Stochastic resonance of quantum discord

Chee Kong Lee,[1,*] Leong Chuan Kwek,[1,2] and Jianshu Cao[3]

[1]*Centre for Quantum Technologies, National University of Singapore, 117543 Singapore*

[2]*National Institute of Education and Institute of Advanced Studies, Nanyang Technological University, 637616 Singapore*

[3]*Department of Chemistry, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

(Received 7 August 2011; published 19 December 2011)

We study the stochastic resonance of quantum discord ("discord resonance") in coupled quantum systems and make a comparison with the stochastic resonance of entanglement ("entanglement resonance"). It is found that the discord resonance is much more robust against dephasing noise and thermal effects than the entanglement resonance. We also show that, unlike the entanglement resonance, the level of dissipation at which the discord resonance occurs is not sensitive to dephasing noise. These results suggest that it is easier to detect the discord resonance in actual experiments, where the dephasing noise and temperature are difficult to control.

DOI: 10.1103/PhysRevA.84.062113 PACS number(s): 03.65.Yz, 03.65.Ud, 42.50.Ct

I. INTRODUCTION

The presence of noise is usually regarded as detrimental to quantum systems, causing rapid destruction of coherence and quantum correlations [1]. Thus it is desirable to minimize the noise level in the manipulation of quantum systems. However, it was recently found that a certain amount of noise can actually be beneficial under certain circumstances. For example, an optimal level of noise can help photosynthetic complexes to achieve near-unity efficiency in the light-harvesting processes [2–6]. Here we focus on one particularly interesting noise-assisted phenomenon, the so-called stochastic resonance where the response of an open system to an external driving field can exhibit resonance-like behavior on the noise strength [7,8]. Recently, it has been shown that the stochastic resonance phenomenon is also present in correlated quantum systems in terms of entanglement and mutual information [9,10]. However, entanglement is not the only measure of the quantum correlations. Another measure is the quantum discord, introduced by Zurek (see [11,12]), which captures the nonclassical properties of the quantum correlations. Quantum discord is believed to be more pervasive than entanglement: there exist separable states with finite discord. Quantum discord has generated much interest in recent years [13–20]. One motivation behind this is the discovery that quantum discord can play a role in the quantum speedup of deterministic quantum computation with one pure qubit (DQC1) [21,22].

Therefore, it is natural to ask whether quantum discord also exhibits the stochastic resonance behavior as observed in entanglement. Our findings in this paper provide an affirmative answer to this question. We show that the resonance of quantum discord ("discord resonance") can behave very differently from the resonance of entanglement ("entanglement resonance"). The level of dissipation at which the discord resonance occurs is not sensitive to the dephasing noise, while that of entanglement resonance is highly dependent on the dephasing noise. At large dephasing noise or high temperature, the discord resonance occurs even when the entanglement is strictly zero. In actual experimental setups, it is difficult to control the dephasing noise and temperature. Thus, quantum discord might be a better candidate for the detection of the stochastic resonance of quantum correlations. These results might also be useful in searching for quantum effects in biological processes [23–25].

This paper is organized as follows. In Sec. II, we provide a brief introduction to quantum discord. The Hamiltonian and decoherence model used are described in Sec. III. In Sec. IV, we present our results and discussions. Finally, we summarize our work in Sec. V.

II. QUANTUM DISCORD

The total correlations of a bipartite system are measured by the quantum mutual information [26]:

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{1}$$

where $\rho_{A(B)}$ and ρ_{AB} are the reduced density matrix of the subsystem $A(B)$ and the density matrix of the total system, respectively. S is the von Neumann entropy, defined as $S(\rho) = -\mathrm{tr}[\rho \log_2 \rho]$. Next, the purely classical correlations are quantified in terms of the maximum amount of information one can obtain about the subsystem A if we make measurements on the subsystem B. This can be shown to be [27]

$$C(\rho_{AB}) = \max_{\{\Pi_B^k\}} \left[S(\rho_A) - S\left(\rho_{AB}|\Pi_B^k\right) \right]. \tag{2}$$

