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National Committee for Fluid Mechanics Films

FILM NOTES for EULERIAN AND LAGRANGIAN DESCRIPTIONS IN FLUID MECHANICS*

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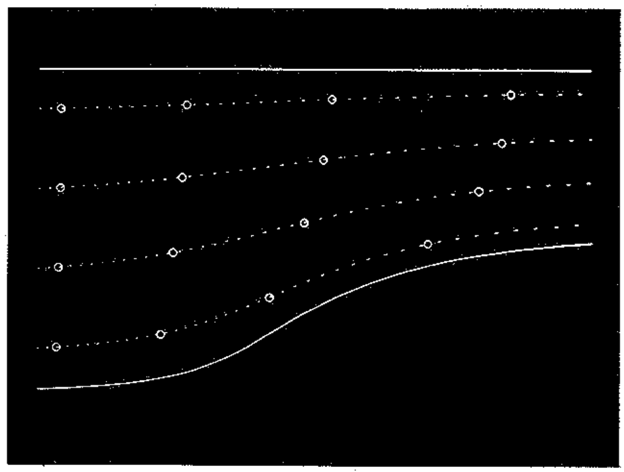
Introduction

In order to calculate forces exerted by moving fluids and to calculate other effects of flows, such as transport, we must be able to describe the dynamics of flow mathematically. To discuss the dynamics, we have to be able to describe the motion itself. The description of motion is called kinematics. We are interested in the kinematics of continuous media, that is, in describing the motion of deformable stuff that fills a region. Specifically, we are interested in describing the displacement, velocity, and acceleration of material points in two kinds of reference frames commonly used in fluid mechanics. We will show how these two descriptions are related to one another.

In addition to moving from place to place, an elementary piece of fluid (a piece small compared to the flow field) is usually distorted and rotated as it goes. Here we focus our attention on the translation.**

** Deformation is dealt with in the NCFMF film DEFORMATION OF CONTINUOUS MEDIA.



1. Computer simulation of steady flow in a contracting channel. The open circles mark moving material points. The dashed lines are the pathlines of the particles.

Figure 1 shows a computer simulation of the flow of water through a contraction. Note that a few typical material points are identified by open circles; we adopt this convention throughout.

* *EULERIAN AND LAGRANGIAN DESCRIPTIONS IN FLUID MECHANICS, a 16-mm B&W sound film, 27 minutes in length, was produced by Education Development Center under the direction of the National Committee for Fluid Mechanics Films, with the support of the National Science Foundation. Additional copies of the film notes and information on purchase and rental of the film may be obtained from the distributor:*

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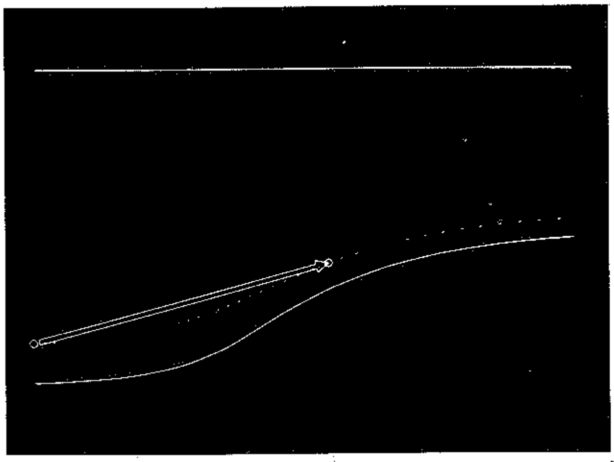
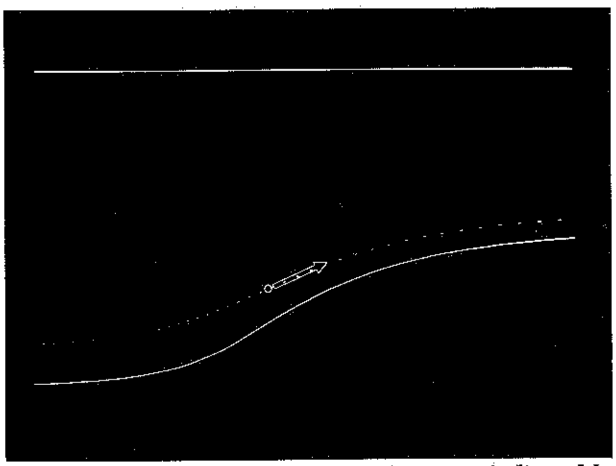
Lagrangian Description

