National Committee for Fluid Mechanics Films

FILM NOTES for DEFORMATION OF CONTINUOUS MEDIA*

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Introduction

The motion of a rigid body can be decomposed into translation of one point in the body and rotation. The motion of a deformable medium can be decomposed into translation of one material point, local rotation, and local distortion of shape, usually called the strain. Local rotation and distortion, together called deformation, is the subject of this discussion. First the deformation that takes place during a finite time will be examined; later the rate of deformation will be discussed. The discussion will be limited to flows at constant density and for the most part to two-dimensional flows (although the ideas carry over with little change to compressible and three-dimensional flows).

Understanding deformation is important to the understanding of the general kinematics of motion. In addition, from a dynamical point of view, deformation and its time derivatives determine the stresses in most continuous media.

Since the deformation of a typical small element of fluid is of primary interest in this film, a reference point at the center of the element will be selected and the relative motion of surrounding points then examined. The time rate of change of position of a line of these points can be represented by velocity profiles, and these will usually have length scales associated with them — related to their curvatures. If a small-enough segment is examined, however, the profile can be represented by straight lines; looking at a still smaller region then gives the same picture. By "local" is meant the largest neighborhood in which the profiles can be represented by straight lines. A local velocity profile will look something like that in Fig. 1 to an observer who himself moves with the reference point.

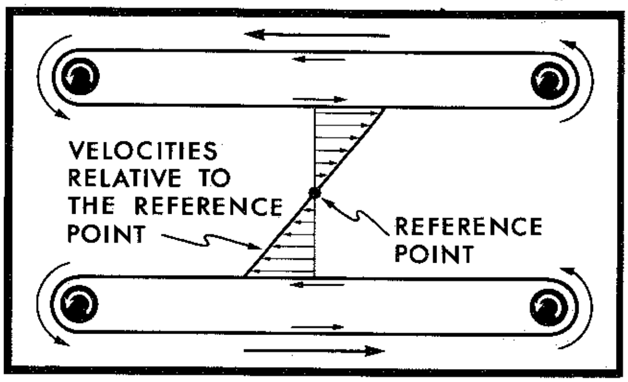

Figure 1

***DEFORMATION OF CONTINUOUS MEDIA, a 16-mm B&W sound film, 38 minutes in length, was produced by Education Development Center (formerly Educational Services Incorporated) under the direction of the National Committee for Fluid Mechanics Films, with the support of the National Science Foundation. Additional copies of the film notes and information on purchase and rental of the film may be obtained from the distributor:**

**Encyclopaedia Britannica Educational Corporation
425 No. Michigan Avenue, Chicago, Illinois 60611**

Characterization of the Flow

The film, DEFORMATION OF CONTINUOUS MEDIA, shows an apparatus consisting of two endless parallel belts immersed in a rectangular tank of glycerine. The belts move in opposite directions at equal speeds to produce a flow like that in Fig. 1. Glycerine is used because its high viscosity establishes the flow quickly, and in a short distance from the ends of the parallel section.

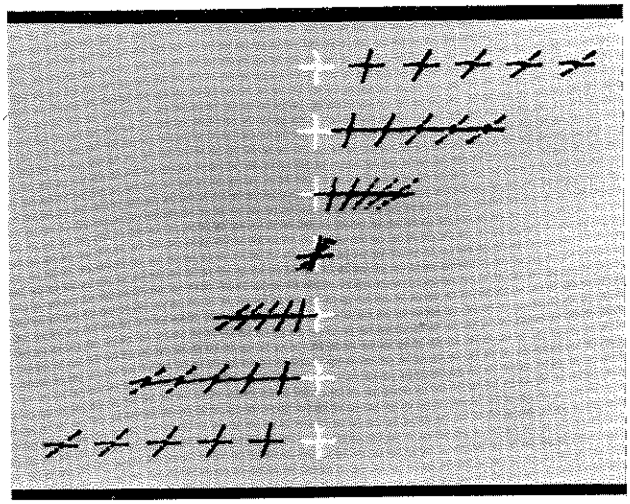

Figure 2

The surface of this flow can be marked with powder; the powder patterns move with the fluid. Figure 2 is a multiple-exposure (at equal time intervals) photograph of an initially vertical row of equally spaced crosses. Note that the point in the middle does not move — it can be taken as the reference point. The distances between successive images of the same point are equal, and the paths are straight lines; this is a steady flow with straight parallel streamlines. Since displacement is proportional to velocity in a steady flow, the velocity profile is evidently a straight line. Such a flow is called a steady rectilinear shear flow, or a steady homogeneous shear. Because straight lines remain straight, this flow evidently has no length scale — similar patterns will distort similarly, no matter what their size. This flow can be used as a model of the local flow.

