Efficiency Loss in Cournot Games

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Abstract

We consider Cournot models of competition, where market participants choose the quantities they demand or supply. We study the loss of aggregate surplus due to the exercise of market power in Cournot competition, for both oligopsony and oligopoly models. We observe that efficiency loss is generally arbitrarily high in Cournot games, but also prove bounds on efficiency loss in several cases of interest. In particular, we show that if multiple consumers with concave utility functions face an affine market supply curve, or if multiple producers with convex cost functions face an affine market demand curve, the aggregate surplus at a Nash equilibrium of the Cournot game is at least $2/3$ of the maximal aggregate surplus; i.e., the efficiency loss is again no more than 33%. We also show that if a monopsonist with concave utility faces a convex market supply curve, or a monopolist with convex cost faces a concave market demand curve, the efficiency loss is again no more than 33%.

1 Introduction

In Cournot competition [5], the strategy of each market participant is the quantity they demand or supply. Cournot games are among the best studied economic models for competition between market participants. Historically the focus on Cournot competition has been on Cournot *oligopoly*, i.e., the competition between multiple firms to satisfy an elastic demand—indeed, this was the original model studied by Cournot in 1838 [5]. (For surveys of this rich topic, see [7, 10, 23].) An analogous model may be formulated for competition between multiple consumers of a resource in elastic supply, known as Cournot *oligopsony*. Such models have been previously considered in the context of labor markets, where a small number of firms compete for an available supply of workers [16]. We consider both Cournot oligopsony and oligopoly models in this paper.

We are interested in computing the welfare loss, or *efficiency loss*, due to market power in Cournot competition. Formally, we consider a partial equilibrium model where the payoff to a consumer is utility less payment, while the payoff to a producer is revenue less cost. We take as our measure of efficiency the Marshallian aggregate surplus, equal to aggregate utility minus aggregate cost. In this paper, we characterize the extent to which aggregate surplus falls from competitive levels due to market power of consumers (in Cournot oligopsony) and producers (in

Cournot oligopoly). We will show that in general, the efficiency loss is arbitrarily high under Cournot competition. However, in certain important special cases, we will find that the efficiency loss is guaranteed to be no larger than 33%.

Efficiency properties of Cournot games have been extensively studied in various contexts. Much attention has been devoted to empirical analysis of the extent to which oligopoly (and more specifically, monopoly) cause welfare losses. Harberger published an early empirical analysis that suggested that welfare loss due to monopoly may be low [11]; Bergson published an influential critique of this work [3]. Other measurements of the efficiency loss due to market power have also been quite important in the literature, particularly market concentration indices such as the *Herfindahl index* [24]. Under appropriate conditions, the Herfindahl index can be analytically related to welfare [6]. However, we note here that these derivations typically require marginal costs of producers to be linear, whereas the results we derive in this paper allow producers to have general convex cost functions.

We expect that in the limit where many market participants compete, the efficiency loss due to market power is mitigated. Indeed, such a *competitive limit* has been discovered under a wide variety of conditions for Cournot oligopolies [12, 18, 19, 22]. However, we note that the Cournot oligopoly models in this paper do not allow for a fixed startup cost for each producer. This is considered an important issue as it can preclude free entry into the market; many of the previous results on competitive limits address this particular concern.

More recently, Anderson and Renault have also considered quantification of efficiency loss for Cournot oligopoly models [1]. The authors discuss the extent to which demand curvature causes efficiency loss; most of their results are phrased in terms of ratios to the aggregate profit of the producers, rather than to the aggregate surplus across the entire economy of consumers and producers. In addition, their results require marginal costs of producers to be constant; by contrast, the most general result of this paper allows arbitrary convex cost functions for producers, but requires affine demand. In general, while market demand may be estimated, cost functions of individual firms will typically be unknown. For this reason, we are interested in bounds on efficiency loss which hold independent of the cost characteristics of the firms.

This paper forms part of a recent line of research focused on quantifying efficiency loss for specific game environments. Results have been developed for network routing [15, 21, 4], network design problems [2, 8, 9], and market mechanisms for resource allocation [14, 13]. The goal of this paper is to establish a similar quantitative understanding of efficiency loss in Cournot games.

The remainder of the paper is organized as follows. In Section 2, we present the basic model of Cournot oligopsony, where multiple consumers of a resource in elastic supply choose the quantity they wish to consume. The price of the resource is then set equal to the marginal cost of the total requested allocation, i.e., so that demand equals supply. The standard interpretation of such a model is that the suppliers form a competitive market, so that the market clears at a price equal to marginal cost. We assume that utility functions are concave, and that the marginal cost of production is convex. We show that in general, the efficiency loss of such a scheme can be arbitrarily high when consumers are price anticipating. However, we consider several special cases and bound the efficiency loss in each of these cases. We show that if N consumers with the same utility function compete for a resource with a differentiable marginal cost function, then the efficiency loss is no more than $1/(2N+1)$ when the consumers are price anticipating. In addition, we establish that if the marginal cost function is not differentiable, the efficiency loss is no more than $1/3$ if exactly one consumer is competing for the resource. This last result bounds the efficiency loss of a model where a *monopsonist* faces a general, convex market supply curve. Finally, if consumers may have arbitrary concave utility functions, we show that the efficiency loss is no more than $1/3$ if the marginal cost function is affine (i.e., the cost function is quadratic). This model captures multiple oligopsonists facing an affine supply curve.

In Section 3, we turn our attention to understanding Cournot oligopoly. We first present the basic model of Cournot oligopoly, where multiple producers choose the quantity they wish to produce, and the price of the resource is then set equal to the marginal utility of the total requested allocation, i.e., so that supply equals demand. Again, the standard interpretation of such a model is that the consumers act as perfectly competitive price takers, so the market clears at the marginal utility of the aggregate. We assume that cost functions are convex, and that the marginal utility function describing the market is concave. As in Section 2, in general the efficiency loss of such a scheme can be arbitrarily high when producers are price anticipating. Nevertheless, we consider several special cases and bound the efficiency loss in each of these cases. Our proof technique proceeds by establishing a formal correspondence between Cournot oligopoly and Cournot oligopsony. We exploit this correspondence to state analogues of all the main results of Section 2 for the case of Cournot oligopoly. We show that if N producers with the same cost function compete for a resource with a differentiable demand curve, then the efficiency loss is no more than $1/(2N+1)$ when the producers are price anticipating; in addition, we establish that if the demand curve is not differentiable, the efficiency loss is no more than $1/3$ if exactly one producer is competing for the resource. This last result bounds the efficiency loss of a model where a *monopolist* faces a general, concave market demand curve. We also show that if producers may have arbitrary cost functions, the efficiency loss is no more than $1/3$ if the marginal utility function is affine. This model captures multiple oligopolists facing an affine market demand curve.

2 Cournot Oligopsony

In this section, we will consider a game where multiple consumers compete for a single resource, and where the strategies of the consumers are their desired quantities; such games are known as *Cournot oligopsonies*. We will find that in general Cournot oligopsonies can yield arbitrarily high efficiency loss, though we will also establish bounds on efficiency loss for several special cases of interest.

Formally, we consider the following model. We assume that N consumers compete for a single resource. We assume that each consumer n has a utility function U_n, and that the production of the resource incurs a cost characterized by a cost function C. We make the the following assumptions.

Assumption 1 *For each n, over the domain $x_n \geq 0$ the utility function $U_n(x_n)$ is concave, nondecreasing, and continuously differentiable (where we interpret $U_n'(0)$ as the right directional derivative of U_n at 0).*

Assumption 2 *There exists a continuous, convex, nondecreasing function $p(q)$ over $q \geq 0$ with $p(0) \geq 0$ and $p(q) \to \infty$ as $q \to \infty$, such that for $q \geq 0$:*

$$C(q) = \int_0^q p(z)dz.$$

In particular, $C(q)$ is convex and nondecreasing.

We note that the results of this section continue to hold even if the utility functions are not necessarily differentiable (as we require in Assumption 1). Differentiability of the utility functions only eases the presentation of the technical arguments, but is not essential to the results.

We assume that both utility and cost are measured in monetary units, so that an efficient allocation is characterized as an optimal solution of the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & \sum_n U_n(x_n) - C\left(\sum_n x_n\right) && (1) \\ \text{subject to} \quad & x_n \geq 0, \quad n = 1, \ldots, N. && (2) \end{aligned}$$

The objective function (1) is the *aggregate surplus* [17]. Since $p(q) \to \infty$ as $q \to \infty$, while U_n only grows at most linearly, it follows that an optimal solution exists. We now consider the following pricing scheme for resource allocation. Each consumer n chooses a desired quantity x_n. Given the vector $\boldsymbol{x} = (x_1, \ldots, x_n)$, a single price $\mu(\boldsymbol{x}) = p(\sum_n x_n)$ is chosen. We first consider the case where, given a price $\mu > 0$, consumer n chooses x_n to maximize:

$$P_n(x_n; \mu) = U_n(x_n) - \mu x_n. \tag{3}$$

Notice that in the previous expression, each consumer is acting as a price taker. Since we are using *marginal cost pricing*, i.e., since $\mu(\boldsymbol{x}) = p(\sum_n x_n)$, we expect that price taking consumers will maximize aggregate surplus at a competitive equilibrium. This is formalized in the following proposition, a special case of the first fundamental theorem of welfare economics [17].

Proposition 1 *Suppose Assumptions 1 and 2 hold. There exists a* competitive equilibrium*, that is, a vector $\boldsymbol{x}$ and a scalar $\mu \geq 0$ such that $\mu = p(\sum_n x_n)$, and:*

$$P_n(x_n; \mu) = \max_{\overline{x}_n \geq 0} P_n(\overline{x}_n; \mu), \quad n = 1, \ldots, N. \tag{4}$$

Any such vector $\boldsymbol{x}$ solves (1)-(2)*. If the functions U_n are strictly concave, such a vector $\boldsymbol{x}$ is unique as well.*

Proposition 1 shows that with marginal cost pricing, and if the consumers of the resource behave as price takers, there exists a vector of quantities $\boldsymbol{x}$ where all consumers have optimally chosen their x_n, with respect to the given price $\mu = p(\sum_m x_m)$; and at this "equilibrium," the aggregate surplus is maximized. However, when the price taking assumption is violated, the model changes into a game and the guarantee of Proposition 1 is no longer valid.

