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PERIODIC REVIEW INVENTORY SYSTEMS WITH CONTINUOUS DEMAND AND DISCRETE ORDER SIZES*

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We study a single product inventory system with nonnegative setup cost in which the demand is a continuous random variable but orders are restricted to be integer valued. Optimal policies, when there are no setup costs, have a nice form. However, we show that, when the setup costs are nonzero, optimal policies may have a very counterintuitive form without any particular structure. We obtain a bound for the increase in costs resulting from the restriction of orders to be integers and define a suboptimal policy whose performance is within that bound.

(INVENTORY SYSTEMS; PERIODIC REVIEW; CONTINUOUS DEMAND; DISCRETE ORDERS)

1. Introduction

We consider a periodic review, stochastic, single product inventory model with nonnegative setup cost in which the demands in each period are independent random variables with known distribution functions. However, although the demand is allowed to take a continuous range of values, we constrain the amounts being ordered to take only integer values. The physical justification for this model is the following: Suppose that a warehouse has a relatively large demand for a particular product but the unit of that product is relatively small. In that case the demand and the stock levels may be accurately described by continuous variables. On the other hand, suppose that, for various reasons, the warehouse may not order arbitrary amounts of this product because it is being sold by a wholesale supplier only in quantities which are integer multiples of a certain minimum quantity. (Consider, for example, a warehouse selling light bulbs which are packed by the wholesale supplier in boxes containing 100 of them.) If the minimum size of an order is of a different order of magnitude from the unit of the product and a substantial fraction of a typical order, then the order levels should be modelled as discrete variables.

An extensive literature exists on the single or multi-product inventory control problem when all variables are continuous (Arrow et al. 1958, Bertsekas 1976, Kalin 1980, Schal 1976, Veinott 1966). In general, optimal policies for the single product problem are well known to be characterized by two numbers s and S, with $s < S$, and have the following form: If the current stock is higher than the threshold s do not order; if lower, order such an amount to reach the target level S. For our model, when the setup cost K is zero, optimal policies are similar to those obtained for the continuous model except that the target level is replaced by a *target interval* (Veinott 1965).

For the case where $K > 0$, however, we show, by means of an example, that optimal policies do not have any nice form, in general. In particular, the optimal order may be an increasing function of the stock level (in the vicinity of some point) which contrasts sharply with the properties of optimal policies for the inventory control problems that have been analyzed in the literature.

In the last section, a particular suboptimal policy is considered and bounds for the distance from optimality of that particular policy are derived. This policy approximates, in some sense, the optimal policy for the inventory control problem with continuous orders and may be computed much more easily than the exactly optimal policy.

2. Model Description

We model our system as follows: Let x_k be the stock available at the beginning of the kth period. Let $u_k \in \mathcal{N}_0 = \{0, 1, \dots\}$ be the quantity ordered (assumed to be immediately delivered) at the beginning of the kth period. Let w_k be a sequence of independent, nonnegative and bounded random variables, denoting the demand

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during the kth period. Let $L_k(\cdot)$ be a function representing the storage cost of the items that remain unsold at the end of the kth period plus the cost due to unfilled demand during the kth period (shortage cost). A particular choice of L_k which is often assumed is

$$L_k(y) = h_k \max\{0, y\} + p_k \max\{0, -y\}, \tag{1}$$

where h_k and p_k are the storage and shortage costs per unit of the product, respectively, and $y = x_k + u_k - w_k$. We only assume in the sequel that L_k is convex and goes to ∞ as $|y|$ goes to ∞. Let $c_k \geq 0$ be the ordering cost per unit of the product and $K \geq 0$ a fixed cost incurred each time a nonzero order is placed. In order to obtain a nontrivial problem we assume that

$$\lim_{x \to -\infty} L_k(x) + c_k x = \infty. \tag{2}$$

If L_k is given by (1) this is equivalent to the assumption $c_k < p_k$.

We assume that excess demand is backlogged and filled as soon as additional inventory becomes available. To represent backlogged demand, we allow x_k to take negative values as well. We then have $x_{k+1} = x_k + u_k - w_k$ and the cost functional to be minimized, for the N-period problem, is $E[\sum_{k=0}^{N-1} c_k u_k + K\delta(u_k) + L_k(x_k + u_k - w_k)]$, where $\delta(u_k) = 1$ if $u_k > 0$, zero otherwise and $E[\cdot]$ denotes expected value. We have implicitly assumed, for simplicity, that the unfilled demand at the end of the Nth period is lost and that the leftover stock x_N has no value. Unlike Veinott (1965), our results remain true if an arbitrary, convex terminal cost $L_N(x_N)$ is assumed.

