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National Committee for Fluid Mechanics Films

FILM NOTES for LOW-REYNOLDS-NUMBER FLOWS*

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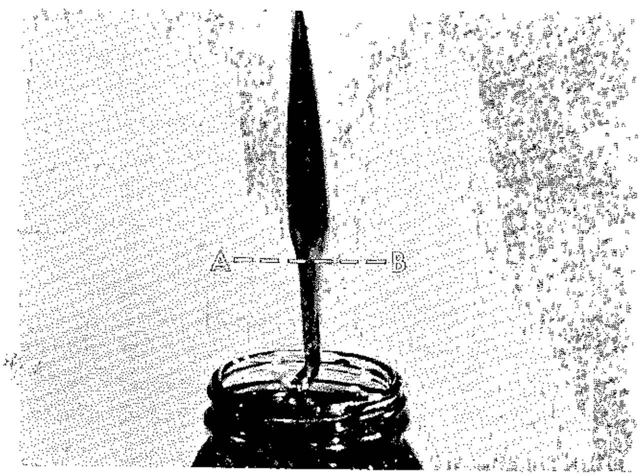
Introduction

Low-Reynolds-number flows are those in which inertia plays only a very small part in the conditions which determine the motion. The Reynolds number of the flow of a fluid which is characterized only by viscosity and density is defined as $R = LV\rho/\mu$. Here L is chosen as a length connected with the solid boundaries of the flow which may be expected to determine the scale of the fluid motion, V is a characteristic velocity, ρ and μ are the density and viscosity of the fluid. In low-Reynolds-number flows, the numerical value of R provides a rough estimate of the relative importance of inertia and viscosity. When R is small, the importance of inertia is small compared with that of viscosity. As examples, the movement of miscroscopic organisms for which L is very small and the movement of glaciers for which V is very small and μ very large are given. In the latter case the flow was made apparent by a line of red flags which was initially straight and was bowed out by the slow motion of the glacier after two years.



1. Honey flowing from a knife drawn out of a jar.

Pulling a knife vertically out of a pot of honey for which μ is large demonstrates two properties of a viscous liquid. It can resist both tangential and tensile stresses. The honey can be lifted by a tangential force exerted by a knife's surface, and the stretching of the stream as it falls gives rise to a tensile stress over a horizontal section such as AB in Fig. 1.

LOW-REYNOLDS-NUMBER FLOWS, a 16-mm color sound film, 33 minutes in length, was produced by Educational Services Incorporated under the direction of the National Committee for Fluid Mechanics Films, with a grant from the National Science Foundation. Information on purchase and rental may be obtained from the distributor:

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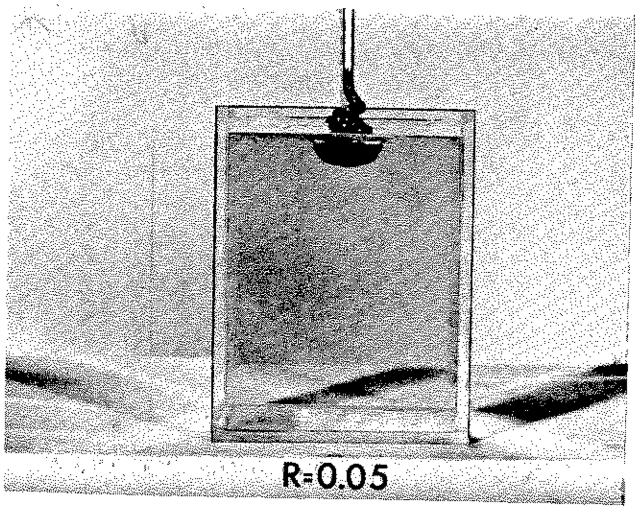


