Rates of Convergence for Quasi-Additive Smooth Euclidean Functionals and Application to Combinatorial Optimization Problems*

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Abstract

Rates of convergence of limit theorems are established for a class of random processes called here quasi-additive smooth Euclidean functionals. Examples include the objective functions of the traveling salesman problem, the Steiner tree problem, the minimum spanning tree problem, the minimum weight matching problem, and a variant of the minimum spanning tree problem with power weighted edges.

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1 Introduction

In Beardwood et al. [1], the authors prove that for any bounded i.i.d. random variables $\{X_i : 1 \leq i < \infty\}$ with values in $\mathbf{R}^d$, $d \geq 2$, the length of the shortest tour through $\{X_1, \ldots, X_n\}$ is asymptotic to $\beta_d n^{(d-1)/d}$ with probability one (the same being true in expectation). This theoretical result has become widely recognized to be at the heart of the probabilistic evaluation of the performance of heuristic algorithms for vehicle routing problems. In fact it is used as the main argument in the probabilistic analysis of partitionning algorithms for the TSP by Karp [6]. Because of these algorithmic applications, results like that of Beardwood et al. have gained considerable practical interest. An important contribution on the subject is contained in Steele [11] in which the author uses the theory of independent subadditive processes to obtain strong limit laws for a class of problems in geometrical probability which exhibit nonlinear growth. Examples include the traveling salesman problem, the Steiner and rectilinear Steiner tree problem, and the minimum weight matching problem. Other problems, such as the minimum spanning tree problem, the minimum 1-tree problem, and some probabilistic versions of the traveling salesman problem and minimum spanning tree problem, have been subsequently treated in different papers (see respectively Steele [12], Goemans and Bertsimas [3], and Jaillet [5]). For most of these problems, the results concern the almost sure convergence of a sequence of normalized random variables, say L_n/n^{α}, to a constant c, as well as the convergence of the normalized means.

Questions about rates of convergence have been raised many times. There are in fact two issues concerning information on the rate of convergence:

1. What is the asymptotic size of $L_n - \mathbf{E}L_n$?
2. What can be said about the rate of convergence of the normalized means $\mathbf{E}L_n/n^{\alpha}$ to c ?

For the traveling salesman problem in the plane ($d = 2$), Rhee and Talagrand [9] (see also [8]) prove that, if the points are uniformly and independently distributed over the unit square, then there is a constant k such that $||L_n - \mathbf{E}L_n||_p \leq k\sqrt{p}$ for each p.

On the other hand, to the best of our knowledge, the second issue has never received a full answer. For the traveling salesman problem in the plane, if one follows the usual subadditivity argument for L_n (see, for example [1, 11]), it is relatively easy to deduce that $\mathbf{E}L_N \geq \beta\sqrt{n} - c$ for a positive constant c, where N has a Poisson distribution with parameter n (N is the number of points corresponding to a Poisson process of intensity n times the Lebesgue measure over $[0,1]^2$). Also it was shown in Karp [6] that $\mathbf{E}L_N \leq \beta\sqrt{n} + 12$.

Our goal is not only to extend this type of result for a general class of random processes in $\mathbf{R}^d$, $d \geq 2$, but also to show that results can be given for the initial random process itself (and not only its Poisson approximation). The material is presented in a general setting, much in the spirit of the paper by Steele [11]. The advantage of this level of generality is that it allows immediate applications to most of the known limit theorems for combinatorial optimization problems.

Section 2 is concerned with the main result of this paper. We first define what we call quasi-additive smooth Euclidean functionals and then show how the properties of these functionals imply limit theorems in expectation together with rates of convergence. Section 3 is mainly concerned with applications. We treat in details the case of the traveling salesman problem, the Steiner tree problem, the minimum spanning tree problem, and the minimum weight matching problem. We then extend the result of Section 2 in order to solve a problem proposed in Steele [12] concerning rates of convergence for the minimum spanning tree problems with power weighted edges.

2 Rates of Convergence for Quasi-Additive Smooth Euclidean Functionals

Let $\{x_i : 1 \leq i < \infty\}$ be an arbitrary infinite sequence of points in $\mathbf{R}^d$, $d \geq 2$, and let $x^{(n)} = \{x_1, x_2, \ldots, x_n\}$ be the first n points of x. Let L denote a non-negative real valued function of the finite subsets of $\mathbf{R}^d$ such that $L(\emptyset) = 0$ and $L(\{y\}) = 0$ for any single set $\{y\}$ of $\mathbf{R}^d$.

Definitions:

1. L is said to be *Euclidean* if $L(\zeta x^{(n)}) = \zeta L(x^{(n)})$ for all positive real ζ, and if $L(x^{(n)} + s) = L(x^{(n)})$ for all $s \in \mathbf{R}^d$.

2. L is said to be (γ_d, ξ_d)*-quasi-additive* if there exist two constants $C_d > 0$ and $D_d > 0$ and two constants $\gamma_d \geq 0$ and $\xi_d > 0$ with $d\gamma_d + \xi_d \leq d - 1$, such that for all positive integer m and any sequence x in $[0, t]^d$, $t > 0$, one has

$$\left| L(x^{(n)}) - \sum_{i=1}^{m^d} L(x^{(n)} \cap Q_i) \right| \leq C_d t m^{d-1} + D_d t n^{\gamma_d} m^{\xi_d} \tag{2.1}$$

whenever $\{Q_i : 1 \leq i \leq m^d\}$ is a partition of the d-cube $[0, t]^d$ into cubes with edges parallel to the axle and of length t/m.

3. L is said to be δ_d*-smooth* if there exist a constant $B_d > 0$ and a constant $\delta_d \geq 0$ such that

$$\left| \mathbf{E}L(X^{(n+1)}) - \mathbf{E}L(X^{(n)}) \right| \leq B_d / n^{\delta_d} \tag{2.2}$$

whenever $\{X_i : 1 \leq i < \infty\}$ are independent and uniformly distributed in $[0, 1]^d$.

The main result of this paper is the following theorem:

Theorem 1 *Suppose L is a (γ_d, ξ_d)-quasi-additive, δ_d-smooth Euclidean functional on $\mathbf{R}^d$. If $\{X_i : 1 \leq i < \infty\}$ are independent and uniformly distributed in $[0,1]^d$, then there is a non-negative finite constant $\beta_d(L)$ and a positive constant K_d such that*

$$\left|\mathbf{E}L(X^{(n)})/n^{(d-1)/d} - \beta_d(L)\right| \leq K_d/n^{\alpha_d}, \tag{2.3}$$

where

$$\alpha_d = \begin{cases} \min\{(d-1)/d, \delta_d - 1/d + 1/2\} & \text{if } d\gamma_d + \xi_d < d-1, \\ \min\{(d-1)/d - \gamma_d, \delta_d - 1/d + 1/2\} & \text{if } d\gamma_d + \xi_d = d-1. \end{cases} \tag{2.4}$$

