Notes on
1.63 Advanced Environmental Fluid Mechanics
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November 22, 2002

6.6 Horton-Rogers-Lapwood instability in a porous layer

Nield & Bejan, Chapter 6 **Convection in Porous Media**
Related: Rayleigh-Bernard Problem (Chandrsekhar, Chapter II, **Hydrodynamic and Hydromagnetic Stability**)

If a layer of viscous fluid is heated from below, instability can occur and leads to convection cells important in meteorology. (Benard Problem).
If a saturated porous layer is heated from below, similar instabilily and convection can occur. This is of basic interest to geothermal convection and suggests more studies of the more complex problem of convection induced by buried nuclear waste.

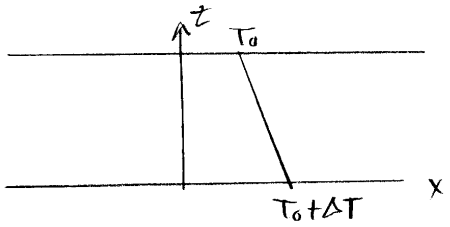


Figure 6.6.1: A saturated porus medium in geothermal gradient

Static state

$$\mathbf{u} = 0, \quad \frac{\partial}{\partial t} = 0$$

Hence

$$T_S = T_o + \Delta T \left(1 - \frac{z}{h}\right) \tag{6.6.1}$$

Static equilibrium:

$$\frac{dp_S}{dz} = -\rho_o g \left[T_o - \beta \Delta T \left(1 - \frac{z}{h}\right)\right]$$

hence

$$p_S = p_o - \rho_o g \left\{T_o z + \frac{\beta \Delta T}{2}\left(\frac{z^2}{h} - 2z\right)\right\} \tag{6.6.2}$$

Consider the perturbed state of small disturbances:

$$\mathbf{u} = 0 + \mathbf{u}', \quad T = T_S + T', \quad P = p_S + p' \tag{6.6.3}$$

then

$$\nabla \cdot \mathbf{u}' = 0 \tag{6.6.4}$$

$$0 = -\nabla p' - \frac{\mu}{k}\mathbf{u}' + \beta g \rho_o T' \mathbf{k} \tag{6.6.5}$$

$$(\rho C)_m \frac{\partial T'}{\partial t} + (\rho C)_f \mathbf{u}' \cdot \nabla T_S = K_m \nabla^2 T' \tag{6.6.6}$$

6.6.1 Non-dimensionalization

. Let $K_m, \kappa_m = K_m/(\rho C)_f$ be the conductivity and diffusivity of the mixture, and k the permeability. Define

$$\begin{aligned} &(x, y, z) \to h(x^*, y^*, z^*), \ t \to \frac{\sigma h^2}{\kappa_m} t^*, \\ &\mathbf{u}' \to \frac{\kappa_m}{h}\mathbf{u}^*, \ T' = \Delta T \theta, \ p' \to \frac{\mu \kappa_m}{k} p^* \end{aligned} \tag{6.6.7}$$

Then, after omitting * for brevity, we get

$$\nabla \cdot \mathbf{u} = 0 \tag{6.6.8}$$

$$0 = -\nabla p - \mathbf{u} + Ra\, \theta \mathbf{k} \tag{6.6.9}$$

$$\boxed{\frac{\partial \theta}{\partial t} - w = \nabla^2 \theta} \tag{6.6.10}$$

where

$$Ra = \frac{\rho_f g k \beta \Delta T h}{\mu \kappa_m} \quad \text{Rayleigh number in a porous medium} \tag{6.6.11}$$

is the Rayleigh number (ratio of buoyancy force to diffusive resistence) of the porous medium. In Benard's problem, Rayleigh number is defined as

$$Ra = \frac{\rho g \Delta T h^4}{\mu \kappa}, \quad \text{Rayleigh number in a pure fluid} \tag{6.6.12}$$

Sometimes one calls

$$D = \frac{k}{h^2}, \text{Darcy number} \tag{6.6.13}$$

Darcy number so that Rayleigh number of a porous medium is the product of the traditional Rayleigh number and Darcy number.

