

Coordination with Local Information

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We study the role of local information channels in enabling coordination among strategic agents. Building on the standard finite-player global games framework, we show that the set of equilibria of a coordination game is highly sensitive to how information is locally shared among different agents. In particular, we show that the coordination game has multiple equilibria if there exists a collection of agents such that (i) they do not share a common signal with any agent outside of that collection and (ii) their information sets form an increasing sequence of nested sets. Our results thus extend the results on the uniqueness and multiplicity of equilibria beyond the well-known cases in which agents have access to purely private or public signals. We then provide a characterization of the set of equilibria as a function of the penetration of local information channels. We show that the set of equilibria shrinks as information becomes more decentralized.

Keywords: coordination; local information; social networks; global games.

Subject classifications: games/group decisions: economics; network/graphs.

Area of review: Special Issue on Information and Decisions in Social and Economic Networks.

History: Received July 2013; revisions received October 2014, January 2015; accepted February 2015. Published online in *Articles in Advance* November 12, 2015.

1. Introduction

Coordination problems lie at the heart of many economic and social phenomena such as bank runs, social uprisings, and the adoption of new standards or technologies. The common feature of these phenomena is that the benefit of taking a specific action to any given individual is highly sensitive to the extent to which other agents take the same action. The presence of such strong strategic complementarities, coupled with the self-fulfilling nature of the agents' expectations, may then lead to coordination failures: individuals may fail to take the action that is in their best collective interest.

Bank runs present a concrete (and classic) example of the types of coordination failures that may arise due to the presence of strategic complementarities. In this context, any given depositor has strong incentives to withdraw her money from the bank if (and only if) she expects that a large fraction of other depositors would do the same. Thus, a bank run may emerge, not because of any financial distress at the bank, but rather as a result of the self-fulfilling nature of the depositors' expectations about other depositors' behavior (Diamond and Dybvig 1983). Similarly, in the context of adoption of new technologies with strong network effects, consumers may collectively settle for an

inferior product simply because they expect other agents to do the same (Argenziano 2008).¹

Given the central role of self-fulfilling beliefs in coordination games, it is natural to expect the emergence of coordination failures to be highly sensitive to the availability and distribution of information across different agents. In fact, since the seminal work of Carlsson and van Damme (1993), which initiated the global games literature, it has been well known that the set of equilibria of a coordination game depends on whether the information available to the agents is public or private: the same game that exhibits multiple equilibria in the presence of public signals may have a unique equilibrium if the information were instead only privately available to the agents.

The global games literature, however, has for the most part only focused on the role of public and private information while ignoring the effects of local information channels in facilitating coordination. This is despite the fact that in many real world scenarios, local information channels play a key role in enabling agents to coordinate on different actions. For instance, it is by now conventional wisdom that protesters in many recent antigovernment uprisings throughout the Middle East used decentralized modes of communication (such as word-of-mouth communications

and Internet-based social media platforms) to coordinate on the time and location of street protests (Ali 2011). Similarly, adopters of new technologies that exhibit strong network effects may rely on a wide array of information sources (such as blogs, professional magazines, expert opinions), not all of which are used by all other potential adopters in the market.

Motivated by these observations, we study the role of local information channels in enabling coordination among strategic agents. Building on the standard finite-player global games framework, we show that the set of equilibria of a coordination game is highly sensitive to how information is locally shared among agents. More specifically, rather than restricting our attention to cases in which information is either purely public or private, we also allow for the presence of *local* signals that are observable to subsets of the agents. The presence of such local sources of information guarantees that some (but not necessarily all) information is common knowledge among a group of agents, with important implications for the determinacy of equilibria. Our main contribution is to provide conditions for uniqueness and multiplicity of equilibria based solely on the pattern of information sharing among agents. Our findings thus provide a characterization of the extent to which coordination failures may arise as a function of which piece of information is available to each agent.

As our main result, we show that the coordination game has multiple equilibria if there exists a collection of agents such that (i) they do not share a common signal with any agent outside of that collection and (ii) their information sets form an increasing sequence of nested sets, which we refer to as a filtration. This result is a consequence of the fact that agents in such a collection face limited *strategic uncertainty*—that is, uncertainty concerning the equilibrium actions—about one another,² which transforms their common signals into a *de facto* coordination device. To see this in the most transparent way, consider two agents whose information sets form a filtration. It is clear that the agent with the larger information set (say, agent 2) faces no uncertainty in predicting the equilibrium action of the agent with the smaller information set (agent 1). Moreover, even though the latter is not informed about other potential signals observed by the former, the realizations of such signals may not be extreme enough to push agent 2 toward either action, thus making it optimal for her to simply follow the behavior of agent 1. In other words, for certain realizations of signals, the agent with the smaller information set becomes pivotal in determining the payoff maximizing action of agent 2. Consequently, the set of signals that are common between the two can effectively function as a coordination device, leading to the emergence of multiple equilibria.

We then focus on a special case of our model by assuming that each agent observes a single signal. We provide an explicit characterization of the set of equilibria in terms of the commonality in agents' information sets and show

that the set of equilibrium strategies enlarges if information is more centralized. In other words, as the signals become more publicly available, the set of states under which agents can coordinate on either action grows. This result is due to the fact that more information concentration reduces the extent of strategic uncertainty among the agents, even if it does not impact the level of fundamental uncertainty.

We then use our characterization results to study the set of equilibria in large coordination games. We show that as the number of agents grows, the game exhibits multiple equilibria if and only if a nontrivial fraction of the agents have access to the same signal. Our result thus shows that if the size of the subsets of agents with common knowledge of a signal does not grow at the same rate as the number of agents, the information structure is asymptotically isomorphic to a setting in which all signals are private. Consequently, all agents face some strategic uncertainty regarding the behavior of almost every other agent, resulting in a unique equilibrium.

Finally, even though our benchmark model assumes that the agents' information sets are common knowledge, we also show that our results are robust to the introduction of small amounts of noise in the information structure of the game. In particular, by introducing uncertainty about the information sets of different agents, we show that the equilibria of this new, perturbed game are close (in a formal sense) to the equilibria of the original game in which all information sets were common knowledge.

In sum, our results establish that the distribution and availability of information play a fundamental role in determining coordination outcomes, thus highlighting the risk of abstracting from the intricate details of information dissemination within coordination contexts. For example, on the positive side, ignoring the distribution and reach of different information sources can lead to wrong predictions about potential outcomes. Similarly, on the normative side (say for example, in the context of a central bank facing a banking crisis), the interventions of policymakers may turn out to be counterproductive if such actions are not informed by the dispersion and penetration of different information channels within the society.

Related Literature. Our paper is part of the by now large literature on global games. Initiated by the seminal work of Carlsson and van Damme (1993) and later expanded by Morris and Shin (1998, 2003), this literature focuses on how the presence of strategic uncertainty may lead to the selection of a unique equilibrium in coordination games. The machinery of global games has since been used extensively to analyze various applications that exhibit an element of coordination, such as currency attacks (Morris and Shin 1998), bank runs (Goldstein and Pauzner 2005), political protests (Edmond 2013), partnership investments (Dasgupta 2007), emergence of financial crises (Vives 2014), and adoption of technologies and products that exhibit strong network effects (Argenziano 2008). For example, Argenziano (2008) uses the global games framework to

study a model of price competition in a duopoly with product differentiation and network effects. She shows that the presence of a sufficient amount of noise in the public information available about the quality of the goods guarantees that the coordination game played by the consumers for any given set of prices has a unique equilibrium, and hence the demand function for each product is well defined.

All the above papers, however, restrict their attention to the case in which all information is either public or private. Our paper, on the other hand, focuses on more general information structures by allowing for signals that are neither completely public nor private and provides conditions under which the presence of such signals may lead to equilibrium multiplicity.

Our paper is also related to the works of Morris et al. (2015) and Mathevet (2014), who characterize the set of equilibria of coordination games in terms of the agents' types while abstracting away from the information structure of the game. Relatedly, Weinstein and Yildiz (2007) provide a critique of the global games approach by arguing that any rationalizable action in a game can always be uniquely selected by properly perturbing the agents' hierarchy of beliefs. Our work, on the other hand, provides a characterization of the set of equilibrium strategies when the perturbation in the beliefs are generated by public, private, or local signals that are informative about the underlying state. Despite being more restrictive in scope, our results shed light on the role of local information in enabling coordination.

A different set of papers studies how the endogeneity of agents' information structure in coordination games may lead to equilibrium multiplicity, thus qualifying the refinement of equilibria proposed by the standard global games argument. For example, Angeletos and Werning (2006), and Angeletos et al. (2006, 2007) show how prices, the action of a policymaker, or past outcomes can function as endogenous public signals that may restore equilibrium multiplicity in settings that would have otherwise exhibited a unique equilibrium. We study another natural setting in which agents may rely on overlapping (but not necessarily identical) sources of information and show how the information structure affects the outcomes of coordination games.

Our paper also belongs to the strand of literature that focuses on the role of local information channels and social networks in shaping economic outcomes. Some recent examples include Acemoglu et al. (2011), Golub and Jackson (2010), Jadbabaie et al. (2012, 2013), and Galeotti et al. (2013). For example, Acemoglu et al. (2011) study how patterns of local information exchange among few agents can have first-order (and long-lasting) implications for the actions of others.

Within this context, however, our paper is more closely related to the subset of papers that studies coordination games over networks, such as Galeotti et al. (2010) and

Chwe (2000). Galeotti et al. (2010) focus on strategic interactions over networks and characterize how agents' interactions with their neighbors and the nature of the game shape individual behavior and payoffs. They show that the presence of incomplete information about the structure of the network may lead to the emergence of a unique equilibrium. The key distinction between their model and ours is in the information structure of the game and the nature of payoffs. Whereas they study an environment in which individuals care about the actions of their neighbors (whose identities they are uncertain about), we study a setting in which network effects are only reflected in the game's information structure: individuals need to make deductions about an unknown parameter and the behavior of the rest of the agents while relying on local sources of information.

Chwe (2000), on the other hand, studies a coordination game in which individuals can inform their neighbors of their willingness to participate in a collective risky behavior. Thus, our work shares two important aspects with that of Chwe (2000): not only do both papers study games with strategic complementarities, but also consider information structures in which information about the payoff-relevant parameters are locally shared among different individuals. Nevertheless, the two papers focus on different questions. Whereas Chwe's main focus is on characterizing the set of networks for which, regardless of their prior beliefs, agents can coordinate on a specific action, our results provide a characterization of how the penetration of different channels of information can impact coordination outcomes.

Outline of the Paper. The rest of the paper is organized as follows: §2 introduces our model. In §3, we present a series of simple examples that capture the intuition behind our results. Section 4 contains our results on the role of local information channels in the determinacy of equilibria. We then provide a characterization of the set of equilibria in §5 and show that the equilibrium set shrinks as information becomes more decentralized. Section 6 concludes. All proofs are presented in the appendix.