2. A low-Reynolds-number "jet" penetrates only slightly into a mass of the same fluid.

Jet Flows

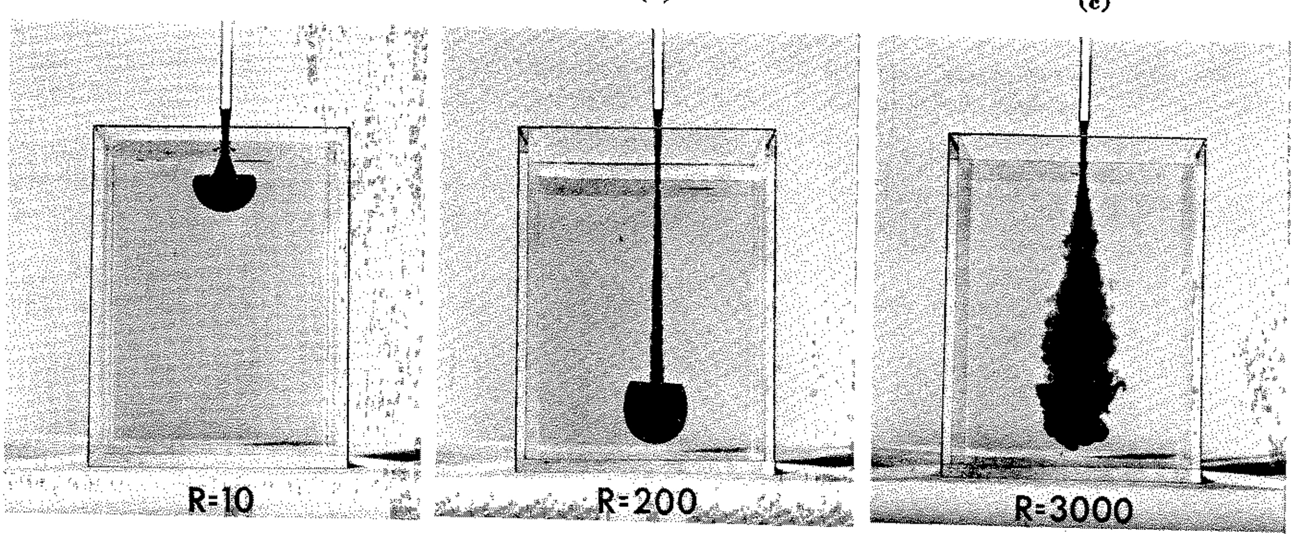
Though the honey stream demonstrates some properties of viscosity, it does not provide a model showing the relative importance of viscosity and inertia. For this purpose the honey experiment is repeated, using a piston driven at a known speed to produce a controlled jet of colored fluid at a known velocity. This jet falls vertically into a transparent box containing the same fluid, but uncolored. Four fluids whose viscosities cover a very wide range are used, and in order to ensure that the differences between the flows observed in the four cases are, and can be seen to be, due only to differences in Reynolds number, the diameters D and velocities V are the same in all four cases. Figs. 2 and 3 show the flows, and the corresponding values of $R = DV\rho/\mu$ are printed below. In Fig. 2 ($R = 0.05$), the fluid is syrup which is so viscous that the vertical velocity is destroyed by the vertical stress in the jet before it reaches the free surface of the fluid in the box, so that it forms a pile which flattens slowly under the influence of gravity. In Fig. 3a ($R = 10$) the fluid is glycerine. The depth to which the jet penetrates before losing its velocity and spreading out into a mushroomlike head is only a few jet diameters. The angle of the cone which forms the stalk of the mushroom is an indication of the rate at which the momentum of the jet is being retarded by viscosity. In Fig. 3b ($R = 200$) the fluid is a mixture of glycerine and water, and the jet penetrates many diameters before being stopped. In fact, it is not stopped until it reaches the bottom of the box. Comparing Figs. 3a and 3b, it appears that the angle which the roughly conical jet assumes when it is retarded by viscosity is of the order of magnitude $1/R$. Fig. 3c shows what happens when the Reynolds number is sufficiently high. At $R = 3000$ the jet has become turbulent.

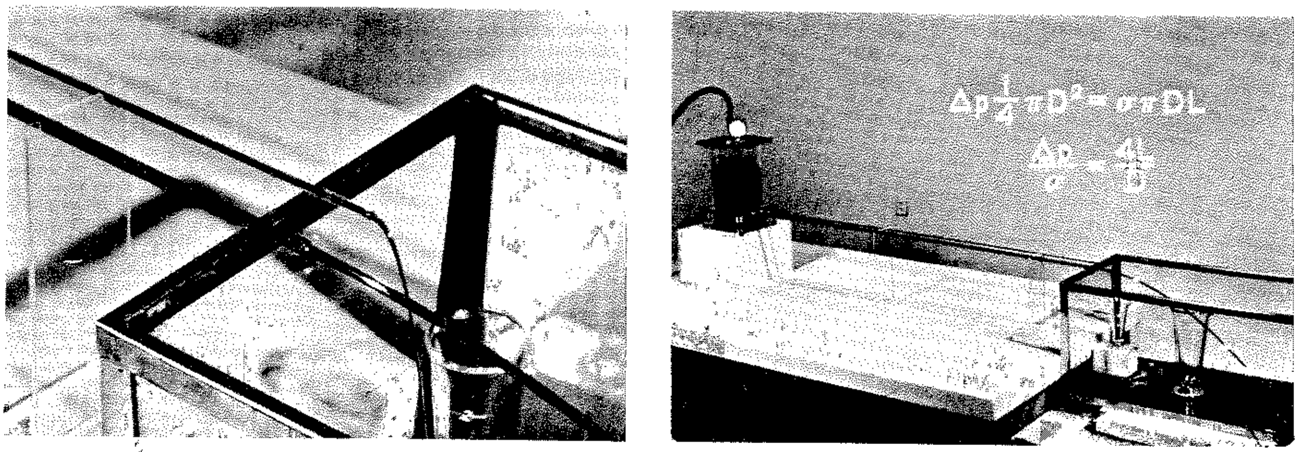
Flow Through Long Tubes

This film is concerned with flow like that shown in Fig. 2, for which R is small compared with 1.0 so that inertia plays no appreciable part in the situation, flow being determined only by the balance of viscous and pressure stresses brought into play by gravity or forces applied at the boundaries. The geometry associated with the flow shown in Fig. 2, however, is too complicated for complete mathematical analysis, and a simpler case is discussed in greater detail — namely, the flow through long tubes of uniform bore. Here, though the Reynolds number is not necessarily small, there is no change in the inertia of the flow as it passes through the tube, so that the results of calculation of the kind used in discussing low-Reynolds-number

3. Comparison of the penetrations of the first two jets with Fig. 2 demonstrates the effect of increasing Reynolds number. The third jet is turbulent.