Consider, then, an alternative model where the consumers of a single resource are price anticipating, rather than price taking, and play a Cournot game to acquire a share of the resource. We use the notation $\boldsymbol{x}_{-n}$ to denote the vector of all quantities chosen by consumers other than n; i.e., $\boldsymbol{x}_{-n} = (x_1, x_2, \dots, x_{n-1}, x_{n+1}, \dots, x_N)$. Then given $\boldsymbol{x}_{-n}$, each consumer n chooses $x_n \geq 0$ to maximize:

$$Q_n(x_n; \boldsymbol{x}_{-n}) = U_n(x_n) - x_n p \left(\sum_m x_m \right). \tag{5}$$

The payoff function Q_n is similar to the payoff function P_n, except that the consumer now anticipates that the price will be set according to $p(\sum_m x_m)$. A *Nash equilibrium* of the game defined by $(Q_1, \dots, Q_N)$ is a vector $\boldsymbol{x} \geq 0$ such that for all n:

$$Q_n(x_n; \boldsymbol{x}_{-n}) \geq Q_n(\overline{x}_n; \boldsymbol{x}_{-n}), \quad \text{for all } \overline{x}_n \geq 0. \tag{6}$$

It is straightforward to show that a Nash equilibrium exists for this game, as we prove in the following result.

Proposition 2 *Suppose that Assumptions 1 and 2 hold. Then there exists a Nash equilibrium $\boldsymbol{x}$ for the game defined by $(Q_1, \dots, Q_N)$.*

Proof. We begin by observing that we may restrict the strategy space of each consumer n to a compact set, without loss of generality. Indeed, for sufficiently large B_n, we will have $U_n(B_n) < B_n p(B_n)$, so that for any vector $\boldsymbol{x}_{-n}$ of quantities chosen by other consumers, consumer n would always be better off choosing $x_n = 0$ rather than $x_n > B_n$. Thus, we may restrict the strategy space of consumer n to the compact interval $S_n = [0, B_n]$ without loss of generality.

Next, note that since p satisfies Assumption 2, $x_n p(\sum_m x_m)$ is convex in $x_n \geq 0$ for any value of $\boldsymbol{x}_{-n}$. This ensures Q_n is concave in $x_n \geq 0$ for all $\boldsymbol{x}_{-n}$.

The game defined by $(Q_1, \dots, Q_N)$ together with the strategy spaces $(S_1, \dots, S_N)$ is now a *concave N-person game*: each payoff function Q_n is continuous in the composite strategy vector $\boldsymbol{x}$, and concave in x_n; and the strategy space of each consumer n is a compact, convex, nonempty subset of $\mathbb{R}$. Applying Rosen's existence theorem [20] (proven using Kakutani's fixed point theorem), we conclude that a Nash equilibrium $\boldsymbol{x}$ exists for this game. □

Because the payoff Q_n is concave in x_n for fixed $\boldsymbol{x}_{-n}$, a vector $\boldsymbol{x}$ is a Nash equilibrium if and only if the following first order conditions are satisfied for each n, where $q = \sum_m x_m$:

$$U_n'(x_n) \leq p(q) + x_n \frac{\partial^+ p(q)}{\partial q}; \tag{7}$$

$$U_n'(x_n) \geq p(q) + x_n \frac{\partial^- p(q)}{\partial q}, \quad \text{if } x_n > 0, \tag{8}$$

where $\partial^+ p(q)/\partial q$ and $\partial^- p(q)/\partial q$ denote the right and left directional derivatives of p, respectively. We will use these conditions to investigate the efficiency loss when consumers are price anticipating. We first show in the following example that, in general, the efficiency loss may be arbitrarily high.

Example 1 Consider a price function p defined as follows:

$$p(q) = \begin{cases} a, & 0 \leq q \leq 1; \\ a + b(q-1), & q \geq 1. \end{cases}$$

Note that this yields:

$$C(q) = \begin{cases} aq, & 0 \leq q \leq 1; \\ \\ aq + \frac{1}{2}b(q-1)^2, & q \geq 1. \end{cases}$$

We assume that $0 < a < 1$, and $b > 1$. We consider a game with $N = 2$ consumers where $U_1(x_1) = x_1$, and:

$$U_2(x_2) = ax_2.$$

In this case, note that aggregate surplus is maximized when $p(q) = 1$, i.e., when $q = 1 + (1-a)/b$; and furthermore, the quantity q should be allocated entirely to consumer 1, since $a < 1$. Thus the maximal aggregate surplus is $U_1(q) - C(q)$, or:

$$1 + \frac{1-a}{b} - a - \frac{a(1-a)}{b} - \frac{(1-a)^2}{2b} = 1 - a + \frac{(1-a)^2}{2b}. \tag{9}$$

On the other hand, we claim that the vector $\boldsymbol{x}$ defined by:

$$x_1 = \frac{1-a}{b};$$
$$x_2 = 1 - \frac{1-a}{b},$$

is a Nash equilibrium. Observe that $q = x_1 + x_2 = 1$, so $p(q) = a$. Furthermore, $\partial^+ p(q)/\partial q = b$, $\partial^- p(q)/\partial q = 0$. It then follows that (7)-(8) hold for both consumers 1 and 2. Since $x_1, x_2 > 0$, these conditions are sufficient to ensure that $\boldsymbol{x}$ is a Nash equilibrium. Note that the aggregate surplus at this Nash equilibrium is:

$$U_1(x_1) + U_2(x_2) - C(q) = \frac{1-a}{b} + a\left(1 - \frac{1-a}{b}\right) - a = \frac{(1-a)^2}{b}.$$

Comparing this expression with (9), it is clear that in the limit where $b \to \infty$, the Nash equilibrium aggregate surplus approaches zero, and the maximal aggregate surplus approaches $1 - a$; thus the ratio of Nash equilibrium aggregate surplus to the maximal aggregate surplus approaches zero. $\square$

Despite this negative result, we now prove a sequence of results characterizing efficiency loss in more limited environments. We start with the following theorem, which shows that as long as the Nash equilibrium is unique and all consumers share the same utility function, the efficiency loss is no more than $1/(2N+1)$. We will then use this result to establish bounds on efficiency loss in several special cases.

Theorem 3 *Suppose that $N \geq 1$ consumers share the same utility function $U_n = U$, such that Assumption 1 holds and $U(0) \geq 0$. In addition, suppose that Assumption 2 holds. Suppose also that the game defined by $(Q_1, \dots, Q_N)$ possesses a unique Nash equilibrium $\boldsymbol{x}$. If $\boldsymbol{x}^S$ is any optimal solution to* (1)-(2)*, then:*

$$\sum_n U_n(x_n) - C\left(\sum_n x_n\right) \geq \left(\frac{2N}{2N+1}\right)\left(\sum_n U_n(x_n^S) - C\left(\sum_n x_n^S\right)\right). \tag{10}$$

Proof. We start with a sequence of three lemmas, which will also be useful in the subsequent development of this paper. The following lemma lets us assume without loss of generality that $\sum_n U_n(x_n^S) - C(\sum_n x_n^S) > 0$.

Lemma 4 *Suppose that Assumptions 1 and 2 hold. Suppose also that $U_n(0) \geq 0$ for all n. Fix any Nash equilibrium $\boldsymbol{x} = (x_1, \dots, x_N)$ of the game defined by $(Q_1, \dots, Q_N)$, and let $\boldsymbol{x}^S$ be any optimal solution to* (1)-(2)*. If $\sum_n U_n(x_n^S) - C(\sum_n x_n^S) = 0$, then $\sum_n U_n(x_n) - C(\sum_n x_n) = 0$, i.e., $\boldsymbol{x}$ is also an optimal solution to* (1)-(2)*.*

Proof of Lemma. Let $q = \sum_n x_n$, and let $q^S = \sum_n x_n^S$. Note that since $\boldsymbol{x}$ is a Nash equilibrium, for each n we must have $U_n(x_n) - x_n p(q) \geq 0$, since otherwise choosing $x_n = 0$ is a profitable deviation for consumer n. Using the optimality of $\boldsymbol{x}^S$ and the convexity of C, we have:

$$\begin{aligned}\sum_n U_n(x_n^S) - C(q^S) &\geq \sum_n U_n(x_n) - C(q)\\ &\geq \sum_n U_n(x_n) - qp(q) \geq 0.\end{aligned}$$

Thus if $\sum_n U_n(x_n^S) - C(q^S) = 0$, then we must have $\sum_n U_n(x_n) - C(q) = 0$ as well. □

Thus, if $\sum_n U_n(x_n^S) - C(\sum_n x_n^S) = 0$, the bound (10) trivially holds. We assume without loss of generality, therefore, that $\sum_n U_n(x_n^S) - C(\sum_n x_n^S) > 0$. Now note that we know $U_n'(x_n) \geq p(\sum_m x_m)$ for all n with $x_n > 0$, from (8). In the next lemma, we show that if $U_n'(x_n) = p(\sum_m x_m)$ for all n with $x_n > 0$, then (10) trivially holds.

Lemma 5 *Suppose that Assumptions 1 and 2 hold. Suppose also that $U_n(0) \geq 0$ for all n. Fix any Nash equilibrium $\boldsymbol{x} = (x_1, \dots, x_N)$ of the game defined by $(Q_1, \dots, Q_N)$, and let $\boldsymbol{x}^S$ be any optimal solution to* (1)-(2)*. If $U_n'(x_n) = p(\sum_m x_m)$ for all n with $x_n > 0$, then $\boldsymbol{x}$ is an optimal solution to* (1)-(2)*.*

On the other hand, if there exists at least one n such that $U_n'(x_n) > p(\sum_m x_m)$, then $\sum_n U_n'(x_n)x_n - C(\sum_m x_m) > 0$.

Proof of Lemma. Let $q = \sum_n x_n$, and let $q^S = \sum_n x_n^S$. First suppose that $\boldsymbol{x} = 0$. In this case we have $U_n'(x_n) \leq p(q)$ for all n (from (7)); since $\boldsymbol{x} = 0$, this is a necessary and sufficient optimality condition for (1)-(2). On the other hand, if $x_n > 0$ for at least one n, and $U_n'(x_n) = p(q)$ for all n with $x_n > 0$, then $\boldsymbol{x}$ is again an optimal solution to (1)-(2) (since $U_n'(x_n) \leq p(q)$ for all n with $x_n = 0$, from (7)).

On the other hand, suppose that $U_n'(x_n) > p(q)$ for at least one consumer n. Then $x_n > 0$ (from (7)). For all other $m \neq n$ with $x_m > 0$, we know $U_m'(x_m) \geq p(q)$ (from (8)). Thus we have $\sum_n U_n'(x_n)x_n > qp(q) \geq C(q)$, where the last inequality follows by convexity; so we conclude $\sum_n U_n'(x_n)x_n - C(q) > 0$. $\square$

In the following lemma, we use linearizations of the utility functions U_n to construct a lower bound on efficiency loss.

