Notes on
1.63 Advanced Environmental Fluid Mechanics
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Ref]: C S Yih, *Dynamics of Nonhomogeneous Fluids*, Macmillan, 1965).
Nield & Bejan: *Convection in Porous Media*

6.5 Geothermal Plume

Consider a steady, two dimensional plume due to a line source of intense heat in a porous medium. From Darcy's law:

$$\frac{\mu}{k}u = -\frac{\partial p}{\partial x} \tag{6.5.1}$$

and

$$\frac{\mu}{k}w = -\frac{\partial p}{\partial z} - \rho g \tag{6.5.2}$$

These are the momentum equations for slow motion in porous medium. Mass conservation requires

$$u_x + w_z = 0 \tag{6.5.3}$$

Energy conservation requires

$$u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = \gamma\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{6.5.4}$$

Equation of state:

$$\rho = \rho_0\left(1 - \beta(T - T_0)\right) \tag{6.5.5}$$

Consider the flow induced by a heat source. Let

$$\vec{q} = (u, w), \quad T = T_0 + T', \quad p = p_o + p' \tag{6.5.6}$$

where p_0 is the hydrostatic pressure satisfying

$$-\frac{\partial p_0}{\partial z} - \rho_0 g = 0.$$

and p', T' are dynamical perturbations. Therefore, (6.5.2) can be written

$$\frac{\mu}{k}w = -\frac{\partial p'}{\partial z} + g\rho_0\beta T'. \tag{6.5.7}$$

Eqn. (6.5.1) can be written

$$\frac{\mu}{k}u = -\frac{\partial p'}{\partial x}. \tag{6.5.8}$$

$$\frac{\mu}{k}w = -\frac{\partial p'}{\partial z} - \rho' g \tag{6.5.9}$$

6.5.1 Boundary layer approximation

Eliminating p' from Eqns. (6.5.7) and (6.5.8), we get

$$\frac{\mu}{k}\left(w_x - u_z\right) = g\rho_0\beta T'_x.$$

Let

$$u = \psi_z, \quad w = -\psi_x$$

then

$$\psi_{xx} + \psi_{zz} = -\frac{g\rho_0\beta k}{\mu}T'_x \tag{6.5.10}$$

For an intense heat source, we expect the plume to be narrow and tall. Let us apply the boundary layer approximation and check its realm of validity later,

$$u \ll w, \qquad \frac{\partial}{\partial x} \gg \frac{\partial}{\partial z}.$$

hence

$$\psi_{xx} \cong -\frac{g\rho_0\beta k}{\mu}T'_x$$

or

$$-\frac{\mu}{k}\psi_x \cong g\rho_0\beta T', \tag{6.5.11}$$

which is the same as ignoring $\partial p'/\partial x$ in Eqn. (6.5.7).

Alternatively , since $u \ll w \;\; \partial p'/\partial x \approx 0$, we muast have $p' \cong p'$ outside the plume. But

$$\frac{\partial p'}{\partial z} = 0$$

outside the plume, hence $\partial p'/\partial z \approx 0$ inside as well.

Applying the B.L. approximation on Eqn. (6.5.4)

$$uT'_x + wT'_z = \gamma T'_{xx} \tag{6.5.12}$$

where λ denotes the thermal difusivity. Using the continuity equation we get

$$(uT')_x + (wT')_z = \gamma T'_{xx}.$$

Integrating across the plume,

$$\frac{\partial}{\partial z}\rho_0 C\int_{-\infty}^{\infty} wT'\,dx = 0 \tag{6.5.13}$$

since $T' = 0$ outside the plume. It follows that

$$\rho_o C\int_{-\infty}^{\infty} wT'\,dx = -\rho_0 C\int_{-\infty}^{\infty} \psi_x T'\,dx = Q = \text{constant}. \tag{6.5.14}$$

6.5.2 Similarity solution

Now let

$$x = \lambda^a x^* \quad z = \lambda^b z^* \quad \psi = \lambda^c \psi^* \quad T' = \lambda^d T^*.$$

From Eqn. (6.5.11)

$$\frac{\mu}{k}\lambda^{c-a}\left(\frac{\partial \psi^*}{\partial x^*}\right) = -g\rho_0\beta\lambda^d T^*.$$

For invariance we require,

$$c - a = d. \tag{6.5.15}$$

$$-\int \frac{\partial \psi^*}{\partial x^*}\frac{T^*}{T_0}\,dx^*\,\lambda^{c-a+a+d} = Q.$$

Therefore,

$$c + d = 0. \tag{6.5.16}$$

From Eqn. (6.5.12)

$$\lambda^{c+d-a-b} = \lambda^{d-2a}.$$

implying,

$$c + a - b = 0. \tag{6.5.17}$$

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$$c = \frac{a}{2}, \quad d = -\frac{a}{2}, \quad b = \frac{3}{2}a.$$