The Strain Ellipse

To examine systematically the distortion of the fluid element whose center is at the reference point, a circle of neighboring points can be marked, equally distant from the reference and equally spaced (Fig. 3). The lines joining the fluid point at the center to each of the marked fluid points are fluid lines.

Figure 4 shows the pattern after 25 seconds; it is not difficult to show that this shape is an ellipse. It is called the *strain ellipse*.

Every other point has been numbered like the

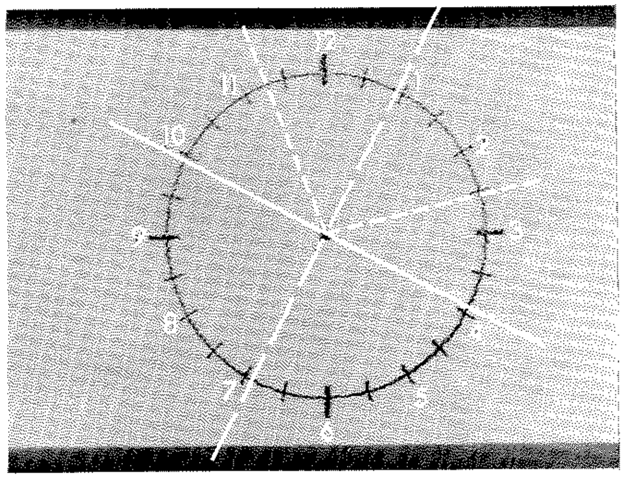

Figure 3

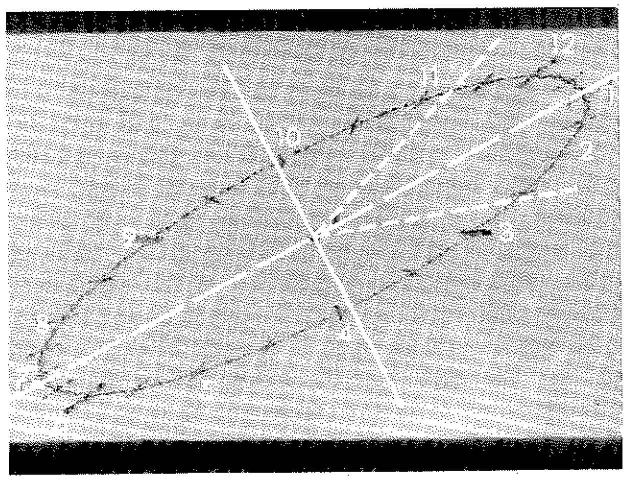

Figure 4

hours on a clock to help analyze the deformation. We will now examine what happens to the different fluid lines (or rays). Comparing Figures 3 and 4, it is evident that there has been both distortion and rotation of the rays, but of greatly variable amount. It is not obvious how to characterize the rotation or the stretching as a whole. For instance, different rays have rotated different amounts: twelve o'clock has rotated clockwise, while three o'clock has not rotated at all.

The Principal Axes

The axes of symmetry of the ellipse represent the directions of extreme strain (these are indicated on Fig. 4, with their initial locations on Fig. 3). That is, the point at the end of the major axis (dashed) has evidently moved farthest from the reference point, while that at the end of the minor axis (solid) has moved closest. These axes appear to be (Fig. 3) at right angles initially, in addition to being at right angles after the deformation.* Therefore they rotate equal

*This can be shown to be mathematically precise for the instants chosen. However, these particular fluid lines are not at right angles at intermediate times.

amounts. No other pair of lines originally perpendicular to each other remains so. These so-called *principal axes* can be used as a basis for the entire analysis of deformation.