(a) (b) (c)



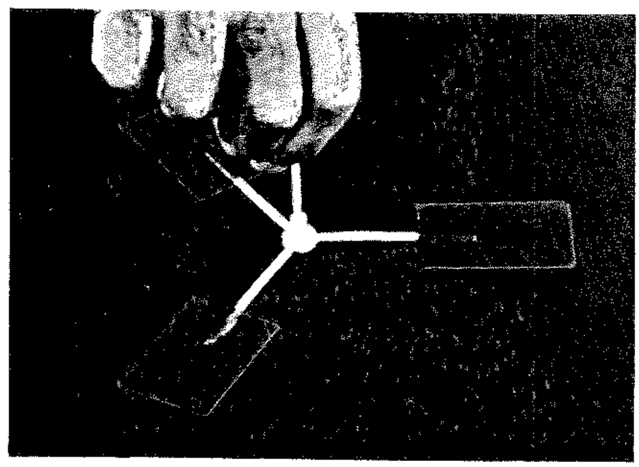


4. Laminar flow through two tubes with inner diameters in the ratio 2:1.

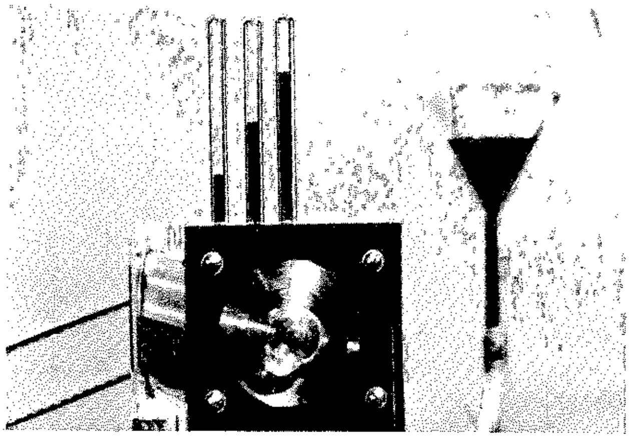
flows are applicable. The two tubes shown in Fig. 4 (left) have the same length, but the bore d of one of them is twice that of the other. Compressed air at the top of the reservoir drives the fluid at constant pressure through the two tubes into two receptacles with marks at one ounce and 16 ounces respectively (Fig. 4, right). Calculation shows that the discharge from the large tube should be 16 times that of the other, so that if the flows into both receptacles are started at the same time, they reach the marks simultaneously. In this experiment the driving pressure $\triangle P$ acting over the area of cross section $\frac{1}{4}\pi d^2$ balances the tangential stress σ acting over an area $\pi d \times L$ so that $\triangle P \frac{1}{4}\pi d^2 = \sigma \pi d L$ or $\triangle P/\sigma = 4L/d$. When the length L is large compared with the diameter d, as it is in the experiment shown, a small tangential stress can produce a large change in pressure. This is the principle on which hydrodynamic lubrication is based.

Hydrodynamic Lubrication

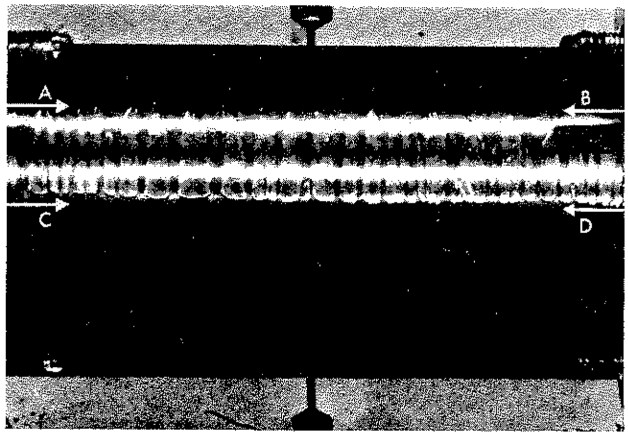
A very simple experiment can demonstrate that the coefficient of friction, which is the ratio of the tangential to the normal force between two solid bodies, can be much reduced by hydrodynamic lubrication. If a sheet of note paper is held horizontally above a smooth polished table and dropped onto it with a small horizontal velocity, it can be made to slide smoothly many times as far as if the table were not smooth. This is because, as the paper settles, the air under it must flow outwards and this outflow produces a tangential stress which enables enough pressure to be built up in a very narrow space between paper and table to support the paper out of contact with the table. This tangential stress is much smaller than solid contact friction stress. When the layer gets so thin that lack of flatness or smoothness of paper or table permits actual contact, the paper stops gliding. In this experiment the layer of air continually decreases in thickness, but if the sheet could be held at a very small angle to the table and moved horizontally with the wide gap in front, air would be swept in there and would be forced to flow out through the narrower parts of the gap. In this way a layer of air would separate paper and table so long as the horizontal motion was maintained. This principle is illustrated by means of the toy shown in Fig. 5, which can be called a teetotum. This consists of three equal flat laminae made of mica, which can be very flat. These are stuck to the lower sides of three light arms which are rigidly set to form a 120° triad from the center of which rises a light vertical rod. The laminae are mounted so that they are inclined upwards at an angle of half a degree or less with a plane normal to the axis rod. The teetotum is spun between finger and thumb in a counterclockwise direction and dropped on a horizontal table. It will spin for a very large number of revolutions, but if it is spun in a clockwise direction it stops instantly. In the film the direction of rotation