2. Model

Our model is a finite-agent variant of the canonical model of global games studied by Morris and Shin (2003).

2.1. Agents and Payoffs

Consider a coordination game played by n agents whose set we denote by $N = \{1, 2, \dots, n\}$. Each agent can take one of two possible actions, $a_i \in \{0, 1\}$, which we refer to as the *safe* and *risky* actions, respectively. The payoff of taking the safe action is normalized to zero, regardless of the actions of other agents. The payoff of taking the risky action, on the other hand, depends on (i) the number of other agents who take the risky action and (ii) an underlying state of the world $\theta \in \mathbb{R}$, which we refer to as *the fundamental*. In particular, the payoff function of agent i is

$$u_i(a_i, a_{-i}, \theta) = \begin{cases} \pi(k, \theta) & \text{if } a_i = 1, \\ 0 & \text{if } a_i = 0, \end{cases}$$

where $k = \sum_{j=1}^n a_j$ is the number of agents who take the risky action and $\pi: \{0, 1, \dots, n\} \times \mathbb{R} \rightarrow \mathbb{R}$ is a function that is Lipschitz continuous in its second argument. Throughout, we impose the following assumptions on the agents' payoff function, which are standard in the global games literature.

ASSUMPTION 1. *Function $\pi(k, \theta)$ is strictly increasing in k for all θ . Furthermore, there exists a constant $\rho > 0$ such that $\pi(k, \theta) - \pi(k - 1, \theta) > \rho$ for all θ and all $k \geq 1$.*

The above assumption captures the presence of an element of coordination between agents. In particular, taking either action becomes more attractive the more other agents take that action. For example, in the context of a bank run, each depositor's incentive to withdraw her deposits from a bank not only depends on the solvency of the bank (i.e., its fundamental) but also increases with the number of other depositors who decide to withdraw. Similarly, in the context of adoption of technologies that exhibit network effects, each user finds adoption more attractive the more widely the product is adopted by other users in the market. Thus, the above assumption simply guarantees that the game exhibits strategic complementarities. The second part of Assumption 1 is made for technical reasons and states that the payoff of switching to the risky action when one more agent takes action 1 is uniformly bounded from below, regardless of the value of θ .

ASSUMPTION 2. *Function $\pi(k, \theta)$ is strictly decreasing in θ for all k .*

That is, any given individual has less incentive to take the risky action if the fundamental takes a higher value. Thus, taking other agents' actions as given, each agent's optimal action is decreasing in the state. For instance, in the bank run context, if the fundamental value θ corresponds to the financial health of the bank, the depositors' withdrawal incentives are stronger the closer the bank is to insolvency. Similarly, in the context of the adoption of network goods, θ can represent the quality of the status quo technology relative to the new alternative in the market: regardless of the market shares, a higher θ makes adoption of the new technology less attractive. Finally, we impose the following assumption:

ASSUMPTION 3. *There exist constants $\underline{\theta}, \bar{\theta} \in \mathbb{R}$ satisfying $\underline{\theta} < \bar{\theta}$ such that*

- (i) $\pi(k, \theta) > 0$ for all k and all $\theta < \underline{\theta}$.
- (ii) $\pi(k, \theta) < 0$ for all k and all $\theta > \bar{\theta}$.

Thus, each agent strictly prefers to take the safe (risky) action for sufficiently high (low) states of the world, irrespective of the actions of other agents. If, on the other hand, the underlying state belongs to the so-called *critical region* $[\underline{\theta}, \bar{\theta}]$, then the optimal behavior of each agent depends on her beliefs about the actions of other agents. Thus, once again, in the bank run context, the above assumption simply means that each depositor finds withdrawing (leaving)

her deposit a dominant action if the bank is sufficiently distressed (healthy).

In summary, agents face a coordination game with strong strategic complementarities in which the value of coordinating on the risky action depends on the underlying state of the world. Furthermore, particular values of the state make either action strictly dominant for all agents.

2.2. Information and Signals

As in the canonical global games model, agents are not aware of the realization of the fundamental. Rather, they hold a common prior belief on $\theta \in \mathbb{R}$, which for simplicity we assume to be the (improper) uniform distribution over the real line. Furthermore, conditional on the realization of θ , a collection $(x_1, \dots, x_m) \in \mathbb{R}^m$ of noisy signals is generated, where $x_r = \theta + \xi_r$. We assume that the noise terms (ξ_1, \dots, ξ_m) are independent from θ and are drawn from a continuous joint probability density function with full support over \mathbb{R}^m .

Not all agents, however, can observe all realized signals. Rather, agent i has access to a nonempty subset $I_i \subseteq \{x_1, \dots, x_m\}$ of the signals, which we refer to as her *information set*.³ This assumption essentially captures that each agent may have access to multiple sources of information about the underlying state. Going back to the bank run example, this means that each depositor may obtain some information about the health of the bank via a variety of information sources (such as the news media, the results of stress tests released by the regulatory agencies, inside information, expert opinions, etc.). The depositor would then use the various pieces of information available to her to decide whether to run on the bank or not.

The fact that agent i 's information set can be any (nonempty) subset of $\{x_1, \dots, x_m\}$ means that the extent to which any given signal x_r is observed may vary across signals. For example, if $x_r \in I_i$ for all i , then x_r is essentially a *public signal* observed by all agents. On the other hand, if $x_r \in I_i$ for some i but $x_r \notin I_j$ for all $j \neq i$, then x_r is a *private signal* of agent i . Any signal that is neither private nor public can be interpreted as a *local* source of information observed only by a proper subset of the agents. The potential presence of such local signals is our point of departure from the canonical global games literature, which only focuses on games with purely private or public signals. Note that in either case, following the realization of signals, the mean of agent i 's posterior belief about the fundamental is simply equal to her Bayesian estimate $\mathbb{E}[\theta | I_i]$.

We remark that even though agents are uncertain about the realizations of the signals they do not observe, the *information structure* of the game—that is, the collection of information sets $\{I_i\}_{i \in N}$ —is common knowledge among them.⁴ Therefore, a pure strategy of agent i is simply a mapping $s_i: \mathbb{R}^{|I_i|} \rightarrow \{0, 1\}$, where $|I_i|$ denotes the cardinality of agent i 's information set.

Finally, we impose the following mild technical assumption, ensuring that the agents' payoff function is integrable.

ASSUMPTION 4. For any collection of signals $\{x_r\}_{r \in H}$, $H \subseteq \{1, \dots, m\}$ and all $k \in \{0, \dots, n\}$,

$$\mathbb{E}[\pi(k, \theta) \mid \{x_r\}_{r \in H}] < \infty.$$

We end this section by remarking that given that the payoff functions have nondecreasing differences in the actions and the state, the underlying game is a Bayesian game with strategic complementarities. Therefore, by Van Zandt and Vives (2007, Theorem 14), the game’s Bayesian Nash equilibria form a lattice whose extremal elements are monotone in types. This lattice structure also implies that the extremal equilibria are symmetric, in the sense that each agent’s strategy only depends on the signals she observes but not on her identity (Vives 2008). Therefore, to characterize the range of equilibria, without loss of generality, we can restrict our attention to symmetric equilibria in threshold strategies.

3. A Simple Example

Before proceeding to our general results, we present a simple example and show how the presence of local signals determines the set of equilibria. Consider a game consisting of $n = 3$ agents and, for expositional simplicity, suppose that

$$\pi(k, \theta) = \frac{k - 1}{n - 1} - \theta. \tag{1}$$

It is easy to verify that this payoff function, which is similar to the one in the canonical finite-player global games literature such as Morris and Shin (2000, 2003), satisfies Assumptions 1–3 with $\underline{\theta} = 0$ and $\bar{\theta} = 1$. We consider three different information structures, contrasting the cases in which agents have access to only public or private information to a case with a local signal. Throughout this section, we assume that the noise terms ξ_i are mutually independent and normally distributed with mean zero and variance $\sigma^2 > 0$.

Public Information. First, consider a case in which all agents observe the same public signal x , that is, $I_i = \{x\}$ for all $i \in \{1, 2, 3\}$. Thus, no agent has any private information about the state. It is easy to verify that under such an information structure, the coordination game has multiple Bayesian Nash equilibria. In particular, for any $\tau \in [0, 1]$, the strategy profile in which all agents take the risky action if and only if $x < \tau$ is an equilibrium, regardless of the value of σ . Consequently, as the public signal becomes infinitely accurate (i.e., as $\sigma \rightarrow 0$), the underlying game has multiple equilibria as long as the underlying state θ belongs to the critical region $[0, 1]$.

Private Information. Next, consider the case in which all agents have access to a different private signal. In particular, suppose that three signals $(x_1, x_2, x_3) \in \mathbb{R}^3$ are realized and that x_i is privately observed by agent i ; that is, $I_i = \{x_i\}$ for all i . As is well known from the global games literature, the coordination game with privately observed signals

has an essentially unique Bayesian Nash equilibrium. To verify that the equilibrium of the game is indeed unique, it is sufficient to focus on the set of equilibria in threshold strategies, according to which each agent takes the risky action if and only if her private signal is smaller than a given threshold.⁵ In particular, let τ_i denote the threshold corresponding to the strategy of agent i . Taking the strategies of agents j and k as given, agent i ’s expected payoff of taking the risky action is equal to $\mathbb{E}[\pi(k, \theta) \mid x_i] = \frac{1}{2}[\mathbb{P}(x_j < \tau_j \mid x_i) + \mathbb{P}(x_k < \tau_k \mid x_i)] - x_i$. For τ_i to correspond to an equilibrium strategy of agent i , she has to be indifferent between taking the safe and the risky actions when $x_i = \tau_i$. Hence, the collection of thresholds (τ_1, τ_2, τ_3) corresponds to a Bayesian Nash equilibrium of the game if and only if for all permutations of i, j , and k , we have

$$\tau_i = \frac{1}{2} \Phi\left(\frac{\tau_j - \tau_i}{\sigma\sqrt{2}}\right) + \frac{1}{2} \Phi\left(\frac{\tau_k - \tau_i}{\sigma\sqrt{2}}\right),$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal. Note that $\theta \mid x_i \sim \mathcal{N}(x_i, \sigma^2)$, and as a result, $x_j \mid x_i \sim \mathcal{N}(x_i, 2\sigma^2)$. It is then immediate to verify that $\tau_1 = \tau_2 = \tau_3 = 1/2$ is the unique solution of the above system of equations. Thus, in the (essentially) unique equilibrium of the game, agent i takes the risky action if she observes $x_i < 1/2$, whereas she takes the safe action if $x_i > 1/2$. Following standard arguments from Carlsson and van Damme (1993) or Morris and Shin (2003), one can show that this strategy profile is also the (essentially) unique strategy profile that survives the iterated elimination of strictly dominated strategies. Consequently, in contrast to the game with public information, as $\sigma \rightarrow 0$, all agents choose the risky action if and only if $\theta < 1/2$. This observation shows that in the limit as signals become arbitrarily precise and all the fundamental uncertainty is removed, the presence of strategic uncertainty among the agents leads to the selection of a unique equilibrium.

