Material dependence of Casimir interaction between a sphere and a plate: First analytic correction beyond proximity force approximation

L. P. Teo*

Department of Applied Mathematics, Faculty of Engineering, University of Nottingham Malaysia Campus, Jalan Broga, 43500 Semenyih, Selangor Darul Ehsan, Malaysia

(Received 21 March 2013; published 23 August 2013)

We derive analytically the asymptotic behavior of the Casimir interaction between a sphere and a plate when the distance between them, d, is much smaller than the radius of the sphere, R. The leading-order and next-to-leading-order terms are derived from the exact formula for the Casimir interaction energy. They are found to depend nontrivially on the dielectric functions of the objects. As expected, the leading-order term coincides with that derived using the proximity force approximation. Numerical results are presented when the dielectric functions are given by the plasma model or the Drude model, with the plasma frequency (for plasma and Drude models) and relaxation frequency (for Drude model) given by the conventional values used for gold metal. It is found that if plasma model is used instead of the Drude model, the error in the sum of the first two leading terms is at most 2%, while the error in θ_1, the ratio of the next-to-leading-order term divided by d/R to the leading-order term, can go up to 4.5%.

DOI: 10.1103/PhysRevD.88.045019 PACS numbers: 12.20.Ds, 11.10.−z

I. INTRODUCTION

Casimir effect is a quantum effect that cannot be ignored in the realm of nanotechnology. It can cause malfunctions of nano devices due to stiction [1–3]. In the last decade, intensive research has been carried out to determine the exact analytic formula for the Casimir effect between two nonplanar objects and its effective numerical computations (see, for example, the references cited in [4]). Prior to this, one could rely only on the proximity force approximation (PFA) to compute an approximation for the Casimir interaction, and there is no way to determine the magnitude of the error in such an approximation.

In the case of the sphere-plate setup, the most popular configuration used in Casimir experiments, there is only one curvature parameter given by the radius of the sphere, R. Hence it is expected that as d, the distance from the sphere to the plate, is much smaller than R, the Casimir interaction energy has an asymptotic expansion of the form

$$E_{\text{Cas}} = E_{\text{Cas}}^{\text{PFA}}\left(1 + \frac{d}{R}\theta_{1,E} + \cdots\right), \tag{1}$$

where $E_{\text{Cas}}^{\text{PFA}}$ is the proximity force approximation to the Casimir interaction energy. It follows that for the Casimir force F_{Cas} and force gradient $\partial F_{\text{Cas}}/\partial d$, one also has expansions of the form

$$\begin{aligned} F_{\text{Cas}} &= F_{\text{Cas}}^{\text{PFA}}\left(1 + \frac{d}{R}\theta_{1,F} + \cdots\right),\\ \frac{\partial F_{\text{Cas}}}{\partial d} &= \frac{\partial F_{\text{Cas}}^{\text{PFA}}}{\partial d}\left(1 + \frac{d}{R}\theta_{1} + \cdots\right). \end{aligned} \tag{2}$$

A few years ago, experiments were set up to measure θ_1 using a micromachined torsional oscillator [5]. This gives a more ernest reason for the theoretical computation of the next-to-leading-order terms of the Casimir interaction. One of the breakthroughs in Casimir research brought by the achievement in explicit functional representation of the Casimir interaction is that it becomes possible to compute analytically the next-to-leading terms, as has been shown in [6–8] for the cylinder-plate configuration, in [9–11] for the sphere-plate configuration, in [12] for the cylinder-cylinder configuration, and in [13] for the sphere-sphere configuration. However, except for [7], all the other works only deal with ideal or nonphysical boundary conditions, i.e., Dirichlet, Neumann, perfectly conducting, infinitely permeable, or Robin boundary conditions. So far, no work has discussed the exact analytical computation of the next-to-leading- order term in the Casimir interaction between a sphere and a plate when both of these objects are made of real materials. The goal of the current work is to deal with this problem.

It should be mentioned that the material-dependent next-to-leading-order term in the Casimir interaction between a sphere and a plate has been studied by Bimonte *et al.* [14] using the method of derivative expansion postulated in [15]. Inspired by the work [16], the authors of [14] proposed that the Casimir energy E_{Cas} between a plane-parallel slab and a gently curved surface should admit a local expansion of the form

$$E_{\text{Cas}}[H] = E_{\text{Cas}}^{\text{PFA}}[H] + \int_{\Sigma_2} d\mathbf{x}\,\alpha(H)\nabla H\cdot\nabla H + \cdots, \tag{3}$$

where $z = H(\mathbf{x})$ is the local vertical distance from a point $\mathbf{x} = (x_1, x_2)$ on the planar surface Σ_2 to the curved surface. The function $\alpha(H)$ is determined by matching (3) with the

*LeePeng.Teo@nottingham.edu.my

perturbative expansion of the Casimir energy in the deformation profile $h(\mathbf{x}) = H(\mathbf{x}) - d$, where d is the smallest distance from the curved surface to the planar surface. However, no explicit analytic formula for the next-to-leading-order term was given. In this work, we compute the next-to-leading-order term from the exact formula for the Casimir interaction between a sphere and a plate made of real materials.

II. THE CASIMIR INTERACTION ENERGY

In this paper, we recall the formula for the Casimir interaction between a sphere and a plate. Assume that the sphere has relative permittivity $\varepsilon_{r,1}$, and the plate has relative permittivity $\varepsilon_{r,2}$. When the thicknesses of the sphere and the plate are larger than their respective skin depths, we can model this configuration by a ball and a semi-infinite space. Let d be the distance from the sphere to the plate, and let $L = d + R$, where R is the radius of the ball.

As shown in [4,17], the electromagnetic Casimir interaction energy of this sphere-plate configuration is given by

$$E_{\mathrm{Cas}} = \frac{\hbar}{2\pi}\int_0^\infty d\omega \mathrm{Tr}\ln(1 - \mathbb{M}(i\omega)), \tag{4}$$

where the trace Tr is

$$\mathrm{Tr} = \sum_{m=0}^\infty \sum_{l=\max\{1,|m|\}}^\infty \mathrm{tr},$$

with tr being the trace over 2×2 matrices. The matrix elements of $\mathbb{M}$ are given by

$$\begin{aligned}\mathbb{M}_{lm,l'm'} = {}& \delta_{m,m'}\frac{(-1)^m\pi}{2}\sqrt{\frac{(2l+1)(2l'+1)}{l(l+1)l'(l'+1)}\frac{(l-m)!(l'-m)!}{(l+m)!(l'+m)!}}\mathbb{T}^{lm}\\ &\times\int_0^\infty d\theta\sinh\theta e^{-2\kappa L\cosh\theta}\begin{pmatrix}\sinh\theta P_l^{m\prime}(\cosh\theta) & -\frac{m}{\sinh\theta}P_l^m(\cosh\theta)\\ -\frac{m}{\sinh\theta}P_l^m(\cosh\theta) & \sinh\theta P_l^{m\prime}(\cosh\theta)\end{pmatrix}\tilde{\mathbb{T}}^\theta\\ &\times\begin{pmatrix}\sinh\theta P_{l'}^{m'\prime}(\cosh\theta) & \frac{m'}{\sinh\theta}P_{l'}^{m'}(\cosh\theta)\\ \frac{m'}{\sinh\theta}P_{l'}^{m'}(\cosh\theta) & \sinh\theta P_{l'}^{m'\prime}(\cosh\theta)\end{pmatrix}.\end{aligned} \tag{5}$$

Here

$$\kappa = \frac{\omega}{c},$$

$P_l^m(x)$ are the associated Legendre functions given by

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l$$

when $m \geq 0$, and

$$P_l^{-m}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x); \tag{6}$$

$\mathbb{T}^{lm}$ and $\tilde{\mathbb{T}}^\theta$ are, respectively, the scattering matrices of the sphere and the plane. They are the following diagonal matrices:

$$\mathbb{T}^{lm} = \begin{pmatrix} T_{lm}^{\mathrm{TE}} & 0\\ 0 & T_{lm}^{\mathrm{TM}}\end{pmatrix}, \qquad \tilde{\mathbb{T}}^\theta = \begin{pmatrix}\tilde{T}_\theta^{\mathrm{TE}} & 0\\ 0 & \tilde{T}_\theta^{\mathrm{TM}}\end{pmatrix}.$$

Let

$$n_i = \sqrt{\varepsilon_{r,i}}, \qquad i = 1, 2.$$

The diagonal elements of $\mathbb{T}^{lm}$ and $\tilde{\mathbb{T}}^\theta$ are, respectively, given by

$$\begin{aligned}T_{lm}^{\mathrm{TE}}(i\omega) &= \frac{I_{l+\frac12}(\kappa R)(\frac12 I_{l+\frac12}(n_1\kappa R) + n_1\kappa R I'_{l+\frac12}(n_1\kappa R)) - I_{l+\frac12}(n_1\kappa R)(\frac12 I_{l+\frac12}(\kappa R) + \kappa R I'_{l+\frac12}(\kappa R))}{K_{l+\frac12}(\kappa R)(\frac12 I_{l+\frac12}(n_1\kappa R) + n_1\kappa R I'_{l+\frac12}(n_1\kappa R)) - I_{l+\frac12}(n_1\kappa R)(\frac12 K_{l+\frac12}(\kappa R) + \kappa R K'_{l+\frac12}(\kappa R))},\\ T_{lm}^{\mathrm{TM}}(i\omega) &= \frac{I_{l+\frac12}(\kappa R)(\frac12 I_{l+\frac12}(n_1\kappa R) + n_1\kappa R I'_{l+\frac12}(n_1\kappa R)) - \varepsilon_{r,1} I_{l+\frac12}(n_1\kappa R)(\frac12 I_{l+\frac12}(\kappa R) + \kappa R I'_{l+\frac12}(\kappa R))}{K_{l+\frac12}(\kappa R)(\frac12 I_{l+\frac12}(n_1\kappa R) + n_1\kappa R I'_{l+\frac12}(n_1\kappa R)) - \varepsilon_{r,1} I_{l+\frac12}(n_1\kappa R)(\frac12 K_{l+\frac12}(\kappa R) + \kappa R K'_{l+\frac12}(\kappa R))},\end{aligned} \tag{7}$$

and

$$\tilde{T}_\theta^{\rm TE} = \frac{\sqrt{n_2^2+\sinh^2\theta}-\cosh\theta}{\sqrt{n_2^2+\sinh^2\theta}+\cosh\theta}, \qquad \tilde{T}_\theta^{\rm TM} = \frac{\sqrt{n_2^2+\sinh^2\theta}-\varepsilon_{r,2}\cosh\theta}{\sqrt{n_2^2+\sinh^2\theta}+\varepsilon_{r,2}\cosh\theta}.$$

Direct numerical computations of the Casimir interaction energy from the formula (4) have been performed in a few works, such as [18,19]. In numerical computations, the infinite matrix $\mathbb{M}$ has to be truncated to a matrix of finite size. A drawback of this direct numerical computation is that when d/R gets smaller, one has to use a truncated matrix of larger size for accuracy, and this is subjected to the capacity of the computer. Currently, numerical computations for plasma and Drude models are limited to $d/R \sim 0.05$ [19]. However, in experiments, we usually have $d/R \sim 0.01$. So, even though in principle, with sufficient computing power, the numerical method can be extended to these values, the analytical computation of the Casimir interaction energy becomes desirable.

III. SMALL SEPARATION ASYMPTOTIC EXPANSION

In this section, we want to derive analytically the small separation asymptotic expansion of the Casimir interaction energy, Casimir force, and the force gradient up to the next-to-leading-order term.