Lemma 6 *Suppose that Assumptions 1 and 2 hold. Suppose also that $U_n(0) \geq 0$ for all n. Fix any quantity vector $\boldsymbol{x} \geq 0$, and let $\boldsymbol{x}^S$ be any optimal solution to* (1)-(2). *Define $\alpha_n = U_n'(x_n)$. If:*

$$\sum_n \alpha_n x_n - C\left(\sum_n x_n\right) > 0,$$

and:

$$\sum_n U_n(x_n^S) - C\left(\sum_n x_n^S\right) > 0,$$

then the following inequality holds:

$$\frac{\sum_n U_n(x_n) - C(\sum_n x_n)}{\sum_n U_n(x_n^S) - C(\sum_n x_n^S)} \geq \frac{\sum_n \alpha_n x_n - C(\sum_n x_n)}{\max_{\overline{q} \geq 0}\left[(\max_n \alpha_n)\overline{q} - C(\overline{q})\right]}. \tag{11}$$

Proof of Lemma. Using concavity, we have:

$$U_n(x_n^S) \leq U_n(x_n) + U_n'(x_n)(x_n^S - x_n). \tag{12}$$

Concavity together with the fact that $U_n(0) \geq 0$ implies:

$$U_n(x_n) - U_n'(x_n)x_n \geq 0.$$

Furthermore, we have:

$$\sum_n \alpha_n x_n^S - C\left(\sum_n x_n^S\right) \leq \max_{\overline{q} \geq 0}\left[(\max_n \alpha_n)\overline{q} - C(\overline{q})\right],$$

as well as:

$$0 < \sum_n \alpha_n x_n - C\left(\sum_n x_n\right) \leq \max_{\overline{q} \geq 0}\left[(\max_n \alpha_n)\overline{q} - C(\overline{q})\right].$$

Thus we have, using (12) for the first inequality:

$$\frac{\sum_n U_n(x_n) - C(\sum_n x_n)}{\sum_n U_n(x_n^S) - C(\sum_n x_n^S)} \geq \frac{\sum_n \big(U_n(x_n) - \alpha_n x_n\big) + \sum_n \alpha_n x_n - C(\sum_n x_n)}{\sum_n \big(U_n(x_n) - \alpha_n x_n\big) + \sum_n \alpha_n x_n^S - C(\sum_n x_n^S)}$$

$$\geq \frac{\sum_n \big(U_n(x_n) - \alpha_n x_n\big) + \sum_n \alpha_n x_n - C(\sum_n x_n)}{\sum_n \big(U_n(x_n) - \alpha_n x_n\big) + \max_{\overline{q}\geq 0}\left[(\max_n \alpha_n)\overline{q} - C(\overline{q})\right]}$$

$$\geq \frac{\sum_n \alpha_n x_n - C(\sum_n x_n)}{\max_{\overline{q}\geq 0}\left[(\max_n \alpha_n)\overline{q} - C(\overline{q})\right]}.$$

This establishes the claim of the lemma. □

We briefly summarize the assumptions and conclusions to this point. Let $q = \sum_n x_n$, and let $q^S = \sum_n x_n^S$. By symmetry, since $U_n = U$ for all n, the unique Nash equilibrium must be given by $x_n = q/N$ for all n. Also by symmetry, we can assume the optimal solution $\boldsymbol{x}^S$ to (1)-(2) is symmetric, since the objective function (1) is concave. We then have $x_n^S = q^S/N$ for all n.

Lemma 4 shows that we can assume without loss of generality that $\sum_n U(x_n^S) - C(q^S) > 0$; and Lemma 5 shows that we can assume without loss of generality that $U'(x_n) > p(q)$ for all n, and that this implies $\sum_n U'(x_n)x_n - C(q) > 0$. In addition, since $\sum_n U'(x_n)x_n - C(q) > 0$, we must have $q > 0$.

If we now apply Lemma 6 with $\alpha = U'(x_n) = U'(q/N)$, we have:

$$\frac{\sum_n U(x_n) - C(q)}{\sum_n U(x_n^S) - C(q^S)} \geq \frac{\alpha q - C(q)}{\max_{\overline{q}\geq 0}\left[\alpha\overline{q} - C(\overline{q})\right]}.$$

We will compute the worst case value of the right hand side over all possible choices of C, under which $\boldsymbol{x}$ is a Nash equilibrium with $x_n = q/N$ and $\alpha = U'(x_n)$.

We now argue as follows. Define a new price function $\overline{p}(\overline{q})$ according to:

$$\overline{p}(\overline{q}) = \begin{cases} p(q), & \overline{q} \leq q; \\ p(q) + \dfrac{(\alpha - p(q))N}{q}(\overline{q} - q), & \overline{q} \geq q. \end{cases} \tag{13}$$

(See Figure 1 for an illustration.) Define $\overline{C}(q) = \int_0^q \overline{p}(z)\ dz$. Note that since $\alpha > p(q)$ and $q > 0$, $\overline{p}$ and $\overline{C}$ satisfy Assumption 2. It is also straightforward to check that the maximum $\max_{\overline{q}\geq 0}[\alpha\overline{q} - \overline{C}(\overline{q})]$ is achieved when $\overline{p}(\overline{q}) = \alpha$, i.e., when $\overline{q} = q + q/N$; and furthermore, it is straightforward to check that at this value of $\overline{q}$ we have $\alpha\overline{q} - \overline{C}(\overline{q}) = (\alpha - p(q))(q + q/(2N))$. On the other hand, $\alpha q - \overline{C}(q) = (\alpha - p(q))q$. Thus we have:

$$\frac{\alpha q - \overline{C}(q)}{\max_{\overline{q}\geq 0}\left[\alpha\overline{q} - \overline{C}(\overline{q})\right]} = \frac{2N}{2N+1}.$$

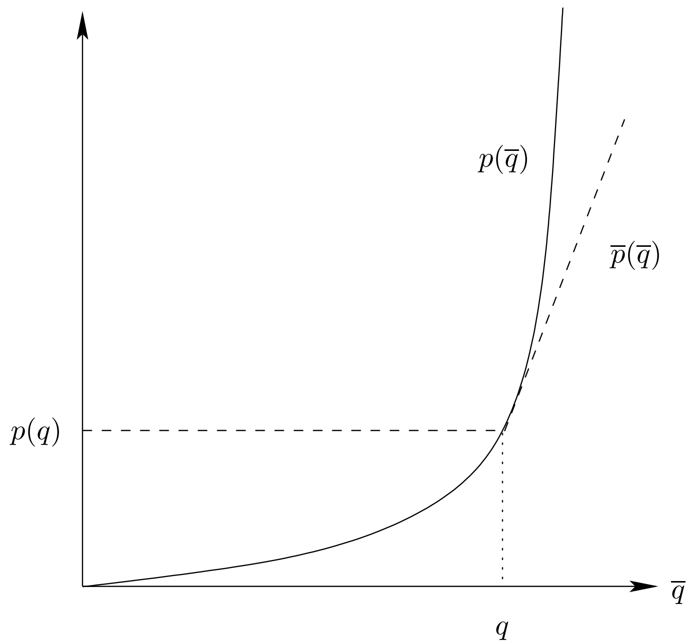

Figure 1: Proof of Theorem 3: Given a price function p (solid line) and total Nash equilibrium quantity q, a new price function $\overline{p}$ (dashed line) is defined according to (13).

To complete the proof of the theorem, therefore, it suffices to show that:

$$\frac{\alpha q - C(q)}{\max_{\overline{q}\geq 0}\left[\alpha\overline{q} - C(\overline{q})\right]} \geq \frac{\alpha q - \overline{C}(q)}{\max_{\overline{q}\geq 0}\left[\alpha\overline{q} - \overline{C}(\overline{q})\right]}.$$

Let q^* be an optimal solution to the maximization $\max_{\overline{q}\geq 0}[\alpha\overline{q} - C(\overline{q})]$. Note that we will have $\alpha \leq p(q^*)$. On the other hand, we have assumed that $\alpha = U'(x_n) > p(q)$. Thus we must have $p(q) < p(q^*)$, i.e., $q < q^*$.

To complete the proof, we define an intermediate price function $\hat{p}(\overline{q})$ as follows:

$$\hat{p}(\overline{q}) = \begin{cases} p(q), & \overline{q} \leq q; \\ p(\overline{q}), & \overline{q} \geq q. \end{cases}$$

Define $\hat{C}(q) = \int_0^q \hat{p}(z)\ dz$; then it is straightforward to check that $\hat{p}$ satisfies Assumption 2. Since p is nondecreasing (Assumption 2), we have $p(\overline{q}) \leq p(q)$ for $\overline{q} \leq q$; and thus if we define $\Delta = \hat{C}(q) - C(q)$, then $\Delta \geq 0$. Furthermore, since we have already shown that $q^* > q$, we have $\hat{C}(q^*) = C(q^*) + \Delta$. Because $\alpha > p(q)$, we have $0 < \alpha q - qp(q) = \alpha q - \hat{C}(q) \leq \alpha q^* - \hat{C}(q^*)$ (where the latter inequality follows since q^* is also seen to be an optimal solution to

$\max_{\overline{q}\geq 0}[\alpha\overline{q} - \hat{C}(\overline{q})]$). Thus we have:

$$\frac{\alpha q - C(q)}{\alpha q^* - C(q^*)} \geq \frac{\alpha q - C(q) - \Delta}{\alpha q^* - C(q^*) - \Delta}$$
$$= \frac{\alpha q - \hat{C}(q)}{\alpha q^* - \hat{C}(q^*)}. \tag{14}$$

We now observe that $\hat{C}(q) = qp(q) = \overline{C}(q)$, so the numerator in the last expression is $\alpha q - \hat{C}(q) = \alpha q - \overline{C}(q)$. On the other hand, from (8) it follows that:

$$\alpha = U'\left(\frac{q}{N}\right) \leq p(q) + \frac{q}{N} \cdot \frac{\partial^+ p(q)}{\partial q}. \tag{15}$$

Rearranging, we conclude that:

$$\frac{\partial^+ \overline{p}(q)}{\partial q} = \frac{(\alpha - p(q))N}{q} \leq \frac{\partial^+ p(q)}{\partial q} = \frac{\partial^+ \hat{p}(q)}{\partial q}. \tag{16}$$

Thus since $\hat{p}$ is convex, we have $\hat{p}(\overline{q}) \geq \overline{p}(\overline{q})$ for $\overline{q} \geq q$; on the other hand, we have $\hat{p}(\overline{q}) = \overline{p}(\overline{q})$ for $\overline{q} \leq q$. Since $q^* > q$, we have $\hat{C}(q^*) \geq \overline{C}(q^*)$, so that $\alpha q^* - \hat{C}(q^*) \leq \alpha q^* - \overline{C}(q^*) \leq \max_{\overline{q}\geq 0}(\alpha\overline{q} - \overline{C}(\overline{q}))$. Combining this inequality with (14) yields:

$$\frac{\alpha q - C(q)}{\alpha q^* - C(q^*)} \geq \frac{\alpha q - \overline{C}(q)}{\max_{\overline{q}\geq 0}\left(\alpha\overline{q} - \overline{C}(\overline{q})\right)} = \frac{2N}{2N+1},$$

as required. □

The preceding theorem can be used to yield bounds on efficiency loss in several special cases. We start with the following corollary, where all consumers share the same linear utility function.