5. The three laminae of the spinning toy are inclined slightly upwards. When spinning counterclockwise the front edges are higher.



6. **Pressure distribution in a journal bearing with minimum gap at the top. Shaft is rotating in a counterclockwise direction.**



7. **Cavitation bubbles in the low-pressure region of a journal bearing.**

is partly obscured by a stroboscopic effect when the teetotum is spinning.

The principle illustrated by the teetotum is employed not only in slipper bearings, but also in the lubrication of a journal bearing in which a cylindrical shaft rotates inside a cylinder of slightly greater diameter. The shaft assumes a slightly eccentric position so that there is a narrowing gap into which the oil is dragged, so producing a high pressure. Beyond the point where the clearance is a minimum, the gap is expanding and the pressure is reduced.

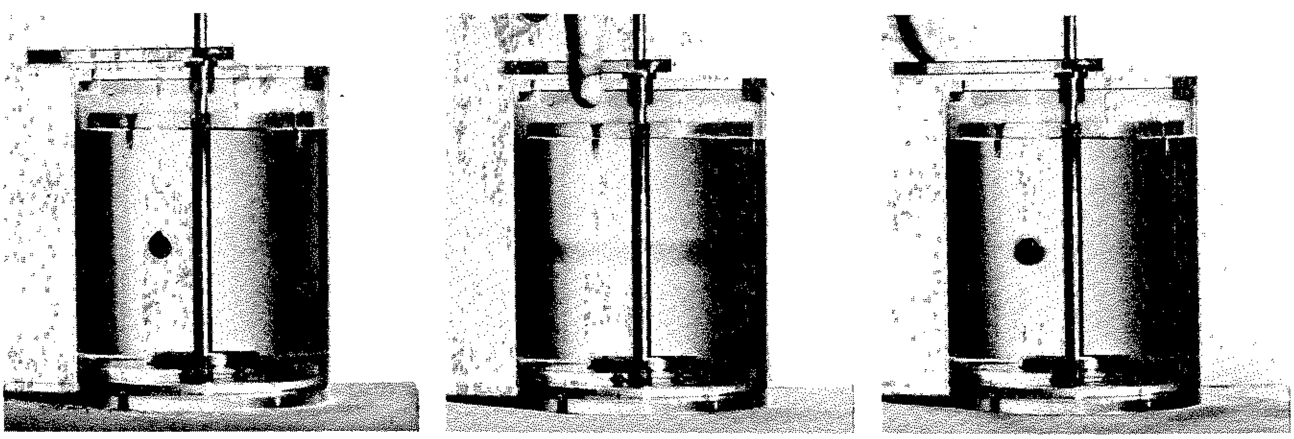
This is illustrated in Fig. 6, which shows a journal bearing with a fixed eccentricity in which the top of the inner cylinder is the position of minimum clearance. Three manometer tubes are connected with holes in the outer cylinder, one just before the minimum clearance, one at this position, and the third as the same distance beyond it. In Fig. 6 the shaft is seen rotating in the counterclockwise direction. The level of the lubricant in the right-hand manometer is just as much above that of the central one as that in the left-hand manometer is below it. The situation is reversed when the direction of rotation is reversed.

Very large pressures which can support large loads can be produced if the fluid is very viscous or the gap very small. The fluid, however, cannot support the large negative pressures which the last experiment might lead us to expect, and cavitation bubbles may occur. These can be seen in Fig. 7, which was taken through a transparent journal and shows the fluid downstream of the minimum clearance at the top of the picture. The bubbles form at the position of minimum pressure marked *AB* in Fig. 7, and extend into the widening gap to the position *CD* where the pressure begins to rise again.

Kinematic Reversibility

The reversibility of the pressure when the direction of rotation is changed in the experiment of Fig. 6 implies reversibility of flow. Some surprising results of this reversibility are shown with the apparatus of Fig. 8, in which the space between two concentric cylinders is filled with glycerine. Dye is introduced

8. **Kinematic reversibility in an annulus, (a) initial dyed element, (b) inner cylinder turned 4 turns forward, (c) inner cylinder turned back 4 turns.**



(a) (b) (c)

