p] = \int_0^{\infty} dy \kappa y \frac{\exp[-(\lambda - \kappa)y]}{[1 - \exp(-\kappa y)]}. \tag{3.12}$$

For  $\kappa \ll \lambda$ ,  $[l_p] = 1/\lambda$  as before [24], while close to the depinning transition it

diverges as  $[l_p] \sim \kappa/(\lambda - \kappa)^2$ . Therefore, the exponent for the divergence of the localization length changes from 1 in the pure system to 2 in the presence of randomness – yet another manifestation of the relevance of randomness in 2d interface systems.

### 3.4. INTERPRETATION AND DISCUSSION

The full power of the replica method is utilized in the example of the depinning transition. It provides the singularities not only of the quench averaged free energy, but also of its higher order cumulants. There are so far no systematic studies of the singular (universal?) behavior of cumulants at a phase transition. The quench-averaged localization length is also calculated and diverges with a new exponent of  $\nu_{\perp} = 2$ . Lipowsky and Fisher [25] have provided a justification for this exponent using the scaling of fluctuations for a single interface subject to randomness.

The above impurity induced transition can be tested numerically [12]. I have also used numerical simulations to study two similar problems that are not amenable to analytic treatment. One is the pinning to a strip of weak bonds in the bulk (instead of on the surface), corresponding to a delta function attractive potential in eq. (3.2). In the absence of randomness the ground state wave function is always localized, and there is no thermally induced depinning transition [24]. However, simulations [12] indicate that such a transition does occur on increasing bond randomness. This is because the driving force for the transition is now energy gain and not entropy. The second problem concerns attracting interfaces, where simulations again show an unpairing transition for strong randomness. This example has a physical realization in steps on crystalline surfaces [24], whose fluctuations and separations may be probed by scanning microscopy. Although at first it appears that Bethe ansatz solutions similar to (3.3) should also be possible in these two cases, I have not succeeded in finding a set of “momenta” that satisfies all boundary conditions.

## 4. The commensurate-to-incommensurate transition

### 4.1. MODEL AND REPLICAS

The commensurate-to-incommensurate transition (CIT) occurs in adsorbed layers when there is a slight mismatch between the periodicity of the substrate and the natural spacing of the adsorbate [28]. Near the transition the incommensurate phase (IC) is composed of large commensurate (C) domains separated by domain walls. At finite temperatures, the statistical mechanics of these domain walls regulates the critical behavior at the CIT. In many adsorbed systems the domain walls form honeycomb networks, as the lattice symmetry results in three easy directions for domain walls [29]. There are also uniaxial systems in which the interfaces are primarily aligned to a single lattice direction, with small transverse fluctuations [30].

The latter are amenable to transfer matrix treatment and form the subject of this section. The most important microscopic property of these domain walls is the high energy cost associated with their overlap, and henceforth they will be treated as *noncrossing*.

Approaching the transition from the C side, the IC phase first appears when the free energy  $f$  of a single interface vanishes. In the IC phase, the noncrossing condition leads to a long-range repulsion between interfaces that determines the average domain size  $l$ . In the absence of impurities, each interface executes a random walk in between collisions with its neighbors. If the domain size is  $l$ , the average distance between collisions in the  $t$  direction is proportional to  $l^2$ . Balancing the free energy gain of  $f$  for each interface, with the entropy loss of order  $1/l^2$  due to collisions [31], leads to a divergence of the domain size with an exponent of  $\frac{1}{2}$  as  $f \rightarrow 0$  at the CIT [32, 33]. This predicted “pure” exponent of  $\frac{1}{2}$  has indeed been observed experimentally for bromine intercalated in graphite [34]. There are several other experimental and theoretical realizations of systems of non-crossing interfaces. Steps on the surface of three-dimensional crystals do not cross, leading to a singular signature in equilibrium crystal shapes [35]. The characteristic exponent of  $\frac{1}{2}$  also describes the ferroelectric transition in the six-vertex model [36], and appears in certain dimer problems [37], and in models of 2d polymer crystalization [38].