Corollary 7 *Suppose that $N \geq 1$ consumers share the same linear utility function $U_n(x_n) = \alpha x_n$, where $\alpha > 0$; in addition, suppose that Assumption 2 holds. If $\boldsymbol{x}^S$ is an optimal solution to* (1)-(2)*, and $\boldsymbol{x}$ is a Nash equilibrium of the game defined by $(Q_1, \ldots, Q_N)$, then:*

$$\sum_n U_n(x_n) - C\left(\sum_n x_n\right) \geq \left(\frac{2N}{2N+1}\right)\left(\sum_n U_n(x_n^S) - C\left(\sum_n x_n^S\right)\right). \tag{17}$$

Proof. The proof follows the proof of Theorem 3, except that the Nash equilibrium may not be unique. The only step which requires modification is the derivation of (15)-(16), which relied on symmetry of the Nash equilibrium.

We argue as follows. Let $\boldsymbol{x}$ be any Nash equilibrium, and let $q = \sum_n x_n$. By Lemma 5, we can again assume without loss of generality that $\alpha q - C(q) > 0$. This implies in turn that $q > 0$. Now recall the optimality condition (7) for each consumer n:

$$\alpha \leq p(q) + x_n \frac{\partial^+ p(q)}{\partial q}.$$

If we consider a consumer n such that $x_n \leq q/N$ (at least one such consumer exists), then the preceding inequality together with the fact that $q > 0$ implies:

$$\frac{(\alpha - p(q))N}{q} \leq \frac{\partial^+ p(q)}{\partial q}.$$

This establishes (16), and the remainder of the proof of Theorem 3 follows. □

Theorem 3 is useful in settings where uniqueness of the Nash equilibrium can be guaranteed. We now apply Theorem 3 in two special cases: first, a *Cournot monopsony*, where only one consumer is purchasing a scarce resource; and second, a Cournot oligopsony where all consumers share the same utility function, and the price function is differentiable.

Corollary 8 *Suppose that there is a single consumer (i.e., $N = 1$), with a utility function U such that Assumption 1 holds; in addition, suppose that Assumption 2 holds. Suppose also that $U(0) \geq 0$. If x^S solves* (1)-(2)*, and x maximizes $U(\overline{x}) - \overline{x}p(\overline{x})$ over $\overline{x} \geq 0$, then:*

$$U(x) - C(x) \geq \left(\frac{2}{3}\right)\left(U(x^S) - C(x^S)\right). \tag{18}$$

This bound is tight, i.e., there exists a choice of U and C such that (18) *holds with equality.*

Proof. The proof relies on the following lemma.

Lemma 9 *Suppose that there is a single consumer (i.e., $N = 1$), with a utility function U such that Assumption 1 holds; in addition, suppose that Assumption 2 holds. Then* at least *one of the following holds: either (1) all optimal solutions to $\max_{\overline{x}\geq 0}[U(\overline{x}) - \overline{x}p(\overline{x})]$ are also optimal solutions to* (1)-(2)*; or (2) there exists a unique optimal solution to $\max_{\overline{x}\geq 0}[U(\overline{x}) - \overline{x}p(\overline{x})]$.*

Proof of Lemma. Suppose there exist $x, \hat{x} \in \arg\max_{\overline{x}\geq 0}[U(\overline{x}) - \overline{x}p(\overline{x})]$ such that $x \neq \hat{x}$. Assume without loss of generality that $x < \hat{x}$; note that this implies $\hat{x} > 0$. By concavity, we have $U'(\hat{x}) \leq U(x)$. By (7), we have $U'(x) \leq p(x) + x\partial^+ p(x)/\partial q$. Since p is nondecreasing and convex, and $x < \hat{x}$, we have $p(x) + x\partial^+ p(x)/\partial q \leq p(\hat{x}) + \hat{x}\partial^- p(\hat{x})/\partial q$. Finally, from (8), we have $p(\hat{x}) + \hat{x}\partial^- p(\hat{x})/\partial q \leq U'(\hat{x})$. Combining these inequalities, we have:

$$U'(\hat{x}) \leq U(x) \leq p(x) + x\frac{\partial^+ p(x)}{\partial q} \leq p(\hat{x}) + \hat{x}\frac{\partial^- p(\hat{x})}{\partial q} \leq U'(\hat{x}).$$

Thus equality must hold throughout; since $x < \hat{x}$, this is only possible if $p(x) = p(\hat{x})$ and $\partial^+ p(x)/\partial x = \partial^- p(\hat{x})/\partial x = 0$. Thus $U'(x) = p(x)$ and $U'(\hat{x}) = p(\hat{x})$, so that both x and $\hat{x}$ are optimal solutions to (1)-(2), as required. □

If all optimal solutions to $\max_{\overline{x}\geq 0}[U(\overline{x}) - \overline{x}p(\overline{x})]$ are also optimal solutions to (1)-(2), then the bound (18) trivially holds. On the other hand, if there exists a unique optimal solution to $\max_{\overline{x}\geq 0}[U(\overline{x}) - \overline{x}p(\overline{x})]$, then we can apply Theorem 3 to conclude that (18) holds.

Finally, to see that the bound is tight, let $U(x) = x$, and let $p(q) = (q-1)^+$ (i.e., $p(q) = 0$ for $0 \le q \le 1$, and $p(q) = q - 1$ for $q \ge 1$); thus $C(q) = 0$ if $0 \le q \le 1$, and $C(q) = (q-1)^2/2$ if $q \ge 1$. Then it is straightforward to verify that $x = 1$ is the unique optimal solution to $\max_{\overline{x}\ge 0}[U(\overline{x}) - \overline{x}p(\overline{x})]$, while $x^S = 2$ is an optimal solution to (1)-(2). Furthermore, we have $U(x) - C(x) = 1$, while $U(x^S) - C(x^S) = 3/2$, matching the bound (18). $\square$

The preceding corollary considered a single consumer. We now consider a model consisting of multiple consumers who share the same utility function.

Corollary 10 *Suppose that $N \ge 1$ consumers share the same utility function $U_n = U$, such that Assumption 1 holds; in addition, suppose that Assumption 2 holds, and that p is differentiable. Suppose also that $U(0) \ge 0$. If $\boldsymbol{x}^S$ is an optimal solution to* (1)-(2)*, and $\boldsymbol{x}$ is a Nash equilibrium of the game defined by $(Q_1, \dots, Q_N)$, then:*

$$\sum_n U_n(x_n) - C\left(\sum_n x_n\right) \ge \left(\frac{2N}{2N+1}\right)\left(\sum_n U_n(x_n^S) - C\left(\sum_n x_n^S\right)\right). \tag{19}$$

Proof. The proof relies on the following lemma.

Lemma 11 *Suppose that Assumptions 1 and 2 hold, and that p is differentiable. Then* at least *one of the following holds: either (1) all Nash equilibria of the game defined by $(Q_1, \dots, Q_N)$ are also optimal solutions to* (1)-(2)*; or (2) there exists a unique Nash equilibrium of the game defined by $(Q_1, \dots, Q_N)$.*

Proof of Lemma. The proof is similar to the proof of Lemma 9. Let $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ be two Nash equilibria such that $\boldsymbol{x} \ne \hat{\boldsymbol{x}}$, and let $q = \sum_n x_n$, and $\hat{q} = \sum_n \hat{x}_n$. Assume without loss of generality that $q \le \hat{q}$. Since $\boldsymbol{x} \ne \hat{\boldsymbol{x}}$, then there must exist a consumer n such that $x_n < \hat{x}_n$; in particular, $\hat{x}_n > 0$. In this case, we have $U_n'(\hat{x}_n) \le U_n'(x_n)$ by concavity. By (7) we have $U_n'(x_n) \le p(q) + x_n p'(q)$. Since p is nondecreasing and convex, $q \le \hat{q}$, and $x_n < \hat{x}_n$, we have $p(q)+x_n p'(q) \le p(\hat{q})+\hat{x}_n p'(\hat{q})$. Since $\hat{x}_n > 0$, by (7)-(8) we have $p(\hat{q}) + \hat{x}_n p'(\hat{q}) = U_n'(\hat{x}_n)$. Combining these relations yields:

$$U_n'(\hat{x}_n) \le U_n'(x_n) \le p(q) + x_n p'(q) \le p(\hat{q}) + \hat{x}_n p'(\hat{q}) = U_n'(\hat{x}_n).$$

Thus equality must hold throughout; since $x_n < \hat{x}_n$, this is only possible if $p(q) = p(\hat{q})$, and $p'(q) = p'(\hat{q}) = 0$. In this case (7)-(8) imply that for all m, we have $U_m'(x_m) = p(q)$ if $x_m > 0$, and $U_m'(x_m) \le p(q)$ if $x_m = 0$; similarly, $U_m'(\hat{x}_m) = p(\hat{q})$ if $\hat{x}_m > 0$, and $U_m'(\hat{x}_m) \le p(\hat{q})$ if $\hat{x}_m = 0$. These are precisely the optimality conditions for (1)-(2), so we conclude that both $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ are optimal solutions to (1)-(2), as required. $\square$

If all Nash equilibria are also optimal solutions to (1)-(2), then the bound (19) trivially holds. On the other hand, if there exists a unique Nash equilibrium, then we can apply Theorem 3 to conclude that (19) holds. $\square$

We note that Lemma 11 did not require all consumers to have the same utility function, and thus holds for any game where the price function p is differentiable. We also note that although a tightness result is not claimed in the preceding corollary, such a result may be established by considering a limit of differentiable price functions which approach the worst case price function $\overline{p}$ defined in the proof of Theorem 3. However, defining such price functions requires additional technical complexity, and does not yield additional insight; thus the argument is omitted.

Note that Corollary 10 also yields a competitive limit theorem [17], since as $N \to \infty$ the efficiency loss approaches zero. Indeed, this result is to be expected, since the consumers are assumed to be identical; thus in the limit of many consumers no single consumer should have a significant impact on the market-clearing price.