One of the technical issues in the analytical computation of the Casimir interaction energy (4) is the appearance of the associated Legendre functions $P_l^m(x)$. First notice that because of the relation (6) and

$$\begin{pmatrix} l-2k & -\frac{m}{\sinh\theta} \\ -\frac{m}{\sinh\theta} & l-2k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} l-2k & \frac{m}{\sinh\theta} \\ \frac{m}{\sinh\theta} & l-2k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{8}$$

the matrix element $\mathbb{M}_{lm,lm}$ (5) is equal to that when m is changed to $-m$. Hence, it is sufficient to consider non-negative m. In this case, one can show that

$$\begin{aligned} P_l^m(\cosh\theta) &= (-1)^m i^m \frac{(l+m)!}{\pi l!}\int_0^\pi d\varphi(\cosh\theta+\sinh\theta\cos\varphi)^l\cos m\varphi \\ &= (-1)^m i^m\frac{(l+m)!}{\pi}\sum_{k=0}^l \frac{1}{k!(l-k)!}e^{(l-2k)\theta}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi\cos^{2l-2k}\varphi\sin^{2k}\varphi e^{2im\varphi}. \end{aligned}$$

Differentiating with respect to θ gives

$$\sinh\theta P_l^{m\prime}(\cosh\theta) = (-1)^m i^m\frac{(l+m)!}{\pi}\sum_{k=0}^l \frac{l-2k}{k!(l-k)!}e^{(l-2k)\theta}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi\cos^{2l-2k}\varphi\sin^{2k}\varphi e^{2im\varphi}.$$

Therefore,

$$\begin{aligned} &\begin{pmatrix} \sinh\theta P_l^{m\prime}(\cosh\theta) & \frac{m}{\sinh\theta}P_l^m(\cosh\theta) \\ \frac{m}{\sinh\theta}P_l^m(\cosh\theta) & \sinh\theta P_l^{m\prime}(\cosh\theta) \end{pmatrix} \\ &= (-1)^m i^m\frac{(l+m)!}{\pi}\sum_{k=0}^l\frac{1}{k!(l-k)!}\begin{pmatrix} l-2k & \frac{m}{\sinh\theta} \\ \frac{m}{\sinh\theta} & l-2k\end{pmatrix} e^{(l-2k)\theta}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi\cos^{2l-2k}\varphi\sin^{2k}\varphi e^{2im\varphi}. \end{aligned}$$

Making a change of variables,

$$\frac{R\omega}{c} = \xi,$$

and expanding the logarithm in (4), we have

$$E_{\rm Cas} = -\frac{\hbar c}{2\pi R}\sum_{s=0}^\infty \frac{1}{s+1}\int_0^\infty d\xi\sum_{m=0}^\infty\left(\prod_{i=0}^s\sum_{l_i=\max 1,|m|}^\infty\right)\mathrm{tr}\left(\prod_{i=0}^s \mathbb{M}_{l_i m, l_{i+1} m}\right),$$

where

$$\mathbb{M}_{l_im,l_{i+1}m} = \frac{1}{2\pi}\sqrt{(2l_i+1)(2l_{i+1}+1)(l_i-m)!(l_{i+1}-m)!(l_i+m)!(l_{i+1}+m)!}\begin{pmatrix} T_{l_i}^{\mathrm{TE}} & 0 \\ 0 & -T_{l_i}^{\mathrm{TM}}\end{pmatrix}$$
$$\times \sum_{k=0}^{l_i}\sum_{k'=0}^{l_{i+1}} \frac{1}{k!(l_i-k)!}\frac{1}{k'!(l_{i+1}-k')!}\int_0^\infty d\theta \sinh\theta e^{-2\xi(1+\varepsilon)\cosh\theta+(l_i+l_{i+1}-2k-2k')\theta}$$
$$\times \begin{pmatrix} \frac{l_i-2k}{\sqrt{l_i(l_i+1)}} & \frac{m}{\sinh\theta\sqrt{l_i(l_i+1)}} \\ \frac{m}{\sinh\theta\sqrt{l_i(l_i+1)}} & \frac{l_i-2k}{\sqrt{l_i(l_i+1)}}\end{pmatrix}\begin{pmatrix} \tilde{T}_\theta^{\mathrm{TE}} & 0 \\ 0 & -\tilde{T}_\theta^{\mathrm{TM}}\end{pmatrix}\begin{pmatrix} \frac{l_{i+1}-2k'}{\sqrt{l_{i+1}(l_{i+1}+1)}} & \frac{m}{\sinh\theta\sqrt{l_{i+1}(l_{i+1}+1)}} \\ \frac{m}{\sinh\theta\sqrt{l_{i+1}(l_{i+1}+1)}} & \frac{l_{i+1}-2k'}{\sqrt{l_{i+1}(l_{i+1}+1)}}\end{pmatrix}$$
$$\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \cos^{2l_i-2k}\varphi \sin^{2k}\varphi e^{2im\varphi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi' \cos^{2l_{i+1}-2k'}\varphi' \sin^{2k'}\varphi' e^{2im\varphi'}, \tag{9}$$

with

$$T_{l_i}^{*} = \frac{I_{l_i+\frac12}(\xi)(\frac12 I_{l_i+\frac12}(n_1\xi)+n_1\xi I'_{l_i+\frac12}(n_1\xi)) - \alpha_1^* I_{l_i+\frac12}(n_1\xi)(\frac12 I_{l_i+\frac12}(\xi)+\xi I'_{l_i+\frac12}(\xi))}{K_{l_i+\frac12}(\xi)(\frac12 I_{l_i+\frac12}(n_1\xi)+n_1\xi I'_{l_i+\frac12}(n_1\xi)) - \alpha_1^* I_{l_i+\frac12}(n_1\xi)(\frac12 K_{l_i+\frac12}(\xi)+\xi K'_{l_i+\frac12}(\xi))}.$$

Here $*=$ TE or TM, and $\alpha_1^{\mathrm{TE}}=1$, $\alpha_1^{\mathrm{TM}}=\varepsilon_{r,1}$. The minus signs on $T_{l_i}^{\mathrm{TM}}$ and $\tilde{T}_\theta^{\mathrm{TM}}$ in (9) come from the two matrices,

$$\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix},$$

in (8). Let

$$e = \frac{d}{R}.$$

In the following, we make a shift of parameters,

$$l_i \mapsto l + l_i, \qquad \theta \mapsto \theta_0 + \theta,$$

where

$$l := l_0, \qquad \sinh\theta_0 = \frac{l}{\xi}.$$

When $e \ll 1$, the main contributions to the Casimir interaction energy come from the terms with

$$l \sim \frac{1}{e}, \qquad l_i \sim \frac{1}{\sqrt{e}}, \qquad m \sim \frac{1}{\sqrt{e}}, \qquad \xi \sim \frac{1}{e}, \qquad \theta \sim e.$$

In the small e expansion below, we will count the order of l, l_i, m, ξ, and θ as $1/e$, $1/\sqrt{e}$, $1/\sqrt{e}$, $1/e$ and e respectively. Making a change of variables,

$$\xi = \frac{l\sqrt{1-\tau^2}}{\tau},$$

and replacing each of the summations over l, l_i, and m by a corresponding integration (which is the leading term in the Abel-Plana summation formula), we have

$$E_{\mathrm{Cas}} \approx -\frac{\hbar c}{2\pi R}\sum_{s=0}^\infty \frac{1}{s+1}\int_0^1 \frac{d\tau}{\tau^2\sqrt{1-\tau^2}}\int_0^\infty dl l\int_{-\infty}^\infty dm\left(\prod_{i=1}^s\int_{-\infty}^\infty dl_i\right)\mathrm{tr}\left(\prod_{i=0}^s \mathbb{M}_{(l+l_i)m,(l+l_{i+1})m}\right), \tag{10}$$

where $l_0=0$ and $l_{s+1}=0$ by default. The integration over θ is from $-\theta_0$ to ∞, which can be approximated by an integration from $-\infty$ to ∞, since θ is of order e and θ_0 is of order 1.

Now we perform the small e expansion of (9). Writing $\cos\varphi$ as $\exp(-\ln\sec\varphi)$ and using the fact that

$$\ln\sec\varphi = \frac{\varphi^2}{2}+\frac{\varphi^4}{12}+\cdots,$$

we have the following small e expansion:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \cos^{2(l+l_i)-2k}\varphi\sin^{2k}\varphi e^{2im\varphi} \approx \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi\varphi^{2k}\left(1-\frac{\varphi^2}{6}\right)^{2k}\exp\left(-(l+l_i-k)\varphi^2-\frac{l+l_i-k}{6}\varphi^4\right)e^{2im\varphi}$$
$$\approx \frac{1}{l^{k+1/2}}\int_{-\infty}^\infty d\varphi\varphi^{2k}\left(1-\frac{k\varphi^2}{3l}\right)\exp\left(-\frac{l+l_i-k}{l}\varphi^2-\frac{l+l_i-k}{6l^2}\varphi^4+\frac{2im\varphi}{\sqrt{l}}\right)$$
$$\approx \frac{1}{l^{k+1/2}}\int_{-\infty}^\infty d\varphi\varphi^{2k}(1+\mathcal{A}_{i,2})\exp(\mathcal{B}_{i,1}+\mathcal{B}_{i,2})\exp\left(-\varphi^2+\frac{2im\varphi}{\sqrt{l}}\right). \tag{11}$$

In the second line, we have performed a rescaling $\varphi \mapsto \varphi/\sqrt{l}$ so that the main contribution to the integration over φ comes from φ that are ~ 1. Here and in the following, for any $\mathcal{X}$, $\mathcal{X}_{i,1}$ and $\mathcal{X}_{i,2}$ are, respectively, terms of order $\sqrt{e}$ and e. When these terms do not depend on i, i would be omitted. Changing l_i to l_{i+1} and k to k' in (11), we obtain a similar expansion:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi' \cos^{2(l+l_{i+1})-2k'} \varphi' \sin^{2k'} \varphi' e^{2im\varphi'} \approx \frac{1}{l^{k'+1/2}} \int_{-\infty}^{\infty} d\varphi' \varphi'^{2k} (1 + \mathcal{C}_{i,2}) \exp(\mathcal{D}_{i,1} + \mathcal{D}_{i,2}) \exp\left(-\varphi'^2 + \frac{2im\varphi'}{\sqrt{l}}\right).$$

Next, we can use Stirling's formula,

$$\ln n! = \left(n + \frac{1}{2}\right) \ln n - n + \frac{1}{2} \ln 2\pi + \frac{1}{12n} + \cdots,$$

to obtain an expansion,

$$\frac{1}{l^{k+k'}} \frac{\sqrt{(l + l_i - m)!(l + l_{i+1} - m)!(l + l_i + m)!(l + l_{i+1} + m)!}}{(l + l_i - k)!(l + l_{i+1} - k')!} \approx \exp\left(\frac{m^2}{l} + \mathcal{H}_{i,1} + \mathcal{H}_{i,2}\right).$$

On the other hand, we have

$$\frac{1}{2l} \sqrt{(2l + 2l_i + 1)(2l + 2l_{i+1} + 1)} \approx (1 + \mathcal{G}_{i,1} + \mathcal{G}_{i,2}).$$

For the terms involving θ, expanding in small e gives

$$\begin{aligned}
&\sinh(\theta + \theta_0) \approx \sinh\theta_0 \left(1 + \theta \coth\theta_0 + \frac{\theta^2}{2}\right) \\
&\quad \approx \frac{\tau}{\sqrt{1 - \tau^2}} (1 + \mathcal{E}_{i,1} + \mathcal{E}_{i,2}); \\
&\exp(-2\xi(1 + \varepsilon)\cosh(\theta + \theta_0) + (2l + l_i + l_{i+1} - 2k - 2k')(\theta + \theta_0)) \\
&\quad \approx \exp((2l + l_i + l_{i+1} - 2k - 2k')\theta_0) \exp\left(-2\xi(1 + \varepsilon)\sinh\theta_0 \left(\coth\theta_0 + \theta + \frac{\theta^2}{2}\coth\theta_0 + \frac{\theta^3}{6} + \frac{\theta^4}{24}\coth\theta_0\right)\right. \\
&\qquad \left. + (2l + l_i + l_{i+1} - 2k - 2k')\theta\right) \\
&\quad \approx \left(\frac{1 - \tau}{1 + \tau}\right)^{k + k' - \frac{l_i + l_{i+1}}{2} - l} \exp\left(-\frac{2l}{\tau} - \frac{l\theta^2}{\tau} - \frac{2el}{\tau} + (l_i + l_{i+1})\theta + \mathcal{F}_{i,1} + \mathcal{F}_{i,2}\right); \\
&\qquad \times \begin{pmatrix} \frac{l + l_i - 2k}{\sqrt{(l + l_i)(l + l_i + 1)}} & \frac{m}{\sinh(\theta + \theta_0)\sqrt{(l + l_i)(l + l_i + 1)}} \\ \frac{m}{\sinh(\theta + \theta_0)\sqrt{(l + l_i)(l + l_i + 1)}} & \frac{l + l_i - 2k}{\sqrt{(l + l_i)(l + l_i + 1)}} \end{pmatrix} \\
&\quad \approx \begin{pmatrix} 1 + \mathcal{L}_{i,2} & \mathcal{M}_1 \\ \mathcal{M}_1 & 1 + \mathcal{L}_{i,2} \end{pmatrix}, \qquad \begin{pmatrix} \frac{l + l_{i+1} - 2k'}{\sqrt{(l + l_{i+1})(l + l_{i+1} + 1)}} & \frac{m}{\sinh(\theta + \theta_0)\sqrt{(l + l_{i+1})(l + l_{i+1} + 1)}} \\ \frac{m}{\sinh(\theta + \theta_0)\sqrt{(l + l_{i+1})(l + l_{i+1} + 1)}} & \frac{l + l_{i+1} - 2k'}{\sqrt{(l + l_{i+1})(l + l_{i+1} + 1)}} \end{pmatrix} \\
&\quad \approx \begin{pmatrix} 1 + \mathcal{N}_{i,2} & \mathcal{M}_1 \\ \mathcal{M}_1 & 1 + \mathcal{N}_{i,2} \end{pmatrix}.
\end{aligned}$$