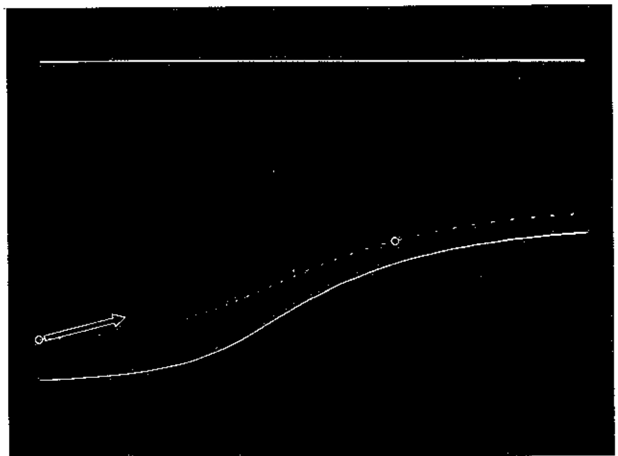
In elementary mechanics, we are accustomed to describing the position of a material point as a function of time, using a vector drawn from some arbitrary location to indicate the displacement. We will use open vectors, (Fig. 2), to indicate velocity and displacement relating to the material points. In a given motion, we can compute velocity and acceleration of such a point at each instant. In Fig. 3, we indicate the velocity by a vector attached to the point. In a continuous fluid, of course, we have an infinity of mass points and we have to find some way of tagging them for identification. A convenient way, though not the only one, is to pick some arbitrary reference time (which we will call the initial time) and identify the material point by its location at the time. Mathematically, we would say that the velocity is a function of initial position and time. To accord with this description, in Fig. 4 the vector is shown attached to the initial position. We could show the vector attached to the moving point, or use both, if we were displaying the motion of a group of points whose vectors do not interfere with one another (Fig. 5). To display the whole motion, and in more complicated situations, we avoid interference by showing the vector only at the initial location, as in Fig. 6. To describe the whole



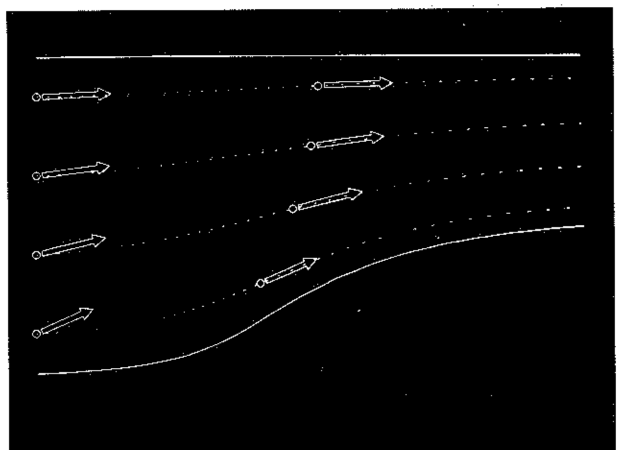
2. The open vector indicates *displacement* of a material point from its initial location.



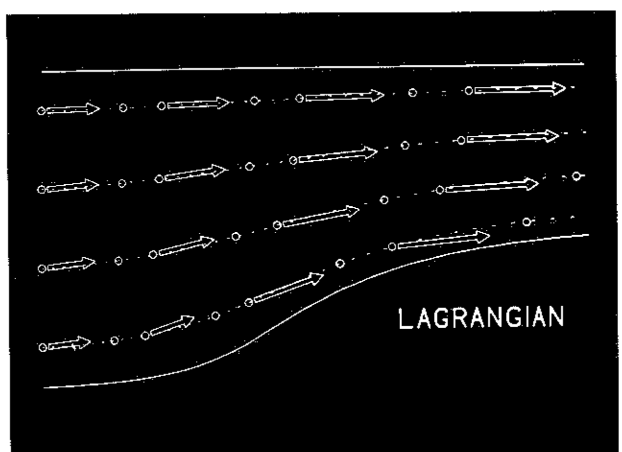
3. The velocity of a moving material point is indicated by an open vector attached to it.



4. The velocity of a moving material point is here indicated by an open vector attached to its initial position.



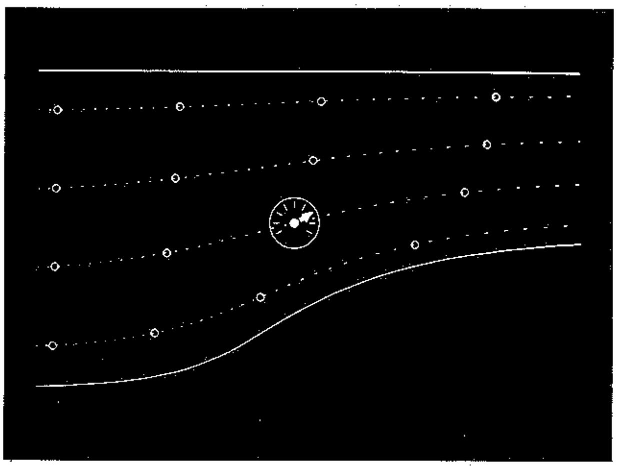
5. Velocity vectors displayed both at the moving points and at their initial locations.



6. The velocities of material points distributed throughout the flow field are displayed at the respective initial positions of the points.