Corollaries 7, 8, and 10 present bounds on efficiency loss under various restrictions on utility functions and the price function p. Although we have assumed differentiability of the utility function, this assumption is not essential, as previously discussed; it only eases presentation of the technical arguments. By contrast, differentiability of the price function p is essential to the proof of Corollary 10. In particular, in considering the statements of Corollaries 7, 8, and 10, one might expect a more general result to hold: if N consumers share the same utility function U and Assumption 1 is satisfied, and the price function p satisfies Assumption 2 (but is not necessarily differentiable), then the efficiency loss is no more than $1/(2N+1)$ when consumers are price anticipating. Such a result would be a generalization of Corollaries 7, 8, and 10.

However, the efficiency loss can be arbitrarily high if the price function is not differentiable, even if all consumers share the same utility function. The main reason for this negative result is that when the price function is not differentiable, there may exist highly inefficient Nash equilibria which are not symmetric among the players. We present an example here of such a situation.

Example 2 Let the number of consumers be $N > 1$, and let the price function be $p(q) = (q-1)^+$. Let $C(q) = \int_0^q p(z)\ dz$ be the associated cost function; note that p and C satisfy Assumption 2. Define $\alpha = 1/N^2$ and $\hat{x} = (\alpha + 1)/N$. We then define the piecewise linear utility function U as follows:

$$U(x) = \begin{cases} \alpha x, & \text{if } x \leq \hat{x}; \\ \alpha \hat{x}, & \text{if } x \geq \hat{x}. \end{cases}$$

Then U is concave and continuous. Note that U is not differentiable, but as discussed above, this feature is inessential to the argument; a similar example can be constructed with a differentiable utility function U, at considerably higher technical expense.

We now claim that if $x_n^S = \hat{x}$ for all n, then $\boldsymbol{x}^S$ is a solution to (1)-(2). To see this, note that $q^S = \sum_n x_n^S = N\hat{x} = \alpha + 1$; and thus $p(q^S) = \alpha$. On the other hand, we have:

$$\frac{\partial^+ U(x_n^S)}{\partial x_n} = \frac{\partial^+ U(\hat{x})}{\partial x} = 0 < \alpha = p(q^S);$$
$$\frac{\partial^- U(x_n^S)}{\partial x_n} = \frac{\partial^- U(\hat{x})}{\partial x} = \alpha = p(q^S).$$

These are necessary and sufficient optimality conditions for $\boldsymbol{x}^S$ to be a solution to (1)-(2), as required. Note that the aggregate surplus at this solution is $\sum_n U(x_n^S) - C(q^S) = N\alpha\hat{x} - \alpha^2/2 = \alpha^2/2 + \alpha$.

Next, let $x_n = \alpha$ for $n = 2, \ldots, N$, and $d_1 = 1 - (N-1)\alpha$. Note that $q = \sum_n x_n = 1$, and thus $p(q) = 0$ and:

$$\frac{\partial^- p(q)}{\partial q} = 0; \quad \frac{\partial^+ p(q)}{\partial q} = 1.$$

We claim $\boldsymbol{x}$ is a Nash equilibrium; note that $\boldsymbol{x}$ is not symmetric among the players. Using the definitions of $\hat{x}$ and α, it is straightforward to establish that $0 < \alpha < \hat{x} < 1 - (N-1)\alpha$ as long as $N > 1$. Thus, in particular, U is differentiable at x_n for all n, and $U'(x_1) = 0$, while $U'(x_n) = \alpha$. Now we observe that:

$$p(q) + x_1 \frac{\partial^- p(q)}{\partial q} = 0 = U'(x_1) < 1 - (N-1)\alpha = p(q) + x_1 \frac{\partial^+ p(q)}{\partial q};$$

$$p(q) + x_1 \frac{\partial^- p(q)}{\partial q} = 0 < U'(x_n) = \alpha = p(q) + x_1 \frac{\partial^+ p(q)}{\partial q}, \quad n = 2, \ldots, N.$$

Thus, the sufficient conditions (7)-(8) are satisfied, so we conclude $\boldsymbol{x}$ is a Nash equilibrium. At this Nash equilibrium, the aggregate surplus is $\sum_n U(x_n) - C(q) = \alpha\hat{x} + (N-1)\alpha^2$. If we now substitute $\alpha = 1/N^2$ and $\hat{x} = (\alpha+1)/N = 1/N^3 + 1/N$, the ratio of Nash equilibrium aggregate surplus to the maximal aggregate surplus reduces to:

$$\frac{1/N^5 + 1/N^3 + (N-1)/N^4}{1/(2N^4) + 1/N^2}.$$

As $N \to \infty$, the preceding ratio approaches zero. $\square$

The preceding example highlights an important issue in market modeling: results on the performance of the market can be very sensitive under assumptions of symmetry among the participants. In particular, one might expect that little difference exists in market performance whether the price function is differentiable or not; nevertheless, the preceding example shows that efficiency loss can become arbitrarily high if the price function is not differentiable.

To avoid such singular effects, we now search instead for a result that holds regardless of the utility functions of the consumers. Of course, such a result cannot hold for all price functions. In particular, we prove in the following theorem that if the price function is affine, the resulting efficiency loss is no more than $1/3$ of the maximal aggregate surplus, regardless of the utility functions of the consumers.

Theorem 12 *Suppose that Assumption 1 holds, and that* $p(q) = aq + b$ *for some* $a > 0, b \geq 0$*. Suppose also that* $U_n(0) \geq 0$ *for all* n*. If* $\boldsymbol{x}^S$ *is any solution to* (1)-(2)*, and* $\boldsymbol{x}$ *is any Nash equilibrium of the game defined by* $(Q_1, \ldots, Q_N)$*, then:*

$$\sum_n U_n(x_n) - C\left(\sum_n x_n\right) \geq \left(\frac{2}{3}\right)\left(\sum_n U_n(x_n^S) - C\left(\sum_n x_n^S\right)\right). \tag{20}$$

Furthermore, this bound is tight: for every $a > 0$, $b \geq 0$, and $\delta > 0$, there exists a choice of N and a choice of (linear) utility functions U_n, $n = 1, \ldots, N$, such that a Nash equilibrium $\boldsymbol{x}$ exists with:

$$\sum_n U_n(x_n) - C\left(\sum_n x_n\right) \leq \left(\frac{2}{3} + \delta\right)\left(\sum_n U_n(x_n^S) - C\left(\sum_n x_n^S\right)\right). \tag{21}$$

Proof. The proof follows in two steps. Using Lemma 6, we first show that the worst case occurs when the utility functions of the consumers are linear. We then optimize over all games with linear utility functions to determine the worst case efficiency loss.

Let $\boldsymbol{x}$ be a Nash equilibrium. As in the proof of Theorem 3, using Lemmas 4 and 5 we can assume without loss of generality that $\sum_n U_n(x_n^S) - C(\sum_n x_n^S) > 0$ and $\sum_n U_n'(x_n)x_n - C(\sum_n x_n) > 0$. If we replace the utility function U_n by $\overline{U}_n$ for each n, where $\overline{U}_n(\overline{x}_n) = (U_n'(x_n))\overline{x}_n$, then $\boldsymbol{x}$ continues to be a Nash equilibrium, since the optimality conditions (7)-(8) still hold. Applying Lemma 6, therefore, we see that the ratio of Nash equilibrium aggregate surplus to the maximal aggregate surplus can only be reduced if we replace U_n by $\overline{U}_n$ for all n.

Thus we assume without loss of generality that the utility functions of all consumers are linear, i.e., $U_n(x_n) = \alpha_n x_n$. Since we have assumed $\sum_n \alpha_n x_n - C(\sum_n x_n) > 0$, we know that $\alpha_n > 0$ for at least one n. Thus, by replacing α_n by $\alpha_n/(\max_m \alpha_m)$, and $C(\cdot)$ by $C(\cdot)/(\max_n \alpha_n)$, we can also assume without loss of generality that $\max_n \alpha_n = 1$. Furthermore, by relabeling if necessary, we can assume that $\alpha_1 = 1$. Note that after rescaling, the new price function p is still affine but may have a different slope.

Since we have restricted attention to settings where $\sum_n \alpha_n x_n - C(\sum_n x_n) > 0$, we must also have $\sum_n x_n > 0$. Thus, from (8) and the fact that $\max_n \alpha_n = 1$ we must have $1 > p(q) = aq + b$; in particular, this implies that $b < 1$.

We start by computing the maximal aggregate surplus under these assumptions. Since the price function is $p(q) = aq + b$, the maximal aggregate surplus is achieved when $p(q^S) = 1$, i.e., when $q^S = (1 - b)/a$; this quantity is entirely allocated to consumer 1. The maximal aggregate surplus is thus:

$$\frac{1-b}{a} - \frac{(1-b)^2}{2a} - \frac{b(1-b)}{a} = \frac{(1-b)^2}{2a}.$$

Since the maximal aggregate surplus is fixed as $(1-b)^2/(2a)$, by (7)-(8) the worst case game is identified by solving the following optimization problem (with unknowns $x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_n, q$):

$$\begin{aligned}
\text{minimize} \quad & \sum_{n=1}^{N} \alpha_n x_n - C(q) && (22)\\
\text{subject to} \quad & \alpha_n \le p(q) + x_n p'(q), \quad n = 1, \ldots, N; && (23)\\
& \alpha_n \ge p(q) + x_n p'(q), \quad \text{if } x_n > 0, \;\; n = 1, \ldots, N; && (24)\\
& \sum_{n=1}^{N} x_n = q; && (25)\\
& \alpha_1 = 1; && (26)\\
& 0 \le \alpha_n \le 1, \quad n = 2, \ldots, N; && (27)\\
& x_n \ge 0, \quad n = 1, \ldots, N. && (28)
\end{aligned}$$

The objective function is the aggregate surplus associated with a Nash equilibrium allocation $\boldsymbol{x}$. The conditions (23)-(24) are equivalent to the Nash equilibrium conditions established in (7)-(8). The constraint (25) ensures that the total allocation made at the Nash equilibrium is equal to q. The constraints (26)-(27) follow since we have restricted without loss of generality to games where $\alpha_1 = \max_n \alpha_n = 1$. The constraint (28) ensures the quantity allocated to each consumer is nonnegative.