Here,

$$\mathcal{M}_1 = \frac{m\sqrt{1 - \tau^2}}{l\tau}$$

is of order $\sqrt{e}$. We do not need the term that is of order e for the off-diagonal terms of these matrices as they won't contribute to the next-to-leading-order term of the Casimir interaction energy. Finally, the small e expansions of $\tilde{T}^*_{\theta+\theta_0}$ are the same as the small θ expansions:

$$\tilde{T}^*_{\theta+\theta_0}=\frac{\sqrt{n_2^2+\sinh^2(\theta+\theta_0)}-\alpha_2^*\cosh(\theta+\theta_0)}{\sqrt{n_2^2+\sinh^2(\theta+\theta_0)}+\alpha_2^*\cosh(\theta+\theta_0)}=(-1)^{\mathrm{sgn}^*}\tilde{T}_0^*(1+\theta\mathcal{K}_1^*+\theta^2\mathcal{K}_2^*),$$

where $\alpha_2^{\mathrm{TE}}=1$, $\alpha_2^{\mathrm{TM}}=\varepsilon_{r,2}$, $\mathrm{sgn}^{\mathrm{TE}}=0$, $\mathrm{sgn}^{\mathrm{TM}}=1$,

$$\tilde{T}_0^{\mathrm{TE}}=\frac{\sqrt{\varepsilon_{r,2}(1-\tau^2)+\tau^2}-1}{\sqrt{\varepsilon_{r,2}(1-\tau^2)+\tau^2}+1},\qquad \tilde{T}_0^{\mathrm{TM}}=\frac{\varepsilon_{r,2}-\sqrt{\varepsilon_{r,2}(1-\tau^2)+\tau^2}}{\varepsilon_{r,2}+\sqrt{\varepsilon_{r,2}(1-\tau^2)+\tau^2}},\qquad \mathcal{K}_1^{\mathrm{TE}}=-\frac{2\tau}{\sqrt{\varepsilon_{r,2}(1-\tau^2)+\tau^2}},$$

$$\mathcal{K}_2^{\mathrm{TE}}=-\frac{\varepsilon_{r,2}(1-\tau^2)}{(\varepsilon_{r,2}(1-\tau^2)+\tau^2)^{3/2}}+\frac{2\tau^2}{\varepsilon_{r,2}(1-\tau^2)+\tau^2},\qquad \mathcal{K}_1^{\mathrm{TM}}=\frac{2\varepsilon_{r,2}\tau(1-\tau^2)}{\sqrt{\varepsilon_{r,2}(1-\tau^2)+\tau^2}(\varepsilon_{r,2}+\tau^2)},$$

$$\mathcal{K}_2^{\mathrm{TM}}=\frac{\varepsilon_{r,2}^2(1-\tau^2)^2}{(\varepsilon_{r,2}(1-\tau^2)+\tau^2)^{3/2}(\varepsilon_{r,2}+\tau^2)}-\frac{\tau^2(-\varepsilon_{r,2}^2\tau^2+\varepsilon_{r,2}^2+\varepsilon_{r,2}+1)}{(\varepsilon_{r,2}(1-\tau^2)+\tau^2)(\varepsilon_{r,2}+\tau^2)}+\frac{\tau^2(\varepsilon_{r,2}\sqrt{\varepsilon_{r,2}(1-\tau^2)+\tau^2}+1)^2}{(\varepsilon_{r,2}(1-\tau^2)+\tau^2)(\sqrt{\varepsilon_{r,2}(1-\tau^2)+\tau^2}+\varepsilon_{r,2})^2}.$$

Notice that $\tilde{T}_0^*$, $\mathcal{K}_1^*$, $\mathcal{K}_2^*$ only depend on $\varepsilon_{r,2}$ and τ. They are independent of l_i and e.

Gathering the expansions obtained above, we can write

$$\begin{aligned}\mathbb{M}_{(l+l_i)m,(l+l_{i+1})m}\approx{}&\frac{1}{\pi}\begin{pmatrix}T_{l+l_i}^{\mathrm{TE}} & 0\\ 0 & -T_{l+l_i}^{\mathrm{TM}}\end{pmatrix}\sum_{k=0}^\infty\frac{1}{k!}\sum_{k'=0}^\infty\frac{1}{k'!}\frac{\tau}{\sqrt{1-\tau^2}}\left(\frac{1-\tau}{1+\tau}\right)^{k+k'-\frac{l_i+l_{i+1}}{2}-l}\\&\times\int_{-\infty}^\infty d\theta\exp\left(\frac{m^2}{l}-\frac{2l}{\tau}-\frac{l\theta^2}{\tau}-\frac{2el}{\tau}+(l_i+l_{i+1})\theta\right)\int_{-\infty}^\infty d\varphi\varphi^{2k}\exp\left(-\varphi^2+\frac{2im\varphi}{\sqrt{l}}\right)\\&\times\int_{-\infty}^\infty d\varphi'\varphi'^{2k}\exp\left(-\varphi'^2+\frac{2im\varphi'}{\sqrt{l}}\right)(1+\mathcal{O}_{i,1}+\mathcal{O}_{i,2})\begin{pmatrix}1+\mathcal{L}_{i,2} & \mathcal{M}_1\\ \mathcal{M}_1 & 1+\mathcal{L}_{i,2}\end{pmatrix}\\&\times\begin{pmatrix}\tilde{T}_0^{\mathrm{TE}}(1+\theta\mathcal{K}_1^{\mathrm{TE}}+\theta^2\mathcal{K}_2^{\mathrm{TE}}) & 0\\ 0 & \tilde{T}_0^{\mathrm{TM}}(1+\theta\mathcal{K}_1^{\mathrm{TM}}+\theta^2\mathcal{K}_2^{\mathrm{TE}})\end{pmatrix}\begin{pmatrix}1+\mathcal{N}_{i,2} & \mathcal{M}_1\\ \mathcal{M}_1 & 1+\mathcal{N}_{i,2}\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}&\exp(\mathcal{B}_{i,1}+\mathcal{B}_{i,2}+\mathcal{D}_{i,1}+\mathcal{D}_{i,2}+\mathcal{F}_{i,1}+\mathcal{F}_{i,2}+\mathcal{H}_{i,1}+\mathcal{H}_{i,2})(1+\mathcal{A}_{i,2})(1+\mathcal{C}_{i,2})(1+\mathcal{E}_{i,1}+\mathcal{E}_{i,2})(1+\mathcal{G}_{i,1}+\mathcal{G}_{i,2})\\&\quad\approx 1+\mathcal{O}_{i,1}+\mathcal{O}_{i,2}.\end{aligned}$$

Notice that $\mathcal{M}_1$, $\mathcal{K}_1^*$, $\mathcal{K}_2^*$ are independent of k, k', φ, φ' and θ. Performing the summation over k and k' using the formulas

$$\sum_{k=0}^\infty\frac{v^k}{k!}=e^{-v},\qquad\sum_{k=0}^\infty k\frac{v^k}{k!}=ve^{-v},\qquad\sum_{k=0}^\infty k^2\frac{v^k}{k!}=(v^2+v)e^{-v},$$

we obtain an expansion of the form

$$\begin{aligned}\mathbb{M}_{(l+l_i)m,(l+l_{i+1})m}\approx{}&\frac{1}{\pi}\begin{pmatrix}T_{l+l_i}^{\mathrm{TE}} & 0\\ 0 & -T_{l+l_i}^{\mathrm{TM}}\end{pmatrix}\frac{\tau}{\sqrt{1-\tau^2}}\left(\frac{1-\tau}{1+\tau}\right)^{-\frac{l_i+l_{i+1}}{2}-l}\int_{-\infty}^\infty d\theta\exp\left(\frac{m^2}{l}-\frac{2l}{\tau}-\frac{l\theta^2}{\tau}-\frac{2el}{\tau}+(l_i+l_{i+1})\theta\right)\\&\times\int_{-\infty}^\infty d\varphi\exp\left(-\frac{2\tau}{1+\tau}\varphi^2+\frac{2im\varphi}{\sqrt{l}}\right)\int_{-\infty}^\infty d\varphi'\exp\left(-\frac{2\tau}{1+\tau}\varphi'^2+\frac{2im\varphi'}{\sqrt{l}}\right)(1+\mathcal{P}_{i,1}+\mathcal{P}_{i,2})\\&\times\begin{pmatrix}\tilde{T}_0^{\mathrm{TE}}(1+\theta\mathcal{K}_1^{\mathrm{TE}}+\theta^2\mathcal{K}_2^{\mathrm{TE}}+\mathcal{R}_2)+\tilde{T}_0^{\mathrm{TM}}\mathcal{M}_1^2 & (\tilde{T}_0^{\mathrm{TE}}+\tilde{T}_0^{\mathrm{TM}})\mathcal{M}_1\\ (\tilde{T}_0^{\mathrm{TE}}+\tilde{T}_0^{\mathrm{TM}})\mathcal{M}_1 & \tilde{T}_0^{\mathrm{TM}}(1+\theta\mathcal{K}_1^{\mathrm{TM}}+\theta^2\mathcal{K}_2^{\mathrm{TM}}+\mathcal{R}_2)+\tilde{T}_0^{\mathrm{TE}}\mathcal{M}_1^2\end{pmatrix}.\end{aligned}$$

The $\mathcal{R}_2$ term comes from $\mathcal{L}_{i,2}$ and $\mathcal{N}_{i,2}$. The Gaussian integrations over φ and φ' can be performed straightforwardly and give

$$\mathbb{M}_{(l+l_i)m,(l+l_{i+1})m} \approx \frac{1}{2}\begin{pmatrix} T^{\rm TE}_{l+l_i} & 0 \\ 0 & -T^{\rm TM}_{l+l_i}\end{pmatrix}\frac{1+\tau}{\sqrt{1-\tau^2}}\left(\frac{1-\tau}{1+\tau}\right)^{-\frac{l_i+l_{i+1}}{2}-l} \times \int_{-\infty}^{\infty} d\theta \exp\left(-\frac{m^2}{l\tau}-\frac{2l}{\tau}-\frac{l\theta^2}{\tau}-\frac{2el}{\tau}+(l_i+l_{i+1})\theta\right)(1+\mathcal{Q}_{i,1}+\mathcal{Q}_{i,2}) \times \begin{pmatrix} \tilde{T}_0^{\rm TE}(1+\theta\mathcal{K}_1^{\rm TE}+\theta^2\mathcal{K}_2^{\rm TE}+\mathcal{U}_2)+\tilde{T}_0^{\rm TM}\mathcal{M}_1^2 & (\tilde{T}_0^{\rm TE}+\tilde{T}_0^{\rm TM})\mathcal{M}_1 \\ (\tilde{T}_0^{\rm TE}+\tilde{T}_0^{\rm TM})\mathcal{M}_1 & \tilde{T}_0^{\rm TM}(1+\theta\mathcal{K}_1^{\rm TM}+\theta^2\mathcal{K}_2^{\rm TM}+\mathcal{U}_2)+\tilde{T}_0^{\rm TE}\mathcal{M}_1^2\end{pmatrix}.$$