Here we examine the effects of quenched impurities on the transition. There are two forms of randomness that can occur in adsorbed layers. Defects in the substrate act as local random fields coupling to the order parameter in the domains. This type of randomness has been studied by Villain [39], and leads to a destruction of ordered phases. There can also be impurity atoms that are free to move with the adsorbate film without exchanging positions with their neighbors. Such impurities are sources of local dilations and compressions that attract or repel the interface without coupling to the order parameter. They lead to reentrant melting at sufficiently low temperatures on a smooth substrate [40, 41]. On a smooth substrate there is a long-range strain field ( $u_{ij} \sim 1/r^2$ ) associated with an impurity. Close to a CIT on the IC side, however, the large regions of registered adsorbate lead to a strain field that falls off exponentially. Consequently, the assumption of short-range interactions between impurities and domain walls is justified. Similar approximations apply to the steps on a crystal surface. The defects in this case are either impurities, or dislocations terminating on the surface. Bond randomness in the six-vertex model of ferroelectrics leads to similar effects.

Subject to these assumptions the IC phase is modeled as a collection of many interfaces in the random bond model introduced in sect. 2. It is sometimes convenient to switch between a grand canonical description of domain walls with a “chemical potential”  $f$  per interface, and a canonical ensemble of  $N$  interfaces. Again the partition function  $Z_N$  is calculated from a transfer matrix  $\mathcal{T}_N$  for the  $N$  interfaces.  $\mathcal{T}_N$  must incorporate the important non-crossing restriction:  $x_1(t) < x_2(t) < \dots < x_N(t)$  for all  $t$ . As originally pointed out by Pokrovski and Talapov

[32], and extended by several other authors [33, 42–44], this condition is most easily enforced by regarding the configurations of domain walls as world lines of  $N$  one-dimensional fermions. The Pauli exclusion principle then automatically prevents crossings. In the continuum limit  $\mathcal{F}_N(t) = \exp[-\mathcal{H}_N(t)]$ , with

$$\mathcal{H}_N(t) = \int dx \{ [\mu(x, t) - 2\gamma] c^\dagger(x) c(x) - \gamma c^\dagger(x) \partial^2 c(x) / \partial x^2 \}, \quad (4.1)$$

where  $c(x)$  and  $c^\dagger(x)$  are fermion fields. The hamiltonian  $\mathcal{H}_N(t)$  is the natural extension of the one particle  $\mathcal{H}(t)$  in eq. (2.3) to a field theory. For a uniform  $\mu$  the free energy  $f_N = -\ln Z_N$  is easily obtained from the ground-state energy of this hamiltonian. Let  $L$  and  $T$  denote the extent of the system in the  $x$  and  $t$  directions respectively. Then for a density  $r = 1/l$  of domain walls

$$f(r)/LT = (\mu - 2\gamma)r + \frac{1}{3}\gamma\pi^2 r^3. \quad (4.2)$$

(This result is obtained by filling the energy band  $\varepsilon(k) = (\mu - 2\gamma) + \gamma k^2$  up to the Fermi level of  $k_F = \pi r$ .) The grand canonical description is obtained by minimizing  $f(r)$  with respect to  $r$ . For  $\gamma < \gamma_c = \frac{1}{2}\mu$ ,  $f(r)$  is minimized for  $r = 0$  describing the C phase with no interfaces. In the IC phase ( $\gamma > \gamma_c$ ),  $r$  vanishes as  $(\gamma - \gamma_c)^{1/2}$ , indicating a divergence of the domain size [32] with the previously mentioned exponent of  $\frac{1}{2}$ .

With quenched impurities, the hamiltonian (4.1) becomes  $t$ -dependent, and describes fermions subject to a chemical potential that is random in *both* space and time. (A potential that is random in space *only* is familiar in the context of localization. It corresponds to strips of correlated randomness in 2d, which if the distribution is gaussian destroys the CIT.) Replicating the problem results in  $n$  species of interfaces. While the  $N$  interfaces in each replica do not cross, the bond averaging procedure results in an attractive interaction on contact between any two interfaces belonging to different replicas. In the transfer matrix formalism the non-crossing condition is incorporated by regarding the interfaces in each replica as world lines of a set of fermions (one of  $n$  species or colors). The replicated transfer matrix is  $t$ -independent, and the continuum limit can be expressed as  $\mathcal{F}_{nN} = \exp(-\mathcal{H}_{nN})$  with