The conditional entropy of A is $S(\rho_{AB}|\Pi_B^k) = \sum_k p_k S(\rho_k)$, where $\rho_k = \mathrm{tr}_B[\Pi_B^k \rho_{AB} \Pi_B^k]/p_k$ and $p_k = \mathrm{tr}[\Pi_B^k \rho_{AB} \Pi_B^k]$. The maximum is taken over the set of positive operator-valued measurements (POVM) on the subsystem B, $\{\Pi_B^k\}$. For two qubits, it is shown that projective measurement is the measurement that maximizes the classical correlations [28]. It can be written as $\Pi_B^k = \mathbb{1} \otimes |k\rangle\langle k|$ $(k = a,b)$, where

$$\begin{aligned} |a\rangle &= \cos\theta|g\rangle + e^{i\phi} \sin\theta|e\rangle; \\ |b\rangle &= \sin\theta|g\rangle - e^{i\phi} \cos\theta|e\rangle. \end{aligned} \tag{3}$$

Finally, the quantum discord is defined as the difference between the total correlations and the purely classical correlations,

$$D(\rho_{AB}) = I(\rho_{AB}) - C(\rho_{AB}), \tag{4}$$

thus capturing the quantum correlations between two systems.

*cqtlck@nus.edu.sg

III. MODEL

We study a chain of N identical coupled qubits, each of which is independently driven by an external field of frequency ω and strength Ω. The Hamiltonian of the system can be written as (we set $\hbar = 1$)

$$H(t) = \omega_0 \sum_{j=1}^{N} \sigma_j^+ \sigma_j^- + J \sum_{j=1}^{N-1} (\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+) + \Omega \cos(\omega t) \sum_{j=1}^{N} \sigma_j^x, \tag{5}$$

where $\sigma_i^+ = |e\rangle\langle g|$ is the usual creation operator of the ith qubit.

The qubits are also subject to local dissipation and dephasing noise due to the coupling to the environment. The evolution of the system is governed by the Linblad master equation within the usual Born-Markov approximation [29,30]

$$\dot{\rho}(t) = i[\rho, H(t)] + \frac{\Gamma}{2} \sum_{j=1}^{N} [2\sigma_j^+ \sigma_j^- \rho \sigma_j^+ \sigma_j^- - \{\sigma_j^+ \sigma_j^-, \rho\}] + \frac{\gamma}{2} \sum_{j=1}^{N} (\bar{n}+1)[2\sigma_j^- \rho \sigma_j^+ - \{\sigma_j^+ \sigma_j^-, \rho\}] + \frac{\gamma}{2} \sum_{j=1}^{N} \bar{n}[2\sigma_j^+ \rho \sigma_j^- - \{\sigma_j^- \sigma_j^+, \rho\}], \tag{6}$$

where Γ and γ are the dephasing rate and dissipation rate, respectively. Here $\bar{n} = (e^{\omega_0/k_B T} - 1)^{-1}$ is the average quanta number in the environment. We next move to the rotating frame under the unitary transformation $U(t) = \exp(-i\omega_0 t \sum_{j=1}^{N} \sigma_j^+ \sigma_j^-)$. Assuming rotating wave approximation and zero detuning, $\omega_0 = \omega$, we have the Hamiltonian in the interaction picture, $H_I = \omega_0 \sum_{j=1}^{N} \sigma_j^+ \sigma_j^- + J \sum_{j=1}^{N-1} (\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+) + \frac{\Omega}{2} \sum_{j=1}^{N} \sigma_j^x$. The other terms in the master equation, Eq. (6), remain unchanged under the transformation.

IV. RESULTS AND DISCUSSIONS

We first consider the simplest case where $N = 2$ and $T = 0$. The analytical solution to the steady state can only be obtained in certain limiting cases. At zero dissipation, $\gamma = 0$, the steady state is maximally mixed, $\rho(\infty) = \mathbb{1}/4$, and obviously has zero entanglement and quantum discord. At the other limit, $\Gamma = 0$, the steady-sate solution in the basis $\{|ee\rangle, |eg\rangle, |ge\rangle, |gg\rangle\}$ is given by

