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**e-companion**

**ONLY AVAILABLE IN ELECTRONIC FORM**

# Electronic Companion—“Efficiency of Scalar-Parameterized Mechanisms” by Ramesh Johari and John N. Tsitsiklis, *Operations Research*, DOI 10.1287/opre.1080.0638.

# Omitted Proofs

*Step 2, Proof of Theorem 1: A user's payoff is concave if he is price taking.* The condition that a uniform market-clearing price must exist implies that for any fixed $\theta > 0$, the range of $D(\mu, \theta)$ must contain $(0, \infty)$ as $\mu$ varies in $(0, \infty)$. Now suppose that for fixed $\theta > 0$, there exist $\mu_1, \mu_2 > 0$ with $\mu_1 \neq \mu_2$ such that $D(\mu_1, \theta) = D(\mu_2, \theta) = d$, where $d > 0$. Let $C = 2d$ and let $R = 2$. Then for $\boldsymbol{\theta} = (\theta, \theta)$, there cannot exist a unique market-clearing price $p_D(\boldsymbol{\theta})$; so we conclude that $D(\cdot, \theta)$ is monotonic, and strictly monotonic in the region where it is nonzero.

Let $I \subset (0, \infty)$ be the set of $\theta > 0$ such that $D(\mu, \theta)$ is monotonically nondecreasing in $\mu$. From the preceding paragraph, we conclude that if $\theta \in (0, \infty) \setminus I$, then $D(\mu, \theta)$ is necessarily monotonically nonincreasing in $\mu$. Further, if $\theta \in I$, then $D(\mu, \theta) \to \infty$ as $\mu \to \infty$, and $D(\mu, \theta) \to 0$ as $\mu \to 0$; on the other hand, if $\theta \in (0, \infty) \setminus I$, then $D(\mu, \theta) \to 0$ as $\mu \to \infty$, and $D(\mu, \theta) \to \infty$ as $\mu \to 0$.

Suppose $I \neq (0, \infty)$ and $I \neq \emptyset$; then choose $\theta \in \partial I$, the boundary of $I$. Choose a sequence $\theta_n \in I$ such that $\theta_n \to \theta$; and choose another sequence $\hat{\theta}_n \in (0, \infty) \setminus I$ such that $\hat{\theta}_n \to \theta$. Fix $\mu_1, \mu_2$ with $0 < \mu_1 < \mu_2$, such that $D(\mu_1, \theta) > 0$ and $D(\mu_2, \theta) > 0$. Then we have $D(\mu_1, \theta_n) \leq D(\mu_2, \theta_n)$, and $D(\mu_1, \hat{\theta}_n) \geq D(\mu_2, \hat{\theta}_n)$. Taking limits as $n \to \infty$, we get $D(\mu_1, \theta) \leq D(\mu_2, \theta)$, and $D(\mu_1, \theta) \geq D(\mu_2, \theta)$, so that $D(\mu_1, \theta) = D(\mu_2, \theta)$. But this is not possible, since $D(\cdot, \theta)$ must be strictly monotonic in the region where it is nonzero. Thus $I = (0, \infty)$ or $I = \emptyset$.

We will use Step 1 to show $D(\mu, \theta)$ is concave in $\theta \geq 0$ for fixed $\mu > 0$. Since $D(\mu, \theta)$ is continuous, it suffices to show that $D(\mu, \theta)$ is concave for $\theta > 0$. Suppose not; fix $\theta > 0$, $\overline{\theta} > 0$, and $\delta \in (0, 1)$ such that:

$$D(\mu, \delta\theta + (1 - \delta)\overline{\theta}) < \delta D(\mu, \theta) + (1 - \delta) D(\mu, \overline{\theta}). \tag{EC.1}$$