Let $J_k^*(x)$ be the optimal cost-to-go starting from the beginning of the kth period, the stock level being x and let

$$J_k(x,u) = c_k u + K\delta(u) + E\big[L_k(x + u - w_k) + J_{k+1}^*(x + u - w_k)\big]. \tag{3}$$

The dynamic programming equations are

$$J_k^*(x) = \inf_{u \in \mathcal{N}_0} J_k(x,u). \tag{4}$$

By means of a standard inductive dynamic programming argument, it is possible to show (when $K = 0$) that there exists a sequence $\{S_k\}$ of real numbers such that the policy $u_k^*(x)$ given, as a function of the stock level x_k, by

$$u_k^*(x_k) = \begin{cases} n & \text{if } x_k \in [S_k - n, S_k - n + 1), \quad n = 1, 2, \dots, \\ 0 & \text{if } x_k \geq S_k, \end{cases}$$

is optimal. The proof essentially consists of showing inductively that $J_k^*(x)$ has the following property: $J_k^*(x + 1) - J_k^*(x)$ is a nondecreasing function of x.

The policy prescribed above has an appealing form: There exists a target set $[S_k, S_k + 1)$ which is a unit interval and orders are placed so that the stock level falls into that set. If the current stock is larger than the threshold S_k, no order is placed. This policy is the same as the one derived in Veinott (1965) under slightly more restrictive conditions and for this reason we omit the proof. If the minimum size of the orders is decreased, the length of the target set decreases accordingly and, in the limit, we would recover the solution of the standard problem with continuous orders and zero setup cost.

3. Optimal Policies When $K > 0$

Optimal policies for inventory control problems usually obey the following rule: If it is optimal to order an amount u when the stock level is x, then the optimal order from a level $y < x$ should be at least as much as u. In more precise terms, there exists an optimal policy $\{u_k^*(x), k = 0, 1, \ldots, N-1\}$ such that $u_k^*(x)$ is a nonincreasing function of x, for any k. This property holds for the standard inventory model with continuous orders and nonnegative setup cost, since it is known that optimal policies are (s, S) policies. The only requirement is that K does not vary with time. The same property holds when the orders are restricted to be integer valued and the fixed cost is zero (see the previous section). It is, therefore, rather surprising that this property no longer holds for our model when the fixed cost is nonzero. We present below a three-stage example for which no optimal policy is monotonic.

EXAMPLE. Consider a three-stage model with time-invariant parameters and deterministic demand, given by:

(a) $L_k(y) = L(y) = |y|$, $\forall k$. That is, $L_k(\cdot)$ has the form (1) with $h_k = p_k = 1$.

(b) $K = 2.2$, $c_k = 0$, $\forall k$.

(c) The demand w_k is equal to 0.6 (for each time k), with probability 1.

For the last stage ($k = 2$), it is easy to see that, for $x > -1.6$, it is optimal to order nothing. Therefore, $J_2^*(x) = L(x - 0.6) = |x - 0.6|$, $x > -1.6$.

To find an optimal policy at time $k = 1$, we form $E[L(x - w) + J_2^*(x - w)] = |x - 0.6| + |x - 1.2|$, $x > -1$ (see Figure 1). We can see from Figure 1 that an optimal policy is given by

$$u_1^*(x) = \begin{cases} 0, & x \geqslant -0.65, \\ 2, & x \in (-1, -0.65). \end{cases}$$

The cost-to-go function $J_1^*(x)$ is plotted in Figure 2, for $x > -1$.

We now evaluate an optimal policy at time 0, for the states $x = -0.05$ and $x = -0.2$. Using (3), we have

$J_0(x = -0.05, u = 0) = 0 + 0.65 + J_1^*(-0.65) = 3.75,$

$J_0(x = -0.05, u = 1) = 2.2 + 0.35 + J_1^*(0.35) = 3.65,$

$J_0(x = -0.05, u = 2) = 2.2 + 1.35 + J_1^*(1.35) = 4.45.$

Higher values of u do not need to be considered and we obtain $u_0^*(-0.05) = 1$. Also,

$J_0(x = -0.2, u = 0) = 0 + 0.8 + J_1^*(-0.8) = 3.6,$

$J_0(x = -0.2, u = 1) = 2.2 + 0.2 + J_1^*(0.2) = 3.8,$

$J_0(x = -0.2, u = 2) = 2.2 + 1.2 + J_1^*(1.2) = 4.0.$

Again, higher values of u do not need to be considered and we conclude that $u_0^*(-0.2) = 0$. This shows that no optimal policy u_0^* can be monotonic.