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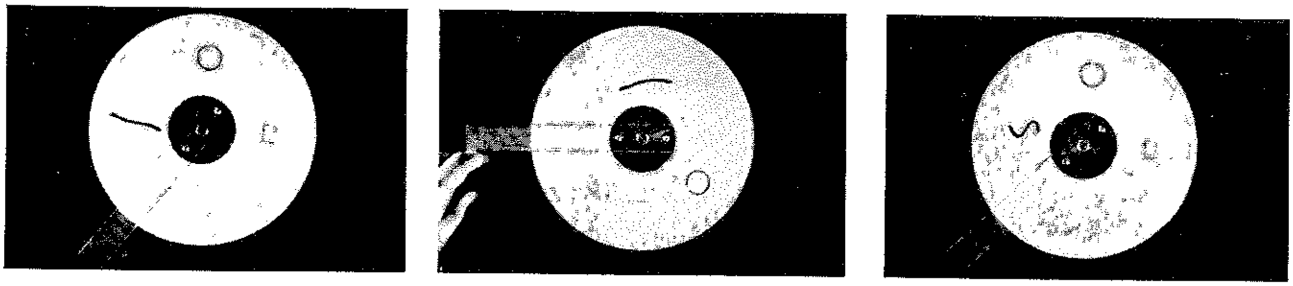


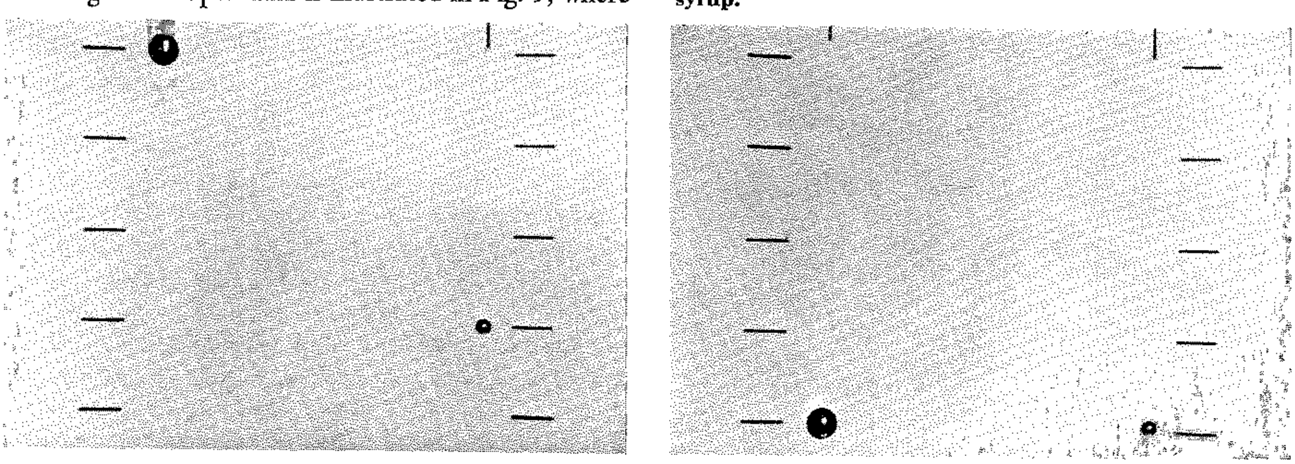
9. Kinematic reversibility is achieved for the dyed fluid square and the rigid ring, but is not achieved for the bit of yarn.

into the annulus which forms a compact colored volume (Fig. 8a). The inner cylinder is turned through, say, *N* revolutions. When observed from the side, the colored area seems to mix with the uncolored glycerine (Fig. 8b), just as milk mixes with tea when stirred in a cup, but on reversal of the motion the dye suddenly collects into a compact mass when the cylinder has been turned exactly *N* turns in reverse (Fig. 8c). To understand the reason for this peculiar behavior, one can observe what happens when looking through the fluid in a direction parallel to the axis of rotation. One sees the dyed areas being drawn out into long, thin streaks. On reversal of the motion of the boundary, every particle retraces exactly the same path on its return journey as on the outward journey, and at every point its speed is the same fraction of the boundary speed as it was at the same point on its outward journey, so that when the boundary has returned to its original position every particle in the fluid has also done so and the original pattern of dye is reproduced. Of course, molecular diffusion, which is irreversible, is negligible during the time of this experiment.

The motion of a rigid body suspended in a fluid is also reversible, but that of a flexible body is not, because when the stresses in a flexible body are reversed it changes its shape. This is illustrated in Fig. 9, where the rigid body is a small plastic ring with a gap to mark its orientation. In Fig. 9a the gap is in the 12 o'clock position. The flexible body, consisting of a piece of wool, is on the left and a rectangular pattern of dyed fluid on the right. The inner cylinder was then rotated in a clockwise direction (Fig. 9b), the dye, ring and wool moved round in a clockwise direction, and the ring rotated in a counterclockwise direction about its center. The wool remained nearly straight, because it was in fluid which was moving in such a way as to stretch it. The dye almost disappeared. After the motion was reversed until the inner cylinder was in its original position (Fig. 9c), the rigid ring returned to its original position and orientation. The rectangle of dye reconstituted itself, but the wool curled up because on the return path the viscous stresses gave rise to a compressive stress along its length which naturally made it collapse.