Note this implies in particular that either $D(\mu, \theta) > 0$ or $D(\mu, \overline{\theta}) > 0$. We assume without loss of generality that $D(\mu, \theta) > 0$. Let $C^R = RD(\mu, \theta)$, and let $\boldsymbol{\theta}^R = (\theta, \ldots, \theta) \in (\mathbb{R}^+)^R$. To emphasize the dependence of the market-clearing price on the capacity, we will let $p_D(\overline{\boldsymbol{\theta}}; C)$ denote the market-clearing price when the composite strategy vector is $\overline{\boldsymbol{\theta}}$ and the capacity is $C$. We will show that for any $\theta' > 0$, if $\mu^R = p_D(\boldsymbol{\theta}^{R-1}, \theta'; C^R)$, then $\mu^R \to \mu$ as $R \to \infty$. First note that by definition, we have $D(\mu^R, \theta') + (R-1)D(\mu^R, \theta) = RD(\mu, \theta)$; or, rewriting, we have:

$$\frac{1}{R} D(\mu^R, \theta') + \left(1 - \frac{1}{R}\right) D(\mu^R, \theta) = D(\mu, \theta). \tag{EC.2}$$

Now note that as $R \to \infty$, the right hand side remains constant. Suppose that $\mu^R \to \infty$. Since $I = (0, \infty)$ or $I = \emptyset$, either $D(\mu^R, \theta'), D(\mu^R, \theta) \to 0$, or $D(\mu^R, \theta'), D(\mu^R, \theta) \to \infty$; in either case, the equality (EC.2) is violated for large $R$. A similar conclusion holds if $\mu^R \to 0$ as $R \to \infty$. Thus we do not have $\mu^R \to 0$ or $\mu^R \to \infty$ as $R \to \infty$. Choose a convergent subsequence, such that $\mu^{R_k} \to \hat{\mu}$, where $\hat{\mu} \in (0, \infty)$. From (EC.2), we must have $D(\hat{\mu}, \theta) = D(\mu, \theta)$. But as established above, since $D(\cdot, \theta)$ is strictly monotonic in the region where it is nonzero, this is only possible if $\hat{\mu} = \mu$. We conclude that the following three limits hold:

$$\begin{aligned}
\lim_{R \to \infty} p_D(\boldsymbol{\theta}^R; C^R) &= \mu; \\
\lim_{R \to \infty} p_D(\boldsymbol{\theta}^{R-1}, \overline{\theta}; C^R) &= \mu; \\
\lim_{R \to \infty} p_D(\boldsymbol{\theta}^{R-1}, \delta\theta + (1-\delta)\overline{\theta}; C^R) &= \mu.
\end{aligned}$$

The remainder of the proof is straightforward. From (EC.1), for $R$ sufficiently large, we must have:

$$\begin{aligned}
&D(p_D(\boldsymbol{\theta}^{R-1}, \delta\theta + (1-\delta)\overline{\theta}; C^R), \delta\theta + (1-\delta)\overline{\theta}) \\
&\qquad < \delta D(p_D(\boldsymbol{\theta}^R; C^R), \theta) + (1-\delta) D(p_D(\boldsymbol{\theta}^{R-1}, \overline{\theta}; C^R), \overline{\theta}).
\end{aligned}$$

This violates the conclusion of Step 1, so we conclude $D(\mu, \theta)$ is concave in $\theta \geq 0$ given $\mu > 0$. A similar argument shows that $\mu D(\mu, \theta)$ is convex in $\theta$, by using the fact that $p_D(\boldsymbol{\theta}) D(p_D(\boldsymbol{\theta}), \theta_r)$ must be convex in $\theta_r$ for nonzero $\boldsymbol{\theta}$. Combining these results yields the desired conclusion.