$$\begin{aligned} &\rho_{11} = \frac{\Omega^4}{z}, \quad \rho_{22} = \frac{a\Omega^2}{z}, \quad \rho_{33} = \frac{a\Omega^2}{z}, \\ &\rho_{44} = 1 - \rho_{11} - \rho_{22} - \rho_{33}, \\ &\rho_{12} = \rho_{13} = -i\frac{\Omega^3 \gamma}{z}, \quad \rho_{14} = \frac{-\Omega^2\gamma^2 + i2J\Omega^2\gamma}{z}, \\ &\rho_{23} = \frac{\Omega^2\gamma^2}{z}, \quad \rho_{24} = \rho_{34} = -\frac{2J\Omega\gamma^2 + ia\Omega\gamma}{z}, \end{aligned} \tag{7}$$

where $a = \Omega^2 + \gamma^2$ and $z = 4a\Omega^2 + 4J^2\gamma^2 + \gamma^4$.

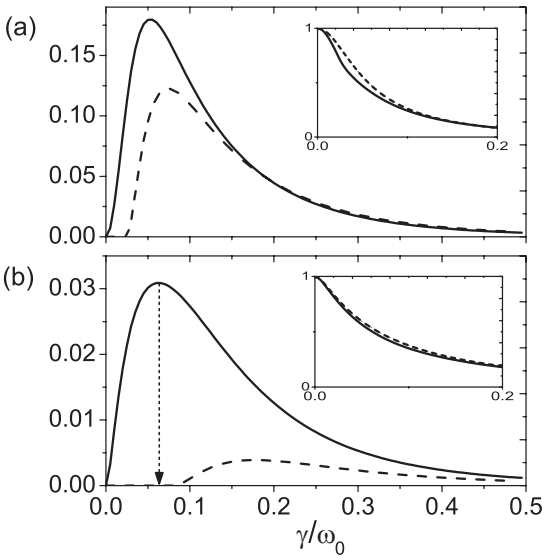

FIG. 1. Steady-state quantum discord (solid lines) and entanglement (dashed lines) between the qubits plotted against dissipation rate γ for $N = 2$, $J = 0.2\omega_0$, $\Omega = 0.1\omega_0$, and $\bar{n} = 0$. Both quantum correlations exhibit resonance behaviors. Insets are the quantum discord (solid lines) and the entanglement (dashed lines) between qubit 1 and the environment, according to Eq. (8). (a) At $\Gamma = 0$. (b) At $\Gamma = 0.15\omega_0$, the discord resonance occurs when the entanglement is strictly zero, as pointed by the dotted arrow.

We use the entanglement of formation as the measure of entanglement [32,33]. The entanglement and quantum discord of the state in Eq. (7) are plotted against dissipation strength in Fig. 1(a). Both the quantum discord and entanglement exhibit stochastic resonance behavior: they first increase as dissipation rises, reaching a maximum at an optimal dissipation level, then decrease monotonically with dissipation. This rather counterintuitive phenomenon can be easily understood by a "cooling" mechanism: The steady state becomes maximally mixed in the limit of zero dissipation $\gamma \to 0$, as all the coherence is lost to the heat bath. As dissipation increases, the off-diagonal terms become nonzero, and the coherence of the system is restored, which is essential for nonzero entanglement and discord. This is due to that fact that the zero-temperature bath "cools" the joint system toward the individual ground state, $|gg\rangle\langle gg|$. The coherence is then restored by the application of the continuous coherent excitation. The correlations can then be reestablished by the qubit-qubit interaction. The balance between the dissipation and the continuous coherent excitation provides the steady-state correlations between the qubits. However, if the dissipation is too strong, the system approaches the separable ground state; consequently, both the entanglement and discord decrease to zero asymptotically.