Taking the curl of (6.6.9),

$$\nabla \times \mathbf{u} = Ra(\mathbf{i}\theta_y - \mathbf{j}\theta_x) \tag{6.6.14}$$

Note that the z compoent of the vorticity vector is zero.

Taking the curl again and using

$$\nabla \times \nabla \times \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}$$

we get

$$\nabla^2 \mathbf{u} = -Ra\left[\mathbf{i}\theta_{xz} + \mathbf{j}\theta_{yz} - \mathbf{k}(\theta_{xx} + \theta_{yy})\right]$$

Taking the z component, we get

$$\boxed{\nabla^2 w = Ra\nabla_H^2 \theta} \tag{6.6.15}$$

where

$$\nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{6.6.16}$$

is the horizontal Laplacian.

Equations (6.6.10) and (6.6.15) couple the two unknowns w and θ. The boundary conditions are

$$w = \theta = 0, \quad z = 0, 1 \tag{6.6.17}$$

After they are solved the other velocity components and pressure can be found.

6.6.2 Solution for sinusoidal disturbances

Let

$$(w, \theta) = (W(z), \Theta(z)) \exp(ilx + imy - i\omega t) \tag{6.6.18}$$

and

$$D = \frac{d}{dz}$$

then from (6.6.10),

$$-i\omega\Theta - W = (D^2 - a^2)\Theta \tag{6.6.19}$$

and from (6.6.15)

$$(D^2 - a^2)W = -a^2 Ra\Theta \tag{6.6.20}$$

where

$$a^2 = \ell^2 + m^2 \tag{6.6.21}$$

The boundary conditions are :

$$W = \Theta = 0, \quad z = 0, 1 \tag{6.6.22}$$

Because the equations and the boundary conditons are homogeneous, the problem for W and θ is an eigenvalue problem.

Note that l, m, a are related to dimensional wave numbers by

$$l = k_x h = \frac{2\pi h}{L_x}, \quad m = k_y h = \frac{2\pi h}{L_y}, a = kh = 2\pi h \left(\frac{1}{L_x^2} + \frac{1}{L_y^2}\right)^{1/2} \quad \text{with} \quad k = \sqrt{k_x^2 + k_y^2} \tag{6.6.23}$$

6.6.3 Principle of exchange of stabilities

We shall first show that ω must be purely imaginary. Multiplying (6.6.20) by W^* and integrating from $z = 0$ to $z = 1$, we get, after partial integration and using the boundary conditions,

$$-\int_0^1 (|DW|^2 + a^2|W|^2)dz = -a^2 Ra \int_0^1 W^*\Theta dz \tag{6.6.24}$$

Similarly we multipy (6.6.19) by Θ^* and integrating from $z = 0$ to $z = 1$, and get

$$-\int_0^1 (|D\Theta|^2 + a^2|\Theta|^2)dz = -i\omega \int |\Theta|^2 dz - \int_0^1 W\Theta^* dz \tag{6.6.25}$$

Taking the complex conjugate of the second equation

$$-\int_0^1 (|D\Theta|^2 + a^2|\Theta|^2)dz = i\omega^* \int |\Theta|^2 dz - \int_0^1 W^*\Theta dz \tag{6.6.26}$$

Eqs (6.6.24) and (6.6.26) can be combined by eliminating the cross product terms, ,

$$-\int_0^1 (|D\Theta|^2 + a^2|\Theta|^2)dz = -i\omega^* \int |\Theta|^2 dz - \frac{1}{a^2 Ra}\int_0^1 (|DW|^2 + a^2|W|^2)dz \tag{6.6.27}$$