Local Information. Finally, consider the case where only two signals (x_1, x_2) are realized and the agents’ information sets are $I_1 = \{x_1\}$ and $I_2 = I_3 = \{x_2\}$; that is, agent 1 observes a private signal whereas agents 2 and 3 have access to the same local source of information. All the information available to agents 2 and 3 is common knowledge between them, which distinguishes this case from the canonical global game model with private signals.

To determine the extent of equilibrium multiplicity, once again it is sufficient to focus on the set of equilibria in threshold strategies. Let τ_1 and $\tau_2 = \tau_3$ denote the thresholds corresponding to the strategies of agents 1, 2, and 3, respectively. If agent 1 takes the risky action, she obtains an expected payoff of $\mathbb{E}[\pi(k, \theta) \mid x_1] = \mathbb{P}(x_2 < \tau_2 \mid x_1) - x_1$. On the other hand, the expected payoff of taking the risky action to agent 2 (and, by symmetry, agent 3) is given by

$$\mathbb{E}[\pi(k, \theta) \mid x_2] = \frac{1}{2}[\mathbb{P}(x_1 < \tau_1 \mid x_2) + \mathbb{1}\{x_2 < \tau_2\}] - x_2,$$

where $\mathbb{1}\{\cdot\}$ denotes the indicator function. Thus, for thresholds τ_1 and τ_2 to correspond to a Bayesian Nash equilibrium, it must be the case that

$$\tau_1 = \Phi\left(\frac{\tau_2 - \tau_1}{\sigma\sqrt{2}}\right). \quad (2)$$

Furthermore, agents 2 and 3 should not have an incentive to deviate, which requires that their expected payoffs have to be positive for any $x_2 < \tau_2$ and negative for any $x_2 > \tau_2$, leading to the following condition:

$$2\tau_2 - 1 \leq \Phi\left(\frac{\tau_1 - \tau_2}{\sigma\sqrt{2}}\right) \leq 2\tau_2. \quad (3)$$

It is easy to verify that the pair of thresholds (τ_1, τ_2) simultaneously satisfying (2) and (3) is not unique. In particular, as $\sigma \rightarrow 0$, for every $\tau_1 \in [1/3, 2/3]$, there exists some τ_2 close enough to τ_1 such that the profile of threshold strategies (τ_1, τ_2, τ_2) is a Bayesian Nash equilibrium. Consequently, as the signals become very precise, the underlying game has multiple equilibria as long as $\theta \in [\frac{1}{3}, \frac{2}{3}]$.

Thus, even though there are no public signals, the presence of common knowledge between a proper subset of the agents restores the multiplicity of equilibria. In other words, the local information available to agents 2 and 3 serves as a coordination device, enabling them to predict one another's actions. The presence of strong strategic complementarities in turn implies that agent 1 will use his private signal as a predictor of how agents 2 and 3 coordinate their actions. Nevertheless, because of the presence of some strategic uncertainty between agents 2 and 3 on the one hand and agent 1 on the other, the set of rationalizable strategies is strictly smaller compared to the case where all three agents observe a public signal. Therefore, the local signal (partially) refines the set of equilibria of the coordination game, though not to the extent that would lead to uniqueness.

4. Local Information and Equilibrium Multiplicity

The examples in the previous section show that the set of equilibria in the presence of local signals may not coincide with the set of equilibria under purely private or public signals. In this section, we provide a characterization of the role of local information channels in determining the equilibria of the coordination game presented in §2. Before presenting our main result, we define the following concept:

DEFINITION 1. The information sets of agents in $C = \{i_1, \dots, i_c\}$ form a *filtration* if $I_{i_1} \subseteq I_{i_2} \subseteq \dots \subseteq I_{i_c}$.

Thus, the information sets of agents in set C constitute a filtration if they form a nested sequence of increasing sets. This immediately implies that the signals of the agent with the smallest information set is common knowledge among all agents in C . We have the following result:

THEOREM 1. Suppose that Assumptions 1–4 are satisfied. Also, suppose that there exists a subset of agents $C \subseteq N$ such that

- (a) The information sets of agents in C form a filtration.
- (b) $I_i \cap I_j = \emptyset$ for any $i \in C$ and $j \notin C$.

Then the coordination game has multiple Bayesian Nash equilibria.

The above result thus shows that, in general, the presence of a cascade of increasingly rich observations guarantees equilibrium multiplicity. Such filtration provides some degree of common knowledge among the subset of agents in C , reduces the extent of strategic uncertainty they face regarding each other's behavior, and as a result leads to the emergence of multiple equilibria. Therefore, in this sense, Theorem 1 generalizes the standard, well-known results in the global games literature to the case when agents have access to local information channels that are neither public nor private.

We remark that even though the presence of local signals may lead to equilibrium multiplicity, the set of Bayesian Nash equilibria does not necessarily coincide with that of a game with purely public signals. Rather, as we show in the next section, the set of equilibria crucially depends on the number of agents in C as well as the information structure of other agents.

To see the intuition underlying Theorem 1, suppose that the information sets of agents in some set C form a filtration. It is immediate that there exists an agent $i \in C$ whose signals are observable to all other agents in C . Consequently, agents in C face no uncertainty (strategic or otherwise) in predicting the equilibrium actions of agent i . Furthermore, the signals in $I_j \setminus I_i$ provide agent $j \in C \setminus \{i\}$ with information about the underlying state θ beyond what agent i has access to. Nevertheless, there exist realizations of such signals such that the expected payoff of taking the risky action to any agent $j \in C \setminus \{i\}$ would be positive if and only if agent i takes the risky action. In other words, for such realizations of signals, agent i 's action becomes pivotal in determining the payoff maximizing actions of all other agents in C . Consequently, signals in I_i can essentially serve as a coordination device among C 's members, leading to multiple equilibria.

Finally, note that condition (b) of Theorem 1 plays a crucial role in the above argument. In particular, it guarantees that agents in C effectively face an induced coordination game among themselves, in which they can use the signals in I_i as a coordination device.

4.1. The Role of Information Filtrations

In this subsection, we use a series of examples to exhibit the main intuition underlying Theorem 1 and highlight the role of its different assumptions.

EXAMPLE 1. Consider a two-player game with linear payoffs $\pi(k, \theta) = k - 1 - \theta$ as in (1). Furthermore, suppose

that the realized signals (x_1, x_2) are normally distributed, with agents' information sets given by

$$I_1 = \{x_1\}, \quad I_2 = \{x_1, x_2\}.$$

It is immediate that the above information structure satisfies the assumptions of Theorem 1.

As before, focusing on threshold strategies is sufficient for determining the set of equilibria. Let τ_1 denote the threshold corresponding to agent 1's strategy. The fact that x_1 is a signal that is observable to both agents implies that agent 2 can predict the equilibrium action of agent 1 with certainty and change her behavior accordingly. On the other hand, the expected payoff of agent 2 when she takes the risky action is given by $a_1 - \mathbb{E}[\theta | x_1, x_2] = a_1 - (x_1 + x_2)/2$. These two observations together imply that the best response of agent 2 would be a threshold strategy (τ_2, τ'_2) constructed as follows: if $x_1 < \tau_1$, then agent 2 takes the risky action if and only if $(x_1 + x_2)/2 < \tau_2$, whereas if $x_1 \geq \tau_1$, agent 2 takes the risky action if and only if $(x_1 + x_2)/2 < \tau'_2$.

Therefore, the expected payoff of taking the risky action to agent 2 is equal to

$$\mathbb{E}[\pi(k, \theta) | x_1, x_2] = \mathbb{1}\{x_1 < \tau_1\} - \frac{1}{2}(x_1 + x_2).$$

The fact that agent 2 has to be indifferent between taking the safe and the risky actions at her two thresholds implies that $\tau_2 = 1$ and $\tau'_2 = 0$. On the other hand, the expected payoff of taking the risky action to agent 1 is given by

$$\begin{aligned} \mathbb{E}[\pi(k, \theta) | x_1] &= \mathbb{P}(x_2 < 2\tau_2 - x_1 | x_1)\mathbb{1}\{x_1 < \tau_1\} \\ &\quad + \mathbb{P}(x_2 < 2\tau'_2 - x_1 | x_2)\mathbb{1}\{x_1 \geq \tau_1\} - x_1. \end{aligned}$$

Given that τ_1 captures the threshold strategy of agent 1, the above expression has to be positive if and only if $x_1 < \tau_1$, or equivalently:

$$\Phi\left(-\frac{\tau_1}{\sigma\sqrt{2}}\right) \leq \tau_1 \leq \Phi\left(\frac{2 - \tau_1}{\sigma\sqrt{2}}\right).$$

It is immediate to verify that the value of τ_1 that satisfies the above inequalities is not unique, thus guaranteeing the multiplicity of equilibria.

To interpret this result, note that whenever $\mathbb{E}[\theta | x_1, x_2] \in (0, 1)$, agent 2 finds it optimal to simply follow the behavior of agent 1, regardless of the realization of signal x_2 . This is because (i) within this range, agent 2's belief about the underlying state θ lies inside the critical region; and (ii) agent 2 can perfectly predict the action of agent 1, making agent 1 pivotal and hence leading to multiple equilibria.

It is important to note that the arguments following Theorem 1 and the above example break down if agent 1 has access to signals that are not in the information set of agent 2, even though there are some signals that are common knowledge between the two agents. To illustrate this

point, consider a three-agent variant of the above example, where the agents' information sets are given by

$$I_1 = \{x_2, x_3\}, \quad I_2 = \{x_3, x_1\}, \quad I_3 = \{x_1, x_2\}. \tag{4}$$

Thus, even though there is one signal that is common knowledge among any given pair of agents, the information sets of no subset of agents form a filtration. We have the following result:

PROPOSITION 2. *Suppose that agents' information sets are given by (4). There exists $\bar{\sigma}$ such that if $\sigma > \bar{\sigma}$, then the game has an essentially unique equilibrium.*

Thus, the coordination game has a unique equilibrium despite the fact that any pair of agents shares a common signal. This is because, unlike Theorem 1 and Example 1 above, every agent faces some strategic uncertainty about all other agents in the game, hence guaranteeing that no collection of signals can serve as a coordination device among a subset of agents.