*Step 5, Proof of Theorem 1: $B$ is an invertible, differentiable, strictly increasing, and concave function on* $(0, \infty)$. Note from (10) that:

$$B(p_D(\boldsymbol{\theta})) = \frac{\sum_{r=1}^R \theta_r}{C}. \tag{EC.3}$$

We immediately see that $B$ must be invertible on $(0,\infty)$; it is clearly onto, as the right hand side of (EC.3) can take any value in $(0,\infty)$. Furthermore, if $B(p_1)=B(p_2)=\gamma$ for some prices $p_1, p_2 > 0$, then choosing $\boldsymbol{\theta}$ such that $\sum_{r=1}^R \theta_r / C = \gamma$, we find that $p_D(\boldsymbol{\theta})$ is not uniquely defined. Thus $B$ is one-to-one as well, and hence invertible. Finally, note that since $D$ is differentiable, $B$ must be differentiable as well.

We let $\Phi$ denote the differentiable inverse of $B$. We will show that $\Phi$ is strictly increasing and convex. We first note that for nonzero $\boldsymbol{\theta}$ we have:

$$p_D(\boldsymbol{\theta}) = \Phi\left(\frac{\sum_{r=1}^R \theta_r}{C}\right).$$

Let

$$w_r(\boldsymbol{\theta}) = p_D(\boldsymbol{\theta}) D(p_D(\boldsymbol{\theta}), \theta_r) = \Phi\left(\frac{\sum_{s=1}^R \theta_s}{C}\right)\left(\frac{\theta_r}{\sum_{s=1}^R \theta_s} C\right). \tag{EC.4}$$

By Step 1, $w_r(\boldsymbol{\theta})$ is convex in $\theta_r > 0$. By considering strategy vectors $\boldsymbol{\theta}$ for which $\boldsymbol{\theta}_{-r} = \mathbf{0}$, it follows that $\Phi$ is convex.

It remains to be shown that $\Phi$ is strictly increasing. Since $\Phi$ is invertible, it must be monotonic; and thus $\Phi$ is either strictly increasing or strictly decreasing. To simplify the argument, we assume that $\Phi$ is twice differentiable.[10] We twice differentiate $w_r(\boldsymbol{\theta})$, given in (EC.4). Letting $\mu = \sum_{s=1}^R \theta_s / C$, we have for nonzero $\boldsymbol{\theta}$:

$$\frac{\partial^2 w_r}{\partial \theta_r^2}(\boldsymbol{\theta}) = \Phi''(\mu)\frac{\theta_r}{C^2 \mu} + \frac{2\sum_{s\neq r}\theta_s}{C^2\mu^3}\left(\mu\Phi'(\mu) - \Phi(\mu)\right). \tag{EC.5}$$

Consider some nonzero $\boldsymbol{\theta}_{-r}$, and take the limit as $\theta_r \to 0$. The limit of the left-hand side in (EC.5) is nonnegative, by the convexity of $w_r(\boldsymbol{\theta})$ in $\theta_r > 0$. The limit of the first term in the right-hand side of (EC.5) is zero. Since $\Phi(\mu) > 0$, it follows that $\Phi'(\mu) > 0$, so that $\Phi$ is strictly increasing. This establishes the desired facts regarding $B$.

[10] While the most direct argument uses twice differentiability of $\Phi$, it is possible to make a similar argument even if $\Phi$ is only once differentiable, by arguing only in terms of increments of $\Phi$.

*Step 6, Proof of Theorem 1: Let $(C,R,\mathbf{U})$ be a utility system. A vector $\boldsymbol{\theta}\geq 0$ is a Nash equilibrium if and only if at least two components of $\boldsymbol{\theta}$ are nonzero, and there exists a nonzero vector $\mathbf{d}\geq 0$ and a scalar $\mu>0$ such that $\theta_r=\mu d_r$ for all $r$, $\sum_{r=1}^R d_r = C$, and the following conditions hold:*

$$U_r'(d_r)\left(1-\frac{d_r}{C}\right)=\Phi(\mu)\left(1-\frac{d_r}{C}\right)+\mu\Phi'(\mu)\left(\frac{d_r}{C}\right), \quad \text{if } d_r>0;$$