Measurement of Rotation

In Figures 3 and 4 the initial and final locations of a pair of dotted lines on either side of the major axis are shown. Initially they are symmetrically placed with respect to the major axis. The line on the right evidently does not rotate as much as the major axis, while the one on the left rotates more. Since the lines are symmetrically placed with respect to the major axis after the rotation, the *average* of the rotation of these two lines must be the same as that of the major axis. All the points on the ellipse can be paired this way; hence, the principal axes rotate an amount that is the average of the amounts rotated by all the fluid rays emanating from the reference point. Specifying the angle through which the principal axes have rotated is thus a convenient way of specifying the average rotation of the deforming fluid element.

The Reciprocal Strain Ellipse

The numbered marks on the periphery of the circle are a somewhat artificial way of finding the initial position of the principal axes. It would be more satisfying to find an intrinsic way of marking the initial position. Such a way can be found by asking the question "What initial pattern will turn into a circle after the deformation?" Since running this flow backward is very much like running it forward, it can be anticipated that probably an ellipse with a backward orientation will turn into a circle.

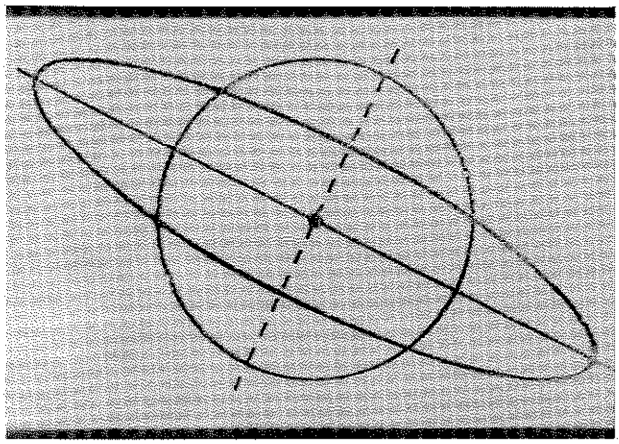

Figure 5

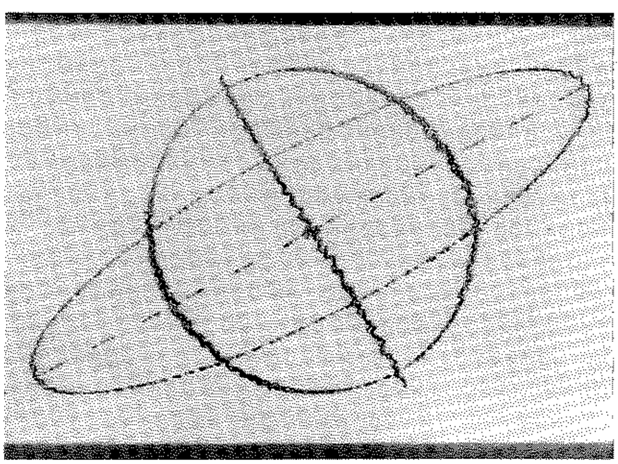

Figure 6

In Fig. 5 is shown a pattern consisting of the old circle plus a backward ellipse oriented about the initial position of the principal axes. Figure 6 shows what happens after deformation. The circle, of course, turns into the strain ellipse, and the backward ellipse turns into a circle. Notice that the axes of the strain ellipse have grown out of the axes of the backward ellipse. That is not surprising, since the minor axis of the strain ellipse marks the point that *has* moved closest, while the major axis of the backward ellipse marks the point that *will* move closest, and vice-versa. The backward ellipse is called the *reciprocal strain ellipse* and the initial principal axes are its principal axes. Thus, to discover the original principal axes of any strain ellipse, we merely flip it over into the reciprocal strain ellipse, and look at the principal axes of the latter. The angle between the minor axis of the reciprocal strain ellipse and the major axis of the strain ellipse gives the average angle of rotation during the deformation.

Properties of the Principal Axes

Two characteristics of the principal axes have already been noted. First, *they each rotate an amount equal to the average rotation of the fluid;* and second, *they lie in the directions of extreme strain.* There are two more important properties associated with deformation in the absence of rotation. Such deformation without rotation is called pure strain. To study it, a pattern based on lines parallel to the axes of the reciprocal strain ellipse is convenient.