Though both quantum correlations exhibit stochastic resonance behavior, the discord resonance differs significantly from the entanglement resonance. First, discord is always positive for any finite dissipation, while the state remains separable below a certain threshold of dissipation. This situation is similar to the Werner state, $\rho_w = (1-p)\frac{\mathbb{1}}{4} + p\,|\Phi^+\rangle\langle\Phi^+|$, where $|\Phi^+\rangle = (|ee\rangle + |gg\rangle)/\sqrt{2}$. The discord of the Werner state is always positive for $p > 0$, while the state remains

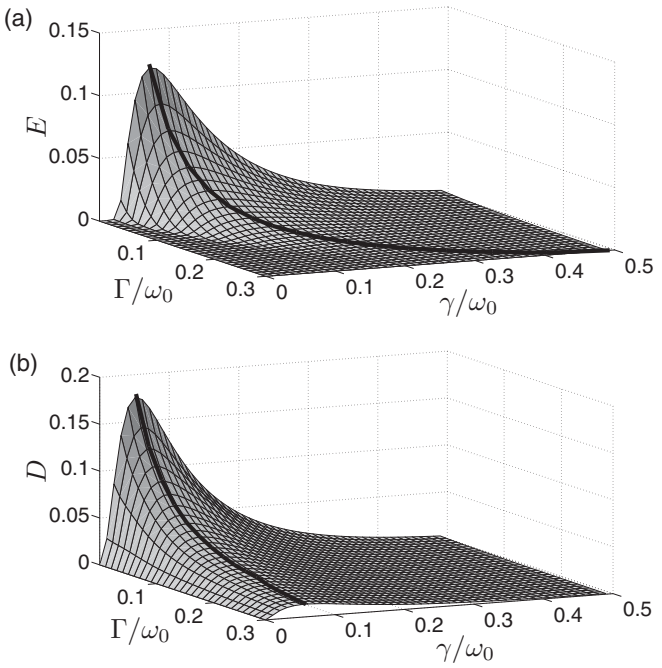

FIG. 2. Steady-state entanglement E and discord D plotted as a function of dissipation rate γ and dephasing noise Γ for $N = 2$, $J = 0.2\omega_0$, $\Omega = 0.1\omega_0$, and $\bar{n} = 0$. At a fixed dissipation rate, the steady-state entanglement and discord decay monotonically as a function of dephasing rate. On the other hand, the steady-state entanglement and discord exhibit a nonmonotonic behavior as a function of the dissipation rate, a signature of stochastic resonance. It is also observed that the entanglement resonance [thick line in (a)] occurs at increasing γ as Γ increases, while the discord resonance [thick line in (b)] is less sensitive to Γ: it always occurs at $\gamma \approx 0.05\omega_0 - 0.07\omega_0$.

separable for $p < 1/3$. It is observed that the entanglement resonance occurs at increasing dissipation with increasing dephasing noise [see Fig. 2 (a)], while the discord resonance is not sensitive to dephasing noise: it occurs at about the same level of dissipation, regardless of dephasing noise [see Fig. 2(b)]. It is also found that the discord resonance always occurs at smaller dissipation than the entanglement resonance. If the dephasing noise is strong enough, the discord resonance even occurs while the entanglement is strictly zero, as seen in Fig. 1(b). In fact, entanglement remains zero for all values of dissipation when the dephasing noise is too strong (not shown), and yet discord resonance can still be observed. The above situations can be vividly illustrated in the pictorial representation of the set of states in Fig. 3.

At zero temperature, the qubits and the environment form a tripartite pure state. Therefore, the entanglement and quantum discord between qubit 1 and the environment can be respectively written as [16,34]

$$\begin{aligned} E(\rho_{1,E}) &= D(\rho_{1,2}) - S(\rho_{1,E}|\rho_E), \\ D(\rho_{1,E}) &= E(\rho_{1,2}) - S(\rho_{1,E}|\rho_E), \end{aligned} \tag{8}$$

where $E(\rho_{1,2})$ and $D(\rho_{1,2})$ are the entanglement and quantum discord between the qubits, respectively. $S(\rho_{1,E}|\rho_E) = S(\rho_{1,E}) - S(\rho_E)$ is the conditional entropy between qubit 1 and the environment. Since the total system is in a pure state, we have $S(\rho_{1,E}) = S(\rho_2)$ and $S(\rho_E) = S(\rho_{1,2})$. The above