In addition to the presence of a subset of agents C whose information sets form a filtration, Theorem 1 also requires that the information set of no agent outside C contain any of the signals observable to agents in C . To clarify the role of this assumption in generating multiple equilibria, once again consider the above three-player game but instead suppose that agents' information sets are given by

$$I_1 = \{x_1\}, \quad I_2 = \{x_1, x_2\}, \quad I_3 = \{x_2\}. \tag{5}$$

It is immediate to verify that the conditions of Theorem 1 are not satisfied, even though there exists a collection of agents whose information sets form a filtration. We have the following result:

PROPOSITION 3. *Suppose that agents' information sets are given by (5). Then the game has an essentially unique equilibrium. Furthermore, as $\sigma \rightarrow 0$, all agents choose the risky action if and only if $\theta < 1/2$.*

Thus, as fundamental uncertainty is removed, information structure (5) induces the same (essentially) unique equilibrium as the case in which all signals are private. This is despite the fact that the information sets of agents in $C = \{1, 2\}$ form a filtration.

To understand the intuition behind this result, it is instructive to compare the above game with the game described in Example 1. Notice that in both games, agents with the larger information sets do not face any uncertainty (strategic or otherwise) about predicting the actions of agents with smaller information sets. Nevertheless, the above game exhibits a unique equilibrium, whereas the game in Example 1 has multiple equilibria. The key distinction lies in the different roles played by the extra pieces of information available to the agents with larger information sets. Recall that in Example 1, there exists a set of realizations of signals for which agent 2 finds it optimal to

imitate the action of agent 1. Therefore, under such conditions, agent 2 uses x_2 solely as a means of obtaining a more precise estimate of the underlying state θ . This, however, is in sharp contrast with what happens under information structure (5). In this case, signal x_2 plays a second role in the information structure of agent 2: it not only provides an extra piece of information about θ but also serves as a perfect predictor of agent 3's equilibrium action. Thus, even if observing x_2 does not change the mean of agent 2's posterior belief about θ by much, it still provides her with information about the action that agent 3 is expected to take in equilibrium. This extra piece of information, however, is not available to agent 1. Thus, even as $\sigma \rightarrow 0$, agents 1 and 3 face some strategic uncertainty regarding the equilibrium action of one another and, as a consequence, regarding that of agent 2. The presence of such strategic uncertainties implies that the game with information structure (5) would exhibit a unique equilibrium.

We conclude our argument by showing that even though the conditions of Theorem 1 are sufficient for equilibrium multiplicity, they are not necessary. To see this, once again consider a variant of the game in Example 1 but instead assume that there are four agents whose information sets are given by

$$I_1 = \{x_1\}, \quad I_2 = \{x_1, x_2\}, \quad I_3 = \{x_2\}, \quad I_4 = \{x_2\}, \quad (6)$$

Note that the above information structure does not satisfy the conditions of Theorem 1. Nevertheless, one can show that the game has multiple equilibria:

PROPOSITION 4. *Suppose that agents' information sets are given by (6). Then the game has multiple Bayesian Nash equilibria.*

The importance of the above result is highlighted when contrasted with the information structure (5) and Proposition 3. Note that in both cases, agent 2 has access to the signals available to all other agents. However, in information structure (6), signal x_2 is also available to an additional agent, namely, agent 4. The presence of such an agent with an information set identical to that of agent 3 means that agents 3 and 4 face no strategic uncertainty regarding one another's actions. Therefore, even though they may be uncertain about the equilibrium actions of agents 1 and 2, there are certain realizations of signal x_2 under which they find it optimal to imitate one another's action, thus leading to the emergence of multiple equilibria.

5. Local Information and the Extent of Multiplicity

Our analysis thus far was focused on the dichotomy between multiplicity and uniqueness of equilibria. However, as the example in §3 shows, even when the game exhibits multiple equilibria, the set of equilibria depends on how information is locally shared between different agents.

In this section, we provide a characterization of the set of all Bayesian Nash equilibria as a function of the information sets of different agents. Our characterization quantifies the dependence of the set of rationalizable strategies on the extent to which agents observe common signals.

To explicitly characterize the set of equilibria, we restrict our attention to a game with linear payoff functions given by (1). We also assume that $m \leq n$ signals, denoted by $(x_1, \dots, x_m) \in \mathbb{R}^m$, are realized, where the noise terms (ξ_1, \dots, ξ_m) are mutually independent and normally distributed with mean zero and variance $\sigma^2 > 0$. Furthermore, we assume that each agent observes only one of the realized signals; that is, for any given agent i , her information set is $I_i = \{x_r\}$ for some $1 \leq r \leq m$. Finally, we denote the fraction of agents that observe signal x_r by c_r , and let $c = (c_1, \dots, c_m)$. Since each agent observes a single signal, we have $c_1 + \dots + c_m = 1$. Note that as in our benchmark model in §2, we assume that the allocation of signals to agents is deterministic, prespecified, and common knowledge. Our next result provides a simple characterization of the set of all rationalizable strategies as $\sigma \rightarrow 0$.

THEOREM 5. *Let s_i denote a threshold strategy of agent i . As $\sigma \rightarrow 0$, strategy s_i is rationalizable if and only if*

$$s_i(x) = \begin{cases} 1 & \text{if } x < \underline{\tau} \\ 0 & \text{if } x > \bar{\tau}, \end{cases}$$

where

$$\underline{\tau} = 1 - \bar{\tau} = \frac{n}{2(n-1)}(1 - \|c\|_2) \leq \frac{1}{2}$$

and $\|c\|_2$ denotes the Euclidean norm of vector c .

The above theorem shows that the distance between the thresholds of the “largest” and “smallest” rationalizable strategies depends on how information is locally shared between different agents. More specifically, a smaller $\|c\|_2$ implies that the set of rationalizable strategies would shrink.

Note that $\|c\|_2$ is essentially a proxy for the extent to which agents observe common signals: it takes a smaller value whenever any given signal is observed by fewer agents. Hence, a smaller value of $\|c\|_2$ implies that agents would face higher strategic uncertainty about one another's actions, even when all the fundamental uncertainty is removed as $\sigma \rightarrow 0$. As a consequence, the set of rationalizable strategies shrinks as the Euclidean norm of c decreases. In the extreme case that agents' information is only in the form of private signals (that is, when $m = n$ and $c_r = 1/n$ for all r), the upper and lower thresholds coincide ($\underline{\tau} = \bar{\tau} = 1/2$), implying that the equilibrium strategies are essentially unique. This is indeed the case that corresponds to maximal level of strategic uncertainty. On the other hand, when all agents observe the same public signal (i.e., when $m = 1$ and $\|c\|_2 = 1$), they face no strategic uncertainty about each other's actions, and hence all

undominated strategies are rationalizable. Thus, to summarize, the above two extreme cases coincide with the standard results in the global games literature. Theorem 5 above generalizes those results by characterizing the equilibria of the game in the intermediate cases in which signals are neither fully public nor private.

Recall that since the Bayesian game under consideration is monotone supermodular in the sense of Van Zandt and Vives (2007), there exist a greatest and a smallest Bayesian Nash equilibrium, both of which are in threshold strategies. Moreover, by Milgrom and Roberts (1990), all profiles of rationalizable strategies are “sandwiched” between these two equilibria. Therefore, Theorem 5 also provides a characterization of the set of equilibria of the game, showing that a higher level of common knowledge, captured via a larger value for $\|c\|_2$, implies a larger set of equilibria. Note that if $m < n$, by construction, there are at least two agents with identical information sets. Theorem 5 then implies that $\underline{\tau} < 1/2 < \bar{\tau}$, which means that the game exhibits multiple equilibria, an observation consistent with Theorem 1.

A simple corollary to Theorem 5 implies that with $m < n$ sources of information, the set of Bayesian Nash equilibria is largest when $m - 1$ agents each observe a private signal and $n - m + 1$ agents have access to the remaining signal. In this case, common knowledge of signals among such a large group of agents minimizes the extent of strategic uncertainty and hence leads to the largest set of equilibria. On the other hand, the set of equilibria shrinks whenever the sizes of the sets of agents with access to the same signal are more equalized. In particular, the case in which $c_r = 1/m$ for all r corresponds to the highest level of intergroup strategic uncertainty, leading to the greatest extent of refinement of rationalizable strategies.

5.1. Large Coordination Games

Recall from Theorem 1 that the existence of two agents i and j with identical information sets is sufficient to guarantee equilibrium multiplicity, irrespective of the number of agents in the game or how much other agents care about coordinating with i and j . In particular, no matter how insignificant and uninformed the two agents are, the mere fact that i and j face no uncertainty regarding each other's behavior leads to equilibrium multiplicity. On the other hand, as Theorem 5 and the preceding discussion show, even under information structures that lead to multiplicity, the set of equilibria still depends on the extent to which agents observe common signals. To further clarify the role of local information in determining the size of the equilibrium set, we next study large coordination games.

Formally, consider a sequence of games $\{G(n)\}_{n \in \mathbb{N}}$ parametrized by the number of agents, in which each agent i can observe a single signal, and assume that the noise terms in the signals are mutually independent and normally distributed with mean zero and variance $\sigma^2 > 0$. We have the following corollary to Theorem 5.

PROPOSITION 6. *The sequence of games $\{G(n)\}_{n \in \mathbb{N}}$ exhibits an (essentially) unique equilibrium asymptotically as $n \rightarrow \infty$ and in the limit as $\sigma \rightarrow 0$ if and only if the size of the largest set of agents with a common observation grows sublinearly in n .*

Thus, as the number of agents grows, the game exhibits multiple equilibria if and only if a nontrivial fraction of the agents have access to the same signal. Even though such a signal is not public—in the sense that it is not observed by all agents—the fact that it is common knowledge among a nonzero fraction of the agents implies that it can function as a powerful enough coordination device and hence induce multiple equilibria. On the other hand, if the size of the largest subset of agents with common knowledge of a signal does not grow at the same rate as the number of agents, information is diverse and effectively private: any agent faces strategic uncertainty regarding the behavior of most other agents, even as all the fundamental uncertainty is removed ($\sigma \rightarrow 0$). Consequently, as the number of agents grows, the set of equilibrium strategies of each agent collapses to a single strategy.

5.2. Robustness to Uncertainty in the Information Structure

So far, we have assumed that the information structure of the game is common knowledge. In this subsection, we relax this assumption and show that our results are robust to a small amount of uncertainty regarding the information structure of the game.

To capture this idea formally, consider a game G^* with information structure $Q^* = \{I_1, \dots, I_n\}$ that is common knowledge among the agents, where each signal is observed by more than a single agent. Furthermore, consider a perturbed version of G^* in which the information structure is drawn according to a probability distribution μ that is independent from the realization of θ and the signals. In particular, the likelihood that a certain information structure Q is realized is given by $\mu(Q)$.⁶ This means that as long as μ is not degenerate, agents are not only uncertain about the realization of the underlying state θ and other agents' signals, but they may also have incomplete information about one another's information sets.