Figure 7 shows such a pattern, and Fig. 8 a picture of the same pattern after deformation. During the deformation, the lines that were parallel remained parallel but the spacing changed. The angle between the two sets of lines also changed. While the deformation progressed, these sets of lines were not at right angles to each other. However, after the deformation the lines are at right angles again. It is evident that *the displacement of a point in the direction of one axis is not a function of position along the other axis.* This is a third important property of the principal axes. A line initially parallel to a principal axis is parallel to the same principal axis after deformation; note, however, that the distance between any two points on such a line changes during the deformation.

The principal axes can also be defined as those lines which undergo no shear deformation, since they are

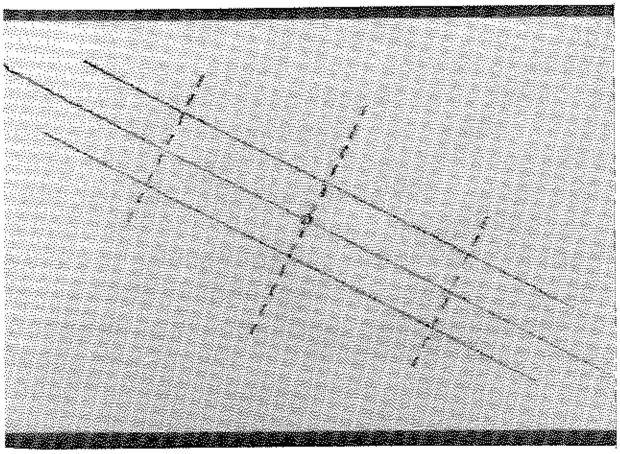

Figure 7

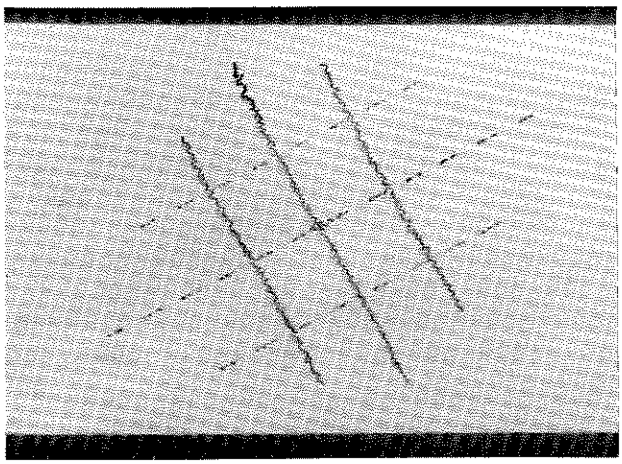

Figure 8

mutually perpendicular both before and after the deformation. In Fig. 9, three sets of mutually perpendicular lines are shown, the principal axes (A, B) and two other crosses. During the deformation all the crosses were sheared, but after deformation (Fig. 10) the principal axes (A, B) are again at right angles, while the other pairs of lines, which also started out at right angles, are no longer mutualy perpendicular. This is the fourth major characteristic of the principal axes: *principal axes are the only pair of lines that are mutually perpendicular before and after the deformation.*

Figure 11 is a multiple-exposure photograph taken of the patterns at equally spaced times. The reciprocal strain ellipses corresponding to each of these instants can be formed in this flow by reversing Fig. 11. Such a mirror-image reversal has been superimposed on Fig. 11 to make Fig. 12. (The major axes of the successive strain ellipses and the minor axes of the corresponding reciprocal strain ellipses have been indicated for future reference.) It is evident that as the strain ellipse rotates one way, the reciprocal strain ellipse rotates the other way, so that the initial location of material points on the principal axes is different at every instant. Put another way, this means that the principal axes are made up of different material points at every instant.

To describe the deformation as a function of time it is necessary to give the rotation, and the strain, at every time and for every neighborhood in the fluid. According to the description developed above, this