Falling Bodies and Sedimentation

The resistance of similar solid bodies moving at low Reynolds numbers through a fluid are proportional to their linear dimensions. The weight of a sphere is proportional to the cube of its diameter. When it is falling through the fluid, it weight is supported by the fluid resistance, so it will fall at a rate proportional to the square of its diameter. Fig. 10

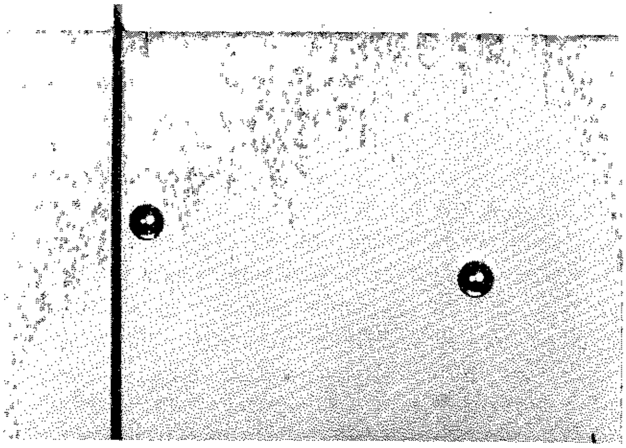


(a) (b)

10. Two balls with $\frac{3}{4}$-in. and $\frac{3}{8}$-in. diameters fall in syrup.

shows two brass balls of diameters ¾ inch and ⅜ inch falling in syrup. They can be released mechanically. Background marks at equal intervals make it possible to demonstrate their relative falling speeds. The bigger ball is released when the smaller one has traversed ¾ of the distance to the lowest mark (Fig. 10a). The two balls reach this mark simultaneously (Fig. 10b).

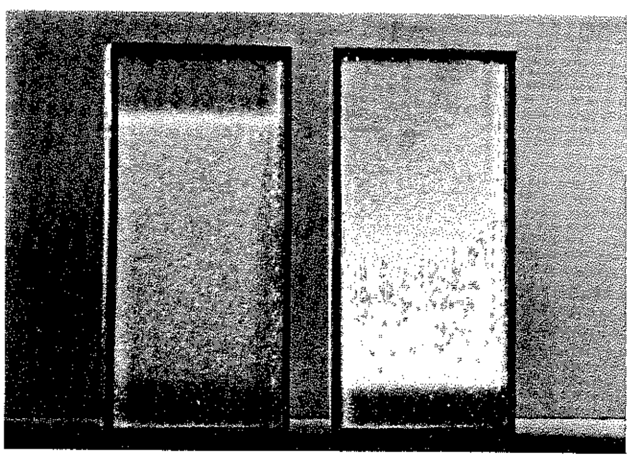
At low Reynolds numbers the disturbance produced by a moving ball extends many diameters. Beads suspended in a fluid at considerable distance from it are moved, conversely, a solid body such as a wall



11. The sphere near the wall falls more slowly than an identical one far from the wall.

can affect its rate of fall. Fig. 11 shows two identical balls which were released simultaneously from the same height. The one nearer the vertical wall falls more slowly, but owing to the reversibility of low-Reynolds-number flow it remains at a constant distance from it.

The retarding effect of neighbors makes a dispersed suspension of particles fall more slowly than a single one. Thus, when a suspension of particles falls in a fluid it develops a sharply defined top. This happens

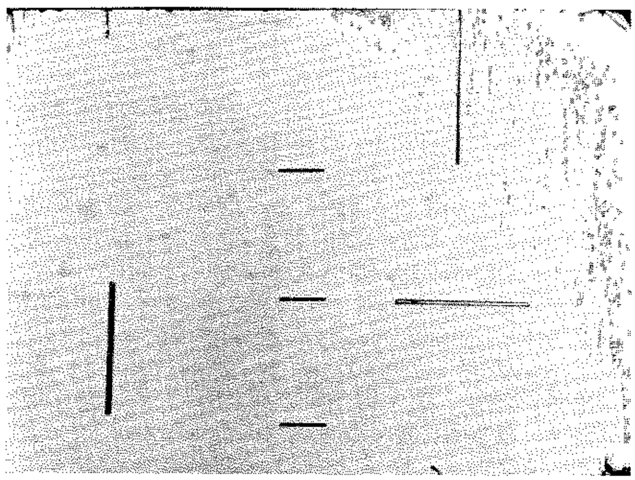


12. Two boxes of assorted beads left to settle at the same time. The box at left has a larger particle density.

even for a suspension of particles of assorted sizes, as in this experiment. A particle which has a terminal velocity rather lower than its neighbors does not get left behind, because if it did it would find itself isolated and would fall faster and catch up the rest. Fig. 12 shows two boxes, each containing fluid and sediment. They were both shaken and left to settle at the same moment. The left-hand box has many particles and settles much more slowly than the right-hand box, which has few.