We start by assuming that $q > 0$ is fixed, and optimize only over $\boldsymbol{x}$ and $\boldsymbol{\alpha}$. In this case, we start by noting that we may assume without loss of generality that $\alpha_n = p(q) + x_n p'(q)$ for all consumers $n = 2, \ldots, N$. Indeed, if $(\boldsymbol{\alpha}, \boldsymbol{x})$ is a feasible solution and $x_n > 0$ for some $n = 2, \ldots, N$, then (23)-(24) imply that $\alpha_n = p(q) + x_n p'(q)$. On the other hand, if $x_n = 0$ for some $n = 2, \ldots, N$, we can set $\alpha_n = p(q) = aq + b$; this preserves feasibility, but does not impact the term $\alpha_n x_n$ in the objective function (22). We can therefore restrict attention to feasible solutions for which:

$$\alpha_n = p(q) + x_n p'(q) = aq + b + a x_n, \quad n = 2, \ldots, N. \tag{29}$$

Having done so, observe that the constraint (27) that $\alpha_n \le 1$ may be written as:

$$x_n \le \frac{1 - aq - b}{a}, \quad n = 2, \ldots, N.$$

Finally, the constraint (27) that $\alpha_n \ge 0$ becomes redundant, as it is guaranteed by the fact that $a > 0$, $b \ge 0$, and $q > 0$.

It follows from (27) together with (22)-(23) that a candidate solution satisfying (25) can only exist if $x_1 > 0$, in which case we have $1 = p(q) + x_1 p'(q) = aq + b + a x_1$, so that $x_1 = (1 - aq - b)/a$. In particular, we conclude immediately that for a feasible solution to exist, we must have $0 < (1 - aq - b)/a \leq q$. This yields the following reduced optimization problem:

$$\begin{aligned}
\text{minimize} \quad & \frac{1 - aq - b}{a} + \sum_{n=2}^{N} (aq + b + ax_n)x_n - C(q) && (30) \\
\text{subject to} \quad & \sum_{n=2}^{N} x_n = q - \frac{1 - aq - b}{a}; && (31) \\
& x_n \leq \frac{1 - aq - b}{a}, \quad n = 2, \ldots, N; && (32) \\
& x_n \geq 0, \quad n = 2, \ldots, N. && (33)
\end{aligned}$$

The objective function (30) is equivalent to (22) upon substitution for α_n (assuming equality in (24)) and x_1 (also by requiring equality in (24)). The constraint (31) is equivalent to the allocation constraint (25); and the constraint (32) ensures $\alpha_n \leq 1$, as required in (27).

For fixed $q > 0$, the resulting problem is symmetric in the quantities x_n for $n = 2, \ldots, N$. It is clear that a feasible solution exists if and only if:

$$\frac{q}{N} \leq \frac{1 - aq - b}{a} \leq q. \tag{34}$$

In this case the following symmetric solution is feasible:

$$x_n = \frac{q - (1 - aq - b)/a}{N - 1}.$$

Furthermore, since the objective function (30) is strictly convex, this symmetric solution must in fact be optimal. If we substitute in the objective function (30), the resulting optimal value is strictly decreasing as N increases; the worst case occurs as $N \to \infty$, and the optimal objective value (30) becomes:

$$\frac{1 - aq - b}{a} + (aq + b)\left(q - \frac{1 - aq - b}{a}\right) - C(q) = \frac{1 - b}{a} - q + (aq + b)\left(2q - \frac{1 - b}{a}\right) - \frac{aq^2}{2} - bq.$$

Furthermore, the feasibility requirements (34) on a, b, and q become $0 < (1 - aq - b)/a \leq q$, or upon rearranging, $(1 - b)/2 \leq aq < 1 - b$.

Until now we have kept the price function and the total quantity q fixed, and found the worst case game. We now optimize over all possible choices of price function p (i.e., over $a > 0$ and $b \geq 0$), as well as over possible Nash equilibrium quantities (i.e., over $q > 0$). Recall that the maximal aggregate surplus is $(1-b)^2/(2a)$. Thus, the worst case ratio is identified by the following optimization problem over q, a, and b:

$$\begin{aligned}
\text{minimize} \quad & \frac{(1 - b)/a - q + (aq + b)\,(2q - (1 - b)/a) - aq^2/2 - bq}{(1 - b)^2/(2a)} \\
\text{subject to} \quad & (1 - b)/2 \leq aq \leq 1 - b, \quad a > 0,\ b \geq 0,\ q > 0.
\end{aligned}$$

If we divide numerator and denominator of the objective function by q, and let $\overline{a} = aq$, then this problem becomes equivalent to the following problem:

$$\begin{array}{ll} \text{minimize} & \dfrac{(1-b)/\overline{a} - 1 + (\overline{a}+b)\left(2 - (1-b)/\overline{a}\right) - \overline{a}/2 - b}{(1-b)^2/(2\overline{a})} \\ \text{subject to} & (1-b)/2 \le \overline{a} \le 1-b, \quad \overline{a} > 0,\ b \ge 0. \end{array}$$

By substituting $x = \overline{a}/(1-b)$ and differentiating, it is straightforward to establish that the minimum value of this optimization problem occurs at any pair $\overline{a}$ and b satisfying the constraints, such that $\overline{a}/(1-b) = 2/3$. One such pair is given by $\overline{a} = 1/3$, and $b = 0$. At any such solution, the minimum objective value is equal to $2/3$. This establishes (20).

We now show (21), for a fixed price function $p(q) = aq + b$ with $a > 0$ and $b \ge 0$. To see this, choose the utility functions so that:

$$\begin{aligned} U_1(x_1) &= \left(\frac{3a}{2} + b\right) x_1; \\ U_n(x_n) &= \left(a + b + \frac{a}{2(N-1)}\right) x_n, \quad n = 2, \dots, N. \end{aligned}$$

Let $\overline{x} = 1/(2(N-1))$. It is then straightforward to check that for sufficiently large N, if $x_1 = 1/2$ and $x_n = \overline{x}$ for $n = 2, \dots, N$, the allocation $\boldsymbol{x}$ is a Nash equilibrium. Furthermore, the maximum aggregate surplus is achieved by choosing q^S so that $3a/2 + b = p(q^S) = aq^S + b$, so $q^S = 3/2$, $x_1^S = q^S = 3/2$, and $x_n^S = 0$ for $n = 2, \dots, N$. Thus the ratio of Nash equilibrium aggregate surplus to maximal aggregate surplus is:

$$\frac{(3a/2+b)(1/2) + (a+b+a\overline{x})(1/2) - a/2 - b}{(3a/2+b)(3/2) - (a/2)(3/2)^2 - b(3/2)} = \frac{3/2 + \overline{x}}{9/4}.$$

Now as $N \to \infty$, this ratio approaches $2/3$, as required. □

Note that while the first part of the proof makes it appear as if the worst case occurs when the price function satisfies $a/(1-b) = 2/3$, in fact by an appropriate choice of utility functions the worst case efficiency loss is always *exactly* $1/3$ for *any* affine price function.

3 Cournot Oligopoly

In this section, we will consider a game where multiple producers compete to satisfy demand for a single good, and where the strategies of the producers are their desired output quantities; such games are known as *Cournot oligopolies*. We will find that in general Cournot oligopolies can yield arbitrarily high efficiency loss, though we will also establish bounds on efficiency loss for several special cases of interest. Our main technique is to establish a corresponding Cournot oligopsony model for any Cournot oligopoly model. While such correspondences are typical between oligopoly and oligopsony models, we choose our correspondence carefully to ensure that aggregate surplus values, efficient allocations, and Nash equilibria are unchanged between the two models; this allows derivation of efficiency loss results as simple analogues of all the main results of Section 2.

Formally, we consider the following model. We assume that N producers compete to satisfy the demand for a single good. We assume that each producer n has a cost function C_n which gives the cost of production as a function of the amount produced. We also assume that producing q units of output yields an aggregate utility to the consumers of $U(q)$. We make the following assumptions.

Assumption 3 *For each n, over the domain $x_n \geq 0$ the cost function $C_n(x_n)$ is convex, nondecreasing, and continuously differentiable (where we interpret $C_n'(0)$ as the right directional derivative of C_n at 0). In addition, $C_n(0) = 0$.*

Assumption 4 *There exists a continuous, nonincreasing function $p(q)$ over $q \geq 0$ such that for $q \geq 0$:*

$$U(q) = \int_0^q p(z)dz.$$

The function $p(q)$ has the following properties:

1. *$p(0) > 0$;*
2. *$p(q)$ is concave with finite directional derivatives for $q \geq 0$; and*
3. *$p(q) \to -\infty$ as $q \to \infty$.*

Thus $U(q)$ is concave. We let $q_{\max} > 0$ denote the unique quantity at which $p(q_{\max}) = 0$.

We use $U(q)$ to characterize the aggregate utility of the consumers. We restrict attention to the setting where the marginal utility $p(q) = U'(q)$ (i.e., the inverse demand curve) is concave and decreasing, from $p(0) > 0$ to $p(q_{\max}) = 0$. In general in oligopoly models, the inverse demand curve is either undefined for $q \geq q_{\max}$, or defined as $p(q) = 0$ for $q \geq q_{\max}$. These formulations have the undesirable feature that the inverse demand curve p is not necessarily globally concave after such a transformation. For analytical simplicity, therefore, we have allowed p to become negative after $q_{\max}$ (cf. Condition 3 in Assumption 4). We make this assumption essentially without loss of generality: the revenue to producers would be zero at any aggregate production quantity $q \geq q_{\max}$, so that in considering either fully efficient production vectors or Nash equilibria it is straightforward to check that we can restrict attention to vectors $\boldsymbol{x}$ such that $\sum_n x_n \leq q_{\max}$. In particular, all the efficiency loss results of this section continue to hold for a model where we define $p(q) = 0$ for $q \geq q_{\max}$.

We note that Assumption 4 implies several basic facts about both p and U. Since $p(0) > 0$ while $p(q_{\max}) = 0$, p is strictly decreasing and negative for $q \geq q_{\max}$. Thus $U(q)$ is strictly decreasing for $q > q_{\max}$.

We assume that both utility and cost are measured in monetary units, so that an efficient allocation is characterized by solving the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & U\left(\sum_n x_n\right) - \sum_n C_n(x_n) && (35) \\ \text{subject to} \quad & x_n \geq 0, \quad n = 1, \ldots, N. && (36) \end{aligned}$$

As before, the objective function (35) is the *aggregate surplus*. Since $U(q)$ is strictly decreasing for $q > q_{\max}$ and the objective function is continuous, it follows that an optimal solution exists; in fact, we may conclude any optimal solution $\boldsymbol{x}^S$ satisfies $\sum_n x_n^S \leq q_{\max}$.