It should be stressed that the surprising behavior of optimal policies occurs in the above example even though the model is stationary. Moreover, since cost-to-go functions depend continuously on h, p, it is easy to see that the nonmonotonicity of optimal policies is retained, even if h or p are slightly perturbed in either direction. Consequently, there exist examples displaying nonmonotonicity, with either $h > p$ or $h < p$.

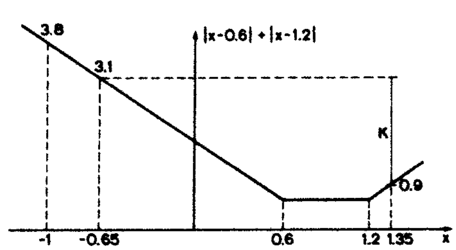

FIGURE 1

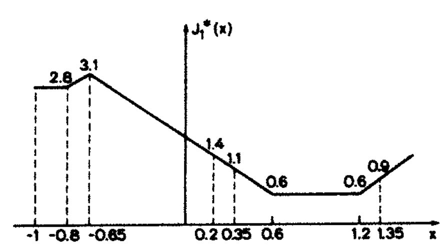

FIGURE 2

4. Suboptimal Policies and Comparison with the Continuous Model

In this section we compare the optimal cost-to-go functions of the models with continuous and discrete orders. In the course of this comparison, we evaluate the performance of a particular (suboptimal) policy for the discrete model. We allow K to take any nonnegative value.

We assume that the slope of L_k is bounded and let A, B be nonnegative constants such that $-A\Delta \leqslant L_k(x+\Delta) - L_k(x) \leqslant B\Delta$, $\forall x \in R$, $\Delta > 0$. In particular, if L_k is given by (1), we may let $A = \max_k\{p_k\}$, $B = \max_k\{h_k\}$. Let $J_k^*(x_k)$ be as defined in §2 and let $\hat{J}_k^*(x)$ be the optimal cost-to-go, starting at the kth stage, when the control variable u is allowed to take any nonnegative real value. Clearly then, $\hat{J}_k^*(x) \leqslant J_k^*(x)$, $\forall x \in R$. Let $\hat{u}_k(x)$ be an optimal policy for the continuous problem. We augment the state of the system by introducing a new state variable $y \in R$ with dynamics $y_{k+1} = y_k + \hat{u}_k(y_k) - w_k$, $k = 0, 1, \ldots, N-1$, initialized by $y_0 = x_0$. So defined, $\{y_k\}$ is the trajectory that would be followed if the optimal continuous order policy was used. Fix some $\lambda \in [0, 1)$. We define a policy $u_k^\lambda(x_k, y_k)$ for the discrete problem as follows:

$u_k^\lambda(x_k, y_k)$ is the unique nonnegative integer such that $x_k + u_k^\lambda(x_k, y_k) \in [y_k + \hat{u}(y_k) - \lambda, y_k + \hat{u}_k(y_k) + 1 - \lambda)$. This policy represents the discrete order policy derived from the continuous order policy by rounding to an integer, either above or below. The rounding is determined by the parameter λ.

Lemma 1. *$u_k^\lambda(x_k, y_k)$ is well defined. When this policy is used, the corresponding trajectory $\{(x_k, y_k) : k = 0, 1, \ldots, N\}$ of the augmented state is such that $x_k \in [y_k - \lambda, y_k + 1 - \lambda)$ for all k.*

Proof. The proof is by forward induction on k. For $k = 0$, $x_0 = y_0$ and $x_0 \in [y_0 - \lambda, y_0 + 1 - \lambda)$. Now suppose that $x_k \in [y_k - \lambda, y_k + 1 - \lambda)$. Then $x_k < y_k + 1 - \lambda$ implies that a unique nonnegative integer $u_k^\lambda(x_k, y_k)$ exists such that $u_k^\lambda(x_k, y_k) + x_k \in [y_k - \lambda + \hat{u}_k(y_k), y_k + 1 - \lambda + \hat{u}_k(y_k))$. Moreover,

$$\begin{aligned} x_{k+1} &= x_k + u_k^\lambda(x_k, y_k) - w_k \in [y_k - \lambda + \hat{u}_k(y_k) - w_k, y_k + 1 - \lambda + \hat{u}_k(y_k) - w_k) \\ &= [y_{k+1} - \lambda, y_{k+1} + 1 - \lambda). \quad \text{Q.E.D.} \end{aligned}$$

Let $J_0^\lambda(x_0)$ be the cost of the N-stage problem corresponding to policy $\{u_k^\lambda(x_k, y_k)\}$ and assume that $c_k = c$ is independent of k.