$\mathcal{U}_2$ comes from $\mathcal{R}_2$, and it is independent of θ. Before performing integration over θ, one is supposed to multiply $(1+\mathcal{Q}_{i,1}+\mathcal{Q}_{i,2})$ into the matrix after it. Up to the terms of order e, we can write

$$1+\mathcal{Q}_{i,1}+\mathcal{Q}_{i,2} \approx (1+\mathcal{Q}_{i,2})(1+\mathcal{Q}_{i,1})$$

and only multiply $(1+\mathcal{Q}_{i,1})$ into the matrix. On the other hand, up to the terms of order e, we can extract the term $\mathcal{U}_2$ of order e out from the matrix. These give

$$\mathbb{M}_{(l+l_i)m,(l+l_{i+1})m} \approx \frac{1}{2}\begin{pmatrix} T^{\rm TE}_{l+l_i} & 0 \\ 0 & -T^{\rm TM}_{l+l_i}\end{pmatrix}\left(\frac{1-\tau}{1+\tau}\right)^{-\frac{l_i+l_{i+1}}{2}-l-\frac{1}{2}}\int_{-\infty}^{\infty} d\theta \exp\left(-\frac{m^2}{l\tau}-\frac{2l}{\tau}-\frac{l\theta^2}{\tau}-\frac{2el}{\tau}+(l_i+l_{i+1})\theta\right)(1+\mathcal{Q}_{i,2}+\mathcal{U}_2) \times \begin{pmatrix} \tilde{T}_0^{\rm TE}(1+\mathcal{Q}_{i,1}+\theta\mathcal{K}_1^{\rm TE}+\theta\mathcal{Q}_{i,1}\mathcal{K}_1^{\rm TE}+\theta^2\mathcal{K}_2^{\rm TE})+\tilde{T}_0^{\rm TM}\mathcal{M}_1^2 & (\tilde{T}_0^{\rm TE}+\tilde{T}_0^{\rm TM})\mathcal{M}_1 \\ (\tilde{T}_0^{\rm TE}+\tilde{T}_0^{\rm TM})\mathcal{M}_1 & \tilde{T}_0^{\rm TM}(1+\mathcal{Q}_{i,1}+\theta\mathcal{K}_1^{\rm TM}+\theta\mathcal{Q}_{i,1}\mathcal{K}_1^{\rm TM}+\theta^2\mathcal{K}_2^{\rm TM})+\tilde{T}_0^{\rm TE}\mathcal{M}_1^2\end{pmatrix}.$$

Performing the Gaussian integration over θ, we have

$$\mathbb{M}_{(l+l_i)m,(l+l_{i+1})m} \approx \frac{\sqrt{\pi\tau}}{2\sqrt{l}}\begin{pmatrix} T^{\rm TE}_{l+l_i} & 0 \\ 0 & -T^{\rm TM}_{l+l_i}\end{pmatrix}\left(\frac{1-\tau}{1+\tau}\right)^{-\frac{l_i+l_{i+1}}{2}-l-\frac{1}{2}}\exp\left(-\frac{m^2}{l\tau}-\frac{2l}{\tau}-\frac{2el}{\tau}+\frac{\tau}{4l}(l_i+l_{i+1})^2\right)(1+\mathcal{S}_{i,2}+\mathcal{U}_2) \times \begin{pmatrix} \tilde{T}_0^{\rm TE}(1+\mathcal{S}_{i,1}+\mathcal{V}_{i,1}^{\rm TE}+\tilde{\mathcal{S}}_{i,2}\mathcal{K}_1^{\rm TE}+\mathcal{V}_{i,2}^{\rm TE})+\tilde{T}_0^{\rm TM}\mathcal{M}_1^2 & (\tilde{T}_0^{\rm TE}+\tilde{T}_0^{\rm TM})\mathcal{M}_1 \\ (\tilde{T}_0^{\rm TE}+\tilde{T}_0^{\rm TM})\mathcal{M}_1 & \tilde{T}_0^{\rm TM}(1+\mathcal{S}_{i,1}+\mathcal{V}_{i,1}^{\rm TM}+\tilde{\mathcal{S}}_{i,2}\mathcal{K}_1^{\rm TM}+\mathcal{V}_{i,2}^{\rm TM})+\tilde{T}_0^{\rm TE}\mathcal{M}_1^2\end{pmatrix}, \tag{12}$$

where

$$\mathcal{S}_{i,j} = \frac{\sqrt{l}}{\sqrt{\pi\tau}}\int_{-\infty}^{\infty} d\theta \exp\left(-\frac{l\theta^2}{\tau}+(l_i+l_{i+1})\theta\right)\mathcal{Q}_{i,j}, \qquad j=1,2,$$
$$\tilde{\mathcal{S}}_{i,2} = \frac{\sqrt{l}}{\sqrt{\pi\tau}}\int_{-\infty}^{\infty} d\theta \exp\left(-\frac{l\theta^2}{\tau}+(l_i+l_{i+1})\theta\right)\theta\mathcal{Q}_{i,1},$$
$$\mathcal{V}_{i,1}^{*} = \frac{\tau}{2l}(l_i+l_{i+1})\mathcal{K}_1^{*}, \qquad \mathcal{V}_{i,2}^{*} = \left(\frac{\tau}{2l}+\frac{\tau^2}{4l^2}(l_i+l_{i+1})^2\right)\mathcal{K}_2^{*}.$$

Next we consider the small e expansions of $T^{*}_{l+l_i}$. Debye asymptotic expansions of modified Bessel functions say that

$$I_\nu(\nu z) \approx \frac{1}{\sqrt{2\pi\nu}}\frac{e^{\nu\eta(z)}}{(1+z^2)^{1/4}}\left(1+\frac{u_1(\tau(z))}{\nu}\right), \qquad \frac{1}{2}I_\nu(\nu z)+\nu z I_\nu'(\nu z) \approx \frac{\sqrt{\nu}e^{\nu\eta(z)}(1+z^2)^{1/4}}{\sqrt{2\pi}}\left(1+\frac{m_1(\tau(z))}{\nu}\right),$$
$$K_\nu(\nu z) \approx \sqrt{\frac{\pi}{2\nu}}\frac{e^{-\nu\eta(z)}}{(1+z^2)^{1/4}}\left(1-\frac{u_1(\tau(z))}{\nu}\right), \qquad \frac{1}{2}K_\nu(\nu z)+\nu z K_\nu'(\nu z) \approx -\sqrt{\frac{\pi\nu}{2}}e^{-\nu\eta(z)}(1+z^2)^{1/4}\left(1-\frac{m_1(\tau(z))}{\nu}\right),$$

where

$$u_1(\tau) = \frac{\tau}{8} - \frac{5\tau^3}{24}, \qquad m_1(\tau) = \frac{\tau}{8} + \frac{7\tau^3}{24}, \qquad \tau(z) = \frac{1}{\sqrt{1+z^2}}, \qquad \eta(z) = \sqrt{1+z^2} + \ln\frac{z}{1+\sqrt{1+z^2}}.$$

Let

$$z = \frac{\xi}{l + l_i + \frac{1}{2}}, \qquad z_1 = n_1 z, \qquad \nu = l + l_i + \frac{1}{2}.$$

Then we find that up to terms of order e, we have

$$T^*_{l+l_i} \approx \frac{e^{2\nu\eta(z)}}{\pi} \frac{\sqrt{1+z_1^2}(1 + \frac{u_1(\tau(z))}{\nu} + \frac{m_1(\tau(z_1))}{\nu}) - \alpha_1^*\sqrt{1+z^2}(1 + \frac{u_1(\tau(z_1))}{\nu} + \frac{m_1(\tau(z))}{\nu})}{\sqrt{1+z_1^2}(1 - \frac{u_1(\tau(z))}{\nu} + \frac{m_1(\tau(z_1))}{\nu}) + \alpha_1^*\sqrt{1+z^2}(1 + \frac{u_1(\tau(z_1))}{\nu} - \frac{m_1(\tau(z))}{\nu})}.$$

In small e expansion,

$$e^{2\nu\eta(z)} \approx C^{l_i - l_{i+1}}\left(\frac{1-\tau}{1+\tau}\right)^{\frac{l_i+l_{i+1}}{2}+l+\frac{1}{2}} \exp\left(\frac{2l}{\tau} - \frac{\tau}{2l}(l_i^2 + l_{i+1}^2) + \mathcal{I}_{i,1} + \mathcal{I}_{i,2}\right).$$

Therefore, we have an expansion of the form

$$\begin{pmatrix} T^{\rm TE}_{l+l_i} & 0 \\ 0 & -T^{\rm TM}_{l+l_i} \end{pmatrix} \approx \frac{C^{l_i - l_{i+1}}}{l}\left(\frac{1-\tau}{1+\tau}\right)^{\frac{l_i+l_{i+1}}{2}+l+\frac{1}{2}} \exp\left(\frac{2l}{\tau} - \frac{\tau}{2l}(l_i^2 + l_{i+1}^2) + \mathcal{I}_{i,1} + \mathcal{I}_{i,2}\right) \times \begin{pmatrix} T_0^{\rm TE}(1 + \mathcal{J}^{\rm TE}_{i,1} + \mathcal{J}^{\rm TE}_{i,2}) & 0 \\ 0 & T_0^{\rm TM}(1 + \mathcal{J}^{\rm TM}_{i,1} + \mathcal{J}^{\rm TM}_{i,2}) \end{pmatrix}, \tag{13}$$

where

$$T_0^{\rm TE} = \frac{\sqrt{\varepsilon_{r,1}(1-\tau^2)+\tau^2} - 1}{\sqrt{\varepsilon_{r,1}(1-\tau^2)+\tau^2} + 1}, \qquad T_0^{\rm TM} = \frac{\varepsilon_{r,1} - \sqrt{\varepsilon_{r,1}(1-\tau^2)+\tau^2}}{\varepsilon_{r,1} + \sqrt{\varepsilon_{r,1}(1-\tau^2)+\tau^2}}, \qquad \mathcal{J}^*_{i,1} = \frac{\tau l_i}{2l}\mathcal{W}^*_1 \qquad \mathcal{J}^*_{i,2} = \frac{\tau^2 l_i^2}{4l^2}\mathcal{W}^*_2 + \frac{\tau}{l}\mathcal{Y}^*_2,$$

with

$$\mathcal{W}_1^{\rm TE} = -\frac{4\tau}{\sqrt{\varepsilon_{r,1}(1-\tau^2)+\tau^2}},$$

$$\mathcal{W}_2^{\rm TE} = \frac{8\tau^2 + 4\tau^4 + 4\varepsilon_{r,1} - 4\varepsilon_{r,1}\tau^4}{(\varepsilon_{r,1}(1-\tau^2)+\tau^2)^{3/2}} + \frac{4(1-\tau^2)^2\left(\varepsilon_{r,1} + \sqrt{\varepsilon_{r,1}(1-\tau^2)+\tau^2}\right)^2}{\tau^2(\varepsilon_{r,1}(1-\tau^2)+\tau^2)\left(\sqrt{\varepsilon_{r,1}(1-\tau^2)+\tau^2}+1\right)^2} - \frac{4(1-\tau^2)(\tau^2+\varepsilon_{r,1})}{\tau^2(\varepsilon_{r,1}(1-\tau^2)+\tau^2)},$$

$$\mathcal{Y}_2^{\rm TE} = -\frac{\tau}{(\varepsilon_{r,1}(1-\tau^2)+\tau^2)^{1/2}} - \frac{8\varepsilon_{r,1}\tau^2 - 3\varepsilon_{r,1} - 5\varepsilon_{r,1}\tau^4 + 9\tau^2 + 5\tau^4}{12(\varepsilon_{r,1}(1-\tau^2)+\tau^2)}, \qquad \mathcal{W}_1^{\rm TM} = \frac{4\varepsilon_{r,1}\tau(1-\tau^2)}{\sqrt{\varepsilon_{r,1}(1-\tau^2)+\tau^2}(\tau^2+\varepsilon_{r,1})},$$

$$\mathcal{W}_2^{\rm TM} = -\frac{\varepsilon_{r,1}(1-\tau^2)(8\tau^2 + 4\tau^4 + 4\varepsilon_{r,1} - 4\varepsilon_{r,1}\tau^4)}{(\varepsilon_{r,1}+\tau^2)(\varepsilon_{r,1}(1-\tau^2)+\tau^2)^{3/2}} + \frac{4(1-\tau^2)^2\varepsilon_{r,1}^2(1+\sqrt{\varepsilon_{r,1}(1-\tau^2)+\tau^2})^2}{\tau^2(\varepsilon_{r,1}(1-\tau^2)+\tau^2)(\sqrt{\varepsilon_{r,1}(1-\tau^2)+\tau^2}+\varepsilon_{r,1})^2} - \frac{4\varepsilon_{r,1}^2(1-\tau^2)^3}{\tau^2(\tau^2+\varepsilon_{r,1})(\varepsilon_{r,1}(1-\tau^2)+\tau^2)},$$

$$\mathcal{Y}_2^{\rm TM} = \frac{\varepsilon_{r,1}(1-\tau^2)\tau}{(\varepsilon_{r,1}+\tau^2)(\varepsilon_{r,1}(1-\tau^2)+\tau^2)^{1/2}} - \frac{7\varepsilon_{r,1}^2\tau^4 - 4\varepsilon_{r,1}^2\tau^2 - 3\varepsilon_{r,1}^2 - 5\varepsilon_{r,1}\tau^6 + 13\varepsilon_{r,1}\tau^4 - 18\varepsilon_{r,1}\tau^2 + 5\tau^6 - 3\tau^4}{12(\varepsilon_{r,1}+\tau^2)(\varepsilon_{r,1}(1-\tau^2)+\tau^2)}.$$

Notice that T_0^*, $\mathcal{W}_1^*$, $\mathcal{W}_2^*$, $\mathcal{Y}_2^*$ only depend on $\varepsilon_{r,1}$ and τ. They are independent of l_i and e.