$$\mathcal{H}_{nN} = \int dx \left\{ \sum_{\alpha} \left[ (\mu) - \frac{1}{2}\sigma^2 - 2\gamma \right] c_{\alpha}^{\dagger} c_{\alpha} - \gamma c_{\alpha}^{\dagger} \partial^2 c_{\alpha} / \partial x^2 \right\} - \sigma^2 \sum_{\alpha < \beta} c_{\alpha}^{\dagger} c_{\alpha} c_{\beta}^{\dagger} c_{\beta}, \quad (4.3)$$

where the fields  $c_{\alpha}^{\dagger}$  and  $c_{\alpha}$  are anticommuting for each color. The hamiltonian  $\mathcal{H}_{nN}$  is now the generalization of (2.6) to  $n$  fermionic species.

## 4.2. THE BETHE ANSATZ

The problem of interacting particles in one dimension has a long history: It was solved for a system of bosons with repulsive interactions by Lieb and Liniger [45]. For two species of fermions the ground state was obtained for repulsive interactions by Yang [14], and for attractive interactions by Gaudin [46]. (The relationship between the two cases has been explored by Takahashi [47].) Finally, Sutherland [15] has laid down the framework for finding ground state wave functions of any symmetry. These solutions have found numerous applications in many branches of physics, but since it is not the purpose of this article to review the Bethe ansatz, the interested reader is referred to a review article by Thacker [10] for further details.

We would like to calculate the ground state energy of the hamiltonian (4.3) for  $n$  species of interacting fermions,  $N$  particles in each species. There are  $(nN)!$  permutations  $Q$ , of the  $(nN)$  particles along a line. Since they interact only on contact, for each arrangement of particles the wave function can be written as a product of plane waves, i.e.

$$\Psi(0 < x_{Q1} < x_{Q2} < \cdots < x_{QnN} < L) = \sum_P [Q, P] \exp[i(k_{P1}x_{Q1} + \cdots + k_{PnN}x_{QnN})] \quad (4.4)$$

where  $Q$  and  $P$  are permutations of  $(nN)$  particles and  $[Q, P]$  are a set of  $[(nN)!]^2$  coefficients. To conserve energy the same set of  $(nN)$  momenta  $\{k_\alpha\}$  has to be used for all orderings  $Q$ . For (4.4) to be an eigenfunction of the hamiltonian  $\mathcal{H}_{nN}$ , the following conditions have to be satisfied:

(i) As two neighboring particles  $x_\alpha$  and  $x_\beta$  are exchanged, the wave function must remain continuous and have the appropriate derivative discontinuity at  $x_\alpha = x_\beta$ . Given  $[Q, P]$ , this condition completely determines  $[Q', P]$  where the permutations  $Q$  and  $Q'$  are related by the interchange of  $x_\alpha$  and  $x_\beta$ . Therefore, only one set of coefficients, say  $[I, P] = \xi_P$  is left undetermined. (The action  $Q\xi_P \equiv [Q, P]$ , thus generates an  $[(nN)!]^2$  representation of the permutation group.)

(ii) The wave function must have the appropriate symmetry, i.e. it must be antisymmetric if two fermions of the same species are interchanged, and symmetric otherwise. This restricts the choice of  $\xi_P$  to one that leads to an irreducible representation with the proper symmetry.

(iii) Finally the requirements imposed by periodic boundary conditions determine the allowed values of the momenta  $\{k_\alpha\}$ .

Sutherland [15] has demonstrated how a wave function subject to these conditions can be constructed. The procedure, involving a nested series of Bethe ansätze, is quite ingenious but not particularly illuminating. Again the interested reader is referred to the original papers [14, 15, 10]. (Sutherland is mainly concerned with repulsive interactions, but the same set of equations applies to attractive interac-

tions.) The final result is a series of coupled integral equations for the density of allowed momenta. At this stage I resort to physical insight to guess the structure of these momenta.