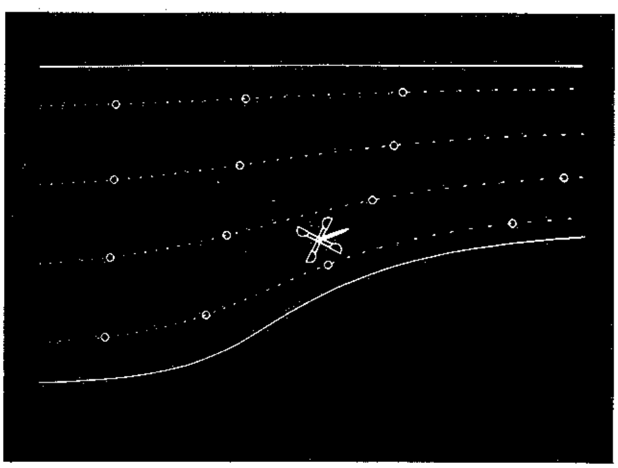
motion, we would have to give the velocity of all the pieces of matter in the flow as a function of time and initial position.

Such a description, in terms of material points, is called a *Lagrangian* description of the flow. The identifying co-ordinates are called *Lagrangian*, or sometimes *material*, co-ordinates. Given the Lagrangian velocity field, we can easily calculate the Lagrangian displacement by integration in time, and the acceleration field by partial differentiation with respect to time.



7. A pressure gauge is attached to one of the moving points.

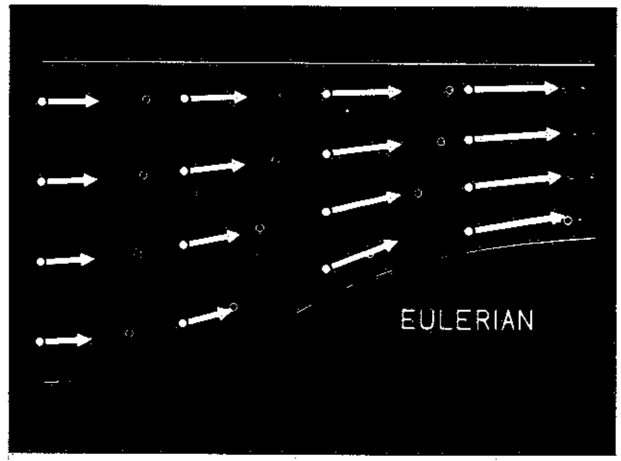
To make what we might call a Lagrangian measurement, we can imagine attaching an instrument like a pressure gauge to a fluid material point (Fig. 7). This sort of measurement is attempted in the atmosphere with balloons of neutral buoyancy. If the balloon does indeed move faithfully with the air, it gives the Lagrangian displacement, i.e. the displacement of an identified fluid "element." Such Lagrangian measurements are actually very difficult, particularly in the laboratory. We usually prefer to make measurements at points fixed in laboratory co-ordinates; it is relatively easy to hold an instrument at a fixed location.



8. An anemometer is placed at a fixed position in the flow field.

Eulerian Description

Classically, the idea of a field, such as an electric, magnetic, or temperature field, is defined by how the response of a test body or probe, like the anemometer in Fig. 8, varies with time at each point in some spatial co-ordinate system. In Fig. 8 the fixed anemometer probes in laboratory co-ordinates. We will always use



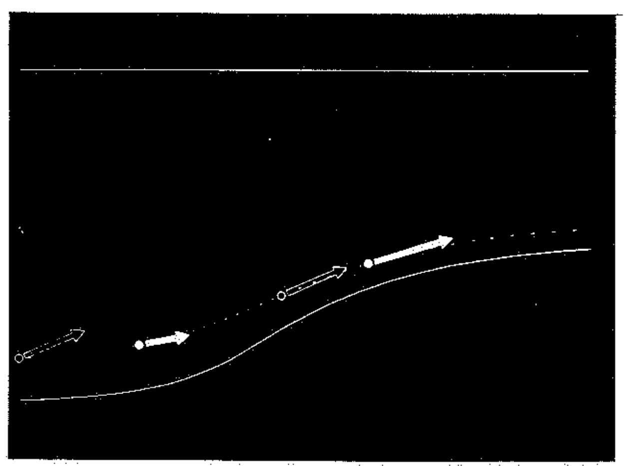
9. The solid vectors indicate the velocities at fixed points in the contraction.

solid points and solid arrows to indicate such probing positions, fixed in our laboratory, and the velocities measured there.

In Fig. 9 we have a grid of points fixed in space with an arrow at each to indicate the velocity at each point. A description like this which gives the spatial velocity distribution in laboratory co-ordinates is called an *Eulerian* description of the flow.