Proof:
Let π denote a Poisson point process in $\mathbf{R}^d$ with constant intensity equal to 1. For any bounded Borel set $A \subset \mathbf{R}^d$, $\pi(A)$ denotes the random set of points in A (almost surely a finite set of points). Then let $\Lambda(t) = L(\pi([0,t]^d))$ and $\phi(t) = \mathbf{E}L(\pi([0,t]^d))$. The theorem is a consequence of the following two lemmas:

1. **Lemma 1** *Suppose L is a (γ_d, ξ_d)-quasi-additive, δ_d-smooth Euclidean functional on $\mathbf{R}^d$. Then there is a non-negative finite constant $\beta_d(L)$ such that*

$$\left|\phi(t) - \beta_d(L)t^d\right| \leq \begin{cases} C_d t & \text{if } d\gamma_d + \xi_d < d-1, \\ C_d t + D_d t^{1+d\gamma_d} & \text{if } d\gamma_d + \xi_d = d-1. \end{cases} \tag{2.5}$$

2. **Lemma 2** *Suppose L is a δ_d-smooth Euclidean functional on $\mathbf{R}^d$. Then we have*

$$\left|\phi(n^{1/d}) - n^{1/d}\mathbf{E}L(X^{(n)})\right| \leq A_d B_d/n^{\nu_d}, \tag{2.6}$$

where $\nu_d = \delta_d - 1/d - 1/2$, and where

$$A_d = \begin{cases} \left((n/(n-1))^{\delta_d}/(1-\delta_d) + 1\right)/\sqrt{2\pi} & \text{if } 0 \leq \delta_d < 1, \\ (\lfloor \delta_d \rfloor + 2)! \left(1/\sqrt{2\pi} + (\lfloor \delta_d \rfloor + 1)/\sqrt{n}\right) + 1/\sqrt{2\pi} & \text{otherwise.} \end{cases} \tag{2.7}$$

Indeed from Lemma 1 and from Lemma 2 we get

$$\left|n^{1/d}\mathbf{E}L(X^{(n)}) - n\beta_d(L)\right| \leq \begin{cases} C_d n^{1/d} + A_d B_d/n^{\nu_d} & \text{if } d\gamma_d + \xi_d < d-1 \\ C_d n^{1/d} + D_d n^{\gamma_d + 1/d} + A_d B_d/n^{\nu_d} & \text{if } d\gamma_d + \xi_d = d-1, \end{cases} \tag{2.8}$$

from which Theorem 1 follows easily.

∎

It remains to prove the two lemmas.

Proof of Lemma 1:
By definition of a Poisson point process, $N = |\pi([0,t]^d)|$ has a Poisson distribution with parameter t^d. From the (γ_d, ξ_d)-quasi-additivity of L,

$$\left|\Lambda(t) - \sum_{i=1}^{m^d} L(\pi(Q_i))\right| \leq C_d t m^{d-1} + D_d t N^{\gamma_d} m^{\xi_d}, \tag{2.9}$$

which implies that

$$\left|\phi(t) - \sum_{i=1}^{m^d} \mathbf{E}L(\pi(Q_i))\right| \leq C_d t m^{d-1} + D_d t \mathbf{E}[N^{\gamma_d}] m^{\xi_d}. \tag{2.10}$$

Now since L is a Euclidean functional $\mathbf{E}L(\pi(Q_i)) = \phi(t/m)$, and since $\gamma_d \leq 1$ $\mathbf{E}[N^{\gamma_d}] \leq \mathbf{E}[N]^{\gamma_d} = t^{d\gamma_d}$. Hence, we have

$$\left|\phi(t) - m^d \phi(t/m)\right| \leq C_d t m^{d-1} + D_d t^{1+d\gamma_d} m^{\xi_d}. \tag{2.11}$$

Setting $t = mu$ and dividing by $(mu)^d$ yields

$$\left|\frac{\phi(mu)}{(mu)^d} - \frac{\phi(u)}{u^d}\right| \leq \frac{C_d}{u^{d-1}} + \frac{D_d}{m^{d-1-d\gamma_d-\xi_d} u^{d-1-d\gamma_d}}. \tag{2.12}$$

Since, by definition, $d - 1 - d\gamma_d - \xi_d \geq 0$, (2.12) yields for all $m = 1, 2, \ldots$

$$\frac{\phi(mu)}{(mu)^d} \leq \frac{\phi(u)}{u^d} + \frac{C_d}{u^{d-1}} + \frac{D_d}{u^{d-1-d\gamma_d}}, \tag{2.13}$$

which implies also that

$$\frac{\phi(m)}{(m)^d} \leq \phi(1) + C_d + D_d < \infty. \tag{2.14}$$

Now, let us show that $\phi(t)$ is a continuous function in t. By definition of a Poisson point process and by scaling property (L is an Euclidean functional) we have:

$$\phi(t) = \sum_{k=0}^{\infty} \mathbf{E}L(tX^{(k)}) \mathbf{P}(N = k) = t \sum_{k=0}^{\infty} \mathbf{E}L(X^{(k)}) e^{-t^d} \frac{t^{dk}}{k!}. \tag{2.15}$$

From the δ_d-smoothness of L, it is easy to see (take $\delta_d = 0$) that there exists a constant $b_1(d)$ such that

$$\mathbf{E}L(X^{(k)}) \leq b_1(d) k. \tag{2.16}$$

This implies that

$$\varphi(t) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \mathbf{E}L(X^{(k)}) \frac{t^{dk}}{k!}, \tag{2.17}$$

is uniformly convergent for all real t, hence is a continuous function in t, as well as ϕ.

From (2.13) and the continuity of ϕ, let us show that there exists a constant $\beta_d(L) \geq 0$ such that

$$\beta_d(L) = \lim_{u\to\infty} \phi(u)/u^d. \tag{2.18}$$

Let $\beta_d(L) = \liminf_{u\to\infty} \phi(u)/u^d$. From the non-negativity of the functional and from (2.14) we have $0 \leq \beta_d(L) < \infty$. From the continuity of ϕ and the definition of a lim inf, one can find, for any $\varepsilon > 0$, an interval $[u_0, u_1]$, $u_1 > u_0$, such that, for all $u \in [u_0, u_1]$,

$$\phi(u)/u^d + C_d/u^{d-1} + D_d/u^{d-1-d\gamma_d} \leq \beta_d(L) + \varepsilon. \tag{2.19}$$

From (2.13) this implies that, for $t \in \cup_{m=1}^{\infty}[mu_0, mu_1]$, we have

$$\phi(t)/(t)^d \leq \beta_d(L) + \varepsilon. \tag{2.20}$$

But, by choosing $m_0 = u_0/(u_1 - u_0)$, the intervals $[mu_0, mu_1]$ are overlapping for $m \geq m_0$, and so (2.20) implies that

$$\forall t \geq m_0 u_0,\ \ \phi(t)/(t)^d \leq \beta_d(L) + \varepsilon. \tag{2.21}$$

This implies that $\limsup_{t\to\infty} \phi(t)/t^d \leq \beta_d(L) + \varepsilon$, and this terminates the proof of (2.18).