Substituting (13) into (12), we have an expansion of the form,

$$\mathbb{M}_{(l+l_i)m,(l+l_{i+1})m} \approx \frac{\sqrt{\pi}}{2} C^{l_i-l_{i+1}} \sqrt{\frac{\tau}{l}} (1 + \mathcal{T}_{i,1} + \mathcal{T}_{i,2} + \mathcal{U}_2) \exp\left(-\frac{m^2}{l\tau} - \frac{2el}{\tau} - \frac{\tau}{4l}(l_i - l_{i+1})^2\right) \times \begin{pmatrix} T_0^{\rm TE}\tilde{T}_0^{\rm TE}\Lambda^{\rm TE} + T_0^{\rm TE}\tilde{T}_0^{\rm TM}\mathcal{M}_1^2 & T_0^{\rm TE}(\tilde{T}_0^{\rm TE} + \tilde{T}_0^{\rm TM})\mathcal{M}_1 \\ T_0^{\rm TM}(\tilde{T}_0^{\rm TE} + \tilde{T}_0^{\rm TM})\mathcal{M}_1 & T_0^{\rm TM}\tilde{T}_0^{\rm TM}\Lambda^{\rm TM} + T_0^{\rm TM}\tilde{T}_0^{\rm TE}\mathcal{M}_1^2 \end{pmatrix}, \tag{14}$$

where

$$\mathcal{T}_{i,1} = I_{i,1}, \qquad \mathcal{T}_{i,2} = I_{i,2} + \mathcal{S}_{i,2} + \frac{1}{2} I_{i,1}^2, \qquad \Lambda^* = 1 + \mathcal{J}_{i,1}^* + \mathcal{S}_{i,1} + \mathcal{V}_{i,1}^* + \mathcal{J}_{i,1}^*\mathcal{V}_{i,1}^* + \mathcal{J}_{i,1}^*\mathcal{S}_{i,1} + \mathcal{J}_{i,2}^* + \tilde{\mathcal{S}}_{i,2}\mathcal{K}_1^* + \mathcal{V}_{i,2}^*.$$

Substituting (14) into (10), and extracting terms up to order e, we have

$$\begin{aligned} E_{\rm Cas} \approx & -\frac{\hbar c}{2\pi^{(s+3)/2}R} \sum_{s=0}^{\infty} \frac{1}{s+1} \frac{1}{2^{s+1}} \int_0^1 d\tau \frac{\tau^{(s+1)/2}}{\tau^2\sqrt{1-\tau^2}} \int_0^{\infty} dl l^{-(s-1)/2} \\ & \times \int_{-\infty}^{\infty} dm \left(\prod_{i=1}^{s} \int_{-\infty}^{\infty} dl_i\right) \exp\left(-\frac{m^2(s+1)}{l\tau} - \frac{2el(s+1)}{\tau} - \frac{\tau}{4l}\sum_{i=0}^{s}(l_i - l_{i+1})^2\right) \\ & \times \Bigg\{ \sum_{*={\rm TE,TM}} [T_0^*\tilde{T}_0^*]^{s+1} \left(1 + \sum_{i=0}^{s}\sum_{j=0}^{s} \mathcal{Z}_{i,1}\mathcal{Z}_{j,1} + \sum_{i=0}^{s} \mathcal{Z}_{i,2} + (s+1)\mathcal{U}_2\right) + X\mathcal{M}_1^2 \\ & + \sum_{*={\rm TE,TM}} [T_0^*\tilde{T}_0^*]^{s+1} \Bigg(\sum_{i=0}^{s}\sum_{j=0}^{s} \mathcal{Z}_{i,1}\mathcal{J}_{j,1}^* + \sum_{i=0}^{s}\sum_{j\neq i} \mathcal{Z}_{i,1}\mathcal{V}_{j,1}^* + \sum_{i=0}^{s} \mathcal{T}_{i,1}\mathcal{V}_{i,1}^* + \sum_{i=0}^{s} \tilde{\mathcal{S}}_{i,2}\mathcal{K}_1^* \sum_{i=0}^{s}\sum_{j=i+1}^{s} \mathcal{J}_{i,1}^*\mathcal{J}_{j,1}^* \\ & + \sum_{i=0}^{s}\sum_{j=i+1}^{s} \mathcal{V}_{i,1}^*\mathcal{V}_{j,1}^* + \sum_{i=0}^{s} \mathcal{J}_{i,1}^* \sum_{j=0}^{s} \mathcal{V}_{j,1}^* + \sum_{i=0}^{s} \mathcal{J}_{i,2}^* + \sum_{j=0}^{s} \mathcal{V}_{i,2}^* \Bigg) \Bigg\}, \end{aligned}$$

where

$$\mathcal{Z}_{i,1} = \mathcal{T}_{i,1} + \mathcal{S}_{i,1}, \qquad \mathcal{Z}_{i,2} = \mathcal{T}_{i,2} + \mathcal{T}_{i,1}\mathcal{S}_{i,1},$$

and

$$X = (s+1)\left\{ (T_0^{\rm TE}\tilde{T}_0^{\rm TM} + T_0^{\rm TM}\tilde{T}_0^{\rm TE}) \frac{[T_0^{\rm TE}\tilde{T}_0^{\rm TE}]^{s+1} - [T_0^{\rm TM}\tilde{T}_0^{\rm TM}]^{s+1}}{T_0^{\rm TE}\tilde{T}_0^{\rm TE} - T_0^{\rm TM}\tilde{T}_0^{\rm TM}} + 2T_0^{\rm TE}\tilde{T}_0^{\rm TE}T_0^{\rm TM}\tilde{T}_0^{\rm TM} \frac{[T_0^{\rm TE}\tilde{T}_0^{\rm TE}]^{s} - [T_0^{\rm TM}\tilde{T}_0^{\rm TM}]^{s}}{T_0^{\rm TE}\tilde{T}_0^{\rm TE} - T_0^{\rm TM}\tilde{T}_0^{\rm TM}} \right\}.$$

We have omitted those terms of order $\sqrt{e}$ since they are odd in one of the l_i and thus would give zero after integration with respect to l_i. It follows that

$$E_{\rm Cas} \approx E_{\rm Cas}^0 + E_{\rm Cas}^1.$$

$E_{\rm Cas}^0$ is the leading-order term that comes from those terms of order e^0, and $E_{\rm Cas}^1$ is the next-to-leading-order term that comes from those terms of order e. $\varepsilon_{r,1}$ and $\varepsilon_{r,2}$ are functions of

$$\omega = \frac{c}{R} \frac{l\sqrt{1-\tau^2}}{\tau}.$$

They are independent of l_i and m. Performing the Gaussian integration over l_i, $1 \leq i \leq s$, and m, we find that

$$E_{\rm Cas}^0 = -\frac{\hbar c}{4\pi R} \sum_{s=0}^{\infty} \frac{1}{(s+1)^2} \int_0^1 \frac{d\tau}{\tau\sqrt{1-\tau^2}} \int_0^{\infty} dl l \exp\left(-\frac{2el(s+1)}{\tau}\right) \sum_{*={\rm TE,TM}} [T_0^*\tilde{T}_0^*]^{s+1}, \tag{15}$$

$$E_{\rm Cas}^1 = -\frac{\hbar c}{4\pi R} \sum_{s=0}^{\infty} \frac{1}{(s+1)^2} \int_0^1 \frac{d\tau}{\tau\sqrt{1-\tau^2}} \int_0^{\infty} dl l \exp\left(-\frac{2el(s+1)}{\tau}\right) \left\{ \sum_{*={\rm TE,TM}} [T_0^*\tilde{T}_0^*]^{s+1}(\mathcal{A} + \mathcal{C}^* + \mathcal{D}^*) + X\mathcal{B} \right\}. \tag{16}$$

The explicit formulas for $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}^*$, and $\mathcal{D}^*$ are given by

$$\begin{aligned}\mathcal{A} &= \frac{e^2 l\tau}{3}((s+1)^3+2(s+1))+\frac{e}{3}((\tau^2-2)(s+1)^2-3\tau(s+1)+2\tau^2-1),\\ &\quad+\frac{\tau^4+\tau^2-12}{12l\tau}(s+1)+\frac{(1+\tau)(1-\tau^2)}{2l\tau}-\frac{\tau(1-\tau^2)}{3l}\frac{1}{s+1},\\ \mathcal{B} &= \frac{1-\tau^2}{2l\tau(s+1)},\qquad \mathcal{C}^* = C_V\mathcal{K}_1^*+C_J\mathcal{W}_1^*,\\ \mathcal{D}^* &= D_{VV}\mathcal{K}_1^{*2}+D_{VJ}\mathcal{K}_1^*\mathcal{W}_1^*+D_{JJ}\mathcal{W}_1^{*2}+\left(\frac{s+1}{2}\frac{\tau}{l}+D_V\right)\mathcal{K}_2^*+D_J\mathcal{W}_2^*+(s+1)\frac{\tau}{l}\mathcal{Y}_2^*,\end{aligned}$$

with

$$\begin{aligned}C_V &= -\frac{e\tau}{3}((s+1)^3+2(s+1))+\frac{1-\tau^2}{6l}(s+1)^2+\frac{\tau}{2l}(s+1)+\frac{1-4\tau^2}{12l},\\ C_J &= -\frac{e\tau}{6}((s+1)^3-(s+1))+\frac{1}{12l}((s+1)^2-1),\qquad D_{VV}=\frac{\tau}{12l}((s+1)^3-2(s+1)^2+2(s+1)-1),\\ D_{JJ} &= \frac{\tau}{48l}((s+1)^3-2(s+1)^2-(s+1)+2),\ D_{VJ}=\frac{\tau}{12l}((s+1)^3-(s+1)),\\ D_V &= \frac{\tau}{6l}(2(s+1)^2-3(s+1)+1),\qquad D_J=\frac{\tau}{12l}((s+1)^2-1).\end{aligned}$$

Using the fact that $\tilde{T}_0^*$, T_0^*, $\mathcal{K}_1^*$, $\mathcal{K}_2^*$, $\mathcal{W}_1^*$, $\mathcal{W}_2^*$, $\mathcal{Y}_2^*$ are independent of e, it is straightforward to take derivative with respect to d. For the Casimir force

$$F_{\mathrm{Cas}} = -\frac{\partial E_{\mathrm{Cas}}}{\partial d},$$

we find that

$$F_{\mathrm{Cas}} \approx F_{\mathrm{Cas}}^0 + F_{\mathrm{Cas}}^1,$$

where F_{Cas}^0 and F_{Cas}^1 are, respectively, the leading-order and next-to-leading-order terms with

$$\begin{aligned}F_{\mathrm{Cas}}^0 &= -\frac{\hbar c}{2\pi R^2}\sum_{s=0}^{\infty}\frac{1}{s+1}\int_0^1\frac{d\tau}{\tau^2\sqrt{1-\tau^2}}\int_0^\infty dl l^2\exp\left(-\frac{2el(s+1)}{\tau}\right)\sum_{*=\mathrm{TE,TM}}[T_0^*\tilde{T}_0^*]^{s+1},\\ F_{\mathrm{Cas}}^1 &= -\frac{\hbar c}{2\pi R^2}\sum_{s=0}^{\infty}\frac{1}{s+1}\int_0^1\frac{d\tau}{\tau^2\sqrt{1-\tau^2}}\int_0^\infty dl l^2\exp\left(-\frac{2el(s+1)}{\tau}\right)\left\{\sum_{*=\mathrm{TE,TM}}[T_0^*\tilde{T}_0^*]^{s+1}(\tilde{\mathcal{A}}+\tilde{\mathcal{C}}^*+\tilde{\mathcal{D}}^*)+X\mathcal{B}\right\}.\end{aligned}$$