We consider the case where the cardinality of any agent's information set is identical across all possible information structures for which $\mu(Q) > 0$. This assumption guarantees that any strategy of a given agent is a mapping from the same subspace of \mathbb{R}^n to $\{0, 1\}$ regardless of the realization of the information structure, thus allowing us to compare agents' strategies for different probability measures μ . We have the following result:⁷

THEOREM 7. *Consider a sequence of probability distributions $\{\mu_s\}_{s=1}^\infty$ such that $\lim_{s \rightarrow \infty} \mu_s(Q^*) = 1$, and let G_s denote the game whose information structure is drawn according to μ_s . Then for almost all equilibria of G^* , there exists a large enough \bar{s} such that for all $s > \bar{s}$, that equilibrium is also a Bayesian Nash equilibrium of G_s .*

The above result thus highlights that even if the information structure of the game is not common knowledge among the agents, the equilibria of the perturbed game would be close to the equilibria of the original game. Consequently, it establishes that our earlier results on the multiplicity of equilibria are also robust to the introduction of a small amount of uncertainty about the information structure of the game.

As a last remark, we note that in the above result, we did not allow agents to obtain extra, side signals about the realization of the information structure of the game. Nevertheless, a similar argument shows that the set of equilibria is robust to small amounts of noise in the information structure of the game, even if agents obtain such informative signals.

6. Conclusions

Many social and economic phenomena (such as bank runs, adoption of new technologies, social uprisings, etc.) exhibit an element of coordination. In this paper, we focused on how the presence of local information channels can affect the likelihood and possibility of coordination failures in such contexts. In particular, by introducing local signals—signals that are neither purely public nor private—to the canonical global game models studied in the literature, we showed that the set of equilibria depends on how information is locally shared among agents. Our results establish that the coordination game exhibits multiple equilibria if the information sets of a group of agents form an increasing sequence of nested sets: the presence of such a filtration removes agents' strategic uncertainty about one another's actions and leads to multiple equilibria.

We also provided a characterization of how the extent of equilibrium multiplicity is determined by the extent to which subsets of agents have access to common information. In particular, we showed that the size of the equilibrium set is increasing in the standard deviation of the fractions of agents with access to the same signal. Our result thus shows that the set of equilibria shrinks as information becomes more decentralized in the society.

On the theoretical side, our results highlight the importance of local information channels in global games by underscoring how the introduction of a signal commonly observed by even a small number of agents may lead to equilibrium multiplicity in an environment that would have otherwise exhibited a unique equilibrium. On the more applied side, our results show that incorporating local communication channels between agents may be of first-order importance in understanding many phenomena that exhibit an element of coordination. In particular, they highlight the risk of abstracting from the intricate details of information dissemination within such contexts. For instance, policy interventions that are not informed by the dispersion and penetration of different information channels among agents may turn out to be counterproductive.

Acknowledgments

The authors are grateful to the editors and two anonymous referees for helpful feedback and suggestions. They also thank Daron Acemoglu, Marios Angeletos, Antonio Penta, Ali Shourideh, and participants at the Workshop on the Economics of Networks, Systems, and Computation and the joint Workshop on Pricing and Incentives in Networks and Systems, where preliminary versions of this work were presented (Dahleh et al. 2011, 2013); the Interdisciplinary Workshop on Information and Decision in Social Networks; and the Cambridge Area Economics and Computation Day. This research was partially supported by the Air Force Office of Scientific Research, Grant FA9550-09-1-0420.

Appendix. Proofs

Notation and Preliminary Lemmas

We first introduce some notation and prove some preliminary lemmas. Recall that a pure strategy of agent i is a mapping $s_i: \mathbb{R}^{|I_i|} \rightarrow \{0, 1\}$, where I_i denotes i 's information set. Thus, a pure strategy s_i can equivalently be represented by the set $A_i \subseteq \mathbb{R}^{|I_i|}$ over which agent i takes the risky action; i.e.,

$$A_i = \{y_i \in \mathbb{R}^{|I_i|}: s_i(y_i) = 1\},$$

where $y_i = (x_r)_{r \in I_i}$ denotes the collection of the realized signals in the information set of agent i . Hence, a strategy profile can equivalently be represented by a collection of sets $A = (A_1, \dots, A_n)$ over which agents take the risky action. We denote the set of all strategies of agent i by \mathcal{A}_i and the set of all strategy profiles by \mathcal{A} . Given the strategies of other agents, A_{-i} , we denote the expected payoff to agent i of the risky action when she observes y_i by $V_i(A_{-i} | y_i)$. Thus, a best response mapping $\text{BR}_i: \mathcal{A}_{-i} \rightarrow \mathcal{A}_i$ is naturally defined as

$$\text{BR}_i(A_{-i}) = \{y_i \in \mathbb{R}^{|I_i|}: V_i(A_{-i} | y_i) > 0\}. \quad (7)$$

Finally, we define the mapping $\text{BR}: \mathcal{A} \rightarrow \mathcal{A}$ as the product of the best response mappings of all agents; that is,

$$\text{BR}(A) = \text{BR}_1(A_{-1}) \times \dots \times \text{BR}_n(A_{-n}). \quad (8)$$

The $\text{BR}(\cdot)$ mapping is monotone and continuous. More formally, we have the following lemmas:

LEMMA 1 (MONOTONICITY). *Consider two strategy profiles A and A' such that $A \subseteq A'$. (We write $A \subseteq A'$ whenever $A_i \subseteq A'_i$ for all i .) Then $\text{BR}(A) \subseteq \text{BR}(A')$.*

PROOF. Fix an agent i and consider $y_i \in \text{BR}_i(A_{-i})$, which by definition satisfies $V_i(A_{-i} | y_i) > 0$. Because of the presence of strategic complementarities (Assumption 1), we have $V_i(A'_{-i} | y_i) \geq V_i(A_{-i} | y_i)$ and, as a result, $y_i \in \text{BR}_i(A'_{-i})$. \square

LEMMA 2 (CONTINUITY). *Consider a sequence of strategy profiles $\{A^k\}_{k \in \mathbb{N}}$ such that $A^k \subseteq A^{k+1}$ for all k . Then*

$$\bigcup_{k=1}^{\infty} \text{BR}(A^k) = \text{BR}(A^\infty),$$

where $A^\infty = \bigcup_{k=1}^{\infty} A^k$.

PROOF. Clearly, $A^k \subseteq A^\infty$ and by Lemma 1, $BR(A^k) \subseteq BR(A^\infty)$ for all k . Thus,

$$\bigcup_{k=1}^{\infty} BR(A^k) \subseteq BR(A^\infty).$$

To prove the reverse inclusion, suppose that $y_i \in BR_i(A_{-i}^\infty)$, which implies that $V_i(A_{-i}^\infty | y_i) > 0$. On the other hand, for any θ and any observation profile (y_1, \dots, y_n) , we have

$$\lim_{k \rightarrow \infty} u_i(a_i, s_{-i}^k(y_{-i}), \theta) = u_i(a_i, s_{-i}^\infty(y_{-i}), \theta),$$

where s^k and s^∞ are strategy profiles corresponding to sets A^k and A^∞ , respectively. Thus, by the dominated convergence theorem,

$$\lim_{k \rightarrow \infty} V_i(A_{-i}^k | y_i) = V_i(A_{-i}^\infty | y_i), \tag{9}$$

where we have used Assumption 4. Therefore, there exists $r \in \mathbb{N}$ large enough such that $V_i(A_{-i}^r | y_i) > 0$, implying that $y_i \in \bigcup_{k=1}^{\infty} BR(A_i^k)$. This completes the proof. \square

Throughout the rest of the proofs, we let $\{R^k\}_{k \in \mathbb{N}}$ denote the sequence of strategy profiles defined recursively as

$$\begin{aligned} R^1 &= \emptyset, \\ R^{k+1} &= BR(R^k). \end{aligned} \tag{10}$$

Thus, any strategy profile A that survives k rounds of iterated elimination of strictly dominated strategies must satisfy $R^{k+1} \subseteq A$. Consequently, A survives the iterated elimination of strictly dominated strategies only if $R \subseteq A$, where

$$R = \bigcup_{k=1}^{\infty} R^k. \tag{11}$$

Proof of Theorem 1

Without loss of generality, let $C = \{1, \dots, \ell\}$ and $I_1 \subseteq I_2 \dots \subseteq I_\ell$, where $\ell \geq 2$. Before proving the theorem, we first define an ‘‘induced coordination game’’ between agents in $C = \{1, \dots, \ell\}$ as follows: suppose that agents in C play the game described in §2, except the actions of agents outside of C are prescribed by the strategy profile R defined in (10) and (11). More specifically, the payoff of taking the risky action to agent $i \in C$ is given by

$$\tilde{\pi}(k, \theta) = \pi\left(k + \sum_{j \notin C} \mathbb{1}\{y_j \in R_j\}, \theta\right),$$

where $k = \sum_{j=1}^{\ell} a_j$ denotes the number of agents in C who take the risky action. As in the benchmark game in §2, the payoff of taking the safe action is normalized to zero.

For this game, define the sequence of strategy profiles $\{R^k\}_{k \in \mathbb{N}}$ as

$$R^1 = \emptyset, \tag{12}$$

$$R^{k+1} = BR(R^k), \tag{13}$$

and set $R' = \bigcup_{k=1}^{\infty} R^k$. Clearly, the strategy profile (R'_1, \dots, R'_ℓ) corresponds to a Bayesian Nash equilibrium of the induced game defined above. Furthermore, it is immediate that $R'_i = R_i$ for all $i \in C$.

We have the following lemma:

LEMMA 3. *The induced coordination game between the ℓ agents in C has an equilibrium strategy profile $B = (B_1, \dots, B_\ell)$ such that $\lambda_{\mathbb{R}^{|I_i|}}(B_i \setminus R'_i) > 0$ for all $i \in C$, where $\lambda_{\mathbb{R}^r}$ is the Lebesgue measure in \mathbb{R}^r and the equilibrium strategy profile R' is given by (12) and (13).*

PROOF. We denote $y_j = \{x_r: x_r \in I_j\}$. Recall that any strategy profile of agents in C can be represented as a collection of sets $A = (A_1, \dots, A_\ell)$, where A_j is the set of realizations of y_j over which agent j takes the risky action. On the other hand, recall that since $I_1 \subseteq I_2 \dots \subseteq I_\ell$, agent j observes the realizations of signals y_i for all $i \leq j$. Therefore, any strategy A_j of agent j can be recursively captured by indexing it to the realizations of signals (y_1, \dots, y_{j-1}) . More formally, given the realization of signals (y_1, \dots, y_ℓ) and the strategies (A_1, \dots, A_{j-1}) , agent j takes the risky action if and only if

$$y_j \in A_j^{(s_1, \dots, s_{j-1})},$$

where

$$s_i = \mathbb{1}\{y_i \in A_i^{(s_1, \dots, s_{i-1})}\}$$

for all $i \leq j - 1$. In other words, j first considers the set of signals (y_1, \dots, y_{j-1}) and, based on their realizations, takes the risky action if $y_j \in A_j^{(s_1, \dots, s_{j-1})}$. Note that in the above notation, rather than being captured by a single set $A_j \subseteq \mathbb{R}^{|I_j|}$, the strategy of agent j is captured by 2^{j-1} different sets of the form $A_j^{(s_1, \dots, s_{j-1})}$. Thus, the collection of sets

$$A = \{A_j^{(s_1, \dots, s_{j-1})}\}_{1 \leq j \leq \ell, s \in \{0, 1\}^{\ell-1}} \tag{14}$$

capture the strategy profile of the agents $\{1, \dots, \ell\}$ in the induced coordination game. Finally, to simplify notation, we let $s^j = (s_1, \dots, s_j)$ and $S_j = \sum_{i=1}^{j-1} s_i$.