Finally by having $m \to \infty$ in (2.12) we get

$$\left|\beta_d(L) - \phi(u)/u^d\right| \leq \begin{cases} C_d/u^{d-1} & \text{if } d\gamma_d + \xi_d < d-1, \\ C_d/u^{d-1} + D_d/u^{\xi_d} & \text{if } d\gamma_d + \xi_d = d-1. \end{cases} \tag{2.22}$$

Multiplying by u^d leads to (2.5).

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Proof of Lemma 2:

We have already seen that since $|\pi([0,t]^d)|$ has a Poisson distribution with parameter t^d and since L is a Euclidean functional we have by scaling property

$$\phi(t) = t\sum_{k=0}^{\infty} \mathbf{E}L(X^{(k)})e^{-t^d}\frac{t^{dk}}{k!} \tag{2.23}$$

So,

$$\left|\phi(t) - t\mathbf{E}L(X^{(n)})\right| \leq t\sum_{k=0}^{\infty}\left|\mathbf{E}L(X^{(k)}) - \mathbf{E}L(X^{(n)})\right| e^{-t^d}\frac{t^{dk}}{k!}. \tag{2.24}$$

From the δ_d-smoothness of L, we have, for all $1 < i < j$,

$$\left|\mathbf{E}L(X^{(j)}) - \mathbf{E}L(X^{(i)})\right| \leq B_d\sum_{k=i}^{j-1}\frac{1}{k^{\delta_d}} = B_d\frac{(j-i)}{A(i,j,\delta_d)}, \tag{2.25}$$

where

$$A(i,j,\delta_d) \stackrel{\text{def}}{=} \frac{(j-i)}{\sum_{u=i}^{j-1} 1/u^{\delta_d}}. \tag{2.26}$$

By replacing (2.25) in (2.24), by setting $t = n^{1/d}$, and by having $f(k) \stackrel{\text{def}}{=} e^{-n}n^k/k!$ for $k \geq 0$, we get

$$\begin{aligned}
\left|\phi(n^{1/d}) - n^{1/d}\mathbf{E}L(X^{(n)})\right| &\leq n^{1/d}\left(\mathbf{E}L(X^{(n)})f(0) + B_d \sum_{k=1}^{\infty} \frac{|n-k|f(k)}{A(k\wedge n, k\vee n, \delta_d)}\right) \\
&= n^{1/d}B_d\psi(n),
\end{aligned} \tag{2.27}$$

where

$$\begin{aligned}
\psi(n) &= \left(\mathbf{E}L(X^{(n)})f(0)/B_d + \sum_{k=1}^{n-1} \frac{(n-k)f(k)}{A(k,n,\delta_d)}\right) + \sum_{k=n+1}^{\infty} \frac{(k-n)f(k)}{A(n,k,\delta_d)} \\
&\stackrel{\text{def}}{=} \psi_1(n) + \psi_2(n).
\end{aligned} \tag{2.28}$$

From the fact that

$$kf(k) = nf(k-1) \text{ for all } k \geq 1, \tag{2.29}$$

and from the fact that, for $k \geq n+1$,

$$A(n,k,\delta_d) \geq \frac{(k-n)}{(k-n)/n^{\delta_d}} = n^{\delta_d}, \tag{2.30}$$

we have

$$\begin{aligned}
\psi_2(n) &= \sum_{k=n+1}^{\infty} \frac{n(f(k-1)-f(k))}{A(n,k,\delta_d)} \\
&\leq \sum_{k=n+1}^{\infty} \frac{n(f(k-1)-f(k))}{n^{\delta_d}} \\
&= \frac{nf(n)}{n^{\delta_d}} \leq \frac{1}{\sqrt{2\pi}n^{\delta_d - 1/2}},
\end{aligned} \tag{2.31}$$

where we have used, for the last inequality, the following rather sharp form of *Stirling's formula* proved in [10]:

$$(n/e)^n\sqrt{2\pi n} \leq (n/e)^n\sqrt{2\pi n}e^{1/(12n+1)} \leq n! \leq (n/e)^n\sqrt{2\pi n}e^{1/(12n)}. \tag{2.32}$$

The evaluation of $\psi_1(n)$ is more complicated and the best bounds are obtained by considering two cases.

Case 1: $0 \leq \delta_d < 1$.

First, from (2.25), we have

$$\mathbf{E}L(X^{(n)}) \leq B_d \frac{(n-1)}{A(1,n,\delta_d)}. \tag{2.33}$$

Also, for this case, we have, for all $n > 1$,

$$\sum_{u=1}^{n-1} \frac{1}{u^{\delta_d}} \leq \int_0^{n-1} \frac{dx}{x^{\delta_d}} = \frac{(n-1)^{1-\delta_d}}{(1-\delta_d)}, \tag{2.34}$$

so that

$$A(1,n,\delta_d) \geq (1-\delta_d)(n-1)^{\delta_d}. \tag{2.35}$$

Hence, using (2.29) and the fact that $f(k) \leq f(n)$ for $k \leq n$, we have

$$\begin{aligned}
\psi_1(n) &= \mathbf{E}L(X^{(n)})f(0)/B_d + \sum_{k=1}^{n-1} \frac{n(f(k)-f(k-1))}{A(k,n,\delta_d)} \\
&\leq \frac{(n-1)f(0)}{A(1,n,\delta_d)} - \frac{nf(0)}{A(1,n,\delta_d)} + \frac{nf(n-1)}{A(n-1,n,\delta_d)} \\
&+ \sum_{k=1}^{n-2} nf(k)\left(\frac{1}{A(k,n,\delta_d)} - \frac{1}{A(k+1,n,\delta_d)}\right) \\
&\leq \frac{nf(n-1)}{A(n-1,n,\delta_d)} + nf(n)\left(\frac{1}{A(1,n,\delta_d)} - \frac{1}{A(n-1,n,\delta_d)}\right) \\
&= \frac{nf(n)}{A(1,n,\delta_d)} \leq \frac{nf(n)}{(1-\delta_d)(n-1)^{\delta_d}} \\
&\leq \frac{(n/(n-1))^{\delta_d}}{(1-\delta_d)\sqrt{2\pi}n^{\delta_d-1/2}}.
\end{aligned} \tag{2.36}$$

Case 2: $\delta_d \geq 1$.
For this case we will use the following bound on $A(k,n,\delta_d)$ $(1 \leq k \leq n-1)$:

$$A(k,n,\delta_d) \geq \frac{(n-k)}{(n-k)/k^{\delta_d}} = k^{\delta_d}. \tag{2.37}$$

We then have

$$\mathbf{E}L(X^{(n)}) \leq B_d \frac{(n-1)}{A(1,n,\delta_d)} \leq B_d(n-1). \tag{2.38}$$