$$U_r'(0)\leq \Phi(\mu), \quad \text{if } d_r=0.$$

*In this case $d_r = D(p_D(\boldsymbol{\theta}),\theta_r)$, $\mu=\sum_{r=1}^R \theta_r/C$, and $\Phi(\mu)=p_D(\boldsymbol{\theta})$.* Suppose that $\boldsymbol{\theta}$ is a Nash equilibrium. Since $Q_r(\theta_r;\boldsymbol{\theta}_{-r})=-\infty$ if $\boldsymbol{\theta}=0$, (from (7)), we must have $\boldsymbol{\theta}\neq 0$. Suppose then that only one component of $\boldsymbol{\theta}$ is nonzero; say $\theta_r>0$, and $\boldsymbol{\theta}_{-r}=0$. Then the payoff to user $r$ is:

$$U_r(C)-\Phi\left(\frac{\theta_r}{C}\right)C.$$

But now observe that by infinitesimally reducing $\theta_r$, user $r$ can strictly improve his payoff (since $\Phi$ is strictly increasing). Thus $\boldsymbol{\theta}$ could not have been a Nash equilibrium; we conclude that at least two components of $\boldsymbol{\theta}$ are nonzero. In this case, from (7), and the expressions in (11) and (EC.4), the payoff $Q_r(\overline{\theta}_r;\boldsymbol{\theta}_{-r})$ to user $r$ is differentiable. When two components of $\boldsymbol{\theta}$ are nonzero, we may write the payoff $Q_r$ to user $r$ as follows, using (11) and (EC.4):

$$Q_r(\theta_r;\boldsymbol{\theta}_{-r})=U_r\left(\frac{\theta_r}{\sum_{s=1}^R\theta_s}C\right)-\Phi\left(\frac{\sum_{s=1}^R\theta_s}{C}\right)\left(\frac{\theta_r}{\sum_{s=1}^R\theta_s}C\right).$$

Differentiating the previous expression with respect to $\theta_r$, we conclude that if $\boldsymbol{\theta}$ is a Nash equilibrium then the following optimality conditions hold for each $r$:

$$F_r(\boldsymbol{\theta})=0 \quad \text{if } \theta_r>0; \tag{EC.6}$$

$$F_r(\boldsymbol{\theta})\leq 0 \quad \text{if } \theta_r=0, \tag{EC.7}$$

where

$$F_r(\boldsymbol{\theta})=U_r'\left(\frac{\theta_r}{\sum_{s=1}^R\theta_s}C\right)\left(\frac{C}{\sum_{s=1}^R\theta_s}-\frac{\theta_r C}{\left(\sum_{s=1}^R\theta_s\right)^2}\right)-\Phi'\left(\frac{\sum_{s=1}^R\theta_s}{C}\right)\left(\frac{\theta_r}{\sum_{s=1}^R\theta_s}\right)$$

$$-\Phi\left(\frac{\sum_{s=1}^R \theta_s}{C}\right)\left(\frac{C}{\sum_{s=1}^R \theta_s} - \frac{\theta_r C}{\left(\sum_{s=1}^R \theta_s\right)^2}\right).$$

These conditions are equivalent to (14)-(15), if we make the substitutions $\mu = \sum_{s=1}^R \theta_s / C$, and $d_r = D(p_D(\boldsymbol{\theta}), \theta_r)$. Furthermore, in this case we have $\mathbf{d} \geq 0$, $\mu > 0$, $\theta_r = \mu d_r$, $\sum_{r=1}^R d_r = C$, and $p_D(\boldsymbol{\theta}) = \Phi(\mu)$.