Relation Between Eulerian and Lagrangian Frames

Although the physical field is the same, the Eulerian and Lagrangian representations are not the same, be-

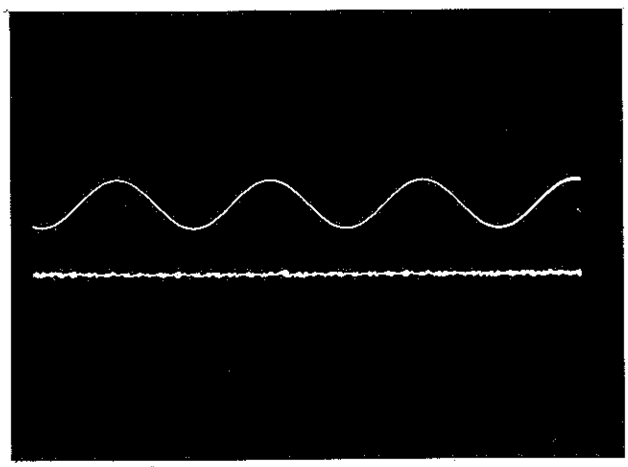


10. The open vectors indicate the Lagrangian velocity of the moving material point. The solid vectors indicate the Eulerian velocities at two fixed points. When the moving point coincides with a fixed point, the Eulerian and Lagrangian velocities coincide.

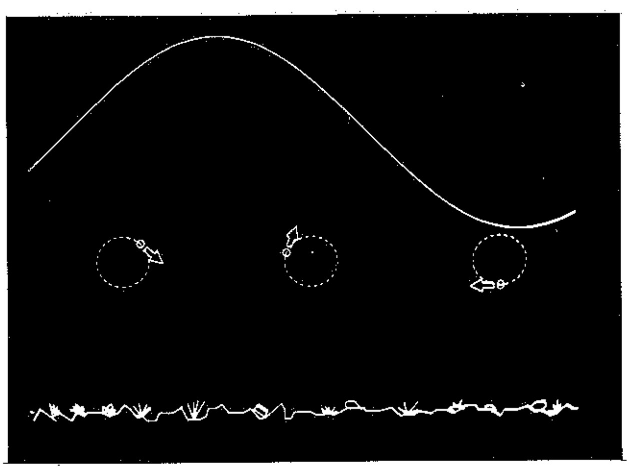
cause the velocity at a point in laboratory co-ordinates does not always refer to the same piece of matter. Different material points are continually streaming through the same laboratory point. The velocity that a fixed probe indicates is the velocity of the material point that is passing through the laboratory point (probe location) at that instant (Fig. 10).

Change of Reference Frame

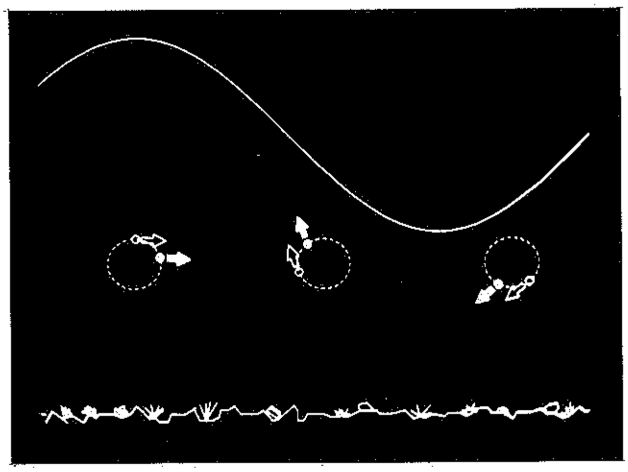
A possible advantage of laboratory co-ordinates (and their Galilean transformations, which are also Eulerian) is that the Eulerian field may be steady in one of these frames. This is illustrated by the case of a simple surface wave. Figure 11 is a computer simulation of the flow under a free-surface gravity wave. To make things clearer, the wave amplitude has been rather exaggerated. Figure 12 is a close-up of the same flow, showing moving material points, and their pathlines. The Lagrangian velocities of the moving points are indicated by arrows attached to the points. In any flow, the Lagrangian field can only be steady if each material point always experiences the same



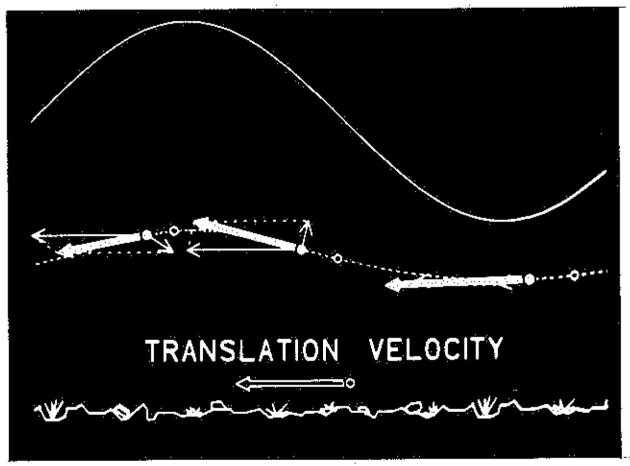
11. Computer simulation of a free-surface gravity wave which moves from left to right.



12. The dashed lines show the pathlines of material points under a moving wave. The open vectors indicate the Lagrangian velocities.