As discussed in sect. 2 the momenta of the  $n$ -boson bound state (representing the replication of a single interface) are equally spaced along the imaginary  $k$  axis; i.e.  $k_\alpha = i\kappa(n + 1 - 2\alpha)$  for  $\alpha = 1, \dots, n$  form an “ $n$ -string” [10]. For  $N$  interfaces in the absence of impurities (the “free-fermion” picture) the momenta  $k_j$  ( $j = 1, \dots, N$ ) are equally spaced along the real axis from  $-k_F$  to  $k_F$ . It is therefore reasonable to assume that the  $(nN)$  momenta for  $\mathcal{H}_{nN}$  are arranged in  $n$  bands as in

$$k_{\alpha,j} = k_j + i(n + 1 - 2\alpha)\kappa \quad (\alpha = 1, \dots, n \text{ and } j = 1, \dots, N). \quad (4.5)$$

A crude representation of this ground state is a set of  $N$  repelling molecules, each molecule being a bound state of  $n$  particles of different species. Of course because of the complex nature of interactions between these molecules, the spacing of the  $N$  momenta  $k_j$  along the real axis is no longer uniform as in the case of free fermions.

A certain amount of algebra confirms that the ansatz (4.5) is consistent with the coupled equations of Sutherland [15]. (Sutherland in fact introduces  $n$  sets of auxiliary momenta: the first set is given by (4.5), while the subsequent sets have similar structure [11] with  $n$  replaced by  $n - 1, n - 2, \dots, 1$ .) There is only one set of equations left, determining the allowed  $k_j$  in a periodic box of size  $L$  from

$$e^{ink_j L} = \prod_{1 \neq j}^N \left( \frac{ik_1 - ik_j + 2\kappa}{ik_1 - ik_j - 2\kappa} \right) \left( \frac{ik_1 - ik_j + 4\kappa}{ik_1 - ik_j - 4\kappa} \right) \dots \left( \frac{ik_1 - ik_j + 2(n-1)\kappa}{ik_1 - ik_j - 2(n-1)\kappa} \right). \quad (4.6)$$

We are interested in the limit  $L \rightarrow \infty$  and  $N \rightarrow \infty$  with the wall density  $r = N/L$  finite. To take this limit, first the differences between logarithms of (4.6) are calculated for all adjacent momenta  $k_j$  and  $k_{j+1}$ , after including a factor of  $\exp(2\pi ij) = 1$  on the right-hand side. The quantity  $[L(k_{j+1} - k_j)]^{-1}$  approaches a continuous function  $\rho(k)$  in the  $L \rightarrow \infty$  limit, and the sum  $\sum_{1 \neq j}$  is replaced by  $\int dk \rho(k)$  in this limit. The density of momenta along the real axis then satisfies the integral equation

$$2\pi\rho(k) = n - \int_{-k_F}^{k_F} dk' \rho(k') F_n(k - k'), \quad (4.7)$$

with a kernel

$$F_n(k) = \sum_{\alpha=1}^{n-1} \frac{4\alpha\kappa}{k^2 + 4\alpha^2\kappa^2}. \quad (4.8)$$

The Fermi wave number  $k_F$  is related to the wall density  $r$  by

$$\int_{-k_F}^{k_F} dk \rho(k) = N/L = r. \tag{4.9}$$

Finally the kinetic energy is calculated from  $\gamma \sum k_{\alpha,j}^2$ , with the momenta given by eq. (4.5). The total energy for the hamiltonian  $\mathcal{H}_{nN}$  then equals

$$E_n(r)/L = n \left\{ ([\mu] - \frac{1}{2}\sigma^2 - 2\gamma - \frac{1}{3}(n^2 - 1)\gamma\kappa^2)r + \gamma \int_{-k_F}^{k_F} dk \rho(k) k^2 \right\}. \tag{4.10}$$

Equations (4.7)–(4.10) represent the complete solution for the ground state at all values of  $n$  and  $N$ . Similar results have appeared in a number of different contexts: by Schlottmann in a study of impurities in the Anderson model [48], and by Koltun and Toso in a model of nuclear matter [49]. The solution depends only on one dimensionless parameter  $\kappa/r = l/l_d$ . For strong attractions, or low densities, from (4.7)

$$\rho = \frac{1}{2\pi} \left( n - \frac{r}{\kappa} \sum_{\alpha=1}^{n-1} \frac{1}{\alpha} \right). \tag{4.11}$$