Here,

$$\begin{aligned}\tilde{\mathcal{A}} &= \frac{e^2 l\tau}{3}((s+1)^3+2(s+1))-\frac{e}{3}(2(s+1)^2+3\tau(s+1)+1),\\ &\quad+\frac{-\tau^4+5\tau^2-12}{12l\tau}(s+1)+\frac{1+\tau-\tau^2}{2l\tau}-\frac{\tau}{6l}\frac{1}{s+1},\\ \tilde{\mathcal{C}}^* &= \widetilde{C}_V\mathcal{K}_1^*+\widetilde{C}_J\mathcal{W}_1^*,\qquad \widetilde{C}_V=-\frac{e\tau}{3}((s+1)^3+2(s+1))+\frac{1}{6l}(s+1)^2+\frac{\tau}{2l}(s+1)+\frac{1}{12l},\\ \widetilde{C}_J &= -\frac{e\tau}{6}((s+1)^3-(s+1))+\frac{(1+\tau^2)}{12l}((s+1)^2-1).\end{aligned}$$

For the force gradient $\partial F_{\mathrm{Cas}}/\partial d$, the leading-order and next-to-leading-order terms are

$$\begin{aligned}\frac{\partial F_{\mathrm{Cas}}^0}{\partial d} &= \frac{\hbar c}{\pi R^3}\sum_{s=0}^{\infty}\int_0^1\frac{d\tau}{\tau^3\sqrt{1-\tau^2}}\int_0^\infty dl l^3\exp\left(-\frac{2el(s+1)}{\tau}\right)\sum_{*=\mathrm{TE,TM}}[T_0^*\tilde{T}_0^*]^{s+1},\\ \frac{\partial F_{\mathrm{Cas}}^1}{\partial d} &= \frac{\hbar c}{\pi R^3}\sum_{s=0}^{\infty}\int_0^1\frac{d\tau}{\tau^3\sqrt{1-\tau^2}}\int_0^\infty dl l^3\exp\left(-\frac{2el(s+1)}{\tau}\right)\left\{\sum_{*=\mathrm{TE,TM}}[T_0^*\tilde{T}_0^*]^{s+1}(\hat{\mathcal{A}}+\hat{\mathcal{C}}^*+\mathcal{D}^*)+X\mathcal{B}\right\},\end{aligned}$$

where

$$\hat{\mathcal{A}} = \frac{e^2 l\tau}{3}((s+1)^3 + 2(s+1)) - \frac{e}{3}((2+\tau^2)(s+1)^2 + 3\tau(s+1) + 1 + 2\tau^2),$$
$$+ \frac{-\tau^4 + 9\tau^2 - 12}{12 l\tau}(s+1) + \frac{1+\tau-\tau^2+\tau^3}{2l\tau},$$
$$\hat{\mathcal{C}}^* = \widehat{C_V}\mathcal{K}_1^* + \widehat{C_J}\mathcal{W}_1^*, \qquad \widehat{C_V} = -\frac{e\tau}{3}((s+1)^3 + 2(s+1)) + \frac{1+\tau^2}{6l}(s+1)^2 + \frac{\tau}{2l}(s+1) + \frac{1+4\tau^2}{12l},$$
$$\widehat{C_J} = -\frac{e\tau}{6}((s+1)^3 - (s+1)) + \frac{(1+2\tau^2)}{12l}((s+1)^2 - 1).$$

Let us compare the leading-order term to the proximity force approximation. The Casimir energy density between a pair of parallel dielectric plates with relative permittivities $\varepsilon_{r,1}$ and $\varepsilon_{r,2}$ is given by the Lifshitz's formula [20],

$$\mathcal{E}_{\mathrm{Cas}}^{\parallel}(d) = \frac{\hbar c}{4\pi^2}\int_0^\infty d\kappa \int_\kappa^\infty dqq \sum_{*=\mathrm{TE,TM}} \ln(1 - r_1^* r_2^* e^{-2qd}), \tag{17}$$

where

$$r_i^{\mathrm{TE}} = \frac{\sqrt{(\varepsilon_{r,i}-1)\kappa^2 + q^2} - q}{\sqrt{(\varepsilon_{r,i}-1)\kappa^2 + q^2} + q}, \qquad r_i^{\mathrm{TM}} = \frac{\varepsilon_{r,i} q - \sqrt{(\varepsilon_{r,i}-1)\kappa^2 + q^2}}{\varepsilon_{r,i} q + \sqrt{(\varepsilon_{r,i}-1)\kappa^2 + q^2}}.$$

The proximity force approximation to the Casimir interaction energy between a sphere and a plate with relative permittivities $\varepsilon_{r,1}$ and $\varepsilon_{r,2}$ is given by

$$E_{\mathrm{Cas}}^{\mathrm{PFA}} = 2\pi R \int_d^\infty du \mathcal{E}_{\mathrm{Cas}}^{\parallel}(u). \tag{18}$$

Expanding the logarithm in (17) and substituting into (18), we find that

$$E_{\mathrm{Cas}}^{\mathrm{PFA}} = -\frac{\hbar c R}{2\pi}\sum_{s=0}^\infty \frac{1}{s+1}\int_d^\infty du \int_0^\infty d\kappa \int_\kappa^\infty dqq e^{-2q(s+1)u} \sum_{*=\mathrm{TE,TM}} [r_1^* r_2^*]^{s+1}$$
$$= -\frac{\hbar c R}{4\pi}\sum_{s=0}^\infty \frac{1}{(s+1)^2}\int_0^\infty d\kappa \int_\kappa^\infty dq e^{-2q(s+1)d} \sum_{*=\mathrm{TE,TM}} [r_1^* r_2^*]^{s+1}.$$

Now, making a change of variables,

$$q = \frac{l}{R\tau}, \qquad \kappa = \frac{l\sqrt{1-\tau^2}}{R\tau},$$

we finally obtain

$$E_{\mathrm{Cas}}^{\mathrm{PFA}} = -\frac{\hbar c}{4\pi R}\sum_{s=0}^\infty \frac{1}{(s+1)^2}\int_0^1 \frac{d\tau}{\tau\sqrt{1-\tau^2}}\int_0^\infty dl l \exp\left(-\frac{2el(s+1)}{\tau}\right)\sum_{*=\mathrm{TE,TM}} [r_1^* r_2^*]^{s+1},$$

where

$$r_i^{\mathrm{TE}} = \frac{\sqrt{\varepsilon_{r,i}(1-\tau^2) + \tau^2} - 1}{\sqrt{\varepsilon_{r,i}(1-\tau^2) + \tau^2} + 1}, \qquad r_i^{\mathrm{TM}} = \frac{\varepsilon_{r,i} - \sqrt{\varepsilon_{r,i}(1-\tau^2) + \tau^2}}{\varepsilon_{r,i} + \sqrt{\varepsilon_{r,i}(1-\tau^2) + \tau^2}}.$$

Comparing to (15), we find that our result for the leading-order term agrees completely with the proximity force approximation.

As mentioned in the Introduction, a method for computing the material-dependent next-to-leading-order term for the sphere-plane configuration has been proposed in [14] based on the validity of the derivative expansion. While we are unable to compare our analytic result for the next-to-leading-order term with the result in [14] directly at the analytic level, the numerical values seem to agree for the case for which results are presented in [14].

IV. PLASMA MODEL

In this section, we consider the special case where the dielectric permittivities of the sphere and the plate are described by the plasma model,

$$\varepsilon_{r,i}(i\omega) = 1 + \frac{\omega_{p,i}^2}{\omega^2},$$

where $\omega_{p,i}$ is the plasma frequency of the material.

In terms of the variables

$$t = \frac{el}{\tau}$$

and τ, we have

$$\varepsilon_{r,i} = 1 + \frac{(\omega_{p,i}d/c)^2}{t^2(1-\tau^2)}. \tag{19}$$

First, consider the case

$$\frac{\omega_{p,i}d}{c} \gg 1.$$

The limit where $\omega_{p,i}d/c \to \infty$, $i = 1, 2$, is the perfect conductor limit. We can compute analytically the asymptotic expansion of the leading- and next-to-leading-order terms in the small parameters

$$a_i = \frac{c}{\omega_{p,i}d}.$$

Specifically, we have

$$E_{\rm Cas} \approx -\frac{\pi^3\hbar cR}{720d^2} \times \left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\beta_{i,j}a_1^ia_2^j + \frac{d}{R}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\lambda_{i,j}a_1^ia_2^j + \cdots\right), \tag{20}$$

TABLE I. The coefficients $\beta_{i,j}$.

β	Exact value	Numerical value
$\beta_{0,0}$	1	1
$\beta_{1,0}$	$-\frac{4}{3}$	−1.3333
$\beta_{2,0}$	$\frac{9}{5}$	1.8
$\beta_{1,1}$	$\frac{18}{5}$	3.6
$\beta_{3,0}$	$-\frac{16}{7}+\frac{32}{735}\pi^2$	−1.8560
$\beta_{2,1}$	$-\frac{48}{7}$	−6.8571
$\beta_{4,0}$	$\frac{25}{9}-\frac{326}{1323}\pi^2$	0.3458
$\beta_{3,1}$	$\frac{100}{9}-\frac{326}{1323}\pi^2$	8.6791
$\beta_{2,2}$	$\frac{50}{3}$	16.6667
$\beta_{5,0}$	$-\frac{36}{11}+\frac{1220}{1617}\pi^2-\frac{379}{32340}\pi^4$	3.0322
$\beta_{4,1}$	$-\frac{180}{11}+\frac{2440}{1617}\pi^2$	−1.4707
$\beta_{3,2}$	$-\frac{360}{11}+\frac{1220}{1617}\pi^2$	−25.2808

where $\beta_{0,0} = 1$ and

$$E_{\rm Cas}^{\rm PFA,PC} = -\frac{\pi^3\hbar cR}{720d^2}$$

is the leading-order approximation to the Casimir interaction energy between a perfectly conducting sphere and a perfectly conducting plate. The exact values of $\beta_{i,j}$ and $\lambda_{i,j}$ for $i+j \le 5$ are listed in Tables I and II. From (15), it is obvious that the leading term is symmetric when we interchange $\varepsilon_{r,1}$ with $\varepsilon_{r,2}$. It follows that

$$\beta_{i,j} = \beta_{j,i} \quad \text{for all } (i,j).$$

Hence, we only list the coefficients of $\beta_{i,j}$ when $i \ge j$ in Table I.

From (20), we have

$$F_{\rm Cas} \approx -\frac{\pi^3\hbar cR}{360d^3}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(i+j+2)}{2}\beta_{i,j}a_1^ia_2^j + \frac{d}{R}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(i+j+1)}{2}\lambda_{i,j}a_1^ia_2^j + \cdots\right), \tag{21}$$

$$\frac{\partial F_{\rm Cas}}{\partial d} \approx \frac{\pi^3\hbar cR}{120d^4}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(i+j+2)(i+j+3)}{6}\beta_{i,j}a_1^ia_2^j + \frac{d}{R}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(i+j+1)(i+j+2)}{6}\lambda_{i,j}a_1^ia_2^j + \cdots\right). \tag{22}$$

Here,

$$F_{\rm Cas}^{\rm PFA,PC} = -\frac{\pi^3\hbar cR}{360d^3}$$

and

$$\frac{\partial F_{\rm Cas}^{\rm PFA,PC}}{\partial d} = \frac{\pi^3\hbar cR}{120d^4}$$

TABLE II. The coefficients $\lambda_{i,j}$.