With the above notation at hand, we now proceed with the proof of the lemma. Note that since the induced coordination game is a game of strategic complementarities, it has at least one equilibrium. In fact, the iterated elimination of strictly dominated strategies in (12) and (13) leads to one such equilibrium $A = R'$. Let A , represented in (14), denote one such strategy profile, where we assume that agents take the safe action whenever they are indifferent between the two actions.

For the strategy profile A to be an equilibrium, the expected payoff of paying the risky action to agent j has to be positive for all $y_j \in A_j^{(s_1, \dots, s_{j-1})}$, and nonpositive for $y_j \notin A_j^{(s_1, \dots, s_{j-1})}$. In other words, if $y_j \in A_j^{(s_1, \dots, s_{j-1})}$, then

$$\begin{aligned} \sum_{(1, s_{j+1}, \dots, s_{r-1})} \mathbb{E}[\tilde{\pi}(1 + S_\ell, \theta) | y_j] \mathbb{P}(y_\ell \in A_\ell^s, y_k \in A_k^{(s^{k-1})} \forall s_k = 1, \\ y_k \notin A_k^{(s^{k-1})} \forall s_k = 0: j < k < \ell | y_j) \end{aligned}$$

$$+ \sum_{(1, s_{j+1}, \dots, s_{r-1})} \mathbb{E}[\tilde{\pi}(S_\ell, \theta) | y_j] \mathbb{P}(y_\ell \notin A_\ell^s, y_k \in A_k^{(s^{k-1})} \forall s_k = 1, \\ y_k \notin A_k^{(s^{k-1})} \forall s_k = 0: j < k < \ell | y_j) > 0;$$

whereas, on the other hand, if $y_j \notin A_j^{(s_1, \dots, s_{j-1})}$, then

$$\begin{aligned} \sum_{(0, s_{j+1}, \dots, s_{r-1})} \mathbb{E}[\tilde{\pi}(1 + S_\ell, \theta) | y_j] \mathbb{P}(y_\ell \in A_\ell^s, y_k \in A_k^{(s^{k-1})} \forall s_k = 1, \\ y_k \notin A_k^{(s^{k-1})} \forall s_k = 0: j < k < \ell | y_j) \end{aligned}$$

$$+ \sum_{(0, s_{j+1}, \dots, s_{r-1})} \mathbb{E}[\tilde{\pi}(S_\ell, \theta) | y_j] \mathbb{P}(y_\ell \notin A_\ell^s, y_k \in A_k^{(s^{k-1})} \forall s_k = 1, \\ y_k \notin A_k^{(s^{k-1})} \forall s_k = 0: j < k < \ell | y_j) \leq 0.$$

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To understand the above expressions, note that in equilibrium, agent j faces no strategic uncertainty regarding the behavior of agents $i \leq j-1$. However, to compute her expected payoff, agent j needs to condition on different realizations of signals (s_{j+1}, \dots, s_ℓ) . Furthermore, she needs to take into account that the realizations of such signals not only affect the belief of agents $k > j$ about the underlying state θ but also determine which set $A_k^{(s^{k-1})}$ they would use for taking the risky action.

Given the equilibrium strategy profile A whose properties we just studied, we now construct a new strategy profile D as follows. Start with agent ℓ at the top of the chain and define a set $D_\ell^{(s_1, \dots, s_{\ell-1})}$ to be such that it satisfies

$$\begin{aligned} \mathbb{E}[\tilde{\pi}(1+S_\ell, \theta) | y_\ell] > 0 & \quad \text{if } y_\ell \in D_\ell^{(s_1, \dots, s_{\ell-1})}, \\ \mathbb{E}[\tilde{\pi}(1+S_\ell, \theta) | y_\ell] \leq 0 & \quad \text{if } y_\ell \notin D_\ell^{(s_1, \dots, s_{\ell-1})}. \end{aligned}$$

Note that for any given s , the set D_ℓ^s is essentially an extension of A_ℓ^s to the whole space $\mathbb{R}^{|\mathcal{I}_\ell|}$ in the sense that the two sets coincide with one another over the subset of signals realizations

$$\{y_\ell: y_i \in A_i^{s_i-1} \text{ if } s_i = 1 \text{ and } y_i \notin A_i^{s_i-1} \text{ if } s_i = 0\}.$$

Similarly, (and recursively) define the sets D_j as extensions of the sets A_j such that they satisfy the following properties: if $y_j \in D_j^{(s_1, \dots, s_{j-1})}$, then

$$\begin{aligned} \sum_{(1, s_{j+1}, \dots, s_{r-1})} \mathbb{E}[\tilde{\pi}(1+S_\ell, \theta) | y_j] \mathbb{P}(y_\ell \in D_\ell^s, y_k \in D_k^{(s^{k-1})} \forall s_k = 1, \\ y_k \notin D_k^{(s^{k-1})} \forall s_k = 0; k > j | y_j) \\ + \sum_{(1, s_{j+1}, \dots, s_{r-1})} \mathbb{E}[\tilde{\pi}(S_\ell, \theta) | y_j] \mathbb{P}(y_\ell \notin D_\ell^s, y_k \in D_k^{(s^{k-1})} \forall s_k = 1, \\ y_k \notin D_k^{(s^{k-1})} \forall s_k = 0; k > j | y_j) > 0; \end{aligned} \quad (15)$$

and, on the other hand, if $y_j \notin D_j^{(s_1, \dots, s_{j-1})}$, then

$$\begin{aligned} \sum_{(0, s_{j+1}, \dots, s_{r-1})} \mathbb{E}[\tilde{\pi}(1+S_\ell, \theta) | y_j] \mathbb{P}(y_\ell \in D_\ell^s, y_k \in D_k^{(s^{k-1})} \forall s_k = 1, \\ y_k \notin D_k^{(s^{k-1})} \forall s_k = 0; k > j | y_j) \\ + \sum_{(0, s_{j+1}, \dots, s_{r-1})} \mathbb{E}[\tilde{\pi}(S_\ell, \theta) | y_j] \mathbb{P}(y_\ell \notin D_\ell^s, y_k \in D_k^{(s^{k-1})} \forall s_k = 1, \\ y_k \notin D_k^{(s^{k-1})} \forall s_k = 0; k > j | y_j) \leq 0. \end{aligned} \quad (16)$$

Note that given that the sets D are simply extensions of sets A , by construction, the strategy profile (D_1, \dots, D_ℓ) also forms a Bayesian Nash equilibrium. In fact, it essentially is an alternative representation of the equilibrium strategy profile A .

Now, consider the expected payoff to agent 1, at the bottom of the chain. In equilibrium, agent 1 should prefer to take the risky action in set D_1 . In other words,

$$f(y_1) > 0 \quad \forall y_1 \in D_1 \quad (17)$$

$$f'(y_1) \leq 0 \quad \forall y_1 \notin D_1, \quad (18)$$

where

$$\begin{aligned} f(y_1) = \sum_{s \in \{0, 1\}^{\ell-1}: s_1=1} \mathbb{E}[\tilde{\pi}(1+S_\ell, \theta) | y_1] \mathbb{P}(y_\ell \in D_\ell^s, \\ y_k \in D_k^{(s^{k-1})} \forall s_k = 1, y_k \notin A_k^{(s^{k-1})} \forall s_k = 0: 1 < k < \ell | y_1) \\ + \sum_{s \in \{0, 1\}^{\ell-1}: s_1=1} \mathbb{E}[\tilde{\pi}(S_\ell, \theta) | y_1] \mathbb{P}(y_\ell \notin D_\ell^s, \\ y_k \in D_k^{(s^{k-1})} \forall s_k = 1, y_k \notin A_k^{(s^{k-1})} \forall s_k = 0: 1 < k < \ell | y_1), \end{aligned}$$

and

$$\begin{aligned} f'(y_1) = \sum_{s \in \{0, 1\}^{\ell-1}: s_1=0} \mathbb{E}[\tilde{\pi}(1+S_\ell, \theta) | y_1] \mathbb{P}(y_\ell \in D_\ell^s, \\ y_k \in D_k^{(s^{k-1})} \forall s_k = 1, y_k \notin A_k^{(s^{k-1})} \forall s_k = 0: k > 1 | y_1) \\ + \sum_{s \in \{0, 1\}^{\ell-1}: s_1=0} \mathbb{E}[\tilde{\pi}(S_\ell, \theta) | y_1] \mathbb{P}(y_\ell \notin D_\ell^s, \\ y_k \in D_k^{(s^{k-1})} \forall s_k = 1, y_k \notin A_k^{(s^{k-1})} \forall s_k = 0: k > 1 | y_1), \end{aligned}$$

are simply determined by evaluating inequalities (15) and (16) for agent 1. Notice that unlike the definition of $f(y_1)$, in the definition of $f'(y_1)$ we have to start from $s_1=0$ because agent 1 is not supposed to take the risky action whenever $y_1 \notin D_1$.

We have the following lemma, the proof of which is provided later on:

LEMMA 4. Let $X = \{y_1: f(y_1) > 0\}$ and $Y = \{y_1: f'(y_1) > 0\}$. Then there exists a set B_1 such that $Y \subseteq B_1 \subseteq X$ and $\lambda_{\mathbb{R}^{|\mathcal{I}_1|}}(B_1 \Delta D_1) > 0$, where $\lambda_{\mathbb{R}^{|\mathcal{I}_1|}}$ refers to the Lebesgue measure and Δ denotes the symmetric difference.