If we let $h_d = \lfloor \delta_d \rfloor$ and $\varepsilon_d = \delta_d - h_d$, we then have

$$\begin{aligned}
\psi_1(n) &= \mathbf{E}L(X^{(n)})f(0)/B_d + n^{-\delta_d}\sum_{k=1}^{n}(n-k)k^{-h_d-1}(k/n)^{1-\varepsilon_d}e^{-n}\frac{n^{k+h_d+1}}{k!} \\
&\leq (n-1)f(0) + n^{-\delta_d}\sum_{k=1}^{n}(n-k)k^{-h_d-1}(k/n)^{1-\varepsilon_d}e^{-n}\frac{n^{k+h_d+1}}{k!}.
\end{aligned} \tag{2.39}$$

Since $(k/n)^{1-\varepsilon_d} \leq 1$ for $k \leq n$, and since $k^{-h_d-1} \leq k!(h_d+2)!/(k+h_d+1)!$ for $k \geq 1$, equation (2.39) leads to

$$\psi_1(n) \leq (n-1)f(0) + n^{-\delta_d}(h_d+2)!\left(\sum_{k=1}^{n-1} e^{-n}\frac{n^{k+h_d+2}}{(k+h_d+1)!}\right)$$

$$
\begin{aligned}
&- \quad n^{-\delta_d}(h_d+2)!\left(\sum_{k=1}^{n-1} e^{-n}\frac{kn^{k+h_d+1}}{(k+h_d+1)!}\right)\\
&= \quad (n-1)f(0) + n^{-\delta_d}(h_d+2)!\left(e^{-n}\frac{n^{n+h_d+1}}{(n+h_d)!} - e^{-n}\frac{n^{h_d+2}}{(h_d+1)!}\right)\\
&+ \quad n^{-\delta_d}(h_d+2)!\left((h_d+1)\sum_{k=1}^{n-1} e^{-n}\frac{n^{k+h_d+1}}{(k+h_d+1)!}\right)\\
&\leq \quad (n-1)f(0) - (h_d+2)nf(0) + n^{-\delta_d}(h_d+2)!\,(nf(n+h_d)+h_d+1)\\
&\leq \quad \frac{(h_d+2)!}{n^{\delta_d-1/2}}\left(\frac{1}{\sqrt{2\pi}} + \frac{h_d+1}{\sqrt{n}}\right), \qquad (2.40)
\end{aligned}
$$

with the use of *Stirling formula* for the last inequality.
Now the proof of Lemma 2 is obtained from (2.27), (2.28), (2.31), (2.36), and (2.40).

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3 Applications

3.1 The Traveling Salesman Problem

The traveling salesman problem (TSP) consists of finding a tour of minimum total length. Let $L_{tsp}(x^{(n)})$ be the length of the shortest tour through $x^{(n)}$. Note that this functional is monotone. The main result for this problem is:

Corollary 3.1 *If $\{X_i : 1 \leq i < \infty\}$ are independent and uniformly distributed in $[0,1]^d$, then there is a constant $\beta_{tsp}(d)$ and a constant $k_{tsp}(d)$ such that*

$$
\left|\mathbf{E}L_{tsp}(X^{(n)})/n^{(d-1)/d} - \beta_{tsp}(d)\right| \leq k_{tsp}(d)/n^{1/d(d-1)}. \qquad (3.1)
$$

From Theorem 1, this corollary will be proved if one can show that L_{tsp} is a $((d-2)/(d-1), 1/(d-1))$-quasi-additive, $1/d$-smooth Euclidean functional on $\mathbf{R}^d$. This assertion is a consequence of the following three lemmas.

Lemma 3.1 *For all positive integers m and any sequence x in $[0,t]^d$, $t > 0$, there is a constant $c_1(d)$ such that one has*

$$
L_{tsp}(x^{(n)}) \leq \sum_{i=1}^{m^d} L_{tsp}(x^{(n)} \cap Q_i) + c_1(d)tm^{d-1}, \qquad (3.2)
$$

whenever $\{Q_i : 1 \leq i \leq m^d\}$ is a partition of the d-cube $[0,t]^d$ into cubes with edges parallel to the axle and of length t/m.

Proof:
The argument, now classical, has its origin in [1] and has been used subsequently in many papers. The proof of Lemma 3.1 is a consequence of the following well-known fact:

Fact 3.1 *There is a constant c_d such that for any $x^{(n)}$ in $[0,t]^d$,*

$$L_{tsp}(x^{(n)}) \leq c_d t n^{(d-1)/d}, \tag{3.3}$$

(the best value of c_d has been successively derived in [13], [2], and [7].)

In order to prove Lemma 3.1, consider now the following tour construction through $x^{(n)}$ in $[0,t]^d$: first construct optimal TSP tours through $x^{(n)} \cap Q_i$ for $1 \leq i \leq m^d$. Then, in each cube Q_i where $x^{(n)} \cap Q_i$ is not empty, choose one point as a representative and finally construct a TSP tour through the set S of all representatives (at most m^d points). The combination of the small TSP subtours together with this TSP tour gives a spanning walk through $x^{(n)}$ of length,

$$\sum_{i=1}^{m^d} L_{tsp}(x^{(n)} \cap Q_i) + L_{tsp}(S). \tag{3.4}$$

One can then delete some arcs and transform this spanning walk into a tour of smaller length so that we get

$$L_{tsp}(x^{(n)}) \leq \sum_{i=1}^{m^d} L_{tsp}(x^{(n)} \cap Q_i) + L_{tsp}(S). \tag{3.5}$$

Finally, from Fact 3.1 we have

$$L_{tsp}(S) \leq c_d t (m^d)^{(d-1)/d}, \tag{3.6}$$

which, replaced in (3.5), leads to (3.1) (with $c_1(d) = c_d$).

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Lemma 3.2 *For all positive integers m and any sequence x in $[0,t]^d$, $t > 0$, one has*

$$\sum_{i=1}^{m^d} L_{tsp}(x^{(n)} \cap Q_i) \leq L_{tsp}(x^{(n)}) + c_2(d) t m^{d-1} + c_3(d) t n^{(d-2)/(d-1)} m^{1/(d-1)}, \tag{3.7}$$

whenever $\{Q_i : 1 \leq i \leq m^d\}$ is a partition of the d-cube $[0,t]^d$ into cubes with edges parallel to the axle and of length t/m.

Proof:
Here again, the argument is classical and has its origin in [1, Lemma 2].

Let T^* be an optimal TSP tour through $x^{(n)}$ and let us suppose that $x^{(n)} \cap Q_i$ is not empty. Let $T_i^* = T^* \cap Q_i$ and let T_{ij} for $1 \leq j \leq \mu_i$ ($\mu_i \leq |x^{(n)} \cap Q_i|$) be the connected components of T_i^* which contain at least an element of $x^{(n)}$. Let y_{1ij} and y_{2ij} be the two endpoints of T_{ij} which intersect the boundary of Q_i. Finally let n_{ik} be the number of these endpoints contained in each face F_{ik}, $1 \leq k \leq 2d$.

Let l_i be the total length of all these connected components. We then have:

$$L_{tsp}(x^{(n)}) \geq \sum_{i=1}^{m^d} l_i. \tag{3.8}$$

Now, from [1, Lemma 2], we know that we can construct a tour through $x^{(n)} \cap Q_i$ by using the connected components T_{ij} together with part of a double circuit going through the $2\mu_i$ endpoints.