On the other hand, suppose that we have found $\boldsymbol{\theta}$, $\mathbf{d}$, and $\mu$ such that the conditions of Step 6 are satisfied. In this case we simply reverse the argument above; since $Q_r(\overline{\theta}_r; \boldsymbol{\theta}_{-r})$ is concave in $\overline{\theta}_r$ (Condition 2 in Definition 4), if at least two components of $\boldsymbol{\theta}$ are nonzero then the conditions (EC.6)-(EC.7) are necessary and sufficient for $\boldsymbol{\theta}$ to be a Nash equilibrium. Furthermore, if $\mathbf{d} \geq 0$, $\mu > 0$, $\theta_r = \mu d_r$, and $\sum_{r=1}^R d_r = C$, then it follows that $\mu = \sum_{s=1}^R \theta_s / C$, $\Phi(\mu) = p_D(\boldsymbol{\theta})$, and $d_r = D(p_D(\boldsymbol{\theta}), \theta_r)$. Thus the conditions (EC.6)-(EC.7) become equivalent to (14)-(15), as required.

*Step 7, Proof of Theorem 1: Let $(C, R, \mathbf{U})$ be a utility system. Then there exists a unique Nash equilibrium.* Our approach will be to demonstrate existence of a Nash equilibrium by finding a solution $\mu > 0$ and $\mathbf{d} \geq 0$ to (14)-(15), such that $\sum_{r=1}^R d_r = C$. If we find such a solution, then at least two components of $\mathbf{d}$ must be nonzero; otherwise, (14) cannot hold for the user $r$ with $d_r = C$. If we define $\boldsymbol{\theta} = \mu \mathbf{d}$, then $\mu = \sum_{s=1}^R \theta_s / C$, so $p_D(\boldsymbol{\theta}) = \Phi(\mu)$; and from (11), we have $d_r = D(p_D(\boldsymbol{\theta}), \theta_r)$. Thus if $\mu > 0$ and $\mathbf{d} \geq 0$ satisfy (14)-(15), then $\boldsymbol{\theta} = \mu \mathbf{d}$ is a Nash equilibrium by Step 6. Consequently, it suffices to find a solution $\mu > 0$ and $\mathbf{d} \geq 0$ to (14)-(15).

We first show that for a fixed value of $\mu > 0$, the equality in (14) has at most one solution $d_r$. To see this, rewrite (14) as:

$$U_r'(d_r)\left(1 - \frac{d_r}{C}\right) - (\mu \Phi'(\mu) - \Phi(\mu))\left(\frac{d_r}{C}\right) = \Phi(\mu).$$

Since $\Phi$ is convex and strictly increasing with $\Phi(\mu) \to 0$ as $\mu \to 0$, we have $\mu \Phi'(\mu) - \Phi(\mu) \geq 0$. Thus the left hand side is strictly decreasing in $d_r$ (since $U_r$ is strictly increasing and concave), from $U_r'(0)$ at $d_r = 0$ to $\Phi(\mu) - \mu \Phi'(\mu) \leq 0$ when $d_r = C$. This implies a unique solution $d_r \in [0, C]$ exists for the equality in (14) as long as $U_r'(0) \geq \Phi(\mu)$; we denote this solution $d_r(\mu)$. If $\Phi(\mu) > U_r'(0)$, then we let $d_r(\mu) = 0$. Observe that as $\mu \to 0$, we must have $d_r(\mu) \to C$, since otherwise we can show that (14) fails to hold for sufficiently small $\mu$.

Next we show that $d_r(\mu)$ is continuous. Since we defined $d_r(\mu) = 0$ if $\Phi(\mu) > U_r'(0)$, and $d_r(\mu) = 0$ if $\Phi(\mu) = U_r'(0)$ from (14), it suffices to show that $d_r(\mu)$ is continuous for $\mu$ such that $\Phi(\mu) \leq U_r'(0)$. But in this case continuity of $d_r$ can be shown using (14), together with the fact that $U_r'$, $\Phi$, and $\Phi'$ are all continuous (the latter because $\Phi$ is concave and differentiable, and hence continuously differentiable). Indeed, suppose that $\mu_n \to \mu$ where $\Phi(\mu) \leq U_r'(0)$, and assume without loss of generality that $d_r(\mu_n) \to d_r$ (since $d_r(\mu_n)$ takes values in the compact set $[0, C]$). Then since $\mu_n$ and $d_r(\mu_n)$ satisfy the equality in (14) for sufficiently large $n$, by taking limits we see that $\mu$ and $d_r$ satisfy the equality in (14) as well. Thus we must have $d_r = d_r(\mu)$, so we conclude $d_r(\mu)$ is continuous.