In view of the above lemma, we now use the set B_1 to construct a new strategy profile B represented as the following collections of sets:

$$B = (B_1, \{D_j^{(s_1, \dots, s_{j-1})}\}_{2 \leq j \leq \ell, s \in \{0, 1\}^{\ell-1}}).$$

In other words, agent 1 takes the risky action if and only if $y_1 \in B_1$. The rest of the agents follow the same strategies as prescribed by strategy profile D except for the fact that rather than choosing their D sets based on whether y_1 belongs to D_1 or not, they choose the sets based on whether $y_1 \in B_1$ or not. By construction, the strategy profile above is an equilibrium. Note that since $Y \subseteq B_1 \subseteq X$, agent 1 is best responding to the strategies of all other agents. Furthermore, given that for $j \neq 1$ we have that $B_j^{(s_j-1)} = D_j^{(s_j-1)}$, inequalities (15) and (16) are also satisfied, implying that agent j also best responds to the behavior of all other agents, hence, completing the proof. \square

We now present the proof of Theorem 1.

Proof of Theorem 1. Lemma 3 establishes that the induced coordination game between agents in C has an equilibrium $B = (B_1, \dots, B_\ell)$ that is distinct from the equilibrium R' derived via iterated elimination of strictly dominated strategies in (12) and (13). We now use this strategy profile to construct an equilibrium for the original n -agent game.

Define the strategy profile $\tilde{B} = (B_1, \dots, B_\ell, R_{\ell+1}, \dots, R_n)$ for the n -agent game, where R_j is defined in (10) and (11). Lemma 3 immediately implies that $\tilde{B} \subseteq \text{BR}(\tilde{B})$. Define the sequence of strategy profiles $\{H^k\}_{k \in \mathbb{N}}$ as

$$H^1 = \tilde{B}$$

$$H^{k+1} = \text{BR}(H^k).$$

Given that $H^1 \subseteq H^2$, Lemma 1 implies that $H^k \subseteq H^{k+1}$ for all k . Thus, $H = \bigcup_{k=1}^{\infty} H^k$ is well defined and by continuity of the BR operator (Lemma 2) satisfies $H = \text{BR}(H)$. As a consequence, H is also a Bayesian Nash equilibrium of the game that, in light of Lemma 3, is distinct from R . Note that R is itself a Bayesian Nash equilibrium of the game because it is a fixed point of the best response operator. \square

Proof of Lemma 4. The definitions of sets X and Y immediately imply that $Y \subseteq D_1 \subseteq X$. On the other hand, the second part of Assumption 1 implies that $\lambda_{\mathbb{R}^{|Y|}}(X \setminus Y) > 0$. Consequently, there exists a set B_1 distinct from D_1 such that $Y \subseteq B \subseteq X$ and $\lambda_{\mathbb{R}^{|Y|}}(B_1 \Delta D_1) > 0$. This completes the proof. \square

Proof of Proposition 2

Recall the sequence of strategy profiles R^k and its limit R defined in (10) and (11), respectively. By Lemma 2, $R = BR(R)$, which implies that $y_i \in R_i$ if and only if $V_i(R_{-i} | y_i) > 0$. We have the following lemma.

LEMMA 5. *There exists a strictly decreasing function $h: \mathbb{R} \rightarrow \mathbb{R}$ such that $V_i(R_{-i} | y_i) = 0$ if and only if $x_j = h(x_i)$, where $y_i = (x_j, x_i)$ and i, j , and l are different.*

PROOF. Using an inductive argument, we first prove that for all k

- (i) $V_i(R_{-i}^k | y_i)$ is continuously differentiable in y_i ,
- (ii) $V_i(R_{-i}^k | y_i)$ is strictly decreasing in both arguments, $(x_j, x_i) = y_i$, and
- (iii) $|\partial V_i(R_{-i}^k | y_i) / \partial x_j| \in [1/2, Q]$,

where $j \neq i$ and

$$Q = \frac{\sigma\sqrt{3\pi} + 1}{2\sigma\sqrt{3\pi} - 2}.$$

The above clearly hold for $k = 1$ because $V_i(\emptyset | y_i) = -(x_j + x_i)/2$. Now suppose that (i)–(iii) are satisfied for some $k \geq 1$. By the implicit function theorem,⁸ there exists a continuously differentiable function $h_k: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$V_i(R_{-i}^k | x_j, h_k(x_j)) = 0,$$

and $-2Q \leq h'_k(x_j) \leq -1/(2Q)$. (Given the symmetry between the three agents, we drop the agent's index for function h_k .) The monotonicity of h_k implies that $V_i(R_{-i}^k | y_i) > 0$ if and only if $x_j < h_k(x_i)$. Therefore,

$$\begin{aligned} V_i(R_{-i}^{k+1} | y_i) &= \frac{1}{2} [\mathbb{P}(y_j \in R_j^{k+1} | y_i) + \mathbb{P}(y_l \in R_l^{k+1} | y_i)] - \frac{1}{2}(x_j + x_i) \\ &= \frac{1}{2} [\mathbb{P}(x_j < h_k(x_i) | y_i) + \mathbb{P}(x_l < h_k(x_i) | y_i)] - \frac{1}{2}(x_j + x_i) \\ &= \frac{1}{2} \left[\Phi\left(\frac{h_k(x_i) - (x_j + x_i)/2}{\sigma\sqrt{3/2}}\right) \right. \\ &\quad \left. + \Phi\left(\frac{h_k(x_j) - (x_j + x_i)/2}{\sigma\sqrt{3/2}}\right) \right] - \frac{1}{2}(x_j + x_i), \end{aligned}$$

which immediately implies that $V_i(R_{-i}^{k+1} | y_i)$ is continuously differentiable and strictly decreasing in both arguments. Furthermore,

$$\begin{aligned} \frac{\partial}{\partial x_j} V_i(R_{-i}^{k+1} | y_i) &= -\frac{1}{2\sigma\sqrt{6}} \phi\left(\frac{h_k(x_i) - (x_j + x_i)/2}{\sigma\sqrt{3/2}}\right) \\ &\quad + \frac{2h'_k(x_j) - 1}{2\sigma\sqrt{6}} \phi\left(\frac{h_k(x_j) - (x_j + x_i)/2}{\sigma\sqrt{3/2}}\right) - \frac{1}{2}, \end{aligned}$$

which guarantees

$$-\frac{1}{2} - \frac{1+2Q}{2\sigma\sqrt{3\pi}} \leq \frac{\partial}{\partial x_j} V_i(R_{-i}^{k+1} | y_i) \leq -\frac{1}{2},$$

completing the inductive argument because

$$\frac{1}{2} + \frac{1+2Q}{2\sigma\sqrt{3\pi}} = Q.$$

Now using (9) and the implicit function theorem once again completes the proof. \square

Proof of Proposition 2. By definition,

$$\begin{aligned} V_i(R_{-i} | y_i) &= \frac{1}{2} [\mathbb{P}(y_j \in R_j | y_i) + \mathbb{P}(y_l \in R_l | y_i)] - \frac{1}{2}(x_j + x_i) \\ &= \frac{1}{2} [\mathbb{P}(V_j(R_{-j} | y_j) > 0 | y_i) \\ &\quad + \mathbb{P}(V_l(R_{-l} | y_l) > 0 | y_i)] - \frac{1}{2}(x_j + x_i), \end{aligned}$$

where $R = BR(R)$. By Lemma 5,

$$\begin{aligned} V_i(R_{-i} | y_i) &= \frac{1}{2} \Phi\left(\frac{h(x_i) - (x_j + x_i)/2}{\sigma\sqrt{3/2}}\right) \\ &\quad + \frac{1}{2} \Phi\left(\frac{h(x_j) - (x_j + x_i)/2}{\sigma\sqrt{3/2}}\right) - \frac{1}{2}(x_j + x_i). \end{aligned}$$

Setting $V_i(R_{-i} | y_i) = 0$ and any solution $y_i = (x_j, x_i)$ of $V_i(R_{-i} | y_i) = 0$ satisfies $h(x_i) = x_j$ and $h(x_j) = x_i$ imply that $x_j + x_i = 1$. Therefore,

$$R_i = \{(x_j, x_i) \in \mathbb{R}^2: \frac{1}{2}(x_j + x_i) < \frac{1}{2}\}.$$

Hence, in any strategy profile that survives iterated elimination of strictly dominated strategies, an agent takes the risky action whenever the average of the two signals she observes is less than $1/2$. A symmetrical argument implies that in any strategy profile that survives the iterated elimination of strictly dominated strategies, the agent takes the safe action whenever the average of her signals is greater than $1/2$. Thus, the game has an essentially unique rationalizable strategy profile and hence an essentially unique Bayesian Nash equilibrium. \square

Proof of Proposition 3

It is sufficient to show that there exists an essentially unique equilibrium in monotone strategies. Note that because of symmetry, the strategies of agents 1 and 3 in the extremal equilibria of the game are identical. Denote the (common) equilibrium threshold of agents 1 and 3's threshold strategies by $\tau \in [0, 1]$, in the sense that they take the risky action if and only if their observation is less than τ . Then the expected payoff of taking the risky action to agent 2 is equal to

$$\mathbb{E}[\pi(k, \theta) | x_1, x_2] = \frac{1}{2} [\mathbb{1}\{x_1 < \tau\} + \mathbb{1}\{x_2 < \tau\} - (x_1 + x_2)].$$

First suppose that $\tau > 1/2$. Then the best response of agent 2 is to take the risky action if either (i) $x_1 + x_2 \leq 1$ or (ii) $x_1, x_2 \leq \tau$ hold. On the other hand, for τ to correspond to the threshold of an equilibrium strategy of agent 1, her expected payoff of taking the risky action has to be positive whenever $x_1 < \tau$. In other words,

$$\begin{aligned} \frac{1}{2} \mathbb{P}(x_2 < \tau | x_1) \\ + \frac{1}{2} [\mathbb{P}(x_1 + x_2 \leq 1 | x_1) + \mathbb{P}(1 - x_1 \leq x_2 \leq \tau | x_1)] \geq x_1 \end{aligned}$$

for all $x_1 < \tau$. As a result, for any $x_1 < \tau$, we have

$$\begin{aligned} \frac{1}{2} \Phi\left(\frac{\tau - x_1}{\sigma\sqrt{2}}\right) + \frac{1}{2} \Phi\left(\frac{1 - 2x_1}{\sigma\sqrt{2}}\right) \\ + \frac{1}{2} \left[\Phi\left(\frac{\tau - x_1}{\sigma\sqrt{2}}\right) - \Phi\left(\frac{1 - 2x_1}{\sigma\sqrt{2}}\right) \right] \geq x_1, \end{aligned}$$

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which simplifies to

$$\Phi\left(\frac{\tau - x_1}{\sigma\sqrt{2}}\right) \geq x_1.$$

Taking the limit of the both sides of the above inequality as $x_1 \uparrow \tau$ implies that $\tau \leq 1/2$. This, however, contradicts the original assumption that $\tau > 1/2$. A similar argument would also rule out the case that $\tau < 1/2$. Hence, τ corresponds to the threshold of an equilibrium strategy of agents 1 and 3 only if $\tau = 1/2$. As a consequence, in the essentially unique equilibrium of the game, agent 2 takes the risky action if and only if $x_1 + x_2 < 1$. This proves the first part of the proposition. The proof of the second part is immediate. \square

Proof of Proposition 4

One again, we can simply focus on the set of equilibria in threshold strategies. Let τ_1 , τ_3 , and τ_4 denote the thresholds of agents 1, 3, and 4, respectively. Note that by symmetry, $\tau_3 = \tau_4$. Also let function f_2 denote the strategy of agent 2, in the sense that agent 2 takes the risky action if and only if $x_2 < f_2(x_1)$.

