On the Existence of a Unique Optimal Threshold Value for the Early Exercise of Call Options

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Abstract

In the case of early exercise of an American-style call option, we consider the issue of the existence of a "treshhold value," namely a boundary which, once exceeded, early exercise is optimal for all values of the underlying asset which exceed that value. We discuss optimal exercise thresholds for call options for two-period models, under alternate contexts: geometric Brownian motion vs. mean-reverting Ornstein-Uhlenbeck process, with and without seasonality, and with time-dependent strike prices. We show that, other than the case of geometric Brownian motion without seasonality, there may exist multiple exercise thresholds.

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We are interested in describing early exercise thresholds of a call option, in a two period model, with respect to the price of the underlying asset. The option may be exercised at times t_1 and t_2. We consider four types of stochastic processes, which, under the risk-neutral measure, are given by:

(i) Geometric Brownian Motion

$dS_t/S_t = (r-\delta)dt + \sigma dW_t$

(ii) Geometric Brownian Motion with Seasonality

$S_t = f_t D_t, \quad dD_t/D_t = (r-\delta)dt + \sigma dW_t$

(iii) Mean reverting process

$S_t = \exp(X_t), \quad dX_t = \kappa(\xi - X_t)dt + \sigma dW_t$

(iv) Mean reverting process with Seasonality

$S_t = f_t \exp(X_t), \quad dX_t = \kappa(\xi - X_t)dt + \sigma dW_t$

where $r, \delta, \sigma, f_t, \kappa$, are non-negative.

In addition, we consider the following payoffs from immediate exercise at times t_1, t_2:

Payoff at time t_2: Call with strike K_2, $h_{t_2}(S) = \max(0, (S-K_2))$

Payoff at time t_1: Call with strike K_1, $h_{t_1}(S) = \max(0, (S-K_1))$

and, in general, allow $K_1 \neq K_2$.

From the payoff at the terminal date, t_2, it is obvious that there exists a unique threshold, namely $S^* = K_2$, such that exercise is optimal for $S_{t_2} > S^*$, and is not optimal for $S_{t_2} < S^*$. We will show that on date t_1 it is possible, in some of the cases, to have multiple exercise regions.[1] In particular, we will show that the optimal exercise policy has at most one threshold in case

[1]We point out that the two period example is particularly simple, since the continuation value at time t_1 is the value of a European option, for which we have closed form expressions for all four cases.

(i), while there may be multiple thresholds in cases (ii), (iii), and (iv). In addition, we will also show that, in cases (iii) and (iv), there is a threshold S^*_{large}, such that, for $S_{t_1} > S^*_{\text{large}}$ it is optimal to exercise the option.

We first prove that in case (i) there is at most one threshold.

Proof. Since the continuation value at time t_1 is the Black-Scholes price of a call with expiration t_2, and strike K_2, it would be straightforward to demonstrate that there is at most one exercise threshold at time t_1. However, to demonstrate that there exists at most a single threshold we will use a more complicated method that can be extended to multiple periods. In particular, we have that the continuation value, at time t_1, is given by

$$V_{\text{cont}} = E(e^{-r(t_2-t_1)} h_{t_2}(S_{t_2})) = E(e^{-r(t_2-t_1)} \max(0, S_{t_2} - K_2))$$

where expectation is with respect to the risk-neutral measure, and is conditional on S_{t_1}. From the equation above, we can calculate the derivative of the continuation value with respect to the price at time t_1, S_{t_1}. From the stochastic process for case (i), we have

$$S_{t_2} = S_{t_1} \exp\left((r - \delta - \sigma^2/2)(t_2 - t_1) + \sigma(W_{t_2} - W_{t_1})\right)$$

and,

$$\frac{d}{dS_{t_1}} E(e^{-r(t_2-t_1)} h_{t_2}(S_{t_2})) = e^{-r(t_2-t_1)} E(\frac{dh_{t_2}}{dS_{t_2}} \frac{dS_{t_2}}{dS_{t_1}}) \leq \max(\frac{dh_{t_2}}{dS_{t_2}}) e^{-r(t_2-t_1)} \frac{E(S_{t_2})}{S_{t_1}} \leq 1$$

where we used that $E(S_{t_2}) = S_{t_1} \exp\left((r-\delta)(t_2 - t_1)\right)$. Since the immediate exercise value has a derivative equal to either zero (for $S_{t_1} < K_1$), or 1 (for $S_{t_1} > K_1$), there can be, at most, one exercise threshold. We note that this proof generalizes for more than two exercise dates, effectively proving that, under geometric Brownian motion, Bermudan call options have single exercise thresholds, independent of their strike structure. □

For case (ii), we can try to proceed as in the proof for case (i). Unfortunately, in this case we have that

$$E(S_{t_2}) = \frac{f_{t_2}}{f_{t_1}} S_{t_1} e^{(r-\delta)(t_2-t_1)}$$

which may lead to a bound that is greater than 1, depending on the values of the seasonality factors f_{t_2} and f_{t_1}. We were able to construct the following example: $f_{t_2} = 2, f_{t_1} = 1, K_2 = 200, K_1 = 100, r = 4\%, \delta = 8\%, \sigma = 5\%, t_2 - t_1 = 1$. In this case one can compute the optimal action and see that it is optimal to wait for $S_{t_1} < 102.5$, and $S_{t_1} > 106.4$, while it is optimal to exercise for $102.6 < S_{t_1} < 106.3$. The intuition of having multiple thresholds is the following: if the seasonality factors were the same, there would be a single threshold, which would be the stock price when the benefit from taking the dividend early would match the benefit of waiting. However, the different seasonality factors change the picture. Now, once the time t_2 option is deep enough into the money, it delivers more shares than the option at time t_1. The dividend rate complicates matters, as it creates a range in which the benefit of having the early dividend is greater than the value of waiting, both in terms of protection, and in terms of the benefit of receiving more shares.