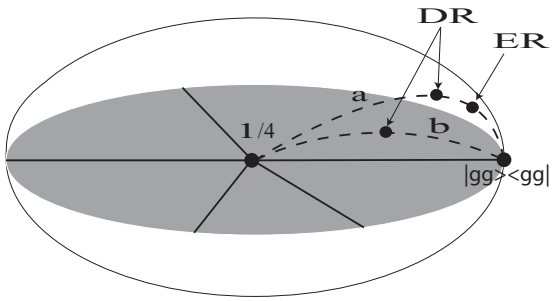

FIG. 3. Two-dimensional schematic representation of the set of all possible states [31], enclosed by the outer circle. It contains the set of separable states (grey area) and nonseparable states (white area). The states with zero discord are connected and have measure zero; they are represented by the black solid lines, with $\rho = \mathbb{1}/4$ at the center. Pure separable states lie at the tips of the star array. The steady state starts from the center, $\mathbb{1}/4$, and goes to $|gg\rangle\langle gg|$ as dissipation increases from zero to infinity (denoted by dashed lines a and b). Line a shows that the discord resonance (DR) occurs at the point farthest from the black lines, whereas the entanglement resonance (ER) occurs at the point farthest from the grey area. Line b shows that for large dephasing noise or high temperature, the state remains in the separable set, while DR can still occur.

quantities are plotted in the insets of Fig. 1. Both quantities decay monotonically as a function of dissipation γ. At $\gamma \to 0$, qubit 1 is maximally entangled with the environment, and thus its reduced density matrix is maximally mixed. As γ increases, the entanglement with the bath decreases as the qubit approaches the ground state.

To gain more insights, we use a simple state that is amenable to analytic analysis. The steady state resembles the Werner state at small dissipation and approaches the ground state at large dissipation. This observation motivates us to propose the following state:

$$\rho = (1 - p_1)\left[(1 - p_1)\frac{\mathbb{1}}{4} + p_1\,\rho_{\text{ent}}\right] + p_1|gg\rangle\langle gg|, \tag{9}$$

where $0 \leqslant p_1 \leqslant 1$ and ρ_{ent} is an entangled state. The parameter p_1 captures the effect of the dissipation; $p_1 = 0$ corresponds to $\gamma = 0$, where the state is maximally mixed. At the limit of $p_1 = 1$, the system is in the ground state, corresponding to $\gamma \to \infty$. We first choose the entangled state to be the Bell

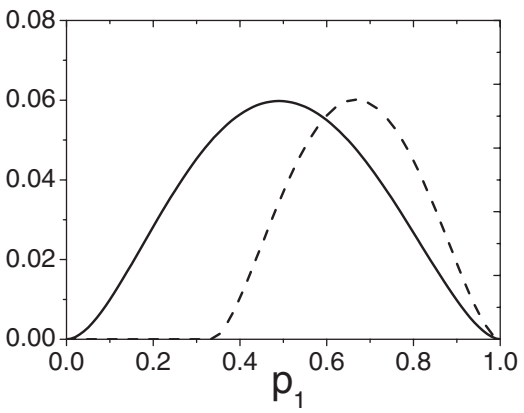

FIG. 4. The quantum discord (solid line) and entanglement (dashed line) of the state given in Eq. (9), with $\rho_{\text{ent}} = |\Phi^+\rangle\langle\Phi^+|$.

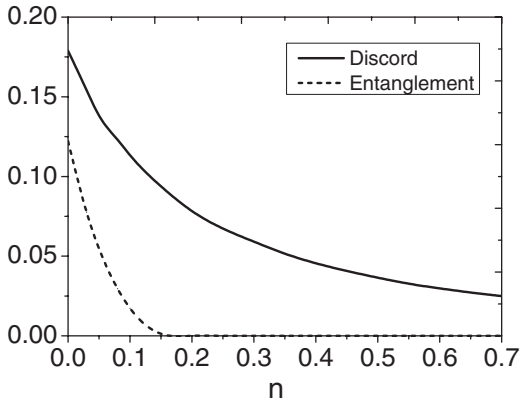

FIG. 5. The peak values of discord resonance (solid line) and entanglement resonance (dashed line) vs the average quanta number (temperature) at $N = 2$, $\gamma = 0.1\omega_0$, $\Gamma = 0$, $J = 0.2\omega_0$, and $\Omega = 0.1\omega_0$. The entanglement resonance vanishes at finite temperature, while the discord resonance persists even at high temperature, with its peak values decreasing asymptotically.

state, $\rho_{\text{ent}} = |\Phi^+\rangle\langle\Phi^+|$. The entanglement and discord of this state as a function of p_1 are plotted in Fig. 4. The resonance behavior is similar to what is observed in our model. The discord resonance occurs at $p_1 = 0.5$, where the values of the off-diagonal terms are maximal. However, the maximum of the entanglements is at a larger value of $p_1 = 2/3$. This choice of entangled state ρ_{ent} makes the maximum values of entanglement and discord comparable.