In view of these we introduce the following similarity variables,

$$\begin{aligned} \eta &= \left(\frac{g}{\nu^2}\right)^{1/9}\frac{x}{z^{2/3}} \\ \frac{\psi}{Q} &= \left(\frac{g}{\nu^2}\right)^{1/9} z^{1/3} f(\eta) \\ \frac{T'}{T_0} &= \left(\frac{g}{\nu}\right)^{1/9} z^{-1/3}\, h(\eta). \end{aligned} \tag{6.5.18}$$

Note that at the center line $\eta = 0$

$$w = -\psi_x \propto z^{1/3} f'(0)(-)z^{-2/3} \sim z^{-1/3} f'(0) \sim z^{-1/3} \tag{6.5.19}$$

$$T' \propto z^{-1/3} h(0) \tag{6.5.20}$$

and

$$b \propto z^{2/3} \tag{6.5.21}$$

Thus the velocity and temperature along the centerline decay as $z^{-1/3}$ and the plume width grows as $z^{2/3}$.

Substituting these into Eqns. (6.5.4) and (6.5.11), we get, after some algebra

$$-\frac{df}{d\eta} = Ah \tag{6.5.22}$$

where

$$A = \frac{T_0 \beta k}{Q} \left(\frac{g^2}{\nu} \right)^{1/3} \tag{6.5.23}$$

and

$$-\frac{d}{d\eta}(fh) = \frac{3\gamma}{Q} \frac{d^2 h}{d\eta^2}. \tag{6.5.24}$$

The boundary conditions are,

$$\begin{aligned} f &= 0 \qquad (\psi = 0) \\ f''(0) &= 0, \quad (w(0,z) = w_{max}) \end{aligned}$$

along the center line, and

$$\begin{aligned} f'(\pm\infty) &= 0 \\ h(\pm\infty) &= 0. \end{aligned}$$

Integrating Eqn. (6.5.24), we get

$$-fh = \frac{3\gamma}{Q} h'.$$

Using Eqn. (6.5.23), we get

$$-ff' = \frac{3\gamma}{Q} f''.$$

Integrating again, we get

$$\frac{6\gamma f'}{Q} = f_0^2 - f^2$$

where $f_0 = f_{\text{max}}$. Let $f = f_0 F$, then

$$1 - F^2 = \frac{6\gamma}{Q f_0} F'$$

which can be integrated to give

$$\frac{Q f_0}{6k}\eta = \frac{1}{2} \ln \frac{1+F}{1-F} = \tan h^{-1} F \qquad 0 \le F < 1.$$

Therefore,

$$F(\eta) = \tanh \frac{Q f_0}{6\gamma}\eta.$$

What is f_0? Use Eqn. (6.5.14)

$$-\int_{-\infty}^{\infty} \frac{df}{d\eta} h \, d\eta = 1$$

since

$$f' = f_0 F' = \frac{Q f_0^2}{6\gamma} \text{sech}^2 \frac{Q f_0}{6\gamma}\eta$$

and

$$h = -\frac{1}{Q} f'.$$

Therefore,

$$\frac{1}{Q}\left(\frac{Qf_0^2}{6\gamma}\right)^2 \int_{-\infty}^{\infty} \sec h^4 \left(\frac{Qf_0}{6\gamma}\eta\right) d\eta = \frac{Q^2 f_0^3}{6kT_0\beta\gamma}\left(\frac{\nu}{g}\right)^{1/3} \int_{-\infty}^{\infty} \mathrm{sech}^4 z dz = 1.$$

since

$$\int_{-\infty}^{\infty} \mathrm{sech}^4 z dz = 4/3.$$

This determines f_0!

$$f_0 = \left(\frac{9\gamma T_0 \beta k}{2Q^2}\right)^{1/3} \left(\frac{g^2}{\nu}\right)^{1/9} \tag{6.5.25}$$

The solution is plotted in Figure 6.5.1 and Figure 6.5.2, after defining

$$\frac{\psi}{\gamma} = 6Z^{1/3} \tanh \frac{X}{Z^{2/3}} \tag{6.5.26}$$

$$\frac{T}{T_0}\frac{Q}{f_0}\left(\frac{6\gamma}{Qf_0}\right)^2 = \frac{1}{Z^{1/3}} \mathrm{sech}^2 \left(\frac{X}{Z^{2/3}}\right) \tag{6.5.27}$$

where

$$X = \left(\frac{Qf_0}{6\gamma}\right)^3 \left(\frac{g}{\nu^2}\right)^{1/3} x, \quad Z = \left(\frac{Qf_0}{6\gamma}\right)^3 \left(\frac{g}{\nu^2}\right)^{1/3} z \tag{6.5.28}$$

The similarity variable is

$$\frac{X}{Z^{2/3}} = \left(\frac{Qf_0}{6\gamma}\right)\left(\frac{g}{\nu^2}\right)^{1/6} \frac{x}{z^{2/3}}$$

Hence the plume width decreases with increasing

$$\left(\frac{Qf_0}{6\gamma}\right)\left(\frac{g}{\nu^2}\right)^{1/6} = \frac{Q}{6\gamma}\left(\frac{9\gamma T_0\beta k}{2Q^2}\right)^{1/3}\left(\frac{g^2}{\nu}\right)^{1/9} \propto Q^{1/3}$$

i.e., increasing heat source strength.