Since all integrals above are real, $-i\omega* = -\omega_i - i\omega_r$ must also be real. We conclude that

$$\omega_r = 0, \quad \text{hence} \quad -i\omega = -\omega_i \tag{6.6.28}$$

Marginal stability (the threshold of instability) occurs at $\omega_r = \omega_i = 0$. If $\omega_i > 0$, the static state is unstable; $\omega_i = 0$, marginally stable; if $\omega_i < 0$, stable. A problem where the eigenfrequency is real so that marginal instabililty occurs when $\omega = 0$ is said to obey the *principle of exchange of stabilities.*

6.6.4 Solution to eigenvalue problem

Consider the situation at marginal stability : $\omega = 0$,

$$(D^2 - a^2)W = -a^2 Ra\Theta \tag{6.6.29}$$

$$-W = (D^2 - a^2)\Theta \tag{6.6.30}$$

Eliminating Θ, we get

$$(D^2 - a^2)^2 W = a^2 RaW \tag{6.6.31}$$

subject to

$$W = 0, \quad D^2 W = 0, \quad z = 0, 1 \tag{6.6.32}$$

Expanding (6.6.31)

$$D^4 W - 2a^2 D^2 W + a^4 W = a^2 RaW \tag{6.6.33}$$

Clearly $D^4W = 0$ on $z = 0, 1$. Differentiating (6.6.33) twice we see that $D^6 = 0$ on $z = 0, 1$. Repeating the process we find

$$D^{(2m)} = 0, \quad m = 1, 2, 3, \cdots, \quad \text{on} \quad z = 0, 1 \tag{6.6.34}$$

Therefore the eigensolution must be

$$W \sim \sin j\pi z \tag{6.6.35}$$

To satisfy (6.6.31) it is necessary that

$$Ra = \frac{[j^2\pi^2 + a^2]^2}{a^2}, \quad \text{for } j = 1, 2, 3..., \tag{6.6.36}$$

which is the eigenvalue condition. For any j, Ra becomes unbounded for both $a^2 \to 0$ and $a^2 \to \infty$ and is curve concave upward in the plane of a^2(abscissa) vs, Ra (ordinate).

The lowest threshold occurs at $j = 1$, and

$$\frac{\partial Ra}{\partial a^2} = 0$$

i.e.,

$$a^2 = \pi^2 \tag{6.6.37}$$

or

$$Ra_c = 4\pi^2 = 39.48 \tag{6.6.38}$$

6.6.5 Possible convection patterns

This is similar to Benard's problem which has been exhaustively studied theoretically and experimentally. There are many possibilities. Let us consider the lowest mode only with $j = 1$.

2-Dimensinal Rolls : $(\ell = \pi, m = 0)$

Take

$$w = \cos \pi x \sin \pi z \tag{6.6.39}$$

then from mass conservation,

$$u = -\sin \pi x \cos \pi z \tag{6.6.40}$$

The dimensionless wavelength is $L_x = L_y = 2$. Along lines $x = 0, \pm n, n = 1, 2, 3, 4, ...$, $u = 0$ but $w \neq 0$. Along $x = 0, \pm 2m$, $w > 0$, hence fluid rises verrically. Along $x = \pm 2m - 1$, $w < 0$ hence fluids sinks vertically. Along $z = 0$ and $1, w = 0$. On the bottom ($z = 0$), $u > 0$ while on the top ($z = 1$), if $0 < x < 1, 2 < x < 3, 4 < x < 5, ...$. The streamlines are shown in Figure 6.6.2.

Experimetal images of rolls in a pure fluid (Benard problem) is shown in Figure 6.6.3.