13. The solid arrows show the Eulerian velocities at points fixed in the laboratory frame of reference. (The Eulerian velocity vectors rotate about the fixed points as the waves pass.)



14. The Eulerian velocities (solid arrows) as seen from a laboratory reference frame moving to the right at the wave speed. In this frame, the fluid has an equal velocity directed to the left, which adds with the material-point velocity in the original frame to produce the resultant Eulerian velocities. The dashed line is the pathline of the particles.

velocity. This degenerate case happens only in a steady parallel flow.

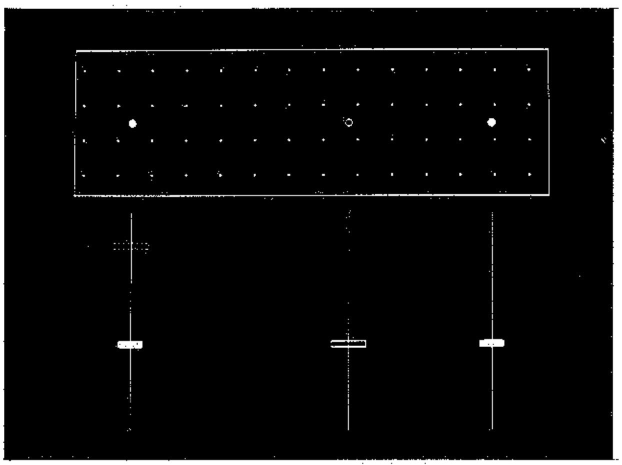
Figure 13 shows the Eulerian description. In this wave motion neither the Eulerian nor the Lagrangian description is steady. In fact, they have an identical appearance. However, in this flow, if the laboratory frame is moved with the wave speed, the Eulerian pattern will become stationary. Figure 14 illustrates the result; the translation velocity is indicated by an arrow at the bottom. The velocities have been resolved into components: one component is the velocity with which the laboratory frame is translating; the other component is the material point velocity in the original frame of reference.

The pathlines are also streamlines in this frame of reference, since the flow is steady. The pathlines resemble the form of the free surface. As the material point passes through each laboratory point, its velocity is instantaneously the same as that of the laboratory point. It is partly this possibility of eliminating time as a variable that makes the Eulerian representation attractive.

Most laws of nature are more simply stated in terms of properties associated with material elements, that is, quantities described in Lagrangian frames. But it is nearly always much easier mathematically, when describing a continuum, to deal with these laws in laboratory co-ordinates. Thus, to write the conservation equations of fluid mechanics, one must be able to transform from one set of co-ordinates to the other. We will discuss first the relation between time derivatives of a scalar field in a flowing fluid, in the two types of co-ordinates.

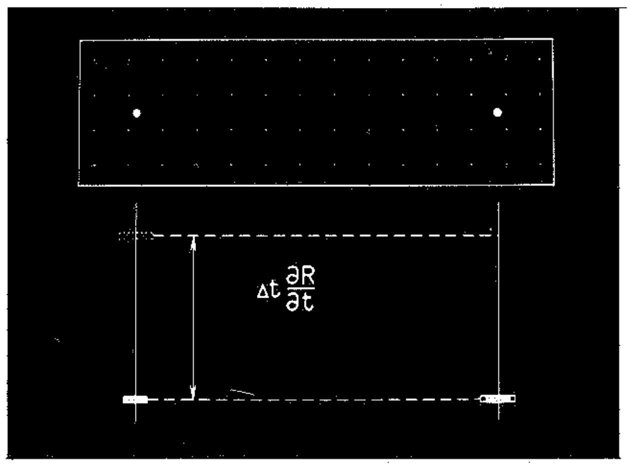
Material Derivative in a Scalar Field

Let us imagine a river in which a radioactive tracer is suddenly and uniformly distributed. Since derivatives measure local changes, let us look at an infinitesimal part of this river. The dots in Fig. 15 symbolize



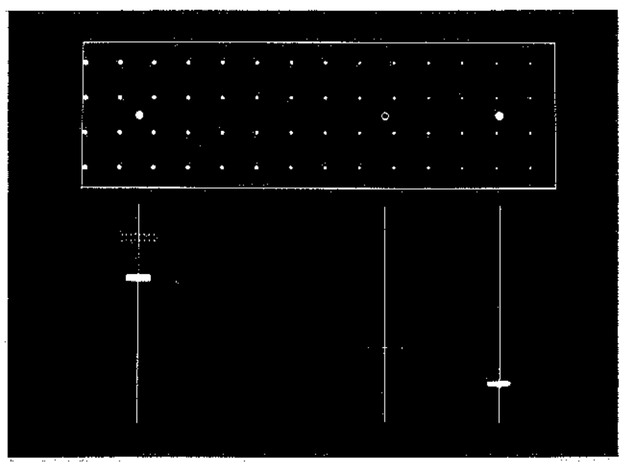
15. An expanded view of an infinitesimally small rectangular area in a river. The dots represent decaying radioactive tracer particles which are uniformly distributed there. The solid circles are laboratory points; the solid bars on the counters below indicate the level of radioactivity at these fixed points. The open circle is moving on a streamline from one laboratory point to the other; the open bar below indicates the level of radioactivity experienced by the moving point.