Case (iii) is similar to case (i), and we can try to perform the same calculation as in the proof of case (i). We have

$$X_{t_2} = X_{t_1} e^{-\kappa(t_2-t_1)} + \xi(1 - e^{-\kappa(t_2-t_1)}) + \int_{t_1}^{t_2} e^{\kappa(s-t_2)} \sigma dW_s$$

and

$$\frac{dS_{t_2}}{dS_{t_1}} = \frac{dS_{t_2}}{dX_{t_2}} \frac{dX_{t_2}}{dX_{t_1}} \frac{dX_{t_1}}{dS_{t_1}}$$

Since $S_{t_i} = \exp(X_{t_i})$, we have that

$$\frac{dS_{t_i}}{dX_{t_i}} = S_{t_i}, \quad i = 1, 2$$

and

$$\frac{dS_{t_2}}{dS_{t_1}} = \frac{S_{t_2}}{S_{t_1}} e^{-\kappa(t_2-t_1)} = \exp\left(X_{t_1}(e^{-\kappa(t_2-t_1)}-1) + \xi(1-e^{-\kappa(t_2-t_1)}) - \kappa(t_2-t_1) + \int_{t_1}^{t_2} e^{\kappa(s-t_2)}\sigma dW_s \right)$$

Finally, we have that

$$\begin{aligned}\frac{d}{dS_{t_1}} E(e^{-r(t_2-t_1)} h_{t_2}(S_{t_2})) &= e^{-r(t_2-t_1)} E(\frac{dh_{t_2}}{dS_{t_2}} \frac{dS_{t_2}}{dX_{t_2}} \frac{dX_{t_2}}{dX_{t_1}} \frac{dX_{t_1}}{dS_{t_1}}) \\ &\leq \max(\frac{dh_{t_2}}{dS_{t_2}}) E(\frac{dS_{t_2}}{dX_{t_2}} \frac{dX_{t_2}}{dX_{t_1}} \frac{dX_{t_1}}{dS_{t_1}}) \\ &\leq \exp\left(-r(t_2-t_1) + X_{t_1}(e^{-\kappa(t_2-t_1)}-1) + \xi(1-e^{-\kappa(t_2-t_1)}) - \kappa(t_2-t_1) \right)\end{aligned} \tag{1}$$

From Equation 1, we have that, for large enough values of the underlying price, the derivative of the contiuation value becomes arbitrarily small. This is due to the mean reversion of the stochastic process and implies that there exists a value S^*_{large}, such that for $S_{t_1} > S^*_{\text{large}}$, it is optimal to exercise the option. Moreover, if the upper bound on the derivative of the continuation value is less than one at the value when it *first* becomes optimal to exercise the option at time t_1, then the threshold is unique.

Unfortunately the expression in Equation 1, is only an upper bound for the derivative of the continuation value, and it is possible for the derivative of the continuation value to have the following behavior: for small values of S_{t_1} it may be smaller than 1 and positive, then, for an intermediate range, it may be greater than one, and eventually it may again become smaller than one. If this behavior occurs, it would be possible to have multiple exercise thresholds. Indeed, we were able to construct such an example, with the following parameters: $\kappa = 0.05, \sigma = 0.7, t_2 - t_1 = 0.08333, e^{\xi} = 120, r = 0.05, K_2 = 100, K_1 = 96$. For these parameters, it is optimal to wait for $S \leq 121$, it is optimal to exercise for $122 \leq S \leq 282$, it is optimal to wait for $283 \leq S \leq 4914$ and it is optimal to exercise for $S \geq 4915$. This example demonstrates the possibility of multiple exercise/no-exercise thresholds. We have the following intuition: we can think of this case as a example of geometric Brownian motion with state-dependent dividends. For low prices the dividend is negative, while for high prices the dividend is positive. Early exercise for low prices occurs due to the difference in strike prices, when even though one has a negative dividend, the time t_1 option may be deep in the money, while the time t_2 option may be still out of the money. Then, as the price rises, the time t_2 option comes into the money, while the dividend is still negative. It then becomes optimal to wait. Finally, for high prices, the dividend becomes large and it again becomes optimal to exercise.[2]

Given the above discussion, it is not surprising that under case (iv) there may also exist multiple thresholds. Indeed, such an example is given for the parameter values $\kappa = 0.05, \sigma = 0.6, t_2 - t_1 = 0.08333, e^{\xi} = 120, r = 0.05, K_2 = 100, K_1 = 96, f_{t_2} = 1.01, f_{t_1} = 1$. For these parameters, it is optimal to wait for $S \leq 119$, it is optimal to exercise for $120 \leq S \leq 188$, it is optimal to wait for $189 \leq S \leq 16633$ and it is optimal to exercise for $S \geq 16634$.

We point out that in cases (iii) and (iv), the existence of multiple thresholds depends on the mean reversion rate κ. If the value of κ is large enough, there exists only a single threshold. In all the examples of swing options we considered we have never encountered a case with multiple thresholds when using the parameter values that were calibrated from the natural gas data.

[2] We have been unable to construct a counter-example in the case where the strike prices are the same, or for cases where the strike price at the terminal date t_2 is lower than the strike price at the initial date t_1; i.e. $K_2 \leq K_1$.