But it is easy to see that a tour through the endpoints can be obtained from a combinaison of subtours through points contained in F_{ik}, $1 \leq k \leq 2d$ (each of dimension $d-1$), together with a tour connecting at most $2d$ points (one representative for each face containing endpoints). So from Fact 3.1 this tour through $x^{(n)} \cap Q_i$ has a length bounded from above by

$$l_i + 2\left(\sum_{k=1}^{2d} c_{d-1}(t/m) n_{ik}^{(d-2)/(d-1)} + c_d(t/m)(2d)^{(d-1)/d}\right). \tag{3.9}$$

This implies that

$$\begin{aligned}
&\sum_{i=1}^{m^d} L_{tsp}(x^{(n)} \cap Q_i) \\
&\quad \leq \sum_{i=1}^{m^d} l_i + 2c_{d-1}(t/m) \sum_{i=1}^{m^d} \sum_{k=1}^{2d} n_{ik}^{(d-2)/(d-1)} + 2(2d)^{(d-1)/d} c_d t m^{d-1}, \qquad (3.10)
\end{aligned}$$

which , together with (3.8) and the fact that the function $z^{(d-2)/(d-1)}$ is concave, gives

$$\begin{aligned}
&\sum_{i=1}^{m^d} L_{tsp}(x^{(n)} \cap Q_i) \\
&\leq L_{tsp}(x^{(n)}) + 2c_{d-1}(t/m)(2dm^d)(2n/2dm^d)^{(d-2)/(d-1)} + 2(2d)^{(d-1)/d} c_d t m^{d-1} \\
&= L_{tsp}(x^{(n)}) + 4d^{1/(d-1)} c_{d-1} t n^{(d-2)/(d-1)} m^{1/(d-1)} + 2(2d)^{(d-1)/d} c_d t m^{d-1}. (3.11)
\end{aligned}$$

Finally, Lemma 3.2 follows from (3.11) by taking $c_2(d) = 2(2d)^{(d-1)/d} c_d$, and $c_3(d) = 4d^{1/(d-1)} c_{d-1}$.

■

Lemma 3.3 *If* $\{X_i : 1 \le i < \infty\}$ *are independent and uniformly distributed in* $[0,1]^d$*, then there is a constant* b_d *such that*

$$\mathbf{E}L_{tsp}(X^{(n)}) \le \mathbf{E}L_{tsp}(X^{(n+1)}) \le \mathbf{E}L_{tsp}(X^{(n)}) + b_d/n^{1/d} \tag{3.12}$$

Proof:
The first inequality is obvious since L_{tsp} is monotone. Now let l_{n+1} denote the distance of X_{n+1} from the nearest of $X_1, \ldots, X_n$. It is then easy to see that

$$L_{tsp}(X^{(n+1)}) \le L_{tsp}(X^{(n)}) + 2l_{n+1}, \tag{3.13}$$

which implies that

$$\mathbf{E}L_{tsp}(X^{(n+1)}) \le \mathbf{E}L_{tsp}(X^{(n)}) + 2E_{n+1}[E^c_{n+1}[l_{n+1}]], \tag{3.14}$$

where E_{n+1} is the expectation over X_{n+1}, and E^c_{n+1} is the conditional expectation over $X^{(n+1)}$ given X_{n+1}.

Let $\mathcal{C}_r$ denote a d-dimensional hypersphere of radius r centered at X_{n+1} and V_r be the volume of $\mathcal{C}_r \cap [0,1]^d$. We then have

$$\begin{aligned} E^c_{n+1}[l_{n+1}] &= \int_0^\infty \mathbf{P}(l_{n+1} > r | X_{n+1}) dr \\ &= \int_0^\infty (1 - V_r)^n dr. \end{aligned} \tag{3.15}$$

Since $(1-z)^n$ is a non-increasing non-negative function of z for $0 \le z \le 1$, and since there exists a constant α such that $V_r \ge \alpha r^d$, (3.15) leads to:

$$\begin{aligned} E^c_{n+1}[l_{n+1}] &\le \int_0^{\alpha^{-1/d}} (1 - \alpha r^d)^n dr \\ &= \frac{\Gamma(1/d)\Gamma(n+1)}{d\alpha^{1/d}\Gamma(n+1+1/d)} \\ &\le \frac{\Gamma(1/d)}{d\alpha^{1/d}} n^{-1/d}. \end{aligned} \tag{3.16}$$

The last inequality follows from the fact that $a_n = \Gamma(n+1)n^{1/d}/\Gamma(n+1+1/d)$ is such that $\lim_{n\to\infty} a_n = 1$ and $a_{n+1}/a_n \ge 1$.

∎

Remarks when $d = 2$:
The best constant in Fact 3.1 is $c(2) = \sqrt{2}$ (see [2]), so that Lemma 3.1 gives

$$L_{tsp}(x^{(n)}) \le \sum_{i=1}^{m^2} L_{tsp}(x^{(n)} \cap Q_i) + \sqrt{2}tm. \tag{3.17}$$

Also, from [6, Theorem 3] one can deduce that

$$L_{tsp}(x^{(n)} \cap Q_i) \le l_i + (3/2)(4t/m). \tag{3.18}$$

Hence, Lemma 3.2 gives

$$\sum_{i=1}^{m^2} L_{tsp}(x^{(n)} \cap Q_i) \leq L_{tsp}(x^{(n)}) + 6tm. \tag{3.19}$$

From (3.17) and (3.19), and defining $\phi_{tsp}(t)$ by $\mathbf{E}L_{tsp}(\pi([0,t]^d))$ (see Section 2) one has from Lemma 1

$$\left|\phi_{tsp}(\sqrt{n})/n - \beta_{tsp}\right| \leq 6/\sqrt{n}. \tag{3.20}$$

Note that this improves the partial result obtained in [6, Theorem 7], which, translated in our notation, says that

$$\phi_{tsp}(\sqrt{n})/n \leq \beta_{tsp} + 12/\sqrt{n}. \tag{3.21}$$

Finally when $d = 2$, one can take $\alpha = 1/2$ in (3.16) so that Lemma 3.3 gives

$$\begin{aligned} \mathbf{E}L_{tsp}(X^{(n+1)}) &\leq \mathbf{E}L_{tsp}(X^{(n)}) + 2\Gamma(1/2)/(2(1/2)^{1/2}n^{1/2}) \\ &= \mathbf{E}L_{tsp}(X^{(n)}) + \sqrt{2\pi}/\sqrt{n}, \end{aligned} \tag{3.22}$$

so, **for the TSP in the plane**, we finally have from (3.20), (3.22), Lemma 1, and Lemma 2, the following version of Corollary 3.1:

$$\begin{aligned} \left|\mathbf{E}L_{tsp}(X^{(n)})/\sqrt{n} - \beta_{tsp}\right| &\leq 6/\sqrt{n} + \sqrt{2\pi}\left((2(n/(n-1))^{1/2} + 1)/\sqrt{2\pi}\right)/\sqrt{n} \\ &= 7/\sqrt{n} + 2/\sqrt{n-1}. \end{aligned} \tag{3.23}$$