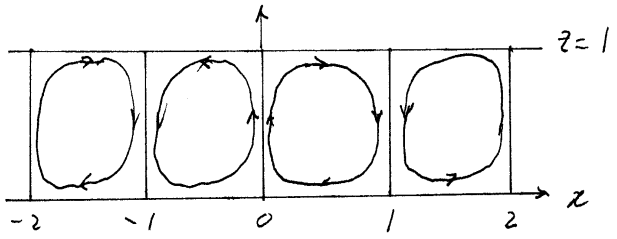


Figure 6.6.2: Rolls in a period

Rectangular cells: $(\ell = m = \pi/\sqrt{2})$.

$$w = \cos\frac{\pi x}{\sqrt{2}}\cos\frac{\pi y}{\sqrt{2}}\sin\pi z \tag{6.6.41}$$

From the z component of the vorticity equation (6.6.14),

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{6.6.42}$$

and from continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z} \tag{6.6.43}$$

By cross differentiation, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 w}{\partial z \partial y} = \frac{\pi^2}{2\sqrt{2}}\sin\frac{\pi x}{\sqrt{2}}\cos\frac{\pi y}{\sqrt{2}}\cos\pi z \tag{6.6.44}$$

and

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 w}{\partial z \partial x} = \frac{\pi^2}{2\sqrt{2}}\cos\frac{\pi x}{\sqrt{2}}\sin\frac{\pi y}{\sqrt{2}}\cos\pi z \tag{6.6.45}$$

These are easily solved to give

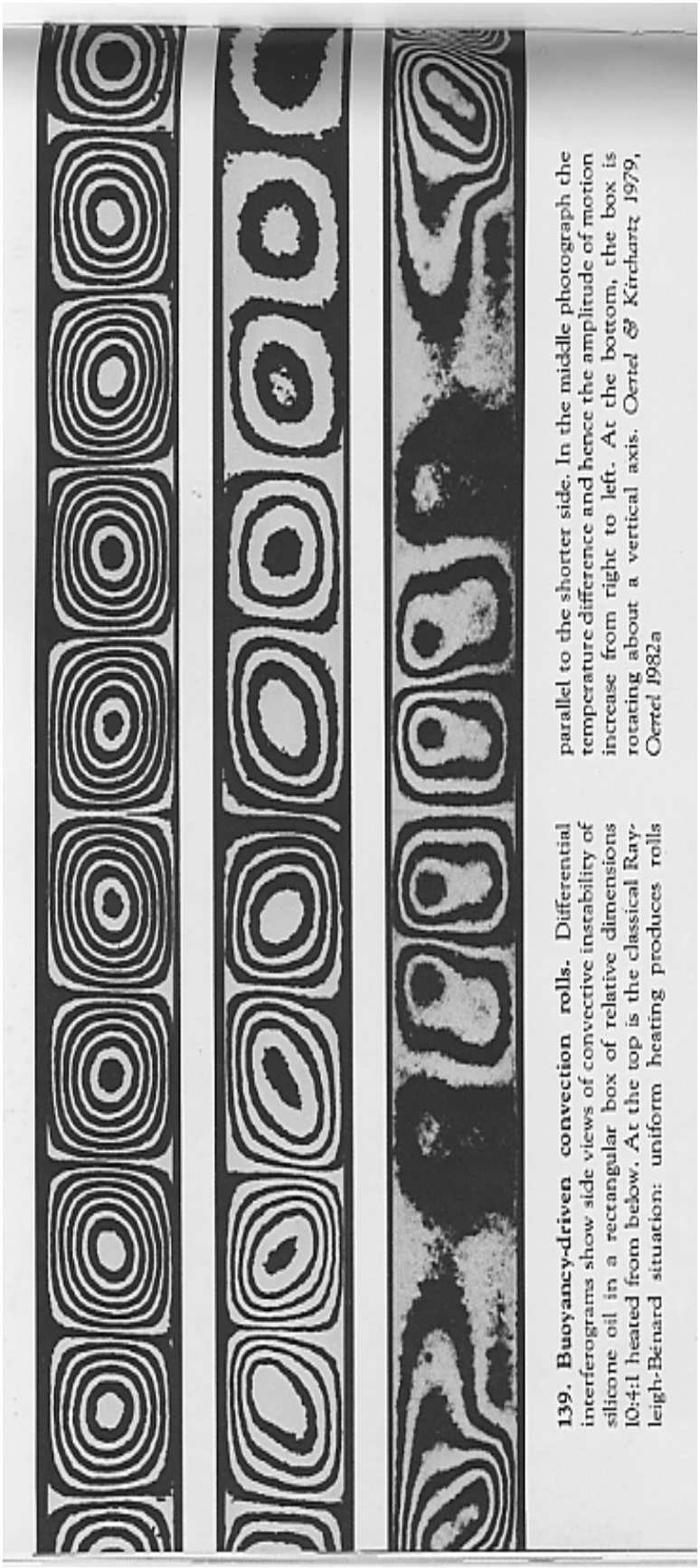
$$u = -\frac{1}{\sqrt{2}}\sin\frac{\pi x}{\sqrt{2}}\cos\frac{\pi y}{\sqrt{2}}\cos\pi z \tag{6.6.46}$$