6.5.3 Checking the boundary layer approximation.

$$\frac{\partial^2 \psi}{\partial x^2} \sim z^{-1}, \quad \frac{\partial^2 \psi}{\partial z^2} \sim z^{-5/3}$$

$$\frac{\partial^2 T'}{\partial x^2} \sim z^{-5/3}, \quad \frac{\partial^2 T'}{\partial z^2} \sim z^{-7/3}$$

hence for large z, B. L. approximation is good.

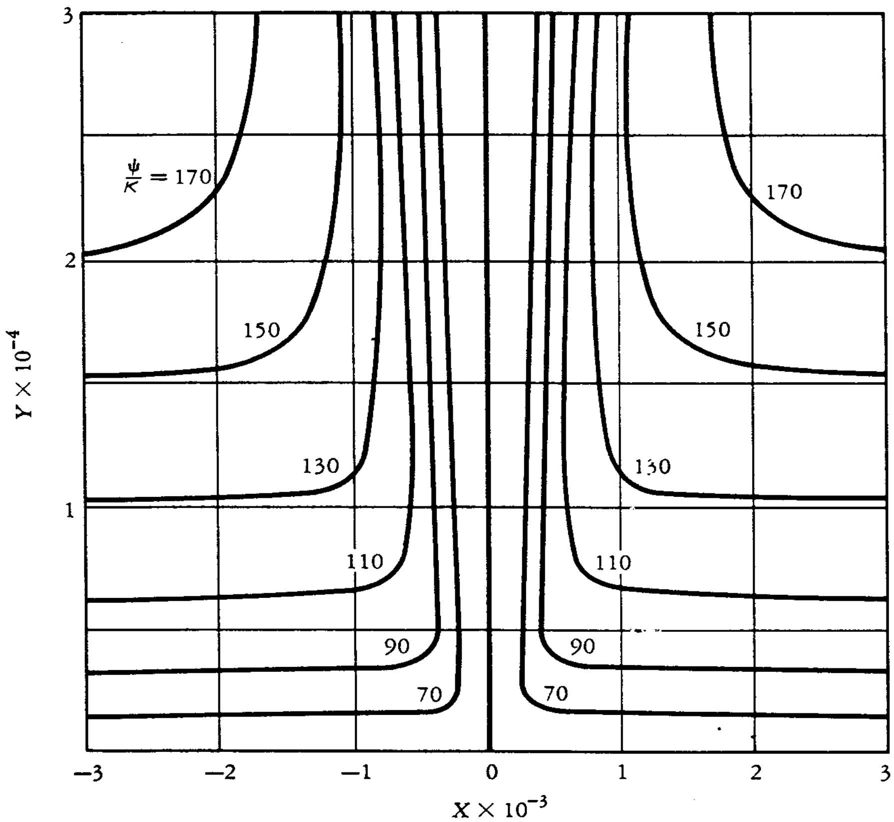

FIGURE 63. Pattern of two-dimensional convection in a porous medium from a boundary source.

Figure 6.5.1: Theoretical solution for a geothermal plume due to Yih

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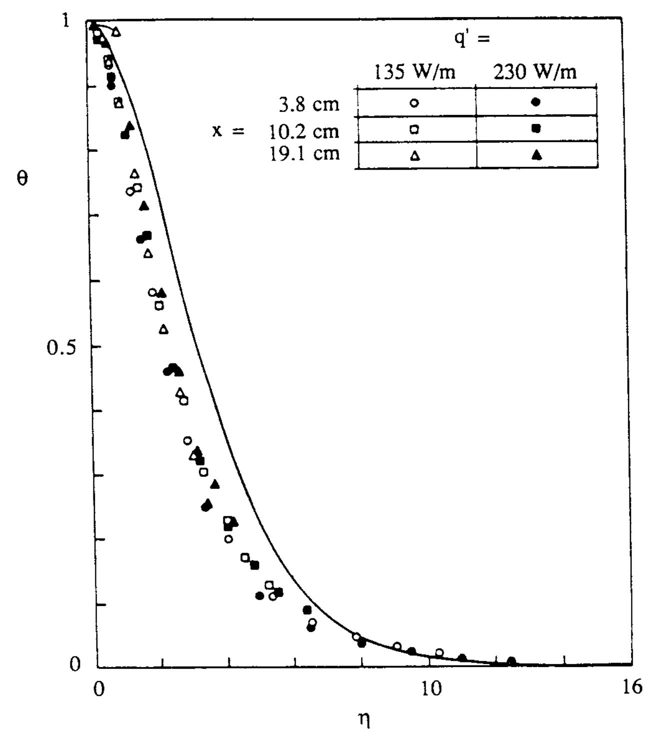

Figure 5.19 Dimensionless temperature profiles for plume rise above a horizontal line source of heat in a porous medium (Lee, 1983, Cheng, 1985a, with permission from Hemisphere Publishing Corporation).

Figure 6.5.2: Comparison of theory and experiment. From Nield and Bejan