We now consider the following pricing scheme. Each producer n chooses a desired output quantity x_n, and a single price $\mu(\boldsymbol{x}) = p(\sum_n x_n)$ is chosen. We first assume that given a price $\mu > 0$, producer n chooses x_n to maximize:

$$R_n(x_n; \mu) = \mu x_n - C_n(x_n). \tag{37}$$

Note that each producer acts as a price taker; since we are employing *marginal utility pricing* (i.e., since $\mu(\boldsymbol{x}) = p(\sum_n x_n)$), we again expect that price taking producers will maximize aggregate surplus at a competitive equilibrium. This is formalized in the following proposition, a special case of the first fundamental theorem of welfare economics [17].

Proposition 13 *Suppose Assumptions 3 and 4 hold. There exists a* competitive equilibrium*, that is, a vector $\boldsymbol{x}$ and a scalar $\mu \geq 0$ such that $\mu = p(\sum_n x_n)$, and:*

$$R_n(x_n; \mu) = \max_{\overline{x}_n \geq 0} R_n(\overline{x}_n; \mu), \quad n = 1, \ldots, N. \tag{38}$$

Any such vector $\boldsymbol{x}$ solves (35)-(36)*. If the functions C_n are strictly convex, such a vector $\boldsymbol{x}$ is unique as well.*

Proposition 13 shows that with marginal utility pricing, and if the producers behave as price takers, there exists a vector of quantities $\boldsymbol{x}$ where all producers have optimally chosen their x_n, with respect to the given price $\mu = p(\sum_n x_n)$; and at this "equilibrium," the aggregate surplus is maximized. However, when the price taking assumption is violated, the model changes into a game and the guarantee of Proposition 13 is no longer valid.

Consider, then, an alternative model where the producers are price anticipating, rather than price taking, and play a Cournot game. We use the notation $\boldsymbol{x}_{-n}$ to denote the vector of all quantities chosen by producers other than n; i.e., $\boldsymbol{x}_{-n} = (x_1, x_2, \ldots, x_{n-1}, x_{n+1}, \ldots, x_n)$. Then given $\boldsymbol{x}_{-n}$, each producer n chooses $x_n \geq 0$ to maximize:

$$T_n(x_n; \boldsymbol{x}_{-n}) = x_n p\left(\sum_m x_m\right) - C_n(x_n). \tag{39}$$

The payoff function T_n is similar to the payoff function R_n, except that the producer now anticipates that the price will be set according to $p(\sum_m x_m)$. A *Nash equilibrium* of the game defined by $(T_1, \ldots, T_N)$ is a vector $\boldsymbol{x} \geq 0$ such that for all n:

$$T_n(x_n; \boldsymbol{x}_{-n}) \geq T_n(\overline{x}_n; \boldsymbol{x}_{-n}), \quad \text{for all } \overline{x}_n \geq 0. \tag{40}$$

It is straightforward to show that a Nash equilibrium exists for this game, using techniques similar to the proof of Proposition 2; see also [18] for more general conditions guaranteeing existence of Nash equilibria.

Proposition 14 *Suppose that Assumptions 3 and 4 hold. Then there exists a Nash equilibrium $\boldsymbol{x}$ for the game defined by $(T_1, \ldots, T_N)$. Furthermore, any Nash equilibrium $\boldsymbol{x}$ satisfies $\sum_n x_n \leq q_{\max}$.*

Proof. The proof is nearly identical to the proof of Proposition 2, so we omit the details. We only check that $\sum_n x_n \leq q_{\max}$ for any Nash equilibrium $\boldsymbol{x}$. If not, consider a producer n with $x_n > 0$; the payoff $T_n(x_n; \boldsymbol{x}_{-n})$ to this producer is negative. On the other hand, this producer can guarantee a payoff of zero by choosing $x_n = 0$. Thus $\boldsymbol{x}$ could not have been a Nash equilibrium, a contradiction. □

Since we have assumed p is concave and nonincreasing, it is straightforward to verify that the payoff T_n of each producer n is concave in the strategy x_n. Thus a vector $\boldsymbol{x}$ is a Nash equilibrium if and only if the following optimality conditions hold for each n, with $q = \sum_n x_n$:

$$C_n'(x_n) \geq p(q) + x_n \frac{\partial^+ p(q)}{\partial q}; \tag{41}$$

$$C_n'(x_n) \leq p(q) + x_n \frac{\partial^- p(q)}{\partial q}, \quad \text{if } x_n > 0. \tag{42}$$

We will now use these conditions to analyze the efficiency loss when producers are price anticipating. We first show in the following example that, in general, the efficiency loss may be arbitrarily high.

Example 3 Fix $a > 1$ and $b > a - 1$, and consider a price function p defined as follows:

$$p(q) = \begin{cases} a, & 0 \leq q \leq 1; \\ a - b(q-1), & q \geq 1. \end{cases}$$

Thus $q_{\max} = 1 + a/b$. Note that this yields:

$$U(q) = \begin{cases} aq, & 0 \leq q \leq 1; \\ aq - \frac{1}{2}b(q-1)^2, & q \geq 1. \end{cases}$$

We consider a game with $N = 2$ producers where $C_1(x_1) = x_1$, and $C_2(x_2) = ax_2$. In this case, since $a > 1$, note that aggregate surplus is maximized when $p(q) = 1$, i.e., when $q = 1 + (a-1)/b$; and furthermore, this quantity should be produced entirely by producer 1. Thus the maximal aggregate surplus is $U(q) - C_1(q)$, or:

$$a\left(1 + \frac{a-1}{b}\right) - \frac{1}{2}b\left(\frac{a-1}{b}\right)^2 - 1 - \frac{a-1}{b} = a - 1 + \frac{(a-1)^2}{2b}. \tag{43}$$

On the other hand, we claim that the vector $\boldsymbol{x}$ defined by:

$$x_1 = \frac{a-1}{b};$$
$$x_2 = 1 - \frac{a-1}{b},$$

is a Nash equilibrium. Observe that $q = x_1 + x_2 = 1$, so $p(q) = a$. Furthermore, $\partial^+ p(q)/\partial q = -b$, $\partial^- p(q)/\partial q = 0$. It then follows that (41)-(42) are satisfied by both producers 1 and 2. Since $b > 0$, these conditions are sufficient to ensure that $\boldsymbol{x}$ is a Nash equilibrium. Note that the aggregate surplus at this Nash equilibrium is $U(q) - C_1(x_1) - C_2(x_2)$, or:

$$a - \frac{a-1}{b} - a\left(1 - \frac{a-1}{b}\right) = \frac{(a-1)^2}{b}.$$

Comparing this expression with (43), it is clear that in the limit where $b \to \infty$, the Nash equilibrium aggregate surplus approaches zero, and the maximal aggregate surplus approaches $a - 1$; thus the ratio of Nash equilibrium aggregate surplus to the maximal aggregate surplus approaches zero. $\square$

As expected, the preceding example appears "symmetric" to Example 1. Indeed, we expect a general correspondence between a given model of oligopoly, and an appropriately defined model of oligopsony. In the remainder of this section, we formally establish this correspondence; we then exploit it to directly establish results on efficiency loss for Cournot oligopoly, from the results already proven for Cournot oligopsony. Although such correspondences are common, the key difficulty in the present development is that we must ensure aggregate surplus, efficient allocations, and Nash equilibria remain unchanged between an oligopoly model and its corresponding oligopsony model.

The following theorem is the main result in this development.

Theorem 15 *Suppose that Assumptions 3 and 4 hold. Define the constant* $\Gamma \geq 0$ *according to:*

$$\Gamma = \max\left\{p(0), C_1'(q_{\max}), \dots, C_N'(q_{\max})\right\}. \tag{44}$$

For each n*, define:*

$$\hat{U}_n'(x_n) = \begin{cases} \Gamma - C_n'(x_n), & \textit{if } 0 \leq x_n \leq q_{\max}; \\ \Gamma - C_n'(q_{\max}), & \textit{if } x_n > q_{\max}. \end{cases} \tag{45}$$

Define the associated utility function $\hat{U}_n(x_n) = \int_0^{x_n} \hat{U}_n'(z)dz$*. In addition, define a new price function* $\hat{p}$ *according to:*

$$\hat{p}(q) = \Gamma - p(q). \tag{46}$$

Define an associated cost function $\hat{C}(q) = \int_0^q \hat{p}(z)\, dz$*. Then:*

1. *Assumption 1 is satisfied by* $(\hat{U}_1, \dots, \hat{U}_N)$*; and Assumption 2 is satisfied by* $\hat{p}$ *and* $\hat{C}$*.*

2. *For any vector $\boldsymbol{x} \geq 0$ such that $x_n \leq q_{\max}$ for all n, there holds:*

$$U\left(\sum_n x_n\right) - \sum_n C_n(x_n) = \sum_n \hat{U}_n(x_n) - \hat{C}\left(\sum_n x_n\right), \tag{47}$$

as well as:

$$T_n(x_n; \boldsymbol{x}_{-n}) = x_n p\left(\sum_m x_m\right) - C_n(x_n) = \hat{U}_n(x_n) - x_n \hat{p}\left(\sum_m x_m\right) = Q_n(x_n; \boldsymbol{x}_{-n}). \tag{48}$$

3. *A vector $\boldsymbol{x}^S$ solves* (1)-(2) *with utility functions $\hat{U}_1, \ldots, \hat{U}_N$ and cost function $\hat{C}$ if and only if $\boldsymbol{x}^S$ solves* (35)-(36)*.*

4. *A vector $\boldsymbol{x}$ is a Nash equilibrium of the game defined by $(T_1, \ldots, T_N)$ if and only if $\boldsymbol{x}$ is a Nash equilibrium of the game defined by $(Q_1, \ldots, Q_N)$ when the utility function of consumer n is $\hat{U}_n$ and the cost function is $\hat{C}$.*

Proof. We prove each of the four claims of the theorem in four separate steps.

Proof of Claim 1. Because C_n is continuously differentiable and convex for each n, we conclude that $\hat{U}_n'$ is continuous and nonincreasing for each n. Furthermore, by definition of Γ, we have $\hat{U}_n'(x_n) \geq 0$ for all x_n. Thus Assumption 1 is satisfied by $(\hat{U}_1, \ldots, \hat{U}_N)$.

Next, observe that $\hat{p}(0) = \Gamma - p(0) \geq 0$; and since $p(q)$ is continuous, concave, and nonincreasing, $\hat{p}(q)$ is continuous, convex, and nondecreasing. Finally, since $p(q) \to -\infty$ as $q \to \infty$, we conclude $\hat{p}(q) \to \infty$ as $q \to \infty$. Thus Assumption 2 is satisfied by $\hat{p}$ and $\hat{C}$.