and

$$v = -\frac{1}{\sqrt{2}}\cos\frac{\pi x}{\sqrt{2}}\sin\frac{\pi y}{\sqrt{2}}\cos\pi z \tag{6.6.47}$$

The streamlines in a horizontal plane is shown in Figure 6.6.4.

Hexagonal cells: See Chandrasekhar. See Figure 6.6.5.



139. Buoyancy-driven convection rolls. Differential interferograms show side views of convective instability of silicone oil in a rectangular box of relative dimensions 10:4:1 heated from below. At the top is the classical Rayleigh-Bénard situation: uniform heating produces rolls parallel to the shorter side. In the middle photograph the temperature difference and hence the amplitude of motion increase from right to left. At the bottom, the box is rotating about a vertical axis. *Oertel & Kirchartz 1979, Oertel 1982a*

Figure 6.6.3: Rolls in a pure fluid layer heated from below. From van Dyke: **An Album of Fluid Motion**

The streamlines of the motion in the horizontal plane are determined by the equation

$$\frac{dx}{dy} = \frac{u}{v} = \frac{L_y}{L_x}\frac{[\sin(2\pi x/L_x)][\cos(2\pi y/L_y)]}{[\cos(2\pi x/L_x)][\sin(2\pi y/L_y)]}. \tag{246}$$

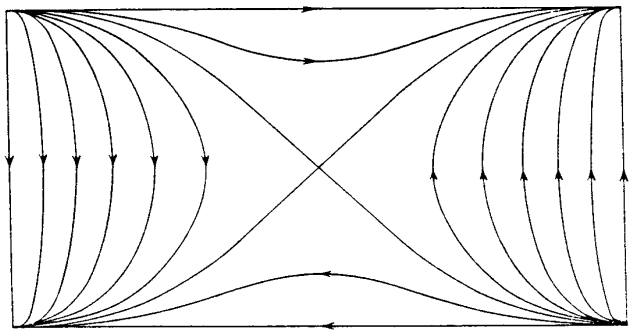


FIG. 5. The streamlines in the horizontal plane for a rectangular cell.