If a nonmaximally entangled state is chosen, the maximum value of entanglement is usually lower than that of discord. One choice is again the Werner state (with $p \to 1 - p$), $\rho_{\text{ent}} = p_2 \frac{\mathbb{I}}{4} + (1 - p_2)|\Phi^+\rangle\langle\Phi^+|$. The parameter p_2 can be regarded as the dephasing noise, with $p_2 = 0(1)$ corresponding to $\Gamma = 0(\infty)$. If one uses negativity as a measure of entanglement [33], it is found that the entanglement is maximum at $p_1^* = \frac{4-3p_2}{6(1-p_2)}$ for $p_2 < 2/3$; the state is separable for $p_2 \geqslant 2/3$. It has been checked numerically that the location of the maximum is the same even if one uses entanglement of formation as the entanglement measure. From the functional form, it can be seen that the entanglement peaks at larger p_1 as p_2 increases, having the same effect as dephasing noise. Numerical results show that the maximum of discord remains at $p_1 = 0.5$ throughout.

At finite temperature, $T \neq 0$, both steady-state discord and entanglement drop since the cooling mechanism is less effective. The peak values of discord resonance and entanglement resonance as a function of average quanta number (temperature) are plotted in Fig. 5. While the entanglement resonance disappears at finite temperature, the discord resonance persists even at high temperature, with its maximum value decreasing asymptotically. This shows that discord resonance is more robust against thermal effects.

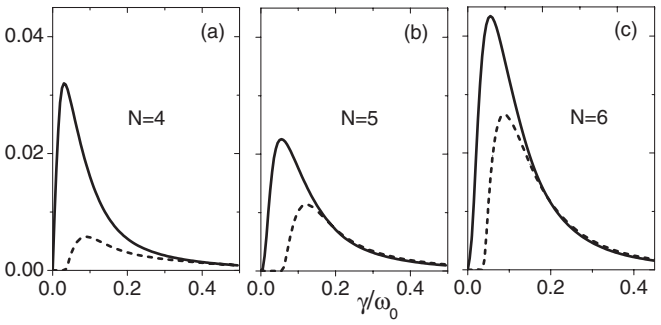

FIG. 6. The steady-state quantum discord (solid lines) and entanglement (dashed lines) between qubit 1 and qubit 2 in spin chains of (a) four, (b) five, and (c) six qubits for $\Gamma = 0$, $J = 0.2\omega_0$, $\Omega = 0.1\omega_0$, and $\bar{n} = 0$.

Finally, we generalize our results to spin chains of four, five, and six qubits. The results are plotted in Fig. 6. It is observed that the behavior of the discord resonance and entanglement resonance is similar to what is observed in a two-qubit chain. Thus, our results should also hold for spin chains of arbitrary length and possibly coupled qubits of any configuration.

V. CONCLUSION

To summarize, we study the stochastic resonance of quantum discord and make a comparison with that of entanglement. It is found that, unlike the entanglement resonance, the level of dissipation at which the discord resonance occurs is not sensitive to dephasing noise. It is also much more robust against dephasing noise and thermal effects than the entanglement resonance. Therefore, it should be easier to detect discord resonance in experiments, where the dephasing noise and temperature might be difficult to control. Our results might find applications in the search for quantum effects in biological processes, where the molecules might be too "warm and wet" for steady-state entanglement to survive [23–25].