The lowest order result describes “molecules” of  $n$  tightly bound particles. This result is modified as the density of particles is increased, and breaks down for  $r \approx \kappa n/f_n$ , where

$$f_n = \sum_{\alpha=1}^{n-1} \frac{1}{\alpha} \xrightarrow{n \text{ large}} \ln(n). \tag{4.12}$$

The kinetic energy per unit length in this limit behaves as

$$\text{KE}/L = \pi^2 \gamma r^3 (1 + 2rf_n/\kappa n)/3n. \tag{4.13}$$

In the high density limit ( $r \gg \kappa$ ) the effects of confinement become pronounced, and  $\rho$  approaches the free particle value of  $1/2\pi$ . The asymptotic kinetic energy density is then  $\frac{1}{3}\pi^2 \gamma r^3$ , as in eq. (4.2).

### 4.3. THE $n \rightarrow 0$ LIMIT

The central result of the Bethe ansatz solution in the integral equation (4.7), which can be rewritten as

$$\int_{-k_F}^{k_F} dk' \rho(k') G_n(k - k') = 1, \tag{4.14}$$



with a kernel

$$\begin{aligned}
 G_n(k) &= \frac{1}{n} \left[ 2\pi\delta(k) + \sum_{\alpha=1}^{n-1} \frac{4\alpha\kappa}{k^2 + 4\alpha^2\kappa^2} \right] \\
 &= \frac{1}{n} \int_{-\infty}^{\infty} dy \exp(iky) \sum_{\alpha=0}^{n-1} \exp(-2\alpha\kappa|y|) \\
 &= \frac{1}{n} \int_{-\infty}^{\infty} dy \exp(iky) \frac{1 - e^{-2n\kappa|y|}}{1 - e^{-2\kappa|y|}}. \tag{4.15}
 \end{aligned}$$

These manipulations are reminiscent of those used in calculating  $[l_p]$  in sect. 3. The final expression can now be continued to  $n \rightarrow 0$ , resulting in

$$\begin{aligned}
 G_0(k) &= \lim_{n \rightarrow 0} G_n(k) = \int_{-\infty}^{\infty} dy \exp(iky) \frac{2\kappa|y|}{1 - e^{-2\kappa|y|}} \\
 &= \frac{\partial}{\partial k} \left[ \frac{2\kappa}{k} + \pi \coth\left(\frac{\pi k}{2\kappa}\right) \right]. \tag{4.16}
 \end{aligned}$$

The above kernel has a  $-4\kappa/k^2$  singularity as  $k \rightarrow 0$  which arises from Fourier transforming  $2\kappa|y|$ . Insertion of a convergence factor of  $\exp(-\beta|y|)$  shows how to get around this singularity in a manner similar to taking principal parts. The  $n \rightarrow 0$  momentum density now satisfies the integral equation

$$\text{P} \int_{-k_F}^{k_F} dk' \rho(k') \frac{\partial}{\partial k} \left[ \frac{2\kappa}{k - k'} + \pi \coth\left(\frac{\pi(k - k')}{2\kappa}\right) \right] = 1. \tag{4.17}$$

The origin of the principal part in eq. (4.17) can be traced back to the  $\Pi_{1 \neq j}$  in (4.6).

It is hard to analytically invert the above kernel in general, but the limiting behaviors of  $\rho(k)$  can be calculated. In the large density limit ( $\kappa/r = l/l_d \ll 1$ ),  $G_0(k) \approx 2\pi\delta(k)$ , and the noninteracting results  $\rho = 1/2\pi$  and the free energy (4.2) are recovered. In the low density limit, or close to the CIT with  $l \gg l_d$ , eq. (4.17) simplifies to

$$\text{P} \int_{-k_F}^{k_F} dk' \rho(k') \frac{\partial}{\partial k} \left( \frac{4\kappa}{k - k'} \right) = 1. \tag{4.18}$$