Theorem 1.

$$\hat{J}_0^*(x_0) \leqslant J_0^*(x_0) \leqslant J_0^\lambda(x_0) \leqslant \hat{J}_0^*(x_0) + (1-\lambda)c + N\max\{\lambda A, (1-\lambda)B\}. \tag{5}$$

Proof. The first inequality follows from the fact that the enlargement of the range of the control variables cannot increase the costs. The second is immediate because policy $\{u_k^\lambda(x_k, y_k)\}$ cannot be better than an optimal policy for the discrete model. We concentrate on the third inequality. We have

$$\hat{J}_0^*(x_0) = E\left[\sum_{k=0}^{N-1} c\hat{u}_k(y_k) + K\delta(\hat{u}_k(y_k)) + L_k(y_k + \hat{u}_k(y_k))\right],$$

$$J_0^\lambda(x_0) = E\left[\sum_{k=0}^{N-1} cu_k^\lambda(x_k, y_k) + K\delta\big(u_k^\lambda(x_k, y_k)\big) + L_k\big(x_k + u_k^\lambda(x_k, y_k)\big)\right].$$

Observe that if $\hat{u}_k(y_k) = 0$, then $u_k^\lambda(x_k, y_k) = 0$. Therefore,

$$\sum_{k=0}^{N-1} K\delta\big(u_k^\lambda(x_k, y_k)\big) \leq \sum_{k=0}^{N-1} K\delta(\hat{u}_k(y_k)), \tag{6}$$

with probability 1. Observe also that

$$y_0 + \sum_{k=0}^{N-1} \hat{u}_k(y_k) = y_N + \sum_{k=0}^{N-1} w_k, \qquad x_0 + \sum_{k=0}^{N-1} u_k^\lambda(x_k, y_k) = x_N + \sum_{k=0}^{N-1} w_k.$$

Since $x_0 = y_0$ and since $x_N - y_N < 1 - \lambda$, we obtain

$$c\sum_{k=0}^{N-1} u_k^\lambda(x_k, y_k) < c\sum_{k=0}^{N-1} \hat{u}_k(y_k) + c(1-\lambda), \tag{7}$$

with probability 1. Finally, using Lemma 1 and the definition of A, B, we obtain

$$L_k\big(x_k + u_k^\lambda(x_k, y_k)\big) \leq L_k(y_k + \hat{u}_k(y_k)) + \max\{A\lambda, (1-\lambda)B\}. \tag{8}$$

Putting inequalities (6), (7) and (8) together, taking expectations and adding them appropriately, we obtain the last inequality in (5). Q.E.D.

The above theorem provides us with a family of bounds for J_0^λ, one for each value of λ. We now look for a tight bound. Assuming that N is large enough so that $NA > c$, the quantity $(1-\lambda)c + N\max\{\lambda A, (1-\lambda)B\}$ achieves its minimum (as λ varies in $[0, 1)$) when $\lambda A = (1-\lambda)B$, i.e. $\lambda = B/(A+B)$. We may then rewrite (5) as

$$0 \leq J_0^*(x_0) - \hat{J}_0^*(x_0) \leq \frac{A}{A+B}c + N\frac{AB}{A+B}. \tag{9}$$

Inequality (9) gives an upper bound for the increase in costs when orders are restricted to be integer valued. Moreover, we have constructed a policy u_k^λ, with $\lambda = B/(A+B)$ which, although suboptimal, stays within that bound. Policy u_k^λ in some sense approximates the optimal policy $\hat{u}_k$ for the continuous model and may be obtained from it in a straightforward manner. The advantage of this approach lies in the fact that (in view of the discussion in §3) an optimal policy u_k^* for the discrete model may be unstructured and harder to compute than $\hat{u}_k$ and u_k^λ.

A similar approach may be taken for infinite horizon problems. Assume that all parameters are time invariant. For the average cost criterion, let g^* and $\hat{g}^*$ be the optimal average costs for the discrete and the continuous problems, respectively. Then, (9) becomes $0 \leq g^* - \hat{g}^* \leq AB/(A+B)$. Finally consider a discounted cost criterion with discount factor $\alpha \in [0, 1)$ and let $J_\alpha^*, \hat{J}_\alpha^*$ be the corresponding infinite time horizon costs for the discrete and continuous the time problems, respectively. Then, the same steps as in the proof of Theorem 1 lead to

$$0 \leq J_\alpha^* - \hat{J}_\alpha^* \leq \left(\frac{1}{1-\alpha}\right)\left(\frac{AB}{A+B}\right).^{1}$$

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