the tracer which is gradually decaying everywhere. The filled-in circles represent two fixed ("laboratory") points which are infinitesimally close together on the same streamline: they only appear to be far apart as a result of our expanded view. Since in this case the tracer was distributed uniformly, the radioactivities at the two laboratory points are the same, but are changing with time. Radiation counters are indicated at the laboratory points. The solid bars on these Eulerian radiation counters show the levels of radioactivity at the two laboratory points. The level experienced by a material point traveling from one laboratory point to



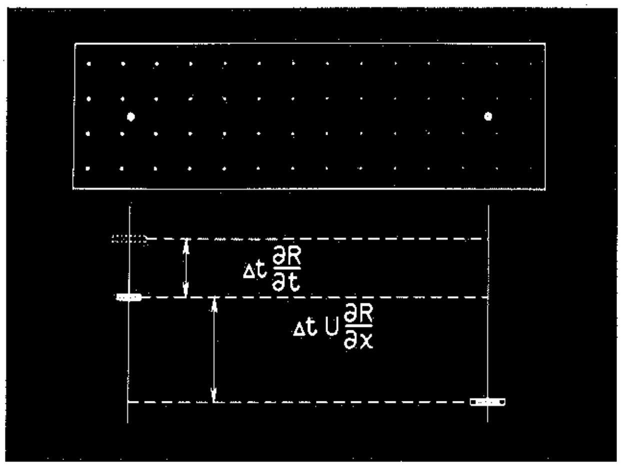
16. Situation at the instant the moving material point coincides with the right-hand laboratory point. $\Delta t \frac{\delta R}{\delta t}$ is the total change since it coincided with the left-hand point.

the other is monitored by watching the open bar on the Lagrangian counter carried by it. The dashed bar represents the value recorded by the Lagrangian counter as the material point passed through the left-hand laboratory point. From the before and after values of the Lagrangian counter (Fig. 16) it is evident that the traveling point sees just the same change that each of the laboratory points sees. This can be written as the time difference multiplied by the rate of change with time, as indicated on the figure.



17. Now the tracer intensity is greater upstream (as shown by larger dots there) and falls off downstream.

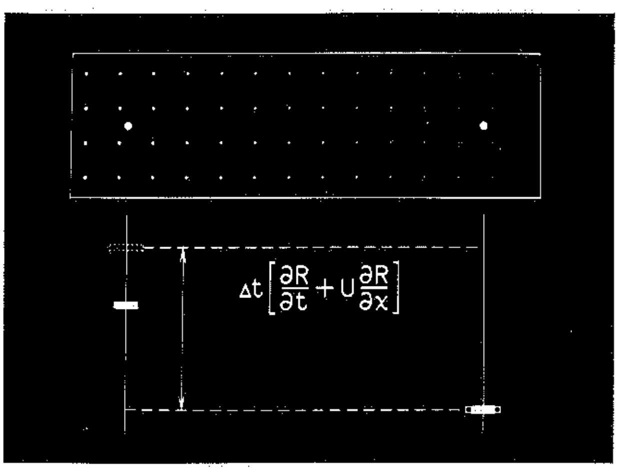
If the tracer is not uniformly distributed, but instead has greater intensity upstream (Fig. 17), both intensities decrease with time as before. Just as before, the only change experienced by a material point is due to decay. The change seen at a fixed laboratory point is not, however, since new material of originally higher intensity is being swept past. To express the change experienced by a material point, but in Eulerian variables, we need two terms (Fig. 18): (1) the change of



18. The total change experienced by a material point as it travels from one laboratory point to the other is the sum of (1) the change with time at either laboratory point (the upper expression), and (2) the intensity difference between the laboratory points at a fixed time (the lower expression).

intensity with time at a fixed point, and (2) the intensity difference between neighboring laboratory points at a fixed time. The total change when the material point has reached the right-hand laboratory point is given by the difference in level between the dashed counter on the left and the Lagrangian counter. The change with time experienced by either laboratory point (they have only infinitesimal separation) is given by the difference in level between the dashed counter and the Eulerian counter on the left, and can be written, as before, as the time difference multiplied by the spatially local rate of change with time.