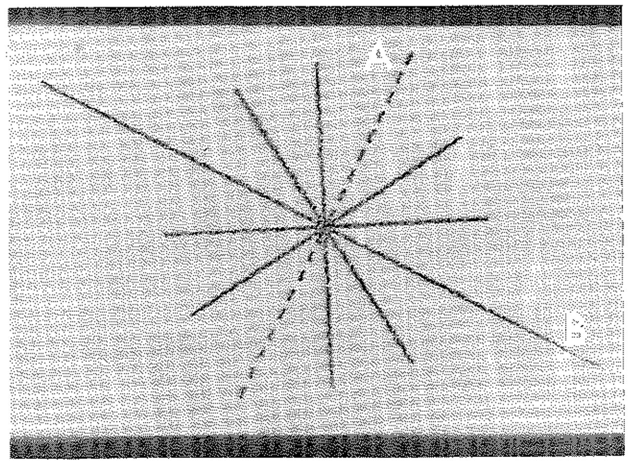

Figure 9

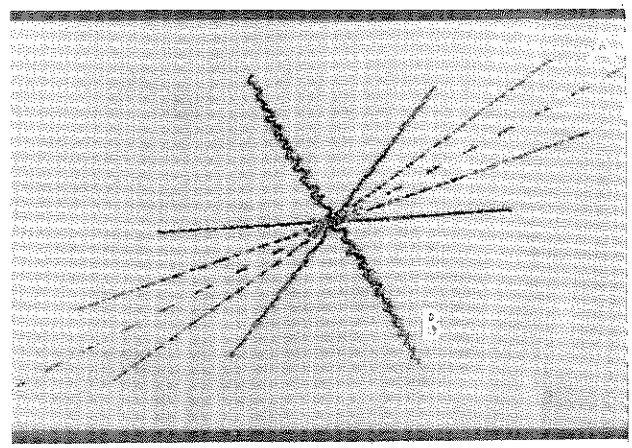

Figure 10

would be in terms of the present location of the points as a function of their initial position. This is called a *Lagrangian* specification. The Lagrangian specification is of great interest as a description of deformation in a solid, and it is also sometimes of interest in a fluid. In a solid the material is tied together and neighboring material points can never get very far away from each other. In a fluid, however, there is no primeval undeformed state to serve as a reference; in an unsteady inhomogeneous flow, the relation between the present state and the initial state can be extremely complicated. For that reason the deformation in a fluid is usually described in terms of the rates of deformation at a particular place, at a particular time, using the present state as a reference; that is, the rate of change from the present state, the rate of change of the rate of change, and so on. Such a specification, involving only laboratory coordinates, is called an *Eulerian* specification. The number of deriv-

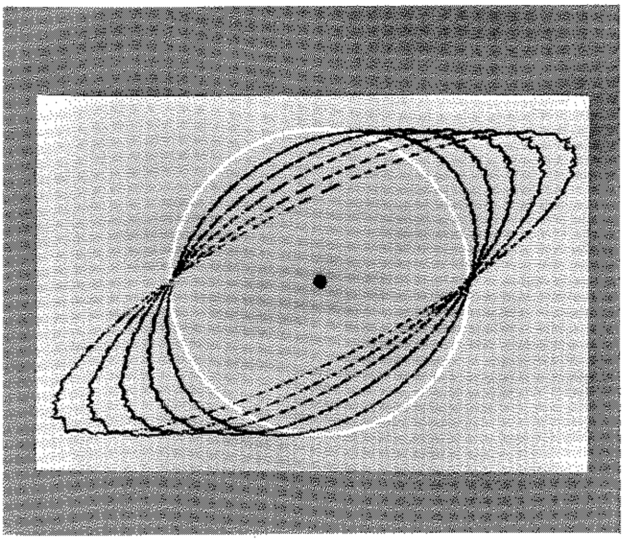

Figure 11

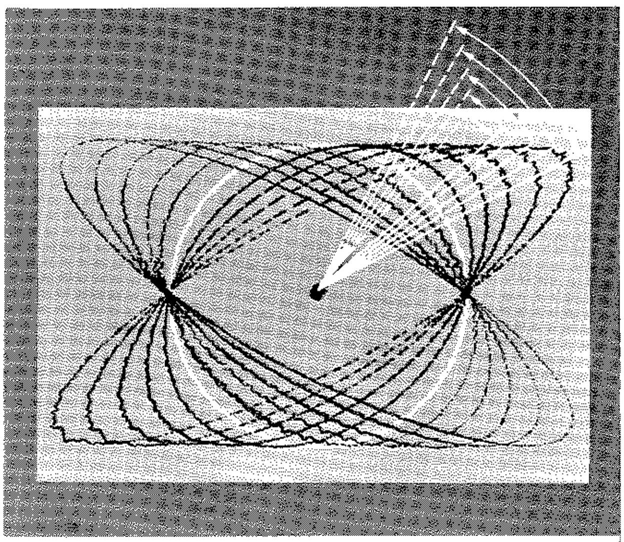

Figure 12

atives that must be determined depends on how complete a specification is needed, and that depends on the use to which it will be put. Incidentally, a very important reason for describing deformation is because stress depends on it. In many fluids stress is quite accurately described by a function only of the present rate of deformation, as for example in those fluids obeying the Navier-Stokes equations (i.e. Newtonian fluids). Our discussion will be limited to *deformation rate.*