This is a variant of Carleman’s equation, and arises in the study of the lift of aerofoils. The kernel can be inverted by a Hilbert transform. However, it is easier to quote the result, and check that it satisfies (4.18). The momentum density function in this limit is

$$\rho(k) = (k_F^2 - k^2)^{1/2} / 4\pi\kappa. \tag{4.19}$$

A proof of this result is given in the appendix. The Fermi wave number  $k_F$  is related to the wall density through

$$r = \int_{-k_F}^{k_F} dk \rho(k) = \int_{-k_F}^{k_F} dk (k_F^2 - k^2)^{1/2} / 4\pi\kappa = k_F^2 / 8\pi\kappa \Rightarrow k_F = (8\pi\kappa r)^{1/2}. \quad (4.20)$$

Similarly the kinetic energy density is

$$\gamma \int_{-k_F}^{k_F} dk k^2 \rho(k) = \gamma \int_{-k_F}^{k_F} dk k^2 (k_F^2 - k^2)^{1/2} / 4\pi\kappa = \gamma k_F^4 / 32\pi\kappa = 2\pi\gamma\kappa r^2. \quad (4.21)$$

(In evaluating the integrals in (4.20) and (4.21), a change of variables to  $k = k_F \sin \theta$  is helpful.) Putting these results together, the quench-averaged free energy obtained from the  $n \rightarrow 0$  limit of  $E_n/n$ , is given by

$$[f(r)]/LT = ([\mu] - 2\gamma - \frac{1}{2}\sigma^2 - \sigma^4/48\gamma)r + \frac{1}{2}\pi\sigma^2 r^2. \quad (4.22)$$

This is to be compared with eq. (4.2) for the non-random case, where the kinetic energy scales as  $r^3$ . In the grand canonical description, minimizing  $[f(r)]$  with respect to  $r$  results in a wall spacing  $l = 1/r$  in the IC phase that diverges as  $(\gamma - \gamma_c)^{-1}$  close to the CIT (i.e. there is a crossover from the pure exponent of  $\frac{1}{2}$  to the random exponent of 1).

#### 4.4. INTERPRETATION AND DISCUSSION

In nonrandom systems the exponent of  $\frac{1}{2}$  for the divergence of domain size  $l$  at the CIT was related to the loss of entropy due to collisions between interfaces. A similar interpretation is possible for the exponent of 1 in random systems, and it turns out that the most important influence is the loss of energy due to the blocking of favorable paths by neighboring walls. Since typical transverse fluctuations of an interface subject to random bonds scale as  $|\Delta x| \sim t^{2/3}$ , the characteristic distance  $t_c$  between collisions scales as  $l^{3/2}$  with the domain size. Each collision is accompanied by a free energy loss scaling as  $t_c^{1/3}$ , and as there are  $T/t_c$  such collisions, the overall free energy loss for a single interface behaves as

$$\Delta f_1(l) \sim T t_c^{-2/3} \sim T l^{-1} \sim T r. \quad (4.23)$$

(This scaling law also holds for an interface confined to a strip of width  $l$ , and has been numerically verified [11].) After multiplying (4.23) by the number of interfaces ( $N = rL$ ), the  $r^2$  dependence of (4.22) is regained, thus providing an independent confirmation of this result.

As discussed in connection with the depinning transition, cumulants of the free energy are also singular at a critical point. To calculate these cumulants, the

complete  $n$  dependence of the free energy  $f_n(r)$  is needed. However, to study the singular parts of the cumulants, it is sufficient to examine the  $r$  dependence of various powers of  $n$  in  $f_n(r)$ . Indeed from eq. (4.15) we find that in the dilute limit the kernel behaves as

$$G_n(k) \sim g_1(n\kappa/k)\kappa/k^2. \quad (4.24)$$

(Here  $g_i(x)$  are used to denote regular functions of their argument.) From eq. (4.14) it then follows that the momentum density scales as

$$\rho_n(k) \sim g_2(n\kappa/k)k/\kappa. \quad (4.25)$$

The  $r$  dependence of the wave number is now calculated from (4.9) as

$$r \sim g_3(n\kappa/k)k_F^2/\kappa \Rightarrow k_F \sim (\kappa r)^{1/2}g_4(n\kappa^{1/2}r^{-1/2}). \quad (4.26)$$