The change due to the intensity difference between the laboratory points at any time is indicated by the difference in level between the two Eulerian counters, and can be written as the distance traveled multiplied by the spatial gradient in the direction traveled. The distance traveled can be written as the time difference



19. The total change is the sum of the two expressions in Fig. 18. The expression in brackets is the *material derivative*.

multiplied by the magnitude of the velocity. The total change (Fig. 19) is the sum of the two changes described. *Material,* or *substantial,* derivative is the name given to the expression multiplying the time difference in Fig. 19. This is simply the time rate of change experienced by the material point as it passes the laboratory point, *expressed in laboratory co-ordinates.* The importance of this point cannot be overemphasized. Since this derivative operator occurs in every Eulerian conservation equation, we often give it a special symbol in fluid mechanics:

$$\frac{DR}{Dt} = \frac{\delta R}{\delta t} + U\,\frac{\delta R}{\delta x}$$

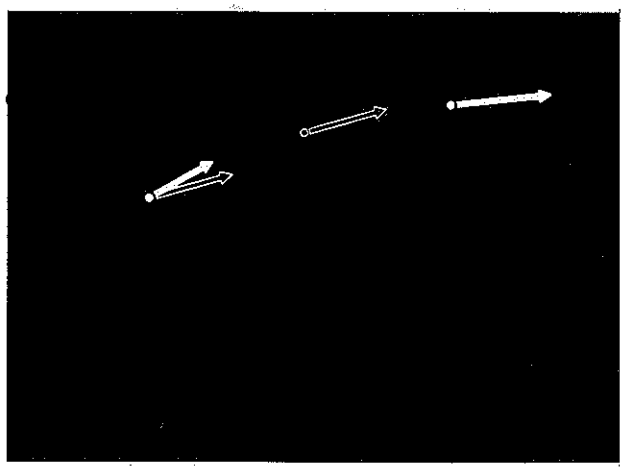
In vector notation, the velocity times the gradient in its direction can be written as the scalar product of velocity and the gradient vector:

$$\frac{DR}{Dt} = \frac{\delta R}{\delta t} + (\mathbf{U}\cdot\nabla)R.$$

Material Derivative in a Steady Vector Field

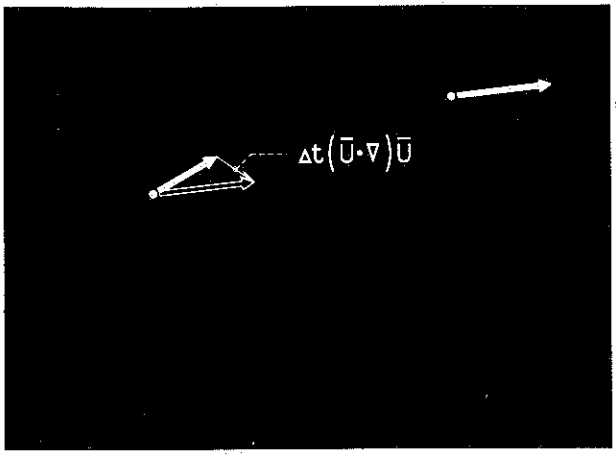
We are also interested in the material derivative of a vector field, such as the velocity, particularly because the material derivative of the velocity expresses the acceleration in a form which we need for the momentum equation in an Eulerian frame.

The expression deduced for the material derivative of a scalar field is correct for each component of a



20. The solid circles are fixed laboratory points infinitesimally close together in a magnified view of an arbitrary steady flow. The solid vectors indicate velocities at the laboratory points. The open vectors indicate the velocity of the moving material point (the open circle).

vector field, but we can also operate on the vector field directly. The two laboratory points in Fig. 20 are an infinitesimal distance apart on the same pathline. The material point travels from one to the other. The material point velocity is indicated by open arrows attached to it, and to its "initial" location, the left-hand laboratory point. Although the flow is steady in the laboratory frame, the moving material point ex-

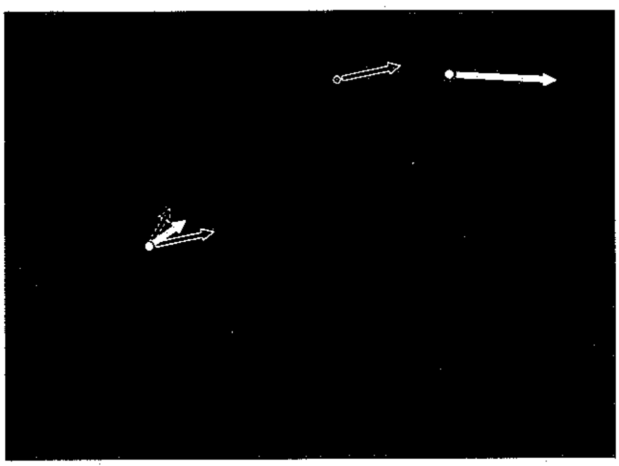


21. When the material point arrives at the right-hand laboratory point, the change in velocity it has experienced is the difference between the Eulerian (solid) vector and the Lagrangian vector at the left-hand laboratory point.

periences change as it travels through regions where the steady velocity is different. The total change is simply the difference between the velocities at the two laboratory points, indicated by the solid Eulerian vectors. The difference between the Eulerian vector and the Lagrangian vector at the left-hand point gives at each instant the change that the material point has experienced. The total change (Fig. 21) when it arrives at the right-hand point, a vector distance Δr away, after a time Δt, is the vector distance traveled times the gradient of the velocity. The distance traveled is just the time difference times the velocity.