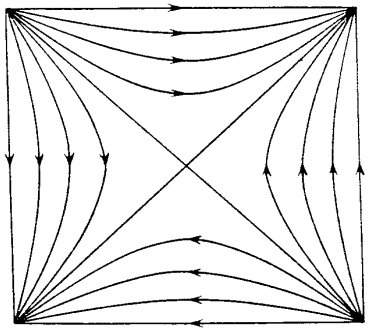
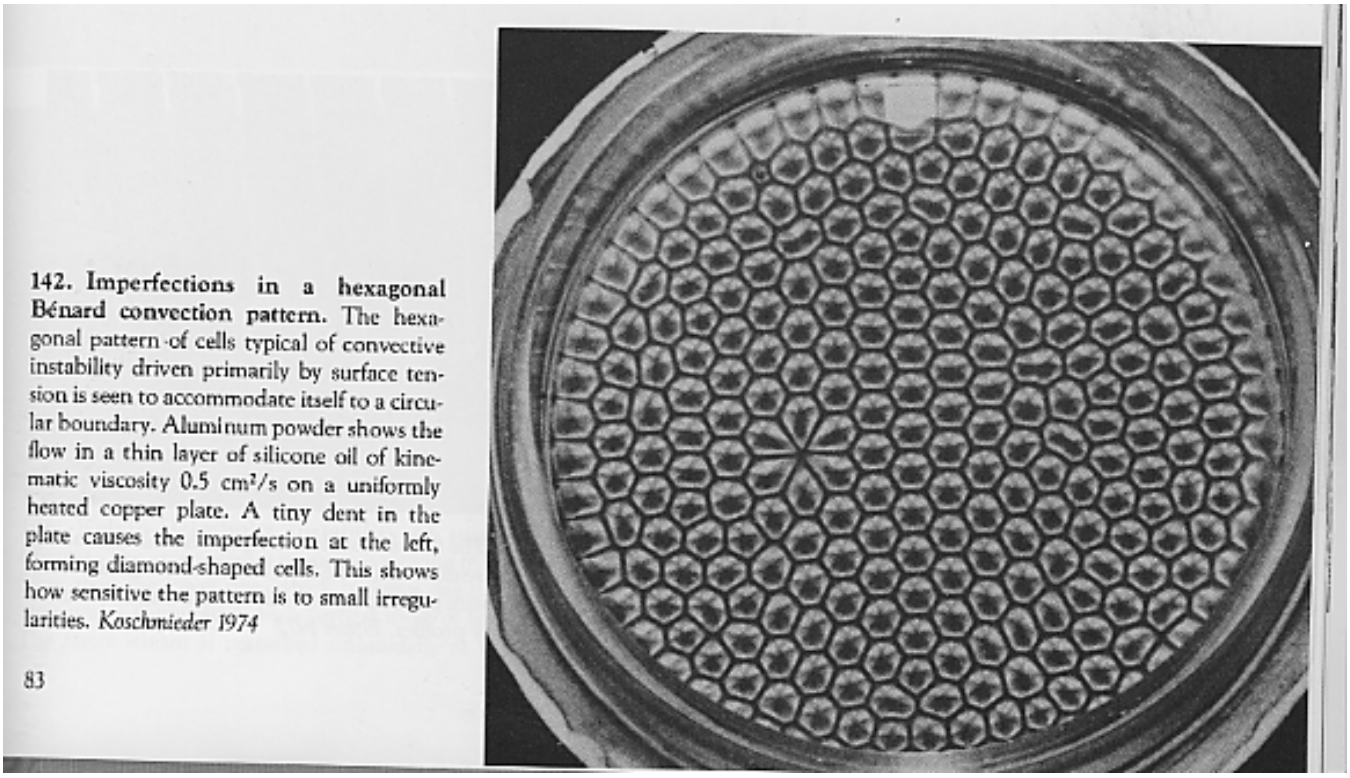


FIG. 6. The streamlines in the horizontal plane for a square cell.

Figure 6.6.4: Rectangular and square cells in a pure fluid heated from below. From Chandrasekhar



142. Imperfections in a hexagonal Bénard convection pattern. The hexagonal pattern of cells typical of convective instability driven primarily by surface tension is seen to accommodate itself to a circular boundary. Aluminum powder shows the flow in a thin layer of silicone oil of kinematic viscosity 0.5 cm^2/s on a uniformly heated copper plate. A tiny dent in the plate causes the imperfection at the left, forming diamond-shaped cells. This shows how sensitive the pattern is to small irregularities. *Koschmieder 1974*

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Figure 6.6.5: Hexagonal cells in a pure fluid headted from below. From van Dyke