Resistance of Long Thin Rods

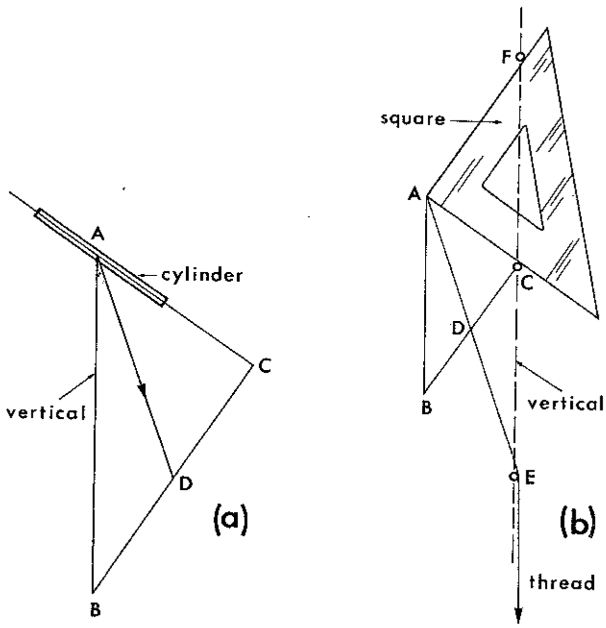
When a body is not spherical, its resistance at low Reynolds numbers is not the same for all directions of motion. A long, thin body of revolution, for instance, has twice the resistance to lateral motion that it has to motion parallel to the axis. This was first proved for the special case of a long ellipsoid, but is true generally. Fig. 13 shows two identical rods which



13. Identical cylinders falling in syrup. When released, the bottom of the vertical cylinder was at the same height as the horizontal cylinder.

were released simultaneously in syrup with the bottom of the vertical rod at the same level as the horizontal rod. This level was that of the uppermost of three equally spaced marks. The photograph was taken when the bottom of the vertical rod had just reached the lowest mark and the horizonal rod was level with the intermediate mark.

When a rod of uniform section and density is released obliquely, it does not change its orientation but drifts sideways. At the terminal speed the net weight is just balanced by the drag, which therefore acts vertically. This drag is the resultant of two forces, one parallel to the long axis and one perpendicular to it. In the triangle of forces ABC (Fig 14a) these are represented by CA and BC, while the weight is represented by AB. Since at low Reynolds number the velocity of a body is proportional to the applied force, and since for long cylinders a force moves the body only half as fast when applied laterally as when applied longitudinally, the triangle of velocity is ACD where D is the midpoint of BC. AD therefore is the direction of motion when the force on the cylinder



14. Geometric construction of the "flight path" of an obliquely oriented cylinder.

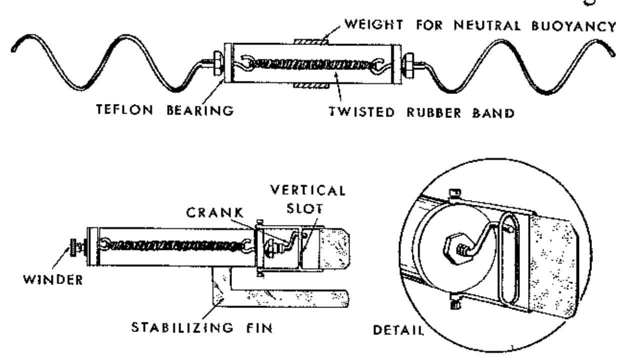
acts in the direction of AB. This geometrical construction can be mechanized, using three drawing pins F, C and E (Fig. 14b) on a vertical line FE of which C is the midpoint. A draftsman's square whose rectangular corner is A can slide around making contact on its two perpendicular sides with pins F and C. The line joining A and E (which in a model could be a thread attached to the corner A of the square) is the direction of motion when the axis of the cylinder lies parallel to AC. The lines AB and CB in Fig. 14b are drawn parallel to AB and CB in Fig. 14a to reveal the geometry of the model. By observing the slope of AE to the line AB as the square is moved, it can be seen that the maximum angle of inclination of the flight path to the vertical is about 19 degrees.

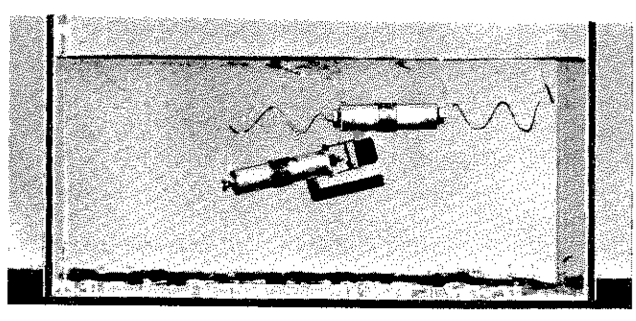
Self-Propelling Bodies

All the familiar types of self-propelling bodies such as airplanes, boats, or fish derive their thrust from the inertial reactions of air or water to their propulsive mechanisms. Even a swimming snake derives its propulsion by sending waves down its body so that each section of it contributes a forward force component by inertial reaction. The relevant Reynolds numbers for these cases are often as high as many millions. Even tadpoles, for which the Reynolds number is of order 10^2 to 10^3, derive their propulsion almost entirely from inertial reaction. On the other hand, microscopic organisms such as the bull sperms shown in Fig. 15, though they make motions like those of tadpoles, have such low Reynolds numbers (of the order 10^{-3} if the over-all size is used in defining Reynolds number or 10^{-5} if the diameter of the tail is chosen) that they cannot derive any appreciable thrust from inertia. They derive their forward thrust from viscous reaction due to oblique motion of thin tails, just in the way that has been demonstrated with an obliquely moving cylinder.