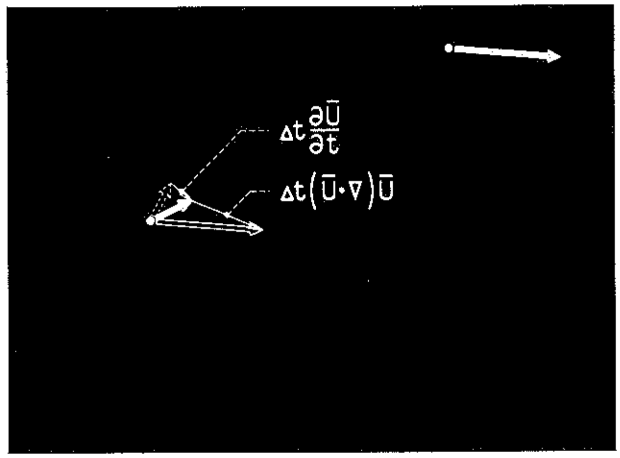
Material Derivative in an Unsteady Vector Field

If the velocity of the entire flow changes with time, the Eulerian vectors (at fixed laboratory points) also change with time (Fig. 22). For clarity, we include as



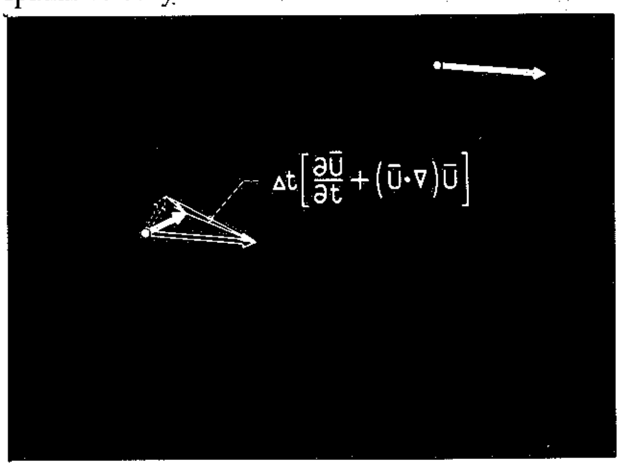
22. The velocity field is here changing with time. The dashed vector shows the initial value at the left-hand laboratory point; the solid vectors show the values at the fixed laboratory points; the open vectors show the velocity of the moving material point.

a dashed vector the initial value of the left-hand Eulerian vector, in addition to placing the Lagrangian vector at the left-hand point. When the material point arrives at the right-hand laboratory point (Fig. 23) the total change it has experienced is the difference be-



23. The material point has arrived at the right-hand laboratory point. The total change has two parts. $\Delta t\ \frac{\delta U}{\delta t}$ is the temporal velocity difference; $\Delta t\ (U\ \cdot\ \nabla)\ U$ is the spatial velocity difference.

tween the dashed vector and the Lagrangian vector. But this can be broken into two parts: the difference between the velocities at the left and right-hand laboratory points at this instant is given by the difference between the Eulerian and Lagrangian vectors on the left. The change each laboratory point has undergone during this time is given by the difference between the dashed and the Eulerian vectors on the left. The spatial velocity difference can be written as before as



24. The total change is the vector sum of the two components in Fig. 23. The *material derivative* is the expression multiplying the time difference.

the time difference times the velocity times the gradient of the velocity. The temporal velocity difference can be written as the time difference times the rate of change with time at a laboratory point. The total change is the vector sum of the two effects (Fig. 24).

The *material* (or *substantial*) derivative is just the expression multiplying the time difference. This is the rate of change seen by the material point as it passes the laboratory point, written in laboratory co-ordinates. The acceleration, more simply written in a Lagrangian frame, has been expressed in an Eulerian frame.

Summary

To summarize: we can "tag" the material points in a flow by using their locations at some reference time, and then give their displacements, velocities, and accelerations as functions of time and initial positions. This is called a *Lagrangian* description. Alternatively, we can choose a "laboratory" co-ordinate system arbitrarily, and probe to find the displacement, velocity, and acceleration at points fixed in that system. This is called an *Eulerian* description, and it has the advantage that some fields are steady in a correctly-chosen frame of this type. In many problems, Eulerian frames are mathematically enormously more convenient, so we nearly always write the conservation equations for a continuum in this system. It has the disadvantage that we are not always referring to the same material point. We can, however, transform between Eulerian and Lagrangian systems by using the fact that displacement and velocity at a laboratory point are the displacement and velocity of the material point that happens to be there.

To express in Eulerian field variables the change experienced by a moving material point, we must take into account not only the change with time of properties at a fixed point, but also the change of properties with position at a fixed time.

Reference

Almost any classical text in fluid mechanics has a brief discussion of this subject. Of these, perhaps the most extensive (though by no means generous) is that in:

Prandtl, L. and O. G. Tietjens, *Fundamentals of Hydro- and Aero-mechanics*, New York, Dover, 1957, 270 pp. S374 Paperbound $1.85.