ACKNOWLEDGMENTS

The Centre for Quantum Technologies is a Research Centre of Excellence funded by the Ministry of Education and the National Research Foundation of Singapore. J.C. acknowledges support from NSF (Grant No. CHE-1112825) and DARPA (Grant No. N99001-10-1-4063). J.C. is also supported as part of the Center for Excitonics, an Energy Frontier Research Center funded by the US Department of Energy, Office of Science, Office of Basic Energy Sciences under Award No. DE-SC0001088. C.K.L. is grateful to Bobby Tan for fruitful discussions.

[1] T. Yu and J. H. Eberly, Science **323**, 598 (2009).
[2] M. B. Plenio and S. F. Huelga, New J. Phys. **10**, 113019 (2008).
[3] M. Mohseni, P. Rebentrost, S. Lloyd, and A. Aspuru-Guzik, J. Chem. Phys. **129**, 174106 (2008).
[4] F. Caruso, A. W. Chin, A. Datta, S. F. Huelga, and M. B. Plenio, J. Chem. Phys. **131**, 105106 (2009).
[5] J. Cao and R. J. Silbey, J. Phys. Chem. A **113**, 13825 (2009).

[6] J. Wu, F. Liu, Y. Shen, J. Cao, and R. J. Silbey, New J. Phys. **12**, 105012 (2010).
[7] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[8] M. Grifoni and P. Hänggi, Phys. Rev. Lett. **76**, 1611 (1996).
[9] S. F. Huelga and M. B. Plenio, Phys. Rev. Lett. **98**, 170601 (2007).
[10] A. Rivas, N. P. Oxtoby, and S. F. Huelga, Eur. Phys. J. B **69**, 51 (2009).
[11] W. Zurek, Ann. Phys. (Berlin) **9**, 855 (2000).
[12] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001).
[13] S. Luo, Phys. Rev. A **77**, 042303 (2008).
[14] L. Mazzola, J. Piilo, and S. Maniscalco, Phys. Rev. Lett. **104**, 200401 (2010).
[15] F. F. Fanchini, T. Werlang, C. A. Brasil, L. G. E. Arruda, and A. O. Caldeira, Phys. Rev. A **81**, 052107 (2010).
[16] F. F. Fanchini, M. F. Cornelio, M. C. de Oliveira, and A. O. Caldeira, Phys. Rev. A **84**, 012313 (2011).
[17] M. F. Cornelio, M. C. de Oliveira, and F. F. Fanchini, Phys. Rev. Lett. **107**, 020502 (2011).
[18] D. Cavalcanti, L. Aolita, S. Boixo, K. Modi, M. Piani, and A. Winter, Phys. Rev. A **83**, 032324 (2011).
[19] V. Madhok and A. Datta, Phys. Rev. A **83**, 032323 (2011).
[20] A. Streltsov, H. Kampermann, and D. Bruß, Phys. Rev. Lett. **106**, 160401 (2011).
[21] A. Datta, A. Shaji, and C. M. Caves, Phys. Rev. Lett. **100**, 050502 (2008).
[22] B. P. Lanyon, M. Barbieri, M. P. Almeida, and A. G. White, Phys. Rev. Lett. **101**, 200501 (2008).
[23] E. M. Gauger, E. Rieper, J. J. L. Morton, S. C. Benjamin, and V. Vedral, Phys. Rev. Lett. **106**, 040503 (2011).
[24] F. Caruso, A. W. Chin, A. Datta, S. F. Huelga, and M. B. Plenio, Phys. Rev. A **81**, 062346 (2010).
[25] M. Sarovar, A. Ishizaki, G. R. Fleming, and K. B. Whaley, Nat. Phys. **6**, 462 (2010).
[26] M. A. Nielson and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2009).
[27] L. Henderson and V. Vedral, J. Phys. A **34**, 6899 (2001).
[28] S. Hamieh, R. Kobes, and H. Zaraket, Phys. Rev. A **70**, 052325 (2004).
[29] H. J. Carmichael, *Statistical Methods in Quantum Optics 1* (Springer, New York, 1999).
[30] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
[31] A. Ferraro, L. Aolita, D. Cavalcanti, F. M. Cucchietti, and A. Acín, Phys. Rev. A **81**, 052318 (2010).
[32] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[33] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[34] M. Koashi and A. Winter, Phys. Rev. A **69**, 022309 (2004).