3.2 The Steiner Tree Problem

The Steiner tree problem (STP) consists of finding a connected graph containing given points which has the least total sum of edge lengths among all such graphs. Let $L_{stp}(x^{(n)})$ be the length of a Steiner tree on $x^{(n)}$. This functional is also monotone. The main result for this problem is:

Corollary 3.2 *If $\{X_i : 1 \leq i < \infty\}$ are independent and uniformly distributed in $[0,1]^d$, then there is a constant $\beta_{stp}(d)$ and a constant $k_{stp}(d)$ such that*

$$\left|\mathbf{E}L_{stp}(X^{(n)})/n^{(d-1)/d} - \beta_{stp}(d)\right| \leq k_{stp}(d)/n^{1/d(d-1)}. \tag{3.24}$$

From Theorem 1, this corollary will be proved if one can show that L_{stp} is a $((d-2)/(d-1), 1/(d-1))$-quasi-additive, $1/d$-smooth Euclidean functional on $\mathbf{R}^d$. This assertion is a consequence of the fact that the functional L_{stp} follows three lemmas similar to Lemma 3.1, 3.2, and 3.3. In fact, from the fact that $L_{stp}(x^{(n)}) \leq L_{tsp}(x^{(n)})$, Lemma 3.1 and Lemma 3.2, as such, are still valid for the STP; also Lemma 3.3 is still valid for the STP (with a constant b_d divided by two).

Remarks when $d = 2$:
In the case of the STP it is easy to see that (3.18) can be replaced by

$$L_{stp}(x^{(n)} \cap Q_i) \leq l_i + 4t/m, \tag{3.25}$$

so **for the STP in the plane**, we have the following version of Corollary 3.2:

$$\begin{aligned} \left| \mathbf{E} L_{stp}(X^{(n)})/\sqrt{n} - \beta_{stp} \right| &\leq 4/\sqrt{n} + \sqrt{\pi/2}\left((2(n/(n-1))^{1/2} + 1)/\sqrt{2\pi}\right)/\sqrt{n} \\ &= 4.5/\sqrt{n} + 1/\sqrt{n-1}. \end{aligned} \tag{3.26}$$

3.3 The Minimum Spanning Tree Problem

The minimum spanning tree problem (MSTP) consists of finding a spanning tree of minimum total length. Let $L_{mstp}(x^{(n)})$ be the length of a shortest spanning tree on $x^{(n)}$. This functional is *not* monotone. The main result for this problem is:

Corollary 3.3 *If $\{X_i : 1 \leq i < \infty\}$ are independent and uniformly distributed in $[0,1]^d$, then there is a constant $\beta_{mstp}(d)$ and a constant $k_{mstp}(d)$ such that*

$$\left| \mathbf{E} L_{mstp}(X^{(n)})/n^{(d-1)/d} - \beta_{mstp}(d) \right| \leq k_{mstp}(d)/n^{1/d(d-1)}. \tag{3.27}$$

From Theorem 1, this corollary will be proved if one can show that L_{mstp} is a $((d-2)/(d-1), 1/(d-1))$-quasi-additive, $1/d$-smooth Euclidean functional on $\mathbf{R}^d$. This assertion is a consequence of the fact that the functional L_{mstp} follows three lemmas similar to Lemma 3.1, 3.2, and 3.3. For the counterpart of Lemma 3.1 the proof can be applied without change; for the counterpart of Lemma 3.2 this is also the case, since, although the MSTP functional is not monotone, the length of the tree construction in each cube Q_i (using the endpoints) is still an upper bound to the length of a optimal spanning tree on $x^{(n)} \cap Q_i$ (the reason being that the boundary of Q_i is convex). Finally the counterpart of Lemma 3.3 has to be modified somewhat and can be expressed as follows.

Lemma 3.4 *If $\{X_i : 1 \leq i < \infty\}$ are independent and uniformly distributed in $[0,1]^d$, then there is two constants a_d and b_d such that*

$$\mathbf{E} L_{mstp}(X^{(n)}) - a_d/n^{1/d} \leq \mathbf{E} L_{mstp}(X^{(n+1)}) \leq \mathbf{E} L_{mstp}(X^{(n)}) + b_d/n^{1/d} \tag{3.28}$$

Proof:
Upper bound:
Let l_{n+1} denote the distance of X_{n+1} from the nearest of $X_1, \ldots, X_n$. It is then easy to see that

$$L_{mstp}(X^{(n+1)}) \leq L_{mstp}(X^{(n)}) + l_{n+1}, \tag{3.29}$$

which implies that

$$\mathbf{E} L_{mstp}(X^{(n+1)}) \leq \mathbf{E} L_{mstp}(X^{(n)}) + E_{n+1}[E^c_{n+1}[l_{n+1}]], \tag{3.30}$$

where E_{n+1} is the expectation over X_{n+1}, and E^c_{n+1} is the conditional expectation over $X^{(n+1)}$ given X_{n+1}. One can then proceed as in the proof of Lemma 3.3.
Lower bound:
The proof uses an argument contained in [12, Lemma 2.3] for completing a tree with a missing point. It goes as follows: let T be an optimal spanning tree through $x^{(n+1)}$ and let y be an element of $V(n+1)$ ($V(n+1)$ is the set of neighbors of x_{n+1} in the graph determined by T) such that $d(x_{n+1}, y)$ is minimal. We get a connected graph spanning $x^{(n)}$ by taking the edges of T, deleting all the edges incident to x_{n+1}, and adding the edges which join y to the other neighbors of x_{n+1}.

Let T_{n+1} be this connected graph; it has a length $l(T_{n+1})$ such that

$$L_{mstp}(x^{(n)}) \leq l(T_{n+1}). \tag{3.31}$$

Now, by construction, we have

$$l(T_{n+1}) \leq L_{mstp}(x^{(n+1)}) + \sum_{j \in V(n+1)} d(y, x_j) - \sum_{j \in V(n+1)} d(x_{n+1}, x_j) \tag{3.32}$$

By the triangle inequality and the definition of y, we have

$$d(y, x_j) \leq 2d(x_{n+1}, x_j). \tag{3.33}$$

From (3.31), (3.32), and (3.33) we get

$$L_{mstp}(x^{(n)}) \leq L_{mstp}(x^{(n+1)}) + \sum_{j \in V(n+1)} d(x_{n+1}, x_j). \tag{3.34}$$

Note that it is a classical result that

$$|V(n+1)| \leq N_d, \tag{3.35}$$

where N_d is the number of spherical caps with angle 60^o which are needed to cover the unit sphere in $\mathbf{R}^d$. Also we know that (it is the counterpart of Fact 3.1) there is a constant c'_d such that

$$\mathbf{E} L_{mstp}(X^{(n+1)}) \leq c'_d n^{(d-1)/d}. \tag{3.36}$$

Now, by symmetry on the X_i's (since $\{X_i : 1 \leq i < \infty\}$ are independent and uniformly distributed in $[0,1]^d$), the edges adjacent to X_{n+1} can assume any ranks (i.e., can be the largest or the smallest of the edges of a minimal spanning tree on $X^{(n+1)}$), and then they are on average bounded by $c'_d/n^{1/d}$. Hence we finally have from (3.35) and (3.36)

$$\mathbf{E}[\sum_{j \in V(n+1)} d(X_{n+1}, X_j)] \leq N_d c'_d / n^{1/d}. \tag{3.37}$$