λ	Exact value	Numerical value
$\lambda_{0,0}$	$-\frac{20}{\pi^2}+\frac{1}{3}$	-1.6931
$\lambda_{1,0}$	$\frac{56}{3}\frac{1}{\pi^2}-\frac{32}{45}$	1.1802
$\lambda_{0,1}$	$\frac{56}{3}\frac{1}{\pi^2}-\frac{14}{45}$	1.5802
$\lambda_{2,0}$	$-\frac{398}{21}\frac{1}{\pi^2}+\frac{401}{315}$	-0.6473
$\lambda_{1,1}$	$-\frac{796}{21}\frac{1}{\pi^2}+\frac{454}{315}$	-2.3993
$\lambda_{0,2}$	$-\frac{398}{21}\frac{1}{\pi^2}+\frac{113}{315}$	-1.5615
$\lambda_{3,0}$	$\frac{410}{21}\frac{1}{\pi^2}-\frac{37}{18}+\frac{286}{6615}\pi^2$	0.3493
$\lambda_{2,1}$	$\frac{410}{7}\frac{1}{\pi^2}-\frac{26}{7}$	2.2202
$\lambda_{1,2}$	$\frac{410}{7}\frac{1}{\pi^2}-\frac{16}{7}$	3.6488
$\lambda_{0,3}$	$\frac{410}{21}\frac{1}{\pi^2}-\frac{79}{126}+\frac{1}{6615}\pi^2$	1.3527
$\lambda_{4,0}$	$-\frac{69824}{3465}\frac{1}{\pi^2}+\frac{35141}{10395}-\frac{28022}{99225}\pi^2$	-1.4484
$\lambda_{3,1}$	$-\frac{279296}{3465}\frac{1}{\pi^2}+\frac{84176}{10395}-\frac{2774}{14175}\pi^2$	-2.0007
$\lambda_{2,2}$	$-\frac{139648}{1155}\frac{1}{\pi^2}+\frac{742}{99}+\frac{32}{11025}\pi^2$	-4.7269
$\lambda_{1,3}$	$-\frac{279296}{3465}\frac{1}{\pi^2}+\frac{43856}{10395}-\frac{46558}{1091475}\pi^2$	-4.3690
$\lambda_{0,4}$	$-\frac{69824}{3465}\frac{1}{\pi^2}+\frac{14981}{10395}-\frac{11962}{1091475}\pi^2$	-0.7087
$\lambda_{5,0}$	$\frac{26732}{1287}\frac{1}{\pi^2}-\frac{150368}{27027}+\frac{4937399}{5675670}\pi^2-\frac{1142}{63063}\pi^4$	3.3627
$\lambda_{4,1}$	$\frac{133660}{1287}\frac{1}{\pi^2}-\frac{35026}{2079}+\frac{773884}{567567}\pi^2$	7.1324
$\lambda_{3,2}$	$\frac{267320}{1287}\frac{1}{\pi^2}-\frac{548024}{27027}+\frac{26212}{51597}\pi^2$	5.7822
$\lambda_{2,3}$	$\frac{267320}{1287}\frac{1}{\pi^2}-\frac{415724}{27027}+\frac{16826}{81081}\pi^2$	7.7116
$\lambda_{1,4}$	$\frac{133660}{1287}\frac{1}{\pi^2}-\frac{256888}{27027}+\frac{19984}{81081}\pi^2$	3.4503
$\lambda_{0,5}$	$\frac{26732}{1287}\frac{1}{\pi^2}-\frac{84218}{27027}+\frac{3329}{62370}\pi^2+\frac{8059}{2522520}\pi^4$	-0.1736

are, respectively, the leading-order approximations to the Casimir force and force gradient between a perfectly conducting sphere and a perfectly conducting plate. Setting $a_1 = a_2 = 0$ in (20)–(22), we obtain

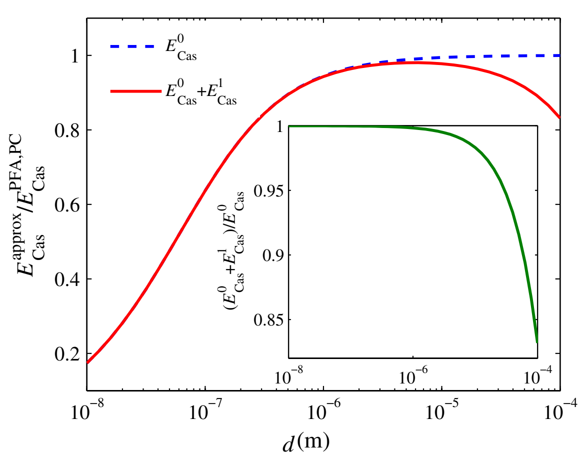

FIG. 1 (color online). The leading-order term of the Casimir interaction energy normalized by $E_{\rm Cas}^{\rm PFA,PC}$ (dashed line) and the sum of the leading- and next-to-leading-order terms normalized by $E_{\rm Cas}^{\rm PFA,PC}$ (solid line). Inset is the ratio of the latter to the former. These are computed using plasma model for gold metal.

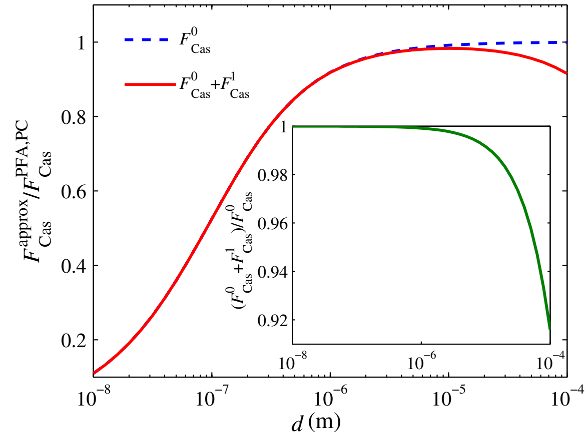

FIG. 2 (color online). The leading-order term of the Casimir force normalized by $F_{\rm Cas}^{\rm PFA,PC}$ (dashed line) and the sum of the leading- and next-to-leading-order terms normalized by $F_{\rm Cas}^{\rm PFA,PC}$ (solid line). Inset is the ratio of the latter to the former. These are computed using plasma model for gold metal.

$$E_{\rm Cas} \approx E_{\rm Cas}^{\rm PFA,PC}\left(1+\frac{d}{R}\left[\frac{1}{3}-\frac{20}{\pi^2}\right]+\cdots\right),$$
$$F_{\rm Cas} \approx F_{\rm Cas}^{\rm PFA,PC}\left(1+\frac{d}{R}\left[\frac{1}{6}-\frac{10}{\pi^2}\right]+\cdots\right),$$
$$\frac{\partial F_{\rm Cas}}{\partial d} \approx \frac{\partial F_{\rm Cas}^{\rm PFA,PC}}{\partial d}\left(1+\frac{d}{R}\left[\frac{1}{9}-\frac{20}{3\pi^2}\right]+\cdots\right),$$

which are well-known results for the leading- and next-to-leading-order terms of the perfectly conducting sphere-plate configuration [11,15].

Next we consider numerical results with $\omega_{p,1} = \omega_{p,2} = 1.3671 \times 10^{16}$ rad/s, which is the plasma frequency for

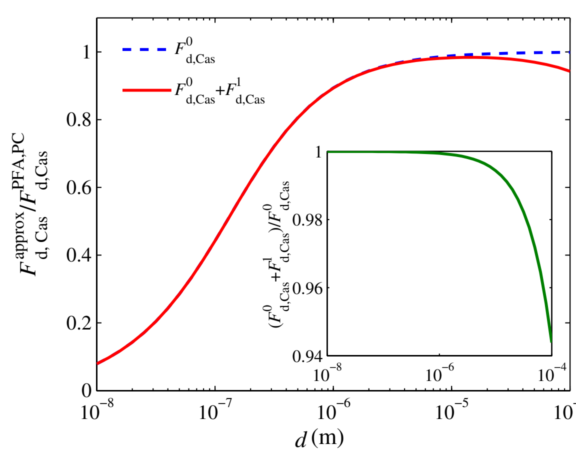

FIG. 3 (color online). The leading-order term of the force gradient normalized by $\partial F_{\rm Cas}^{\rm PFA,PC}/\partial d$ (dashed line) and the sum of the leading- and next-to-leading-order terms normalized by $\partial F_{\rm Cas}^{\rm PFA,PC}/\partial d$ (solid line). Inset is the ratio of the latter to the former. These are computed using plasma model for gold metal.

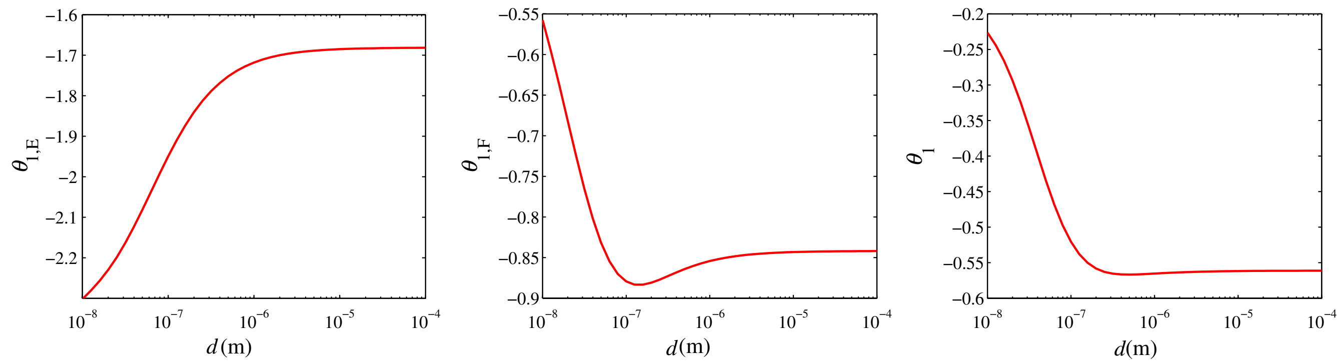

FIG. 4 (color online). $\theta_{1,E}(d)$, $\theta_{1,F}(d)$, and $\theta_1(d)$ computed using plasma model.

gold [21], in the unit system with $\hbar = c = 1$. The radius of the sphere R is taken to be 1 mm. Substituting (19) into the formulas obtained in the previous section, we can compute numerically the leading-order term (the proximity force approximation) and the next-to-leading-order term of the Casimir interaction. In Figs. 1–3, we plot the leading-order term, the sum of the leading-order and next-to-leading-order terms of the Casimir interaction energy, Casimir force, and force gradient, normalized, respectively, by $E_{\rm Cas}^{\rm PFA,PC}$, $F_{\rm Cas}^{\rm PFA,PC}$, and $\partial F_{\rm Cas}^{\rm PFA,PC}/\partial d$, as a function of d for d between 10 nm and 100 μm. From the figures, we notice that when $d/R \sim 0.1$, the corrections to PFA become significant, and they would contribute corrections of about 10%.

To have a better picture about the corrections to the proximity force approximations, define $\theta_{1,E}$, $\theta_{1,F}$, and θ_1 by

$$\theta_{1,E} = \frac{R}{d}\frac{E_{\rm Cas}^1}{E_{\rm Cas}^0}, \qquad \theta_{1,F} = \frac{R}{d}\frac{F_{\rm Cas}^1}{F_{\rm Cas}^0}, \qquad \theta_1 = \frac{R}{d}\frac{\partial F_{\rm Cas}^1/\partial d}{\partial F_{\rm Cas}^0/\partial d},$$

so that (1) and (2) hold.

In Fig. 4, we plot $\theta_{1,E}$, $\theta_{1,F}$, and θ_1 as functions of d for d between 10 nm and 100 μm. As d increases to 100 μm, we find that $\theta_{1,E}$, $\theta_{1,F}$, and θ_1 tend, respectively, to the values $1/3 - 20/\pi^2 = -1.6931$, $1/6 - 10/\pi^2 = -0.8465$, and $1/9 - 20/(3\pi^2) = -0.5644$, which are corresponding values for perfect conductors. This is not surprising since $\omega_{p,i}d/c$ is approximately equal to 5000 when $d = 10^{-4}$. When d is small, the deviations from the limiting values for perfect conductors are very significant. On the other hand, we also notice that $\theta_{1,F}$ and θ_1 are bounded below. θ_1 is a quantity that can be measured experimentally [5]. From Fig. 4, we find that it is bounded below by -0.57.

V. DRUDE MODEL

The Drude dielectric function is given by

$$\varepsilon_{r,i}(i\omega) = 1 + \frac{\omega_{p,i}^2}{\omega(\omega + \gamma_i)},$$

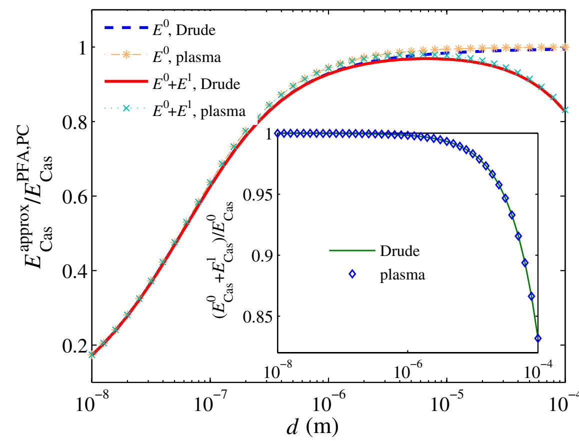

FIG. 5 (color online). The leading-order term and the sum of the leading-order and next-to-leading-order terms of the Casimir interaction energy normalized by $E_{\rm Cas}^{\rm PFA,PC}$. The inset shows the ratio of the latter to the former.