Rate of Rotation

To analyze the present rate of deformation, "the initial instant" of our previous discussion will always be taken as the *present,* that is, the instant of special interest.

The position of the principal axes at the initial instant in this flow can be determined by an examination of Fig. 12; as time is successively closer to zero, the major axis of the strain ellipse and the minor axis of the reciprocal strain ellipse not only approach each other, but both approach 45 degrees. Forty-five degrees and 135 degrees are evidently the initial positions of the principal axes.

Since the amount of rotation at any instant can be measured by the angle between the principal axes of the strain ellipse and those of the reciprocal strain ellipse, a picture like Fig. 12 could be used to infer the rotation rate at the initial instant; it would be necessary to measure the angle between the major axis of the strain ellipse and the minor axis of the reciprocal strain ellipse as a function of time from the initial instant, then to differentiate to obtain the *rate* at the initial instant.

Since the principal axes have at each instant an angular displacement that is the average of that of all the other fluid lines, they have an angular velocity which is the average of that of all the other fluid lines. Thus their rotation rate is related to the moment of momentum, or angular momentum, in the fluid. The *vorticity* in the fluid is exactly twice the rotation rate of the principal axes. The averaging property suggests a simple experiment for exhibiting the average rotation in the fluid.

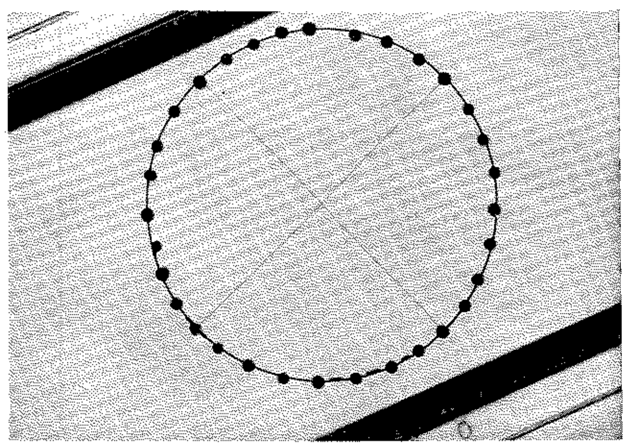

Figure 13

The floats shown in Fig. 13 support a rigid wire ring just above the surface; they are small enough and far enough apart not to seriously affect the flow. The ring may be expected to rotate at the average angular velocity of the rays to each of the floats, because in this situation the clockwise and counterclockwise drags due to the relative motion will be equal. This rotation can be removed by rotating the frame of reference — that is, by rotating the camera — so that the crosshairs on the circle are stationary. The walls of the channel then appear to revolve in the opposite direction.

The Deformation Relative to a Rotating Frame

Having thus picked the correct average angular velocity of the fluid by the foregoing experiment, the wire ring can be removed and a pattern placed on the flow that consists just of the lines that will become the principal axes at the initial instant. Viewed in the rotating framework, Figures 14 and 15 show the pattern slightly before and slightly after the initial instant. At first the figure's major axis is in the 135° direction. The ellipse contracts in this direction, reaches a circle, and expands to an ellipse with major axis in the 45° direction. We can turn the pattern