∎

Remarks when $d = 2$:
In the case of the MSTP (similar to the STP) it is easy to see that (3.18) can be replaced by

$$L_{mstp}(x^{(n)} \cap Q_i) \leq l_i + 4t/m. \tag{3.38}$$

Also the best constant in (3.36) is $c_2' = 1$ (by an argument similar to the one contained in [2]), and $N_2 = 6$. Hence **for the MSTP in the plane**, we have the following version of Corollary 3.3:

$$\begin{aligned} \left|\mathbf{E}L_{mstp}(X^{(n)})/\sqrt{n} - \beta_{mstp}\right| &\leq 4/\sqrt{n} + 6\left((2(n/(n-1))^{1/2} + 1)/\sqrt{2\pi}\right)/\sqrt{n} \\ &= (4 + 6/\sqrt{2\pi})/\sqrt{n} + (12/\sqrt{2\pi})\sqrt{n-1} \\ &< 6.4/\sqrt{n} + 4.8/\sqrt{n-1}. \end{aligned} \tag{3.39}$$

3.4 The Minimum Weighted Matching Problem

The minimum weighted matching problem (MWMP) consists of finding a matching of minimum total length. Let $L_{mwmp}(x^{(n)})$ be the length of a shortest matching on $x^{(n)}$. This functional is *not* monotone. The main result for this problem is:

Corollary 3.4 *If $\{X_i : 1 \leq i < \infty\}$ are independent and uniformly distributed in $[0,1]^d$, then there is a constant $\beta_{mwmp}(d)$ and a constant $k_{mwmp}(d)$ such that*

$$\left|\mathbf{E}L_{mwmp}(X^{(n)})/n^{(d-1)/d} - \beta_{mwmp}(d)\right| \leq k_{mwmp}(d)/n^{(1/d(d-1))\wedge(1/2-1/d)}. \tag{3.40}$$

From Theorem 1, this corollary will be proved if one can show that L_{mwmp} is a $((d-2)/(d-1), 1/(d-1))$-quasi-additive, 0-smooth Euclidean functional on $\mathbf{R}^d$. This assertion is a consequence of the fact that the functional L_{mwmp} follows three lemmas similar to Lemma 3.1, 3.2, and 3.3. From the fact that

$$L_{mwmp}(x^{(n)}) \leq (1/2)L_{tsp}(x^{(n)}), \tag{3.41}$$

the counterpart of Lemma 3.1 can be proved without almost a change (here, the representative of $x^{(n)} \cap Q_i$ is taken to be the point that is possibly left out from a shortest matching of $x^{(n)} \cap Q_i$). For the proof of the counterpart of Lemma 3.2, we can apply the same techniques as in Section 3.1. Finally the counterpart of Lemma 3.3 has to be modified and can be expressed as follows.

Lemma 3.5 *If $\{X_i : 1 \leq i < \infty\}$ are independent and uniformly distributed in $[0,1]^d$, then there is a constant b_d such that*

$$\mathbf{E}L_{mwmp}(X^{(n)}) - b_d \leq \mathbf{E}L_{mwmp}(X^{(n+1)}) \leq \mathbf{E}L_{mwmp}(X^{(n)}) + b_d \tag{3.42}$$

Proof:
Let l_{moy} be the expected value of the distance between two points independently

and uniformly distributed in $[0,1]^d$. Then it easy to see that if n is even, we have the following loose bound

$$\mathbf{E}L_{mwmp}(X^{(n)}) - l_{moy} \leq \mathbf{E}L_{mwmp}(X^{(n+1)}) \leq \mathbf{E}L_{mwmp}(X^{(n)}), \tag{3.43}$$

and if n is odd

$$\mathbf{E}L_{mwmp}(X^{(n)}) \leq \mathbf{E}L_{mwmp}(X^{(n+1)}) \leq \mathbf{E}L_{mwmp}(X^{(n)}) + l_{moy}. \tag{3.44}$$

■

Remarks:
When $d \geq 3$ the rate of convergence for the MWMP is comparable to the other problems. But, when $d = 2$ the bound given in Corollary 3.4 is not very useful and this comes from a very loose result in Lemma 3.5. For the rate of convergence of the Poisson point process in the plane we have the following results:
From (3.41), the counterpart of Lemma 3.1 gives

$$L_{mwmp}(x^{(n)}) \leq \sum_{i=1}^{m^2} L_{mwmp}(x^{(n)} \cap Q_i) + \sqrt{1/2}tm. \tag{3.45}$$

Also, for the same reason, the counterpart of Lemma 3.2 gives

$$\sum_{i=1}^{m^2} L_{mwmp}(x^{(n)} \cap Q_i) \leq L_{mwmp}(x^{(n)}) + 3tm. \tag{3.46}$$

From (3.45) and (3.46) and using the notation of Section 2 (i.e., $\phi_{mwmp}(t) = \mathbf{E}L_{mwmp}(\pi([0,t]^d))$) one finally has

$$\left|\phi_{mwmp}(\sqrt{n})/n - \beta_{mwmp}\right| \leq 3/\sqrt{n}. \tag{3.47}$$

3.5 The Case of Power Weighted Edges

In [12], the author studies the asymptotics of generalizations of the minimum spanning tree problem in which the distance between points are some fixed power of the Euclidean distance. The purpose of this section is to give an answer to a question concerning the rate of convergence of the expectation of the functional.

In order to treat this problem it is useful to generalize Theorem 1 to include the case of what we call quasi-Euclidean functionals. Let us suppose that the power of the Euclidean distance is $0 < \omega < d$. The new definitions are then:

1. L is said to be *ω-quasi-Euclidean* if $L(\zeta x^{(n)}) = \zeta^\omega L(x^{(n)})$ for all positive real ζ, and if $L(x^{(n)} + s) = L(x^{(n)})$ for all $s \in \mathbf{R}^d$.

2. L is said to be $(\omega, \gamma_d, \xi_d)$-*quasi-additive* if there exist two constants $C_d > 0$ and $D_d > 0$ and two constants $\gamma_d \geq 0$ and $\xi_d > 0$ with $d\gamma_d + \xi_d \leq d - \omega$, such that for all positive integer m and any sequence x in $[0,t]^d$, $t > 0$, one has

$$\left| L(x^{(n)}) - \sum_{i=1}^{m^d} L(x^{(n)} \cap Q_i) \right| \leq C_d t^\omega m^{d-\omega} + D_d t^\omega n^{\gamma_d} m^{\xi_d} \tag{3.48}$$

whenever $\{Q_i : 1 \leq i \leq m^d\}$ is a partition of the d-cube $[0,t]^d$ into cubes with edges parallel to the axle and of length t/m.