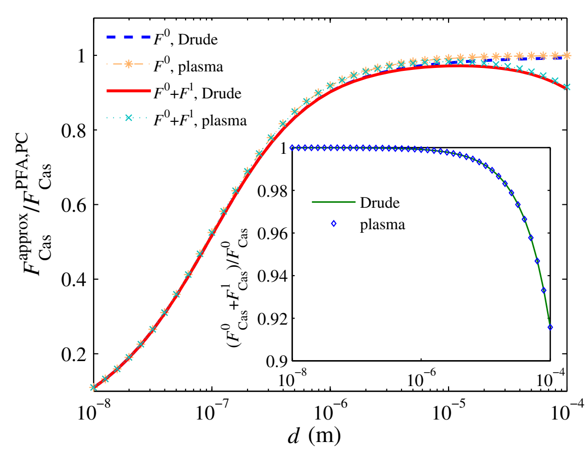

FIG. 6 (color online). The leading-order term and the sum of the leading-order and next-to-leading-order terms of the Casimir force normalized by $F_{\rm Cas}^{\rm PFA,PC}$. The inset shows the ratio of the latter to the former.

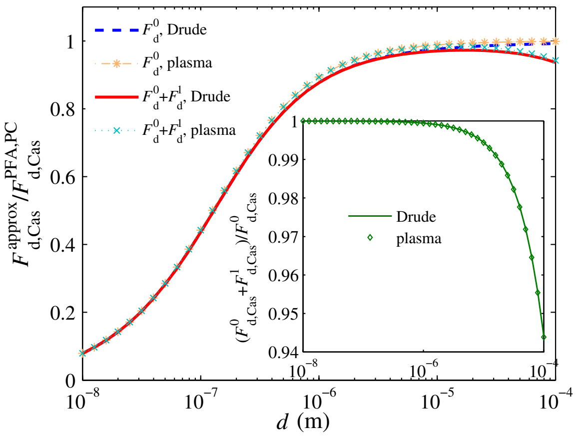

FIG. 7 (color online). The leading-order term and the sum of the leading-order and next-to-leading-order terms of the force gradient normalized by $\partial F_{\mathrm{Cas}}^{\mathrm{PFA,PC}}/\partial d$. The inset shows the ratio of the latter to the former.

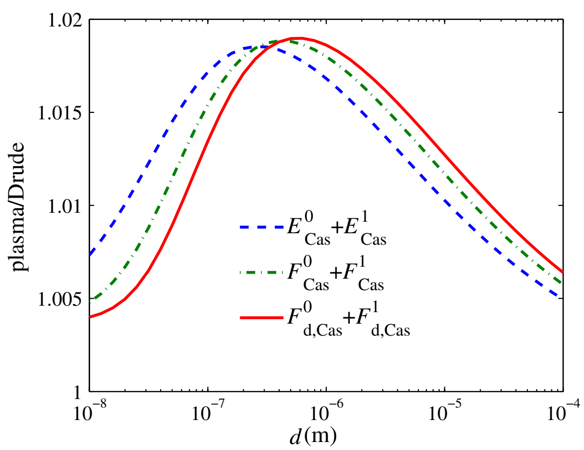

FIG. 8 (color online). The ratio of the plasma model to the Drude model for the sum of the leading-order and next-to-leading-order terms.

where γ_i is the relaxation frequency of the material. In the limit where $\gamma_i \to 0$, the Drude dielectric function becomes the plasma dielectric function.

In terms of the variables t and τ, we have

$$\varepsilon_{r,i} = 1 + \frac{(\omega_{p,i} d/c)^2}{t\sqrt{1-\tau^2}(t\sqrt{1-\tau^2} + \gamma_i d/c)}. \qquad (23)$$

Substituting this into the results obtained in Sec. III, we can compute numerically the leading-order and next-to-leading-order terms of the Casimir interaction for Drude models.

Let us consider the case where $\omega_{p,1} = \omega_{p,2} = 1.3671 \times 10^{16}$ rad/s and $\gamma_1 = \gamma_2 = 5.3165 \times 10^{13}$ rad/s, which are the conventional values used for gold [21].

In Figs. 5–7, we plot the leading-order term, the sum of the leading-order and next-to-leading-order terms of the Casimir interaction energy, Casimir force and force gradient, normalized, respectively, by $E_{\mathrm{Cas}}^{\mathrm{PFA,PC}}$, $F_{\mathrm{Cas}}^{\mathrm{PFA,PC}}$, and $\partial F_{\mathrm{Cas}}^{\mathrm{PFA,PC}}/\partial d$, as a function of d for d between 10 nm and 100 μm. Both the Drude model and the plasma model are plotted on the same graph to show the comparison. To get a better picture, we plot the ratio of the plasma model to the Drude model for the sum of the leading-order and next-to-leading-order terms in Fig. 8. From the figure, we notice that if the plasma model is used instead of the Drude model, the error is at most 2%.

In Fig. 9, we plot $\theta_{1,E}$, $\theta_{1,F}$, and θ_1 for the Drude model and compare to that for the plasma model. As for the plasma model, we notice that for the Drude model, as d increases, $\theta_{1,E}$, $\theta_{1,F}$, and θ_1 tend, respectively, to the limiting values $1/3 - 20/\pi^2 = -1.6931$, $1/6 - 10/\pi^2 = -0.8465$ and $1/9 - 20/(3\pi^2) = -0.5644$, the corresponding values for perfect conductors. When d is small, the deviations from these limiting values are very significant. On the other hand, $\theta_{1,F}$ and θ_1 are also bounded from below.

The ratios of the plasma model to the Drude model for $\theta_{1,E}$, $\theta_{1,F}$, and θ_1 are plotted in Fig. 10. From the figure, we

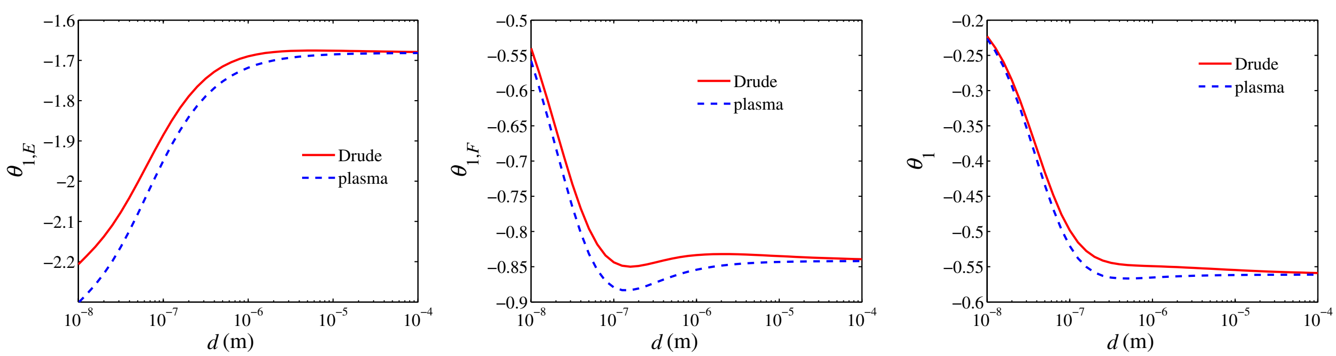

FIG. 9 (color online). $\theta_{1,E}(d)$, $\theta_{1,F}(d)$, and $\theta_1(d)$ computed using the Drude model (solid line), compared to those computed using the plasma model (dashed line).

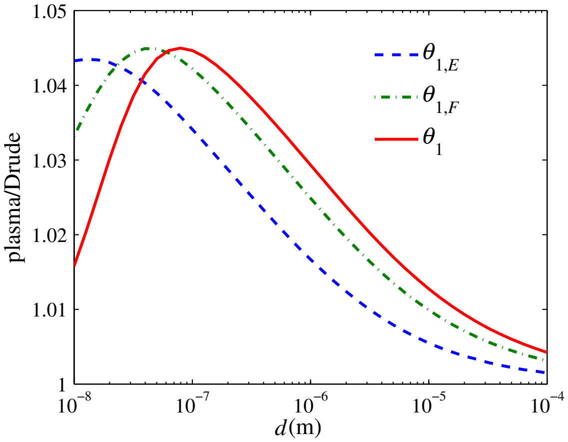

FIG. 10 (color online). The ratio of the plasma model to the Drude model for $\theta_{1,E}$, $\theta_{1,F}$, and θ_1.

find that if the plasma model is used instead of the Drude model, the error is at most 4.5%.

VI. CONCLUSION

Starting from the functional determinant representation of the Casimir interaction energy, we have used the perturbation method to obtain analytically the leading-order and next-to-leading-order terms of the Casimir interaction energy, Casimir force and force gradient for the interaction between a sphere and a plate. The results are written as double integrals over functions of the dielectric permittivities of the objects and are, hence, general. The leading-order terms are shown to be equal to that predicted by the proximity force approximation. The results on the next-to-leading-order terms are new.

With the given dielectric permittivities of the sphere and the plate, the double integrals representing the leading- order and next-to-leading-order terms can be computed numerically, and this is demonstrated for a gold sphere in front of a gold plate, where both plasma and Drude models are used for the dielectric functions of gold. It is observed that even at $d/R \sim 0.1$, the next-to-leading-order term would contribute a correction to the leading-order term of about 10%. Of particular interest is the ratio of the next-to-leading-order term divided by d/R to the leading-order term, denoted by θ_1. It is found that when $\omega_p d/c$ is large enough, $\theta_1(d)$ tends to the corresponding limiting value for perfect conductors. However, when $\omega_p d/c$ is small, the deviation from the limiting perfect conductor value is significant. This signifies that in the nano range, we cannot model real metals by perfect conductors.

A comparison between the plasma model and Drude model shows that their difference is below 2% for the sum of the first two leading-order terms and below 4.5% for the values of $\theta_1(d)$. In fact, this small difference is expected at zero temperature. In this work, we have not considered the thermal effect. When d is small enough such that $2\pi k_B T d/\hbar c$ is $\ll 1$, thermal effect can be neglected. For example, when $T = 300$ K, thermal effect can be neglected when $d \ll 1$ μm. Nevertheless, it would be interesting to examine the behavior of the Casimir interaction at room temperature and its interplay with material properties. This will be addressed in a forthcoming work.

ACKNOWLEDGMENTS

This work is supported by the Ministry of Higher Education of Malaysia under FRGS Grant No. FRGS/1/2013/ST02/UNIM/02/2. I would like to thank M. Bordag for helpful discussions. I would also like to thank the anonymous referee for the useful comments.

[1] F. M. Serry, D. Walliser, and G. J. Maclay, J. Microelectromech. Syst. **4**, 193 (1995).
[2] F. M. Serry, D. Walliser, and G. J. Maclay, J. Appl. Phys. **84**, 2501 (1998).
[3] M. Rose, Photonics Spectra **42**, 77 (2008).
[4] L. P. Teo, Int. J. Mod. Phys. A **27**, 1230021 (2012).
[5] D. E. Krause, R. S. Decca, D. Lopez, and E. Fischbach, Phys. Rev. Lett. **98**, 050403 (2007).
[6] M. Bordag, Phys. Rev. D **73**, 125018 (2006).
[7] M. Bordag, Phys. Rev. D **75**, 065003 (2007).
[8] L. P. Teo, Phys. Rev. D **84**, 025022 (2011).
[9] M. Bordag and V. Nikolaev, J. Phys. A **41**, 164002 (2008).
[10] M. Bordag and V. Nikolaev, Phys. Rev. D **81**, 065011 (2010).
[11] L. P. Teo, M. Bordag, and V. Nikolaev, Phys. Rev. D **84**, 125037 (2011).
[12] L. P. Teo, Phys. Rev. D **84**, 065027 (2011).
[13] L. P. Teo, Phys. Rev. D **85**, 045027 (2012).
[14] G. Bimonte, T. Emig, and M. Kadar, Appl. Phys. Lett. **100**, 074110 (2012).
[15] G. Bimonte, T. Emig, R. L. Jaffe, and M. Kadar, Europhys. Lett. **97**, 50001 (2012).
[16] C. D. Fosco, F. C. Lombardo, and F. D. Mazzitelli, Phys. Rev. D **84**, 105031 (2011).
[17] S. J. Rahi, T. Emig, N. Graham, R. L. Jaffe, and M. Kadar, Phys. Rev. D **80**, 085021 (2009).
[18] A. Canaguier-Durand, P. A. M. Neto, A. Lambrecht, and S. Reynaud, Phys. Rev. Lett. **104**, 040403 (2010).
[19] A. Canaguier-Durand, P. A. M. Neto, A. Lambrecht, and S. Reynaud, Phys. Rev. A **82**, 012511 (2010).
[20] E. M. Lifshitz, Zh. Eksp. Teor. Fiz. **29**, 94 (1956) [Sov. Phys. JETP **29**, 94 (1956)].
[21] M. Bordag, G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, *Advances in the Casimir Effect* (Oxford University, Oxford, 2009).