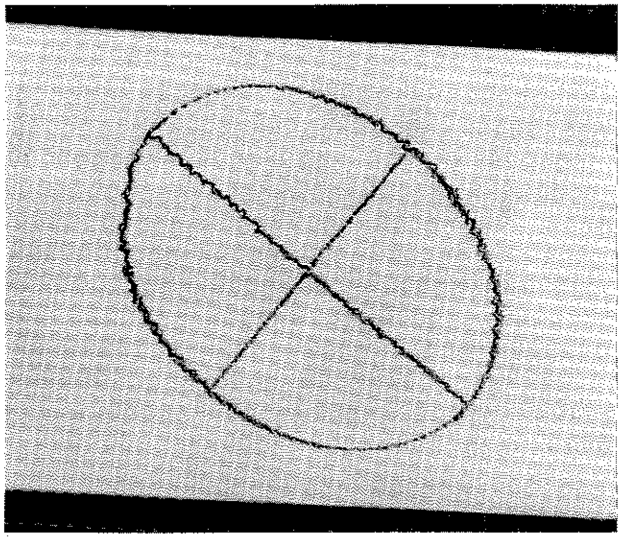

Figure 14

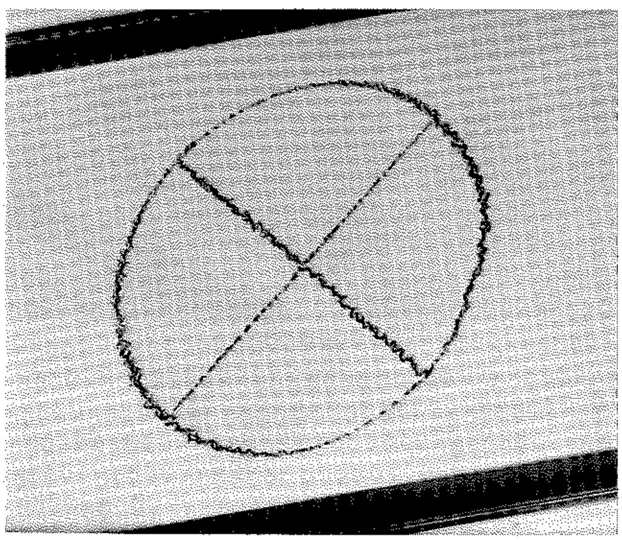

Figure 15

so that the 135° and 45° lines — the principal axes at the initial instant — can be used as coordinate axes. It is only necessary to start the rotation at a different time, relative to the time at which the fluid is marked, and to view the motion in the rotating frame previously established by the wire-ring experiment. Figure 16 (slightly before the initial instant) shows the bulge coming in slightly to the right of the new ordinate, but approaching the ordinate as the figure approaches a circle. As the figure passes out through the circle

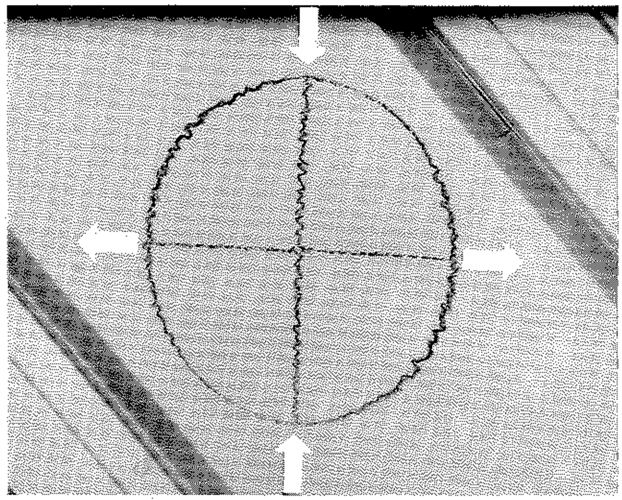

Figure 16

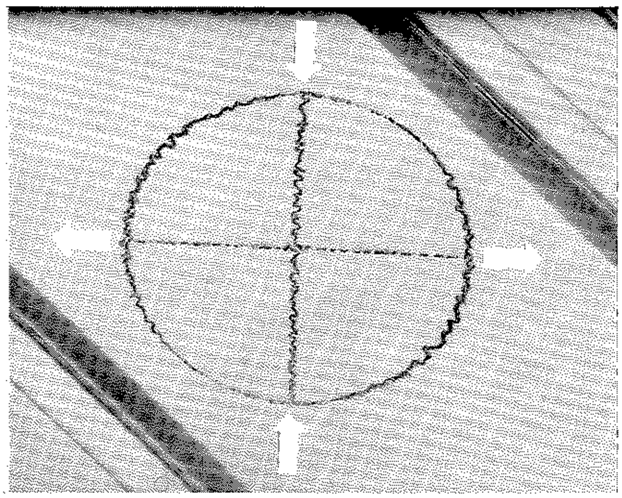

Figure 17

(Fig. 17 — slightly after the initial instant) the bulge moves out along the abscissa.