3. L is said to be δ_d-*smooth* if there exist a constant $B_d > 0$ and a constant $\delta_d \geq 0$ such that

$$\left| \mathbf{E}L(X^{(n+1)}) - \mathbf{E}L(X^{(n)}) \right| \leq B_d / n^{\delta_d} \tag{3.49}$$

whenever $\{X_i : 1 \leq i < \infty\}$ are independent and uniformly distributed in $[0,1]^d$.

We then have the following result.

Theorem 2 *Suppose L is a $(\omega, \gamma_d, \xi_d)$-quasi-additive, δ_d-smooth Euclidean functional on $\mathbf{R}^d$. If $\{X_i : 1 \leq i < \infty\}$ are independent and uniformly distributed in $[0,1]^d$, then there is a non-negative finite constant $\beta_d^{(\omega)}(L)$ and a positive constant $K_\omega(d)$ such that*

$$\left| \mathbf{E}L(X^{(n)})/n^{(d-\omega)/d} - \beta_d^{(\omega)}(L) \right| \leq K_\omega(d)/n^{\alpha_d}, \tag{3.50}$$

where

$$\alpha_d = \begin{cases} \min\{(d-\omega)/d, \delta_d + 1/2 - \omega/d\} & \text{if } d\gamma_d + \xi_d < d - \omega, \\ \min\{(d-\omega)/d - \gamma_d, \delta_d + 1/2 - \omega/d\} & \text{if } d\gamma_d + \xi_d = d - \omega. \end{cases} \tag{3.51}$$

The proof of this result is obtained exactly as for Theorem 1 and therefore is not repeated here. We can use Theorem 2 for the minimal spanning tree with power weighted edges and get the following result. Let $L_{mstp}^{(\omega)}(x^{(n)})$ be the length of a shortest spanning tree on $x^{(n)}$ with power weighted edges ω.

Corollary 3.5 *If $\{X_i : 1 \leq i < \infty\}$ are independent and uniformly distributed in $[0,1]^d$, then there is a constant $\beta_{mstp}^{(\omega)}(d)$ and a constant $k_{mstp}^{(\omega)}(d)$ such that*

$$\left| \mathbf{E}L_{mstp}^{(\omega)}(X^{(n)})/n^{(d-\omega)/d} - \beta_{mstp}^{(\omega)}(d) \right| \leq k_{mstp}^{(\omega)}(d)/n^{(\omega/d(d-1))\wedge(1/2)}. \tag{3.52}$$

From Theorem 2, this corollary will be proved if one can show that $L_{mstp}^{(\omega)}$ is a $(\omega, (d-1-\omega)/(d-1), \omega/(d-1))$-quasi-additive, ω/d-smooth, ω-quasi-Euclidean functional on $\mathbf{R}^d$. Following the same argument as in Corollary 3.3 the proof of this assertion is a consequence of the following fact,

Fact 3.2 *There is a constant $c_\omega(d)$ such that for any $x^{(n)}$ in $[0,t]^d$,*

$$L^{(\omega)}_{mstp}(X^{(n)}) \le c_\omega(d) n^{(d-\omega)/d}, \tag{3.53}$$

which has been proven in [12].

∎

Remarks:
For the MSTP with power weighted edges in the plane, we have the following version of Corollary 3.5:

$$\begin{aligned} \left|\mathbf{E}L^{(\omega)}_{mstp}(X^{(n)})/n^{(2-\omega)/2} - \beta^{(\omega)}_{mstp})\right| &\le (4 + 12(2\sqrt{2}+1)/\sqrt{2\pi})/n^{(\omega\wedge 1)/2} \\ &< 22.4/n^{(\omega\wedge 1)/2}. \end{aligned} \tag{3.54}$$

4 Final Remarks

The result presented in this paper leaves room for further investigations. For example, we have not been able to show that the bound of Theorem 1 is asymptotically best in the sense that $\left|\mathbf{E}L(X^{(n)})/n^{(d-1)/d} - \beta_d(L)\right| = \Omega(1/n^{\alpha_d})$. We have nevertheless given, for practical purposes, the best possible constant K_d involved in the rate $1/n^{\alpha_d}$, but, in general, the difficult question of finding a non trivial lower bound on $\left|\mathbf{E}L(X^{(n)})/n^{(d-1)/d} - \beta_d(L)\right|$ remains opened.

In Jaillet [5], we present general finite-size bounds and limit theorems for probabilistic versions of the traveling salesman problem and of the minimum spanning tree problem. For these problems information about rates of convergence seems more difficult to get, mainly because of the lack of smoothness of the functional. On the other hand the minimum 1-tree problem (and other problems such as routing and facility location) are certainly amenable to the techniques developed in this paper.

Finally, a persistently open question related to the issues of rates of convergence is the possible existence of central limit theorems for the combinatorial optimization problems listed in this paper.

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References

[1] J. Beardwood, J. Halton and J. Hammersley. 1959. The Shortest Path Through Many Points. *Proc. Camb. Phil. Soc.*, **55**, 299–327.

[2] L. Few. 1955. The Shortest Path and the Shortest Road Through n Points. *Mathematika*, **2**, 141–144.

[3] M. Goemans and D. Bertsimas. 1991. Probabilistic Analysis of the Held and Karp Lower Bound for the Euclidean Traveling Salesman Problem. *Math. Oper. Res.*, **16**, 72–90.

[4] S. Halfin. 1990. Personal communication by the referee.

[5] P. Jaillet. 1989. Analysis of Probabilistic Combinatorial Optimization Problems in Euclidean Spaces. *Prépublication du CERMA*, **14**. Centre de Mathématiques Appliquées. ENPC. Paris. France.

[6] R. Karp. 1977. Probabilistic Analysis of Partitioning Algorithms for the Traveling Salesman Problem in the Plane. *Math. Oper. Res.*, **2**, 209–224.

[7] S. Moran. 1982. On the Length of Optimal TSP Circuits in Sets of Bounded Diameter. Technical Report **235**. Computer Science Department. Technion. Haifa.

[8] W. Rhee and M. Talagrand. 1987. Martingale Inequalities and NP-complete Problems. *Math. Oper. Res.*, **12**, 177–181.

[9] W. Rhee and M. Talagrand. 1988. A Sharp Deviation inequality for the Stochastic Traveling Salesman Problem. *Ann. Proba.*, **17**, 1–8.

[10] H. Robbins. 1955. A remark on Stirling's formula. *Amer. Math. Monthly*, **62**, 26–29.

[11] J. Steele. 1981. Subadditive Euclidean Functionals and Nonlinear Growth in Geometric Probability. *Ann. Proba.*, **9**, 365–376.

[12] J. Steele. 1988. Growth Rates of Euclidean Minimal Spanning Trees with Power Weighted Edges. *Ann. Proba.*, **16**, 1767–1787.

[13] S. Verblunsky. 1951. On the Shortest Path Through a Number of Points. *Proc. Amer. Math. Soc.*, **2**, 904–913.