The expected value of taking the risky action to agent 1 is given by

$$\mathbb{E}[\pi(k, \theta) | x_1] = \frac{1}{3}\mathbb{P}(x_2 < f_2(x_1) | x_1) + \frac{2}{3}\mathbb{P}(x_2 < \tau_3 | x_1) - x_1.$$

Thus, for τ_1 to be the equilibrium threshold strategy for agent 1, the above expression has to be zero at $x_1 = \tau_1$:

$$\tau_1 = \frac{1}{3}\Phi\left(\frac{f_2(\tau_1) - \tau_1}{\sigma\sqrt{2}}\right) + \frac{2}{3}\Phi\left(\frac{\tau_3 - \tau_1}{\sigma\sqrt{2}}\right) \quad (19)$$

On the other hand, the expected value of taking the risky action to agents 3 and 4 is given by

$$\mathbb{E}[\pi(k, \theta) | x_2] = \frac{1}{3}\mathbb{P}(x_1 < \tau_1 | x_2) + \frac{1}{3}\mathbb{P}(x_1 < f_2^{-1}(x_2) | x_2) + \frac{1}{3}\mathbb{1}\{x_2 < \tau_3\} - x_2,$$

where f_2^{-1} is the inverse function of f_2 . For τ_3 to be the equilibrium threshold strategy for agents 3 and 4, the above expression has to be positive for $x_2 < \tau_3$ and negative for $x_2 > \tau_3$, which implies

$$\tau_2 - \frac{1}{3} \leq \frac{1}{3}\Phi\left(\frac{\tau_1 - \tau_3}{\sigma\sqrt{2}}\right) + \frac{1}{3}\Phi\left(\frac{f_2^{-1}(\tau_3) - \tau_3}{\sigma\sqrt{2}}\right) \leq \tau_2. \quad (20)$$

It is easy to verify that the triplet (τ_1, f_2, τ_3) that satisfies conditions (19) and (20) simultaneously is not unique. \square

Proof of Theorem 5

As already mentioned, the Bayesian game under consideration is monotone supermodular in the sense of Van Zandt and Vives (2007), which ensures that the set of equilibria has well-defined maximal and minimal elements, each of which is in threshold strategies. Moreover, by Milgrom and Roberts (1990), all profiles of rationalizable strategies are “sandwiched” between these two equilibria. Hence, to characterize the set of rationalizable strategies, it suffices to focus on threshold strategies and determine the smallest and largest thresholds that correspond to Bayesian Nash equilibria of the game.

Denote the threshold corresponding to the strategy of an agent who observes signal x_r with τ_r . The profile of threshold strategies corresponding to thresholds $\{\tau_r\}$ is a Bayesian Nash equilibrium of the game if and only if for all r ,

$$\begin{aligned} \frac{n}{n-1} \sum_{j \neq r} c_j \Phi\left(\frac{\tau_j - x_r}{\sigma\sqrt{2}}\right) - x_r &\leq 0 \quad \text{for } x_r > \tau_r, \\ \frac{nc_r - 1}{n-1} + \frac{n}{n-1} \sum_{j \neq r} c_j \Phi\left(\frac{\tau_j - x_r}{\sigma\sqrt{2}}\right) - x_r &\geq 0 \quad \text{for } x_r < \tau_r, \end{aligned}$$

where the first (second) inequality guarantees that the agent has no incentive to deviate to the risky (safe) action when the signal she observes is above (below) threshold τ_r . Taking the limit as x_r converges to τ_r from above in the first inequality implies that in any symmetric, Bayesian Nash equilibrium of the game in threshold strategies,

$$(n-1)\tau \geq nHc,$$

where $\tau = [\tau_1, \dots, \tau_m]$ is the vector of thresholds and $H \in \mathbb{R}^{m \times m}$ is a matrix with zero diagonal entries, and off-diagonal entries given by $H_{jr} = \Phi((\tau_j - \tau_r)/\sigma\sqrt{2})$. Therefore,

$$\begin{aligned} 2(n-1)c'\tau &\geq nc'(H' + H)c \\ &= nc'(\mathbf{1}\mathbf{1}')c - nc'c, \end{aligned}$$

where $\mathbf{1}$ is the vector of all ones and $\Phi(z) + \Phi(-z) = 1$. Consequently,

$$2(n-1)c'\tau \geq n(1 - \|c\|_2^2).$$

Finally, given that $|\tau_r - \tau_j| \rightarrow 0$ as $\sigma \rightarrow 0$ and $\|c\|_1 = 1$, the left-hand side of the above inequality converges to $2(n-1)\tau^*$ for some constant τ^* ; as a result, the smallest possible threshold that corresponds to a Bayesian Nash equilibrium is equal to

$$\underline{\tau} = \frac{n}{2(n-1)}(1 - \|c\|_2^2).$$

The expression for $\bar{\tau}$ is derived analogously. \square

Proof of Proposition 6

We first prove that if the size of the largest set of agents with a common observation grows sublinearly in n , then asymptotically as $n \rightarrow \infty$, the game has a unique equilibrium. We denote the vector of the fractions of agents observing each signal by $c(n)$ to make explicit the dependence on the number of agents n . Note that if the size of the largest set of agents with a common observation grows sublinearly in n , then

$$\lim_{n \rightarrow \infty} \|c(n)\|_\infty = 0,$$

where $\|z\|_\infty$ is the maximum element of vector z . On the other hand, by Hölder’s inequality,

$$\|c(n)\|_2^2 \leq \|c(n)\|_1 \cdot \|c(n)\|_\infty.$$

Given that the elements of vector $c(n)$ add up to one, $\lim_{n \rightarrow \infty} \|c(n)\|_2 = 0$. Consequently, Theorem 5 implies that thresholds $\underline{\tau}(n)$ and $\bar{\tau}(n)$ characterizing the set of rationalizable strategies of the game of size n satisfy

$$\lim_{n \rightarrow \infty} \underline{\tau}(n) = \lim_{n \rightarrow \infty} \bar{\tau}(n) = 1/2.$$

Thus, asymptotically, the game has an essentially unique Bayesian Nash equilibrium.

To prove the converse, suppose that the size of the largest set of agents with a common observation grows linearly in n , which means that $\|c(n)\|_\infty$ remains bounded away from zero as $n \rightarrow \infty$. Furthermore, the inequality $\|c(n)\|_\infty \leq \|c(n)\|_2$ immediately implies that $\|c(n)\|_2$ also remains bounded away from zero as $n \rightarrow \infty$. Hence, by Theorem 5,

$$\limsup_{n \rightarrow \infty} \underline{\tau}(n) < 1/2$$

$$\liminf_{n \rightarrow \infty} \bar{\tau}(n) > 1/2,$$

guaranteeing asymptotic multiplicity of equilibria as $n \rightarrow \infty$. \square

Proof of Theorem 7

Pick a nonextremal equilibrium in the game with information structure $Q^* = \{I_1, \dots, I_n\}$ and denote it by (A_1, \dots, A_n) , according to which agent i takes the risky action if and only if $y_i \in A_i$. We denote the expected payoff of taking the risky action to agent i under information structure Q^* by $V_i^*(A_{-i} | y_i)$. Given that each signal is observed by at least two agents and that the strategy profile (A_1, \dots, A_n) is a nonextremal equilibrium, there exists $\epsilon > 0$ such that

$$V_i^*(A_{-i} | y_i) > \epsilon \quad \text{for all } y_i \in A_i$$

$$V_i^*(A_{-i} | y_i) < -\epsilon \quad \text{for all } y_i \notin A_i.$$

Now consider the sequence of games $\{G_s\}$ in which the information structure is drawn according to the sequence of probability distributions $\{\mu_s\}$. It is easy to see that in the game indexed by s , agent i 's expected payoff of taking the risky action, when all other agents follow strategies A_{-i} , is given by

$$V_i^{(s)}(A_{-i} | y_i) = \sum_Q \mu_s(Q) V_i^Q(A_{-i} | y_i),$$

where we denote the expected payoff of taking the risky action to agent i under information structure Q by $V_i^Q(A_{-i} | y_i)$. Since $\lim_{s \rightarrow \infty} \mu_s(Q^*) = 1$, it is immediate that the above expression converges to $V_i^*(A_{-i} | y_i)$.

Consequently, given that $V_i^*(A_{-i} | y_i)$ is uniformly bounded away from zero for all y_i , there exists a large enough \bar{s} such that for $s > \bar{s}$, the expected payoff of taking the risky action in game G_s when other agents follow strategies A_{-i} is strictly positive if $y_i \in A_i$, whereas it would be strictly negative if $y_i \notin A_i$. Thus, for any $s > \bar{s}$, the strategy profile (A_1, \dots, A_n) is also a Bayesian Nash equilibrium of game G_s . \square

Endnotes

1. See Morris and Shin (2003) for more examples.
2. Note that strategic uncertainty is distinct from *fundamental uncertainty*. Whereas fundamental uncertainty simply refers to the agents' uncertainty about the underlying, payoff-relevant state of the world, strategic uncertainty refers to their uncertainty concerning the equilibrium actions of one another.
3. With some abuse of notation, we use I_i to denote both the set of actual signals observed by agent i as well as the set of indices of the signals observed by that agent. Zoumpoulis (2014) discusses the special case where the agents are organized in a network, such that each agent observes a signal that pertains to her and the signals of her neighbors.

4. We relax this assumption in §5.2.

5. As already mentioned, this is a consequence of the underlying game being a Bayesian game with strategic complementarities, and hence the extremal equilibria are monotone in types. For a detailed study of Bayesian games with strategic complementarities, see Van Zandt and Vives (2007).

6. Note that since the information set of any given agent is a subset of the finitely many realized signals, μ is a discrete probability measure with finite support.

7. We thank an anonymous referee for suggesting this result.

8. See, for example, Hadamard's global implicit function theorem in Krantz and Parks (2002).

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CORRECTION

In the print version of this article, “Coordination with Local Information” by Munther A. Dahleh, Alireza Tahbaz-Salehi, John N. Tsitsiklis, and Spyros I. Zoumpoulis (*Operations Research*, Vol. 64, No. 3, pp. 622–637), the research article was mislabeled as a technical note. “Technical Note” has been removed from the title in the online version and an erratum will be printed in the July–August issue.