At the instant that the figure passes through the circle, the velocity is inward along the ordinate and outward along the abscissa. Since in the rotating frame of reference the velocity *is* precisely inward along one axis and outward along the other, it is obvious that *the principal axes are the only pair of lines that are not changing direction (relative to one another) at the instant of interest;* this is why this pair of lines can be used to measure the rate of rotation of the fluid region. In addition, at the instant of interest, it is evident that these axes are in the direction of the maximum rates of stretching and shrinking. These axes are called the *principal axes of strain rate.*

Properties of the Principal Axes of Strain Rate

In this rotating reference frame three of the properties of the principal axes have already been seen. At the instant of interest, (1) *they mark the directions of extreme strain rate;* (2) *they rotate with the aver-*

age angular velocity of the fluid; and (3) *they are not only mutually perpendicular instantaneously, but they are the only pair of lines whose included angle is not changing, and therefore they are the only axes for which the shear strain rate is zero.* By analogy with finite strain, a fourth property of these axes may be

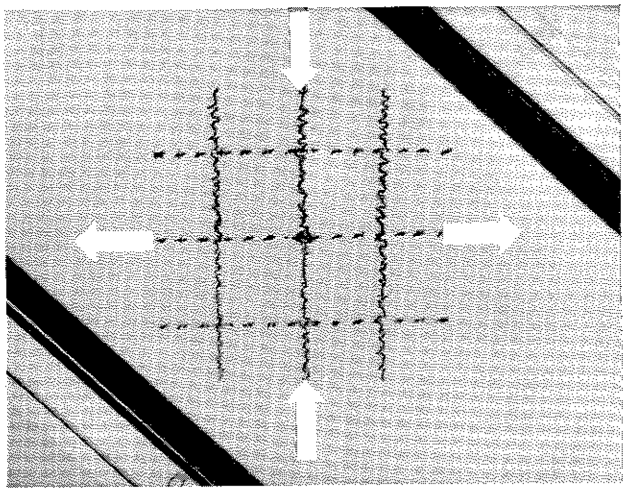

Figure 18

expected. This was seen before by putting lines parallel to the axes. Figure 18 shows such a pattern at the instant of interest. The lines remain parallel as they move toward or away from the axes. This means that the *velocity parallel to one principal axis is not a function of position relative to the other principal axis.*

Conclusions

The analysis of deformation rate is similar to the analysis of deformation itself. They can both be resolved into a rotation and a strain. In both there are mutually-perpendicular principal axes that serve as *average representatives of the angular velocity,* and that *lie in the directions of the extreme stretch,* and *these directions are not changed by the stretch.* In addition, *the motion in the direction of one axis is not a function of position relative to the other axis.* There are differences, of course. The principal axes of strain continually change direction as the deformation progresses. The principal axes of initial strain rate only occupy one position. Generally speaking, deformation rate is easier to describe mathematically than deformation, but that is because an Eulerian specification can be used. The fact that it is easier is not surprising, because it deals only with the first derivative of the deformation at the initial instant. Fortunately, to describe the stress in Newtonian fluids, deformation rate is all that is needed.

To extend these observations to three-dimensional situations, few changes are necessary. Direct and reciprocal strain ellipses become ellipsoids. The velocity profile in a very small region of an incompressible fluid is a plane instead of a line. There are three mutually-perpendicular principal axes for which the strain is purely lineal; two of these axes have the extrema of lineal strain. The reciprocal strain ellipsoid cannot be obtained by simply turning over the strain ellipsoid, but it is defined in an analogous way, i.e. as the surface that turns into a sphere. Deformation can still be analyzed into rotation and strain; deformation rate into angular velocity and strain rate. The principal axes still represent the average rotation. They still are in the directions that are not changed by the stretching. Strain rate in one principal-axis direction is still not a function of position along the other two axes.

No fundamental changes are required to include compressibility; it is necessary only to allow the area (or volume) of the pattern to change with time. The conclusions are the same.

Reference

Long, Robert R., *Mechanics of Solids and Fluids*, Prentice-Hall, Englewood Cliffs, N.J., 1961.