The kinetic energy density scales as  $n\gamma k_F^2 r$ , and hence the non-trivial scaling of  $f_n$  takes the form

$$f_n(r)/LT \sim n\gamma\kappa r^2 g(n\kappa^{1/2}r^{-1/2}). \quad (4.27)$$

Therefore, the  $p$ th cumulant (the coefficient of  $n^p$ ) scales as  $\gamma\kappa^{(p+1)/2}r^{(5-p)/2}$ . Close to the transition  $r$  scales as  $(\gamma - \gamma_c)$ ; and the quench-averaged free energy behaves as  $(\gamma - \gamma_c)^2$ . The second and third cumulants of the free energy have  $(\gamma - \gamma_c)^{3/2}$  and  $(\gamma - \gamma_c)$  singularities respectively. Other cumulants may accidentally be absent as in the case of the depinning transition. This complex behavior of random systems close to a critical point clearly deserves further study.

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## Appendix

### SOLUTION OF THE INTEGRAL EQUATION

In sect. 4 it was stated that the momentum density function  $\rho(k) = (k_F^2 - k^2)^{1/2}/4\pi\kappa$  satisfies the integral equation (4.18). A proof of this result is presented here by substitution,

$$I(k) = \int_{-k_F}^{k_F} dk' \rho(k') G_0(k - k') = \text{P} \int_{-k_F}^{k_F} dk' \frac{\sqrt{k_F^2 - k'^2}}{4\pi\kappa} \frac{\partial}{\partial k} \left[ \frac{4\kappa}{k' - k} \right]. \quad (\text{A.1})$$

One of the singularities is simply removed in the integration by parts, and

$$\begin{aligned}
 I(k) &= \left[ \frac{\sqrt{k_F^2 - k'^2}}{\pi(k' - k)} \right]_{-k_F}^{k_F} + \frac{P}{\pi} \int_{-k_F}^{k_F} dk' \frac{1}{k' - k} \cdot \frac{k'}{\sqrt{k_F^2 - k'^2}} \\
 &= \frac{P}{\pi} \int_{-k_F}^{k_F} \frac{dk'}{\sqrt{k_F^2 - k'^2}} + k \frac{P}{\pi} \int_{-k_F}^{k_F} \frac{dk'}{(k' - k)\sqrt{k_F^2 - k'^2}}. \tag{A.2}
 \end{aligned}$$

The first integral in (A.2) is equal to 1, as can be seen easily by a change of variables to  $k' = k_F \sin \theta$ . To evaluate the second integral, change variables to  $k' = k_F(1 - t^2)/(1 + t^2)$ . Then

$$I'(k) = \frac{P}{\pi} \int_{-k_F}^{k_F} \frac{dk'}{(k' - k)\sqrt{k_F^2 - k'^2}} = \frac{2P}{\pi} \int_0^\infty \frac{dt}{(k_F - k) - (k_F + k)t^2}. \tag{A.3}$$

Since we are interested in  $-k_F \leq k \leq k_F$ , the integral has poles on the real axis at  $t = \pm((k_F - k)/(k_F + k))^{1/2}$ , and

$$I'(k) = \frac{P}{\pi} \int_0^\infty dt \left[ \frac{1}{\sqrt{k_F - k} - \sqrt{k_F + k}t} + \frac{1}{\sqrt{k_F - k} + \sqrt{k_F + k}t} \right] \frac{1}{\sqrt{k_F - k}}. \tag{A.4}$$

The principal part allows integrating over the singularity for  $t > 0$  resulting in

$$I'(k) = \frac{1}{\pi\sqrt{k_F^2 - k^2}} \left[ \ln \left| \frac{\sqrt{k_F - k} + \sqrt{k_F - k}t}{\sqrt{k_F - k} - \sqrt{k_F - k}t} \right| \right]_0^\infty = 0. \tag{A.5}$$

Note that for  $k$  outside the interval  $[-k_F, k_F]$ ,  $I'(k)$  is non-zero, and a further change of variables to  $(k_F + k)t^2 = (k - k_F)\tau^2$  leads to  $I'(k) = -1/\sqrt{k^2 - k_F^2}$  in this range. Putting these results together, we get  $I(k) = 1$  for  $-k_F \leq k \leq k_F$ ; and hence  $\rho(k)$  satisfies eq. (4.18) as required.

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