15. Bull spermatozoa. Each sperm is about .005 cm long.

To illustrate the difference between inertial and viscous propulsion, the two models shown in Fig. 16 were constructed. The "engines" of both consist of twisted rubber bands. In the lower one a tail organ





16. Mechanical swimming models in a vat of syrup.

is made to oscillate by means of a crank, and in the upper one two similar spiral wires, one right-handed and the other left-handed, are driven in opposite directions of rotation by the twisted band to the ends of which they are attached. When the oscillating-tail model is put into the water it swims well, because, as is well known in the case of a boat rudder, such a motion at a large Reynolds number gives rise to a backward flow, the reaction to which propels the boat. The spiral model propels itself much more slowly, because the area of the wires is much smaller than that of the oscillating blades so that it gets only a feeble grip of the water and produces a much smaller backward stream.

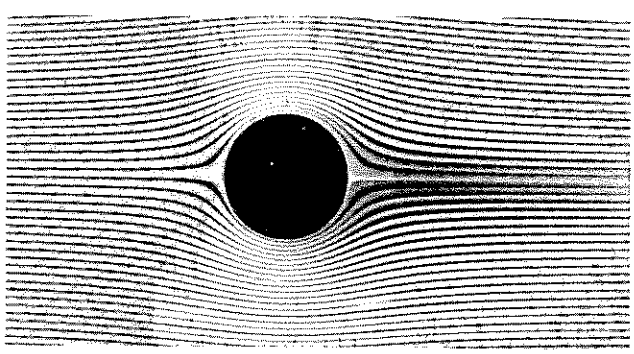
When the oscillating-tail model is wound up and put into viscous syrup, the tail waves backwards and forwards but produces no resultant motion because, owing to the reversibility of low-Reynolds-number flow, the forward motion of the blade is exactly neutralized by the backward motion when it returns through the same position.

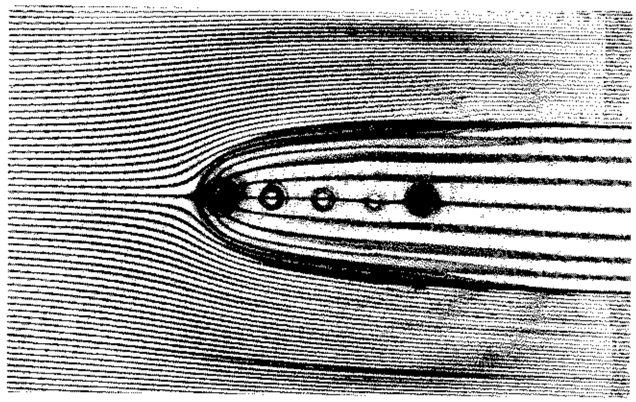
The spiral model swims when put into syrup, because every element of each spiral is behaving like the obliquely moving rod. Since lateral resistance is greater than longitudinal resistance, motion of every element at right angles to the axis of the spiral contributes a resultant longitudinal component.

Hele-Shaw Cell

This apparatus consists of two parallel rectangular glass plates fixed 0.020 inch apart. A viscous fluid is driven through it under pressure applied at one side, the two neighboring sides being sealed. Thus all particles move parallel to the sealed sides when there is no obstruction, though at speeds which depend on the distance from the plates. Colored fluid is injected at points along the injection side so that an observer sees a set of parallel straight lines when the flow is unobstructed. When an obstruction is placed in the cell, the streamlines spread out around it and join together again downstream. The obstruction shown in Fig. 17 is a circular disc of the same thickness as the distance of separation

17. Hele-Shaw flow past a circular disk.





18. Blurring of streamlines of a Hele-Shaw source and uniform flow when the source strength is varied.

of the plates. That the visible streamlines are sharply defined is due to two causes. The first is that motion is steady, so that dye particles follow one another along a fixed streamline. The second is that all particles on a line perpendicular to the sheet move in the same direction, though at different speeds, so that to an eye observing the pattern along a line perpendicular to the glass sheets the streamlines at all depths are superposed when the motion is steady. If the obstacle were to move or change its shape during the experiment, the visible lines of colored particles would no longer be superposed and would appear blurred. This effect is shown in Fig. 18 where the obstacle has been replaced by a source flow. Internal streamlines are marked with red dye emitted from eight small holes at a small radial distance from the source. As long as the rate of delivery at the source is constant the boundary of the fluid originating there is like a fixed obstacle and the streamlines around it are sharply defined. If the rate of delivery is changing, as it was when Fig. 18 was recorded, the colored streams are no longer sharply defined, but they re-establish their definiteness in a new position when the rate of delivery at the source becomes constant at a new value. The interest of the Hele-Shaw cell is that the streamlines observed in this low-Reynolds-number flow have exactly the same shape as those predicted theoretically for two-dimensional flow of a fluid with no viscosity and therefore infinite Reynolds number.

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