We now show that $d_r(\mu)$ is nonincreasing in $\mu$. To see this, choose $\mu_1, \mu_2 > 0$ such that $\mu_1 < \mu_2$. Suppose that $d_r(\mu_1) < d_r(\mu_2)$. Then, in particular, $d_r(\mu_2) > 0$, so (14) holds with equality for $d_r(\mu_2)$ and $\mu_2$. Now note that as we move from $d_r(\mu_2)$ to $d_r(\mu_1)$, the left hand side of (14) strictly increases (since $U_r$ is concave). On the other hand, since $\Phi$ is convex and strictly increasing with $\Phi(\mu) \to 0$ as $\mu \to 0$, we have the inequalities $\mu_2 \Phi'(\mu_2) - \Phi(\mu_2) \geq \mu_1 \Phi'(\mu_1) - \Phi(\mu_1) \geq 0$. From this it follows that the right hand side of (14) strictly decreases as we move from $d_r(\mu_2)$ to $d_r(\mu_1)$ and from $\mu_2$ to $\mu_1$. Thus neither (14) nor (15) can hold at $d_r(\mu_1)$ and $\mu_1$; so we conclude that for all $r$, we must have $d_r(\mu_1) \geq d_r(\mu_2)$.

Thus for each $r$, $d_r(\mu)$ is a nonincreasing continuous function such that $d_r(\mu) \to C$ as $\mu \to 0$, and $d_r(\mu) \to 0$ as $\mu \to \infty$. We conclude there exists at least one $\mu > 0$ such that $\sum_{r=1}^R d_r(\mu) = C$; and in this case $\mathbf{d}(\mu)$ satisfies (14)-(15), so by the discussion at the beginning of this step, we know that $\boldsymbol{\theta} = \mu \mathbf{d}(\mu)$ is a Nash equilibrium.

Finally, we show that the Nash equilibrium is unique. Suppose that there exist two solutions $\mathbf{d}^1 \geq 0, \mu_1 > 0$, and $\mathbf{d}^2 \geq 0, \mu_2 > 0$ to (14)-(15), such that $\sum_{r=1}^R d_r^i = C$ for $i = 1, 2$. Of course, we must have $\mathbf{d}^i = \mathbf{d}(\mu_i)$, $i = 1, 2$. We assume without loss of generality that $\mu_1 \leq \mu_2$; our goal is to show that $\mu_1 = \mu_2$. Since $d_r(\cdot)$ is nonincreasing, we know $d_r(\mu_1) \geq d_r(\mu_2)$ for all $r$. Since $\sum_{r=1}^{R} d_r^i = C$ for $i = 1, 2$, we conclude that $d_r(\mu_1) = d_r(\mu_2)$ for every $r$. Let $r$ be such that $d_r(\mu_1) = d_r(\mu_2) > 0$. Observe that $\Phi(\mu)$ and $\mu\Phi'(\mu)$ are both strictly increasing in $\mu > 0$, since $\Phi$ is strictly increasing and convex. Thus for fixed $d_r > 0$, the equality in (14) has a unique solution $\mu$, so $d_r(\mu_1) = d_r(\mu_2) > 0$ implies $\mu_1 = \mu_2$. Thus (14)-(15) have a unique solution $\mathbf{d} \geq 0$, $\mu > 0$, such that $\sum_{r=1}^{R} d_r = C$. From Step 6, this ensures the Nash equilibrium $\boldsymbol{\theta} = \mu\mathbf{d}$ is unique as well.