Proof of Claim 2. Suppose we are given a vector $\boldsymbol{x}$ such that $x_n \leq q_{\max}$ for all n. Let $q = \sum_n x_n$. We argue as follows:

$$\begin{aligned} U(q) &= \int_0^q p(z)\, dz \\ &= \Gamma q - \int_0^q \hat{p}(z)\, dz \\ &= \Gamma q - \hat{C}(q). \end{aligned}$$

Similarly, since $x_n \leq q_{\max}$ and $C_n(0) = 0$ for each n, we have:

$$\begin{aligned} C_n(x_n) &= \int_0^{x_n} C_n'(z)\, dz \\ &= \Gamma x_n - \int_0^{x_n} \hat{U}_n'(z)\, dz \\ &= \Gamma x_n - \hat{U}_n(x_n). \end{aligned}$$

Finally, since $\hat{p}(q) = \Gamma - p(q)$, we have:

$$x_n p(q) = \Gamma x_n - x_n \hat{p}(q).$$

Thus we have:

$$U(q) - \sum_n C_n(x_n) = \sum_n \hat{U}_n(x_n) - \hat{C}(q),$$

as well as:

$$x_n p(q) - C_n(x_n) = \hat{U}_n(x_n) - x_n \hat{p}(q).$$

Proof of Claim 3. We already know that any optimal solution $\boldsymbol{x}^S$ to (35)-(36) must satisfy $\sum_n x_n \leq q_{\max}$. In light of (47), therefore, to establish Claim 3 it suffices to show the following: any solution $\boldsymbol{x}^S$ to (1)-(2) with utility functions $\hat{U}_1, \ldots, \hat{U}_N$ and cost function $\hat{C}$ satisfies $q^S = \sum_n x_n^S \leq q_{\max}$ as well. Suppose not; then we have $q^S > q_{\max}$, so that $p(q) < 0$. But then $\hat{p}(q^S) > \Gamma$, while $\hat{U}_n'(x_n^S) \leq \Gamma$ for all n. Thus for any user n with $x_n^S > 0$, we have $\hat{U}_n'(x_n^S) < \hat{p}(q^S)$, contradicting the optimality of $\boldsymbol{x}^S$. Thus we must have $\sum_n x_n^S \leq q^S$, and Claim 3 is established.

Proof of Claim 4. The proof is very similar to the proof of Claim 3. In light of (48), it suffices to show that: (a) no producers would ever choose $x_n > q_{\max}$ in the game defined by $(T_1, \ldots, T_N)$; and (b) no consumers would ever choose $x_n > q_{\max}$ in the game defined by $(Q_1, \ldots, Q_N)$. The proof of (a) is similar to the argument given in the proof of Proposition 2: the payoff to a producer n is always negative if $x_n > q_{\max}$, so such a choice is strictly dominated by $x_n = 0$, which yields payoff zero. To prove (b), fix a strategy vector $\boldsymbol{x}$, and let $q = \sum_n x_n$. Note that $\hat{U}_n'(x_n) \leq \Gamma$, while if $x_n > q_{\max}$, then $\hat{p}(q) > \Gamma$ (since $p(q) < 0$). Thus, since $\hat{U}_n$ is concave, nonnegative, and nondecreasing, we have $\hat{U}_n(x_n) \leq \hat{U}_n'(x_n) x_n \leq \Gamma x_n < \hat{p}(q) x_n$. Thus the payoff to consumer n is negative, while choosing $x_n = 0$ yields a payoff to player n of zero. Thus any strategy $x_n > q_{\max}$ is strictly dominated by the choice $x_n = 0$; so we conclude (b) holds as well. Combining (48) with claims (a) and (b), Claim 4 is established. □

Given an oligopoly model, the preceding theorem constructs an oligopsony model which shares all the properties of the oligopoly model—in terms of aggregate surplus, efficient allocations, and Nash equilibria. Using the preceding theorem, we can prove analogues of the main results of Section 2 with little additional effort. We start with the following three results, which follow directly from Corollaries 7, 8, and 10; and Theorem 15. Their proofs are therefore omitted.

Corollary 16 *Suppose that $N \geq 1$ producers share the same cost function $C_n(x_n) = \alpha x_n$, where $\alpha > 0$; in addition, suppose that Assumption 4 holds. If $\boldsymbol{x}^S$ solves* (35)-(36)*, and $\boldsymbol{x}$ is a Nash equilibrium of the game defined by $(T_1, \ldots, T_N)$, then:*

$$U\left(\sum_n x_n\right) - \sum_n C_n(x_n) \geq \left(\frac{2N}{2N+1}\right)\left(U\left(\sum_n x_n^S\right) - \sum_n C_n(x_n^S)\right). \tag{49}$$

Corollary 17 *Suppose that $N = 1$, and producer 1 has a cost function C such that Assumption 3 holds; in addition, suppose that Assumption 4 holds. If x^S solves* (35)-(36)*, and x maximizes $\overline{x}p(\overline{x}) - C(\overline{x})$ over $\overline{x} \geq 0$, then:*

$$U(x) - C(x) \geq \frac{2}{3}\left(U(x^S) - C(x^S)\right). \tag{50}$$

Corollary 18 *Suppose that $N \geq 1$ producers share the same cost function $C_n = C$, such that Assumption 3 holds; in addition, suppose that Assumption 4 holds, and that p is differentiable in the region $(0, q_{\max})$. If $\boldsymbol{x}^S$ solves* (35)-(36)*, and $\boldsymbol{x}$ is a Nash equilibrium of the game defined by $(T_1, \ldots, T_N)$, then:*

$$U\left(\sum_n x_n\right) - \sum_n C_n(x_n) \geq \left(\frac{2N}{2N+1}\right)\left(U\left(\sum_n x_n^S\right) - \sum_n C_n(x_n^S)\right). \tag{51}$$

Corollary 16 is closely related to the results of Anderson and Renault [1]. Specifically, by using equation (12) of [1], it is possible to show that the aggregate surplus at a Nash equilibrium is no worse than a factor $2N/(N+1)^2$ of the maximal aggregate surplus, when N firms share the same linear cost function, and demand satisfies Assumption 4. However, the bound in [1] allows the efficiency loss to approach 100% as $N \to \infty$, whereas the result of Corollary 16 shows that efficiency loss approaches zero as $N \to \infty$ (a competitive limit, as expected).

Note also that Corollary 17 establishes an efficiency loss result for the classical monopoly model [24]. The efficiency loss is no worse than 33%, allowing for both a general convex producer cost function and general concave demand.

The next result is an analogue of Theorem 12.

Theorem 19 *Suppose that Assumption 4 holds, and that $p(q) = b - aq$ for some $a > 0, b > 0$. If $\boldsymbol{x}^S$ is any solution to* (35)-(36)*, and $\boldsymbol{x}$ is any Nash equilibrium of the game defined by $(T_1, \ldots, T_N)$, then:*

$$U\left(\sum_n x_n\right) - \sum_n C_n(x_n) \geq \frac{2}{3}\left(U\left(\sum_n x_n^S\right) - \sum_n C_n(x_n^S)\right). \tag{52}$$

Furthermore, this bound is tight: for every $a > 0$, $b > 0$, and $\delta > 0$, there exists a choice of N and a choice of (linear) cost functions C_n, $n = 1, \ldots, N$, such that a Nash equilibrium $\boldsymbol{x}$ exists with:

$$\sum_n U_n(x_n) - C\left(\sum_n x_n\right) \leq \left(\frac{2}{3} + \delta\right)\left(\sum_n U_n(x_n^S) - C\left(\sum_n x_n^S\right)\right). \tag{53}$$

Proof. Note that if p is affine with negative slope and positive intercept, then $\hat{p}$ as defined in (46) is also affine with positive slope and intercept; thus (52) follows by Theorems 12 and 15.

To establish (53), fix a price function $p(q) = b - aq$ with $a, b > 0$, and define C_n as follows:

$$\begin{aligned} &C_1(x_1) = 0; \\ &C_n(x_n) = \left(\frac{b}{3}\right)\left(\frac{N-2}{N-1}\right)x_n, \quad n = 2, \ldots, N. \end{aligned}$$

Let $\overline{x} = b/(3a(N-1))$. It is then straightforward to check that for sufficiently large N, if $x_1 = b/3a$ and $x_n = \overline{x}$ for $n = 2, \ldots, N$, the allocation $\boldsymbol{x}$ is a Nash equilibrium; to see this, simply note that (42) holds with equality for all n. At this Nash equilibrium, the resulting aggregate surplus is given by:

$$\frac{4b^2}{9a} - \frac{b^2}{9a} \cdot \frac{N-2}{N-1}.$$

Furthermore, the maximum aggregate surplus is achieved by choosing q^S so that $p(q^S) = 0$, so $q^S = b/a$, $x_1^S = q^S = b/a$, and $x_n^S = 0$ for $n = 2, \ldots, N$. This yields maximal aggregate surplus $b^2/(2a)$. Thus as $N \to \infty$, the ratio of Nash equilibrium aggregate surplus to maximal aggregate surplus approaches $2/3$, as required. □

Finally, as in Example 2, there exist oligopoly models where all producers share an identical cost function, and the price function p is not differentiable, for which the efficiency loss can be arbitrarily high at a Nash equilibrium. For completeness we present such an example.

Example 4 Let the number of producers be $N > 1$. We build an example which is analogous to Example 2. Define $\alpha = 1/N^2$ and $\hat{x} = (\alpha + 1)/N$. Let the price function p be given by:

$$p(q) = \begin{cases} \alpha, & \text{if } q \leq 1; \\ \alpha - (q-1), & \text{if } q \geq 1. \end{cases}$$

Note that with this definition we have $q_{\max} = \alpha + 1$. Define a single cost function $C(x)$ by $C(x) = \alpha(x - \hat{x})^+$. We assume all producers share the same cost function C.

It is now straightforward to establish that under the transformation described in Theorem 15, we have $\Gamma = \alpha$ and we recover exactly the model described in Example 2. It thus follows that as $N \to \infty$, the efficiency loss can be arbitrarily high when producers are price anticipating. □

4 Conclusion

This paper has considered models of both Cournot oligopsony and Cournot oligopoly, and established bounds on efficiency loss in both cases—i.e., bounds on the ratio of aggregate surplus at a Nash equilibrium to the maximum possible aggregate surplus. We find that while efficiency loss is generally arbitrarily high, in several special cases of interest the efficiency loss may be bounded. The most interesting results are those which hold independent of the characteristics of the market participants, Theorem 12 and Corollary 19. These results show that for general concave consumer utility functions with affine market supply (for Cournot oligopsony), or for general convex producer cost functions with affine market demand (for Cournot oligopoly), the efficiency loss is no worse than $1/3$ of the maximal aggregate surplus.

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